EXERCISES 1 - Feynman Integrals and Number Theory

Les Diablerets, 08/01/2024

Let G be a scalar Feynman graph. We denote by V_G the collection of its vertices and by E_G the collection of its internal edges. We fix $m_e \in \mathbb{R}_{\geq 0}$, $\forall e \in E_G$, and $q_i \in \mathbb{R}^D$, $i = 1, \dots, n$, where n is the number of external legs and D is the spacetime dimension. By global momentum conservation, $\sum_{i=1}^{m} q_i = 0$. We denote the Schwinger parameters by α_e, $e \in E_G$. h_G is the number of independent loops of G. Recall the definitions:

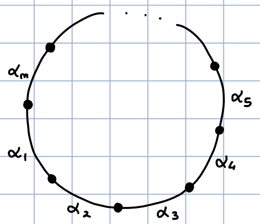
1. $$\psi_G = \sum_{\substack{T \subseteq G \\ \text{spanning} \\ \text{1-tree}}} \prod_{e \notin T} \alpha_e$$ Kirchhoff (or 1st Symanzik) graph polynomial

2. $$\phi_G(q) = \sum_{\substack{T_1 \cup T_2 \subseteq G \\ \text{spanning} \\ \text{2-tree}}} (-1)\, q^{T_1} \cdot q^{T_2} \prod_{e \notin T_1 \cup T_2} \alpha_e$$ 2nd Symanzik graph polynomial

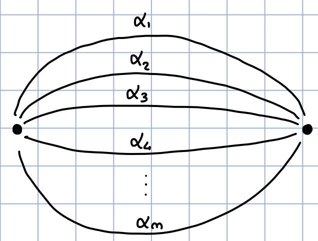
   where $q^{T_1} = - q^{T_2}$ is the total momentum entering T_1 and $q_1 \cdot q_2$ is the Euclidean product $\sum_{k=1}^{D} q_1^{(k)} q_2^{(k)}$

3. $$\Xi_G(m, q) = \phi_G(q) + \Big(\sum_{e \in E_G} m_e^2 \alpha_e\Big) \psi_G$$

ex. 1: Compute ψ_G for $G = L_n$ (n-th one-loop graph) and $G = B_n$ (n-th banana graph).



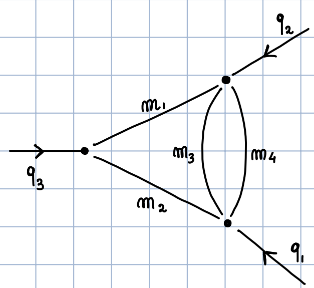
$G = L_n$



$G = B_n$

ex. 2: Compute ψ_G, ϕ_G, Ξ_G for the following graph:

$G =$



ex. 3: For a graph G with n_G edges and v_G vertices, define the Laplacian matrix $\mathcal{L} = (\mathcal{L}_{ij})_{i,j = 1, \dots, v_G}$ such that

$$\mathcal{L}_{ij} = \begin{cases} \sum \alpha_k & \text{if } i = j \text{ and } v_i \overset{e_k}{\bullet\!\!-\!\!-\!\!\bullet} \neq v_i \\ -\sum \alpha_k & \text{if } i \neq j \text{ and } v_i \overset{e_k}{\bullet\!\!-\!\!-\!\!\bullet} v_j \end{cases}$$

Denote by $\mathcal{L}^{(i)}$ the $(v_G - 1) \times (v_G - 1)$ - matrix obtained from $\mathcal{L}$ by deleting the i^{th} row and column.

By the matrix-tree theorem, $\psi_G = \det \mathcal{L}^{(i)} \Big|_{\alpha_e \mapsto \alpha_e^{-1}} \cdot \prod_{e \in E_G} \alpha_e$ for any choice of $i = 1, \dots, v_G$.

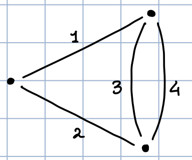
Compute the Laplacian matrix for the same graph G of exercise 2.

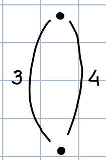
ex. 4 : Let $G \setminus e$ be the graph obtained from G by deleting the edge e.
Let $G /\!\!/ e$ be the graph obtained from G by contracting the edge e if e is not a tadpole and the empty graph otherwise.

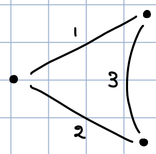
Prove the deletion-contraction identity $\Psi_G = \alpha_e \Psi_{G \setminus e} + \Psi_{G/\!\!/e}$.

ex. 5 : Let $\gamma \subset G$ be a subgraph and G/γ the graph obtained by contracting γ.
Then, $\Psi_G = \Psi_\gamma \Psi_{G/\gamma} + R_{\gamma,G}$ where $R_{\gamma,G}$ is a polynomial of degree strictly greater than $\deg \Psi_\gamma = h_\gamma$ in the variables α_e, $e \in E_\gamma$.

Write the partial factorization of the first graph polynomial Ψ_G for the following choices of G and $\gamma = \gamma_1, \gamma_2$:

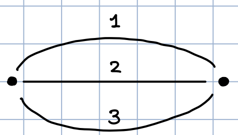
$G =$ 

, $\gamma_1 =$ 

, $\gamma_2 =$ 

ex. 6 : Consider the sunrise graph below.
Compute the intersection of $\sigma = \{(\alpha_1 : \alpha_2 : \alpha_3) \in \mathbb{P}^2(\mathbb{R}) \mid \alpha_i \geqslant 0,\ i = 1,2,3\}$ with the vanishing locus

$$V(\Psi_G) = \{(\alpha_1 : \alpha_2 : \alpha_3) \in \mathbb{P}^2(\mathbb{R}) \mid \Psi_G(\alpha_1, \alpha_2, \alpha_3) = 0\}.$$

$G = B_2 =$ 

ex. 7 (*) **:** Consider two graphs G_1, G_2 and let $G = G_1 \cdot G_2$ be the one-vertex join. Show that $\Psi_G = \Psi_{G_1} \cdot \Psi_{G_2}$.

ex. 8 (*) **:** Let G' be the graph obtained from G by replacing an edge •—e—• with the two edges •—e_1—•—e_2—• .

Show that $\Psi_{G'} = \Psi_G \big|_{\alpha_e = \alpha_{e_1} + \alpha_{e_2}}$.

ex. 9 (*) **:** Show that $\Phi_G(q,m) \big|_{\alpha_e = 0} = \Phi_{G/\!\!/e}(q,m)$.