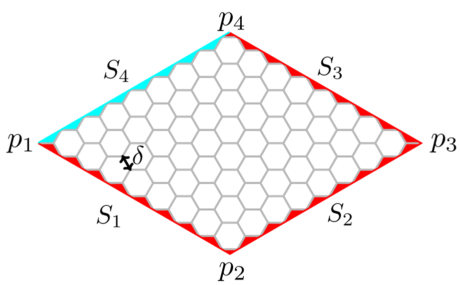
Exercises Day 1

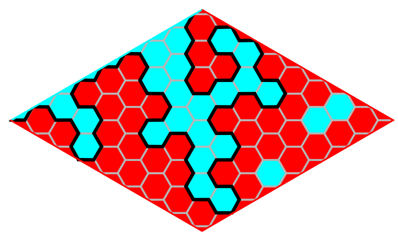
Exercise 1. *(See figures below) In this and a later exercise, we will argue why* SLE(6) *is a natural scaling limit for a percolation interface. Denote by* Λ *the hexagonal lattice, and let* Ω *be a rhombus with corners* p_1, p_2, p_3, p_4 *counterclockwise. Denote by* $\Omega^\delta := \Omega \cap \delta\Lambda$ *the hexagonal* δ*-discretization of* Ω*. Consider a site percolation on* Ω^δ *with Dobrushin boundary conditions:*

- *Color the hexagons on* S_4 *blue, and the hexagons on* S_1, S_2, S_3 *red.*
- *Color each interior hexagon independently either red or blue with equal probability.*

Denote by γ^δ *the interface between the blue and red hexagons started from the point* p_1*.*



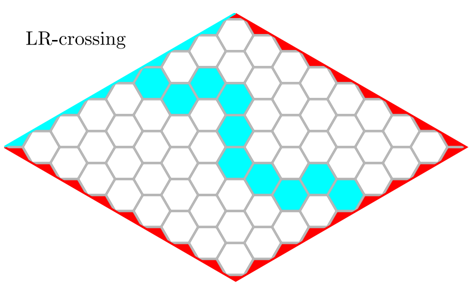
(A) Rhombus Ω and its hexagonal discretization Ω^δ with Dobrushin boundary conditions.



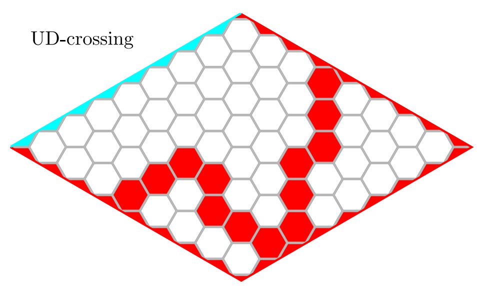
(B) A sample of percolation on Ω^δ. The highlighted interface is the curve γ^δ.

(a) An LR-crossing (UD-crossing) is a path of blue (red) hexagons from S_2 *to* S_4 *(*S_1 *to* S_3*). Show that every configuration has either an LR or UD crossing, but not both. Find a bijection between configurations with LR-crossings and UD-crossings. From this, conclude that both events have equal probability equal to* $\frac{1}{2}$*:*

$$P(\{\textit{There is an LR-crossing}\}) = P\left(\{\textit{There is a UD-crossing}\}\right) = 1/2.$$



(A) LR-crossing



(B) UD-crossing

(b) Show that* γ^δ *is a curve from* p_1^δ *to* p_4^δ*, where* p_j^δ *is a vertex in* Ω^δ *closest to* p_j*. Show that* γ^δ *hits the boundary* S_2 *if and only if there is an LR-crossing.*

(c) Assuming γ^δ *has a conformally invariant scaling limit as* $\delta \downarrow 0$*, argue why it has to be an* SLE(κ) *curve in* Ω *from* p_1 *to* p_4 *for some* $\kappa \geq 0$*. Hint: formulate and check a discrete version of the domain Markov Property for* γ^δ*.*

Solution. (a) We show that if one does not have UD-crossing than one does have an LR-crossing (the other case follows analogously). Suppose that there is no UD-crossing then there is no connected paths of red hexagons connecting S_1 and S_3. But since there are only two colors there must be a connected cluster of blue hexagons from S_4 to S_2. But this gives the existence of a LR-crossing.

Since there is either a LR or an UD-crossing it suffices to show that we can pair each LR crossing with exactly one UD crossing. Before giving the proof, we advice the reader to check the image below. Without loss of generality suppose that we are given a LR-crossing. First switch any color in the interior of the rhombus. This gives a red LR-crossing. Next reflect the rhombus around the straight line connecting p_1^δ and p_3^δ, which yields a (uniquely determined) red UP-crossing. Checking the image again we see that we can reverse this process for an UD-crossing and this finishes ((a)).

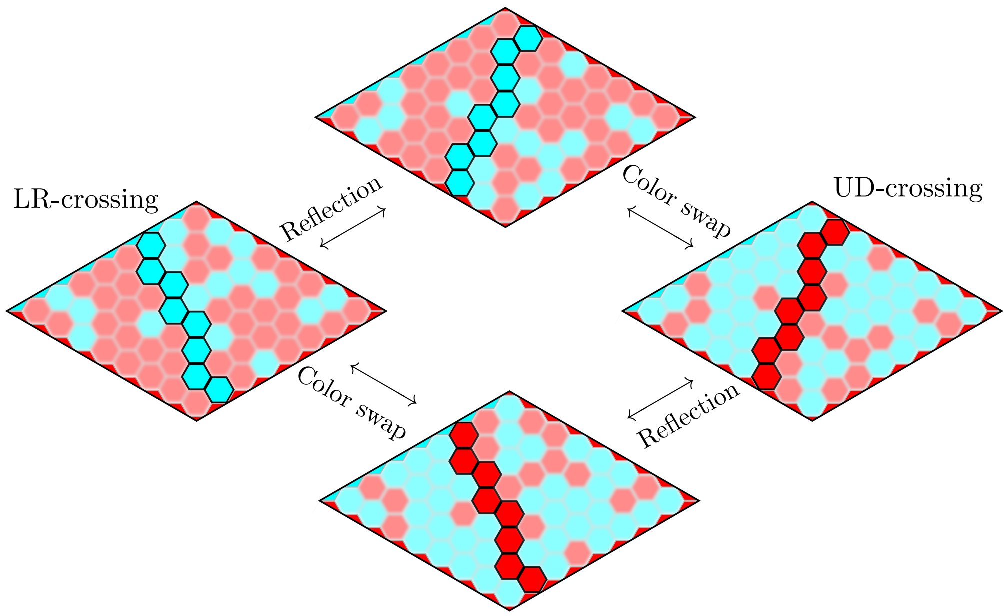


FIGURE 3. Starting with a LR crossing configuration after color swapping and reflecting we end up with one UD-crossing configuration. Also starting with a UD-crossing we can reverse the process to get one LR-crossing.

(b*) Since p_1 and p_4 are contained in the same boundary component of the cluster of blue hexagons (denoted by C) containing S_4, there exists a path $\tilde{\gamma}^\delta$ from p_1 to p_4 along the boundary of C. In particular, on the left side of $\tilde{\gamma}^\delta$ we have blue hexagons from C, hence, since $\tilde{\gamma}^\delta \subset \partial C$, on the right we must not have blue hexagons, hence they are red. This shows that $\tilde{\gamma}^\delta$ is the interface between blue and red hexagons started at p_1, i.e. $\tilde{\gamma}^\delta = \gamma^\delta$. In particular, this implies that $\gamma^\delta \subset \partial C$ hits S_2 if and only if ∂C intersects S_2, which is equivalent of existence of LR-crossing.
(c) Recall that SLE(κ)-curves are characterized by conformal invariance and domain Markov property. Hence, assuming conformal invariance of the scaling limit only the domain Markov property needs to be checked. More precisely, denote by $\Lambda(\Omega^\delta, p_1, p_4)$ the law of the interface curves γ^δ connecting p_1 and p_4. We say that the law of the interface satisfies the domain Markov property if the following holds. Stop the exploration of the interface γ at any of its points p' and call γ' the part of the interface starting at p_1 up to p'. Now the domain Markov property holds if

$$\Lambda(\Omega^\delta, p_1, p_4)\big|\gamma' = \Lambda(\Omega^\delta \setminus \gamma', p', p_4)$$

in other words the conditional law of what remains to be discovered after p' is the same as law of the interface in the remaining slit domain $\Omega^\delta \setminus \gamma'$. Therefore, conditioning γ^δ to γ' is equivalent to conditioning the percolation on $\Omega^\delta \setminus \gamma'$ only to have blue hexagons on the left and red hexagons on the right of γ'; the rest hexagons are colored independently as before. Therefore, the remaining interface $\gamma^\delta \setminus \gamma'$ has exactly the same law as $\Lambda(\Omega^\delta \setminus \gamma', p', p_4)$.

The Virasoro algebra $\mathfrak{Vir}$ is an infinite dimensional Lie algebra with generators $L_k, k \in \mathbb{Z}$ and C satisfying the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}n(n^2-1)\delta_{n,-m}C, \quad \text{for } n, m \in \mathbb{Z},$$
$$[L_n, C] = 0.$$

A $\mathfrak{Vir}$-module V is a highest-weight module if

$$V = \mathfrak{Vir}\, v_{c,h}$$

where $v_{c,h} \in V$ is a highest-weight vector of weight $h \in \mathbb{C}$ and central charge $c \in \mathbb{C}$ which satisfies the following relations:

$$Cv_{c,h} = cv_{c,h}, \qquad L_0 v_{c,h} = hv_{c,h}, \qquad L_n v_{c,h} = 0 \quad \forall n > 0,$$

Such a module admits a PBW-basis

$$V = \operatorname{span}\{L_{-n_1} \dots L_{-n_k} v_{c,h} \mid n_1 \geq n_2 \geq \dots \geq n_k > 0, k \in \mathbb{Z}_{\geq 0}\}$$

Exercise 2. *Show that the L_0 eigenvalue of a given basis vector $v = L_{-n_1} \dots L_{-n_k} v_{c,h} \in V$ is given as follows*

$$L_0 v = \left(h + \sum_{i=1}^{k} n_i\right) v.$$

Solution. We show this via induction on k. For $k = 0$ we have $v = v_{c,h}$, so by assumption we get $L_0 v = hv$, proving the base case. Now suppose that the claim holds for an arbitrary but fixed $k \in \mathbb{Z}_{\geq 0}$ and denote $v = L_{-n_1} \dots L_{-n_{k+1}} v_{c,h}$. Note that we have

$$L_0 L_{-n_1} = \big[L_0, L_{-n_1}\big] + L_{-n_1} L_0 = n_1 L_{-n_1} + L_{-n_1} L_0.$$

Using the calculation above one has

$$L_0 L_{-n_1} \dots L_{-n_{k+1}} = n_1 L_{-n_1} \dots L_{-n_{k+1}} v_{c,h} + L_{-n_1} L_0 \underbrace{L_{-n_2} \dots L_{-n_{k+1}} v_{c,h}}_{:=w}.$$

Now using the induction hypothesis on w together with the identity $v = L_{-n_1} w$ yields that

$$L_0 v = n_1 v + L_{-n_1} \left(h + \sum_{i=2}^{n+1} n_i\right) w = \left(h + \sum_{i=1}^{n+1}\right) v,$$

and this finishes the proof.

Exercise 2 shows that we can write $V = \bigoplus_{\ell \in \mathbb{Z}_{\geq 0}} V_\ell$, where

$$V_\ell = \{v \in V : L_0 v = (h + \ell) v\}$$

is the $(h+\ell)$-eigenspace of L_0. Vectors in V_ℓ are called ℓ-level vectors. An ℓ-level vector $w \in V_\ell$ is called singular if it satisfies

$$L_k w = 0 \qquad \forall k \in \{1, \dots, \ell\}.$$

Exercise 3. *The goal of this and a later exercise is to derive the BPZ equation for the correlation functions of a null field at level 2 and relate them to SLE-theory. Fix the central charge $c \in \mathbb{C}$.*

(a) Using the commutation relations show that $w = L_{-1} v_{c,h}$ is a singular vector at level 1 iff $h = 0$.

(b) *Show that the generic (up to scaling) two-level vector* $w_{c,h} = \left(L_{-2} + aL_{-1}^2\right) v_{c,h}$ *is singular if and only if*

$$h = h_\pm := \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right) \qquad \text{and} \qquad a = a_\pm := \frac{-3}{2\,(2h_\pm + 1)}.$$

(c) *Writing* $a_- = \frac{\kappa}{4}$, *show that* c *and* h_- *can be expressed in terms of* κ *as* $c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$ *and* $h_- = \frac{6-\kappa}{2\kappa}$.

Solution. (a) We already know by the previous exercise that $L_0 w = (1+h)\,w$ and so it is left to determine h by the condition that w is singular iff

$$L_1 w = 0.$$

Using the commutator we rewrite as follows

$$L_1 L_{-1} v_{c,h} = \left(\left[L_1, L_{-1}\right] + L_{-1}L_1\right) v_{c,h} = 0,$$

which from Virasoro algebra commutation relations is given by

$$2L_0 v_{c,h} + L_{-1}L_1 v_{c,h} = 0.$$

Now using the conditions $L_0 v_{c,h} = h v_{c,h}$ and $L_n v_{c,h} = 0$ for $n > 0$ yields that w is singular iff

$$2h v_{c,h} = 0,$$

and this yields that $h = 0$.

(b) We need to find all $a, h \in \mathbb{C}$ such that w satisfies the condition of being singular, i.e.

$$L_k\Big(\left(L_{-2} + aL_{-1}^2\right) v_{c,h}\Big) = 0 \quad k \in \{1, 2\}. \tag{1}$$

Recall that that for any $k > 0$ we have $L_k v_{c,h} = 0$, hence we get

$$\begin{aligned}
L_k L_{-2} v_{c,h} &= [L_{-2}, L_k] v_{c,h} + L_{-2}\underbrace{L_k v_{c,h}}_{=0} \\
&= \left((k+2)L_{k-2} - \frac{C}{2}\delta_{k,2}\right) v_{c,h}, \\
L_k L_{-1}^2 v_{c,h} &= [L_k, L_{-1}^2] v_{c,h} + L_{-1}^2\underbrace{L_k v_{c,h}}_{=0} \\
&= (L_{-1}[L_k, L_{-1}] + [L_k, L_{-1}]L_{-1}) v_{c,h} \\
&= (k+1)(L_{-1}L_{k-1} + L_{k-1}L_{-1}) v_{c,h} \\
&= (k+1)(2L_{-1}L_{k-1} + [L_{k-1}, L_{-1}]) v_{c,h} \\
&= (k+1)(2L_{-1}L_{k-1} + kL_{k-2}) v_{c,h}.
\end{aligned}$$

Plugging the above to the equation (1) for $k = 1$ and $k = 2$ we get

$$\begin{aligned}
\big(3L_{-1} + 2a(2L_{-1}L_0 + L_{-1})\big) v_{c,h} &= 0 \qquad (k = 1), \\
\big(4L_0 - \frac{C}{2} + 3a(2L_{-1}L_1 + 2L_0)\big) v_{c,h} &= 0 \qquad (k = 2).
\end{aligned}$$

Recalling that under Virasoro action one has $L_1 v_{c,h} = 0$, $L_0 v_{c,h} = h v_{c,h}$, and $C v_{c,h} = c v_{c,h}$, the above equations become

$$\begin{aligned}
\big(3 + 2a(2h+1)\big) L_{-1} v_{c,h} &= 0 \qquad (k = 1), \\
\big(4h - \frac{c}{2} + 6ah\big) v_{c,h} &= 0 \qquad (k = 2).
\end{aligned}$$

Since $L_{-1}v_{c,h}$ and $v_{c,h}$ are non-zero vectors, the coefficients in front of them must be zero for the above equations to hold. Solving a and h in terms of c yields

$$h = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right) \quad \text{and} \quad a = \frac{-3}{2\,(2h+1)}.$$

Thus we conclude (as expected) that there are two singular vectors h_+, h_- at level 2 that are given by choosing the corresponding sign in the formula for h, namely

$$w_+ = \left(L_{-2} - \frac{3}{2\,(h_+ + 1)} L_{-1}^2\right) v_{c,h_+}, \tag{2}$$

and

$$w_- = \left(L_{-2} - \frac{3}{2\,(h_- + 1)} L_{-1}^2\right) v_{c,h_-}. \tag{3}$$

This finishes (b).
(c) is just a straightforward exercise of algebraic manipulation.