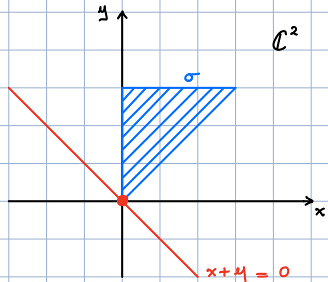
EXERCISES 2 - Feynman Integrals and Number Theory

Les Diablerets, 11/01/2024

ex. 1 Consider the simplex $\sigma = \{(x,y) \in \mathbb{C}^2 : 0 \leq x \leq y \leq 1\}$ and the form $\omega = \dfrac{dx\,dy}{x+y}$.

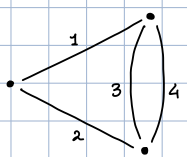
Consider the change of variables $\pi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, $(u,v) \longmapsto (uv, v) = (x,y)$ such that $u = \frac{x}{y}$ and $v = y$.

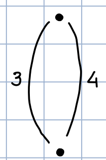
Compute $\pi^*(\omega)$ and $\tilde{\sigma} = \overline{\pi^{-1}(\mathring{\sigma})}$.

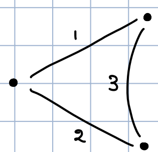


FROM EXERCISES 1 :

ex. 2 : Let $\gamma \subset G$ be a subgraph and G/γ the graph obtained by contracting γ.
Then, the partial factorization formula reads $\Psi_G = \Psi_\gamma \Psi_{G/\gamma} + R_{\gamma,G}$, where $R_{\gamma,G}$ is a polynomial of degree strictly greater than $\deg \Psi_\gamma = h_\gamma$ in the variables α_e, $e \in E_\gamma$.
Write the partial factorization of the first graph polynomial Ψ_G for the following choices of G and $\gamma = \gamma_1, \gamma_2$:

$G =$ 

, $\gamma_1 =$ 

, $\gamma_2 =$ 

ex. 3 Let $f \in \mathbb{Z}[\alpha_1, \dots, \alpha_N]$ and $q = p^m$ prime power. Recall that $[f]_q = |\{(\alpha_1, \dots, \alpha_N) \in \mathbb{F}_q : f(\alpha_1, \dots, \alpha_N) = 0\}|$ and $[f,g]_q = |\{(\alpha_1, \dots, \alpha_N) \in \mathbb{F}_q : f(\alpha_1, \dots, \alpha_N) = 0, \ g(\alpha_1, \dots, \alpha_N) = 0\}|$.
Prove that, if $f = \alpha_1 f^1 + f_1$ where $f^1, f_1 \in \mathbb{Z}[\alpha_2, \dots, \alpha_N]$, then $[f]_q = q\,[f^1, f_1]_q + q^{N-1} - [f^1]_q$.

(*)
ex. 4 Let $f \in \mathbb{Z}[x_1, \dots, x_m]$ homogeneous of degree $\deg f \leq m$. Let S_m be the coefficient of $(x_1 \cdots x_m)^m$ in f^m, $\forall m \geq 0$.
Prove that $[f]_p = (-1)^{m-1} S_{p-1} \pmod p$ for p prime. Follow three steps :

1. Define the indicator function $\mathbb{1}(x) : \mathbb{F}_p \longrightarrow \mathbb{Z}/_p\mathbb{Z}$ such that $\mathbb{1}(x) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod p \\ 1 & \text{if } x \not\equiv 0 \pmod p \end{cases}$.
   Prove that $\mathbb{1}(x) = x^{p-1}$.

2. Prove that $\displaystyle\sum_{x \in \mathbb{F}_p} x^a = \begin{cases} 0 & \text{if } 0 \leq a < p-1 \\ p-1 & \text{if } a = p-1 \end{cases}$.

3. Use $\displaystyle [f]_p = \sum_{\vec{x} \in \mathbb{F}_p^m} 1 - \mathbb{1}(f(\vec{x}))$ to conclude.