1. Exercises Day 2

An SLE(κ)-curve γ on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ can be described in terms of mapping out functions, which are (properly normalized) conformal maps $g_t : H_t \to \mathbb{H}$, where H_t is the unbounded connected component of $\mathbb{H} \setminus \gamma[0,t]$. With the so-called capacity parameterization, g_t satisfies the Loewner differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t},$$

where the driving function is $W_t = \sqrt{\kappa} B_t$, and B is the standard Brownian motion. The domain Markov property together with conformal invariance can be used to show that for every $s \geq 0$, the curve $\gamma_t^s := g_s(\gamma_{s+t}) - W_s$ is also an SLE(κ) curve in $\mathbb{H}$ independent of $\gamma[0,s]$.
The sets $K_t := \overline{\mathbb{H} \setminus H_t}$ are called hulls associated to γ. A swallowing time T_z of $z \in \overline{\mathbb{H}}$ is the first time instance $\gamma[0,t]$ hits z or disconnects z from ∞ in $\overline{\mathbb{H}}$:

$$T_z = \inf\{t \geq 0 : z \in K_t\}.$$

Exercise 1. *This exercise is continuation of Exercise 1 from day one. Let $\kappa \in (4,8)$ and $x \in [0,1]$.*

(a) Let γ be an SLE(κ) curve starting from 0. Show that $P(T_x \leq T_1) = 1$.
(b) Consider the martingale $M_t = P\left(T_x < T_1 \mid \gamma[0,t]\right)$. Using properties of SLE(κ), argue that

$$M_t = \begin{cases} \mathbf{1}\left(T_x < T_1\right) & \text{if } t \geq T_x \wedge T_1, \\ F\left(\frac{g_t(x)-W_t}{g_t(1)-W_t}\right) & \text{if } t < T_x \wedge T_1, \end{cases}$$

where $F(x) = P(T_x < T_1)$.

Solution. (a) The curve γ disconnects 1 from ∞ at time T_1. As γ starts at 0 and $x \in [0,1]$, by topological reasons x also has to be disconnected from ∞ at or before the time T_1 – if this was not the case, γ should hug $\mathbb{R}$ in a neighbourhood of x in a sense that for some $0 \leq s < t \leq T_1$ we have $x \in \gamma(s,t) \subset \mathbb{R}$, which would imply $H_u = H_s$ for $u \in (s,t)$, and consequently $\partial_u g_u(z) = 0$ for $u \in (s,t)$, contradicting capacity parametrization. This shows that $T_x \leq T_1$ almost surely.
(b) If $t \geq T_x \wedge T_1$, the event $T_x < T_1$ is already contained in $\mathcal{F}_t$, hence we get

$$M_t|\{t \geq T_x \wedge T_1\} = P(T_x < T_1 | \mathcal{F}_t \cap \{t \geq T_x \wedge T_1\}) = \begin{cases} 1, & \textbf{if } T_x < T_1, \\ 0, & \text{otherwise}, \end{cases}$$

which is just the indicator $\mathbf{1}(T_x < T_1)$. Next assume $t < T_x \wedge T_1$. By the domain Markov property and conformal invariance, $s \mapsto g_t(\gamma_{t+s}) - W_t =: \gamma_s^t$ is an SLE(κ) curve independent of $\mathcal{F}_t$. When $t < T_x \wedge T_1$, the point x is swallowed before 1 by γ if and only if $g_t(x) - W_t$ is swallowed before $g_t(1) - W_t$ by γ^t. Writing $T_z^t := \inf\{s \geq 0 : z \in K_s^t\}$ the swallowing time of z by γ^t, we get

$$\begin{aligned} M_t|\{t < T_x \wedge T_1\} &= P(T_x < T_1 \,|\, \mathcal{F}_t \cap \{t < T_x \wedge T_1\}) \\ &= P(T^t_{g_t(x)-W_t} < T^t_{g_t(1)-W_t} \,|\, \mathcal{F}_t \cap \{t < T_x \wedge T_1\}). \end{aligned}$$

As γ^t is independent of $\mathcal{F}_t$ (and hence also of the event $\{t < T_x \wedge T_1\} \in \mathcal{F}_t$), we can drop the conditioning on the RHS:

$$M_t|\{t < T_x \wedge T_1\} = P(T^t_{g_t(x)-W_t} < T^t_{g_t(1)-W_t}).$$

By Brownian scaling of SLE(κ), the curve $\gamma'_s = (g_t(1) - W_t)^{-1}\gamma^t_{(g_t(1)-W_t)^2 s}$ is also SLE(κ) distributed. Writing $T'_z := \inf\{s \geq 0 : z \in K'_s\}$ the swallowing time of z by γ' we have $T'_z = (g_t(1) - W_t)^2 T^t_{(g_t(1)-W_t)^{-1}z}$, so in particular $T'_z < T'_w$ is equivalent with $T^t_{(g_t(1)-W_t)^{-1}z} < T^t_{(g_t(1)-W_t)^{-1}w}$ for every $z, w \in \overline{\mathbb{H}}$. Applying this to the above equation yields

$$M_t|\{t < T_x \wedge T_1\} = P(T'_{\frac{g_t(x)-W_t}{g_t(1)-W_t}} < T'_1).$$

Finally, the random variables $(T'_z)_{z\in\overline{\mathbb{H}}}$ have the same joint law as $(T_z)_{z\in\overline{\mathbb{H}}}$, so we can replace T' by T to get

$$M_t|\{t < T_x \wedge T_1\} = P(T_{\frac{g_t(x)-W_t}{g_t(1)-W_t}} < T_1) = F\left(\frac{g_t(x) - W_t}{g_t(1) - W_t}\right),$$

where

$$F(x) = P(T_x < T_1).$$

This finishes (b).

A stochastic process X_t satisfying the following stochastic differential equation (SDE)

$$dX_t = \mu_t dt + \sigma_t dB_t$$

is a local martingale if and only if the finite variation part μ_t is zero: $\mu_t \equiv 0$. If $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuously twice differentiable function, then by Ito's formula the process $f(t, X_t)$ satisfies the following SDE:

$$df(t, X_t) = \left(\partial_1 + \mu_t\partial_2 + \frac{\sigma_t^2}{2}\partial_2^2\right) f(t, X_t)dt + \frac{\sigma_t}{2}\partial_2 f(t, X_t)dB_t. \tag{1}$$

Exercise 2.

(a) Under the assumption $F \in C^2([0, 1])$, apply Equation (1) on $F(\frac{g_t(x)-W_t}{g_t(1)-W_t})$ to conclude that for M_t to be a local martingale, F should satisfy the differential equation

$$F'(x)\left(2\left(x^{-1} - x\right) + \kappa\left(x - 1\right)\right) + \frac{\kappa}{2}F''(x)\left(x - 1\right)^2 = 0, \qquad x \in (0, 1). \tag{2}$$

(b) Solve for F, and then (use the optional stopping theorem ($\mathbb{E}[M_{T_x}] = \mathbb{E}[M_0]$) to) deduce that

$$P(T_x < T_1) = \frac{\int_x^1 (1-u)^{\frac{8}{\kappa}-2} u^{-\frac{4}{\kappa}} du}{\int_0^1 (1-u)^{\frac{8}{\kappa}-2} u^{-\frac{4}{\kappa}} du}.$$

(c) Show that $P(T_{1/2} < T_1) = \frac{1}{2}$ if and only if $\kappa = 6$.

(d) Let $\varphi : \Omega \to \mathbb{H}$ be the conformal map from the rhombus Ω to the upper half plane $\mathbb{H}$ such that when extended continuously on the boundary, φ satisfies*

$$\varphi(p_1) = 0, \qquad \varphi(p_3) = 1, \qquad \varphi(p_4) = \infty.$$

Show that $\varphi(p_2) = \frac{1}{2}$.

(e) Conclude that the only possible conformally invariant scaling limit for the critical percolation interface from Exercise 1 is SLE(6). HINT: *How does an LR-crossing crossing from Exercise 1 relate to $P(T_x < T_1)$?*

Solution. (a) Suppose that $F \in C^2((0, 1))$, and write $\phi(s, u) = \frac{g_s(x)-u}{g_s(1)-u}$, and $f(s, u) = F(\phi(s, u))$. By Equation (1)$f(t, W_t)$ satisfies the following SDE:

$$df(t, W_t) = \left(\partial_1 + \frac{\kappa}{2}\partial_2^2\right) f(t, W_t)dt + \frac{\kappa}{2}\partial_2 f(t, W_t)dB_t.$$

For M_t to be a martingale, the finite variation part of $f(t, W_t)$ has to vanish, which happens if f satisfies the following PDE:

$$\Big(\partial_1 + \frac{\kappa}{2}\partial_2^2\Big) f(s,u) = 0.$$

By applying the chain rule to $f(s,u) = F(\phi(s,u))$ we can write the above differential equation in terms of F and ϕ as follows:

$$F'(\phi(s,u))\left(\partial_s \phi(s,u) + \frac{\kappa}{2}\,\partial_u^2\,\phi(s,u)\right) + \frac{\kappa}{2}F''(\phi(s,u))\left(\partial_u\,\phi(s,u)\,\right)^2 = 0.$$

Calculate the following partial derivatives.

$$\partial_s\phi(s,u) = \frac{\frac{2}{g_s(x)-u}\left(g_s(1)-u\right) - \left(g_s(x)-u\right)\frac{2}{g_s(1)-u}}{\left(g_s(1)-u\right)^2} = \frac{2\left(\phi(s,u)^{-1} - \phi(s,u)\right)}{(g_s(1)-u)^2}.$$

$$\partial_u\phi(s,u) = \frac{g_s(x)-g_s(1)}{\left(g_s(1)-u\right)^2} = \frac{g_s(x)-u+u-g_s(1)}{\left(g_s(1)-u\right)^2} = \frac{1}{\left(g_s(1)-u\right)}\left(\phi(s,u)-1\right)$$

$$\partial_u^2\phi(s,u) = \frac{2\left(g_s(x)-u\right)}{(g_s(1)-u)^3} - \frac{2}{(g_s(1)-u)^2} = \frac{2}{(g_s(1)-u)^2}\left(\phi(s,u)-1\right)$$

Plugging in the partial derivatives from above yields that M_t on $\{t < \min(T_x, T_1)\}$ is a local martingale on iff for all $\phi \in (0,1)$ one has

$$F'(\phi)\left(2\frac{\left(\phi^{-1}-\phi\right)}{(g_t(1)-W_t)^2} + \frac{\kappa}{2}\frac{2}{(g_t(1)-W_t)^2}\left(\phi-1\right)\right) + \frac{\kappa}{2}F''(\phi)\frac{1}{(g_t(1)-W_t)^2}\left(\phi-1\right)^2 = 0.$$

which after multiplication by $(g_t(1)-W_t)^2$ from both sides and rearranging becomes

$$-\frac{4}{\kappa}\left(\frac{\phi^{-1}-\phi}{(\phi-1)^2} + \frac{\kappa}{2(\phi-1)}\right) = \frac{F''(\phi)}{F'(\phi)} = \frac{d}{d\phi}\log(F'(\phi)).$$

Integrating with respect to ϕ and exponentiating gives

$$F'(\phi) = C\left(1-\phi\right)^{\frac{8}{\kappa-2}}\phi^{-\frac{4}{\kappa}},$$

which after another integration becomes

$$F(\phi) = C\int_\phi^1 (1-u)^{\frac{8}{\kappa}-2}u^{-\frac{4}{\kappa}}\,du + B.$$

Applying our initial condition $F(1) = 0$ yields that $B = 0$ and further using $F(0) = 1$ one has $C = (\int_0^1 (1-u)^{\frac{8}{\kappa}-2}u^{-\frac{4}{\kappa}}\,du^{-1})$, which finally concludes that

$$F(\phi) = \frac{\int_\phi^1 (1-u)^{\frac{8}{\kappa}-2}u^{-\frac{4}{\kappa}}\,du}{\int_0^1 (1-u)^{\frac{8}{\kappa}-2}u^{-\frac{4}{\kappa}}\,du}.$$

This concludes (a).

(b) Note that we don't a priori know that the function F in the form of the martingale M_t is twice differentiable, so we need to work "backwards". Let $\phi_t = \frac{g_t(x)-W_t}{g_t(1)-W_t}$, and

$$\tilde{M}_t = \frac{\int_{\phi_t}^1 (1-u)^{\frac{8}{\kappa}-2}u^{-\frac{4}{\kappa}}\,du}{\int_0^1 (1-u)^{\frac{8}{\kappa}-2}u^{-\frac{4}{\kappa}}\,du}.$$

By the previous part, $\tilde{M}_t$ is is a martingale up to the stopping time $T = \inf\{t \geq 0 : \phi_t \in \{0,1\}\}$. It is a fact that

$$\phi_T = 0 \iff T_x < T_1, \qquad \textbf{and} \qquad \phi_T = 1 \iff T_x = T_1,$$

thus we get $\tilde{M}_T = \mathbf{1}(T_x < T_1)$. As $\phi_0 = x$, by optional stopping theorem we thus get

$$P(T_x < T_1) = \mathbb{E}[\tilde{M}_T] = \mathbb{E}[\tilde{M}_0] = \frac{\int_x^1 (1-u)^{\frac{8}{\kappa}-2} u^{-\frac{4}{\kappa}} du}{\int_0^1 (1-u)^{\frac{8}{\kappa}-2} u^{-\frac{4}{\kappa}} du}.$$

(c) By the previous exercise, it suffices to show that $F(1/2) = 1/2$, which by the initial condition $F(0) = 1$ is equivalent to showing $F(1/2) = F(0) - F(1/2)$, hence we are going to prove that

$$\frac{F(\frac{1}{2})}{F(0) - F(\frac{1}{2})} = 1 \qquad \text{iff } \kappa = 6.$$

Observe that

$$\frac{F(\frac{1}{2})}{F(0) - F(\frac{1}{2})} = \frac{\int_{1/2}^1 (1-u)^{\frac{8}{\kappa}-2} u^{-\frac{4}{\kappa}} du}{\int_0^{1/2} (1-u)^{\frac{8}{\kappa}-2} u^{-\frac{4}{\kappa}} du} = \frac{\int_0^{1/2} u^{\frac{8}{\kappa}-2} (1-u)^{-\frac{4}{\kappa}} du}{\int_0^{1/2} (1-u)^{\frac{8}{\kappa}-2} u^{-\frac{4}{\kappa}} du},$$

where in the last equality we made the change of variables $u \mapsto 1-u$ to the top integral. To compare the integrand of the integrals, observe that for every $u \in (0, \frac{1}{2}]$ we have

$$\frac{u^{\frac{8}{\kappa}-2}(1-u)^{-\frac{4}{\kappa}}}{(1-u)^{\frac{8}{\kappa}-2} u^{-\frac{4}{\kappa}}} = \left(\frac{u}{1-u}\right)^{\frac{12}{\kappa}-2} \quad \begin{cases} > 1, & \text{if } \frac{12}{\kappa} - 2 < 0, \\ = 1, & \text{if } \frac{12}{\kappa} - 2 = 0, \\ < 1, & \text{if } \frac{12}{\kappa} - 2 > 0. \end{cases}$$

From this we see that one of the integrands (hence the whole integral) is strictly greater than the other, unless $\frac{12}{\kappa} - 2 = 0$, which happens only for $\kappa = 6$. Since $F(0) = 1$, this implies that for only $\kappa = 6$ we do get $F(\frac{1}{2}) = \frac{1}{2}$.

(d*) We give a complex analysis argument. Denote by ℓ the vertical line connecting p_2 and p_4 in Ω and let $\Gamma_\ell \colon \Omega \to \Omega$ be the reflections along ℓ. This is an anticonformal map fixing ℓ, hence ℓ is the hyperbolic geodesic from p_2 to p_4 in Ω. Since conformal maps preserve hyperbolic geodesics, $\varphi(\ell)$ is the hyperbolic geodesic in $\mathbb{H}$ from $\varphi(p_2)$ to $\varphi(p_4) = \infty$, which is simply the vertical line $L = \{\varphi(p_2) + iy \,|\, y > 0\}$. The map $\psi = \phi^{-1} \circ \Gamma_\ell \circ \phi$ is anticonformal and fixes all points in L, therefore ψ is the reflection of $\mathbb{H}$ along L. Since we also have (after extending each map continuously to the boundary)

$$\psi(1) = \psi(\varphi(p_3)) = \varphi(\Gamma_\ell(p_3)) = \varphi(p_1) = 0,$$

we conclude that $\varphi(p_2) = \frac{1}{2}$.

(e) By exercise (1(c)) we already know that if the scaling limit of the interface is conformally invariant it has to correspond to some SLE(κ), which we now denote by γ. Recall that in exercise (1(b)), we showed that the percolation admits a LR-crossing with probability 1/2 and occurs iff γ^δ hits S_2. Therefore, in the limit as $\delta \downarrow 0$, γ starts at p_1, hits S_2 and terminates at p_4 with probability 1/2. Using conformal invariance of SLE(κ) the probability of this event is the same as γ starting from 0, hitting between 1/2 and 1 and terminating at ∞. This event is given by $\{T_{1/2} < T_1\}$, which has probability 1/2 iff $\kappa = 6$. This finishes (e).

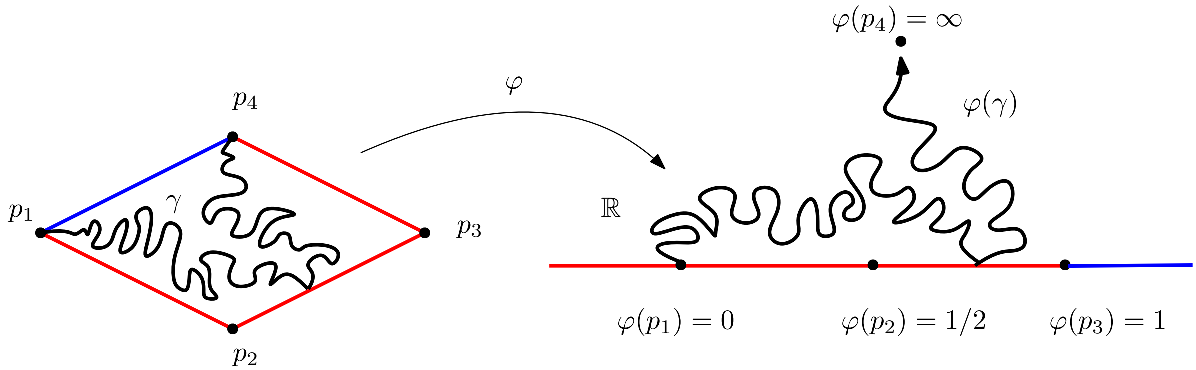

FIGURE 1. The event of percolation having an LR crossing on the left, while having the corresponding SLE event on the right.

The Verma module $M_{c,h}$ of weight h and central charge c is the unique (up to isomorphism) highest weight module satisfying the following universality property: if V is another (non-zero) highest weight module of weight h and central charge c, then (unless $V \cong M_{c,h}$) there exists a singular vector $w \in M_{c,h}$ such that we have an isomorphism

$$V \cong M_{c,h}/(\mathfrak{Vir}\, w)$$

sending a highest weight vector of V to a highest weight vector of $M_{c,h}$.

Physics postulate. Suppose we have a family $(\Phi_\iota(z))_{\iota \in I}$ of primary fields with conformal weights Δ_ι. Physics arguments show that each $\Phi_\iota(z)$ is a highest weight vector of some irreducible Verma module V_{c,Δ_ι}. The Virasoro action on $\Phi_\iota(z)$ manifests in the level of correlation functions as

$$\langle \Phi_{\iota_1}(z_1) \ldots \Phi_{\iota_n}(z_n) L_{-k}^m \Phi_\iota(z) \rangle = (L_{-k}^{(z)})^m \langle \Phi_{\iota_1}(z_1) \ldots \Phi_{\iota_n}(z_n) \Phi_\iota(z) \rangle, \tag{3}$$

for $\iota_1, \ldots, \iota_n \in I,$ and $k, m \in \mathbb{Z}_{\geq 0}$, where $\mathcal{L}_{-k}^{(z)}$ is a first order differential operator given by

$$\mathcal{L}_{-k}^{(z)} = \sum_{i=1}^{n} \left(\frac{(k-1)}{(z_i - z)^k} \Delta_{\iota_i} - \frac{1}{(z_i - z)^{k-1}} \frac{\partial}{\partial z_i} \right).$$

Exercise 3. *This is continuation of Exercises 2 and 3 from day 1. Let $(\Phi_\iota(z))_{\iota \in I}$ be a family of primary fields with central charge c and conformal weights Δ_ι. Denote by V_{c,Δ_ι} the $\mathfrak{Vir}$-module generated by Φ_ι. Assume $\Phi(z)$ is a primary field from the collection with conformal weight $\Delta = h_-$, where h_- is as in Exercise 3.*

(a) Use Exercise 2 to show that the lowest level at which the Verma module $M_{c,\Delta}$ has singular vectors is 2. Conclude that $V_{c,\Delta} \cong M_{c,\Delta}/(\mathfrak{Vir}\, w_{c,\Delta})$, where $w_{c,\Delta}$ is the 2-level singular vector from Exercise 3.

(b) Assuming that the correlation function

$$F_{\iota_1,\ldots,\iota_n}(z_1, \ldots, z_n, z) = \langle \Phi_{\iota_1}(z_1) \ldots \Phi_{\iota_n}(z_n) \Phi(z) \rangle$$

satisfies Equations (3), find a differential operator D expressed in terms of $\mathcal{L}_{-k}^{(z)}$ such that the following differential equation is satisfied:

$$D F_{\iota_1,\ldots,\iota_n}(z_1, \ldots, z_n, z) = 0. \tag{4}$$

Hint: note that $\langle\cdot\rangle$ is a linear operator.

(c) *Assuming translation invariance, i.e.*

$$F_{\iota_1,\ldots,\iota_n}(z_1+\lambda,\ldots,z_n+\lambda,z+\lambda) = F_{\iota_1,\ldots,\iota_n}(z_1,\ldots,z_n,z),$$

show that equation (4) becomes the following BPZ equation:

$$\left[-\frac{3}{2\,(2\Delta+1)}\frac{\partial^2}{\partial z^2} - \sum_{i=1}^{n}\left(\frac{1}{z_i-z}\frac{\partial}{\partial z_i} - \frac{\Delta_{\iota_i}}{(z_i-z)^2}\right)\right]F_{\iota_1,\ldots,\iota_n}(z_1,\ldots,z_n,z) = 0. \tag{5}$$

(d) *Consider the parameterization $c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$ from Exercise 3(c). Assuming $\Delta_{\iota_i} = h_- = \frac{6-\kappa}{2\kappa}$ for every i, write down the BPZ-equation (5) in terms of κ.*

Solution. (a) Since $\Delta \neq 0$, there are no 1-level singular vectors in $M_{c,\Delta}$, while $w_{c,\Delta} = (L_{-2} + a_- L_{-1}^2)v_{c,\Delta}$ is the unique singular level vector of level 2 in $M_{c,\Delta}$. This shows the first claim. Since $M_{c,\Delta}$ contains a 2-level singular vector, it is not irreducible, so in particular $V_{c,\Delta} \neq M_{c,\Delta}$. By the universality property of Verma modules, there thus exists a singular vector $w \in M_{c,\Delta}$ such that $J_{c,\Delta} \cong M_{c,\Delta}/(\mathfrak{Vir}\,w)$. Since singular vectors of highest-weight modules generate non-trivial ideals, irreduciblity of $J_{c,\Delta}$ implies $w_{c,\Delta} \in \mathfrak{Vir}\,w$, and since $w_{c,\Delta}$ is the lowest level singular vector of $M_{c,\Delta}$, we get $w = w_{v,\Delta}$, proving the second claim.

(b) Under the isomorphism $V_{c,\Delta} \cong M_{c,\Delta}/(\mathfrak{Vir}\,w_{c,\Delta})$ we identify $\Phi(z) \cong v_{c,\Delta}$, thus we get

$$(L_{-2} + a_- L_{-1}^2)\Phi(z) \cong (L_{-2} + a_- L_{-1}^2)v_{c,\Delta} = w_{c,\Delta} \in \mathfrak{Vir}\,w_{c,\Delta},$$

therefore $(L_{-2} + a_- L_{-1}^2)\Phi(z) = 0$. By linearity of $\langle\cdot\rangle$, we thus get

$$0 = \langle 0\rangle = \langle \Phi_{\iota_1}(z)\ldots\Phi_{\iota_n}(z)(L_{-2} + a_- L_{-1}^2)\Phi(z)\rangle = (\mathcal{L}_{-2}^{(z)} + a_-(\mathcal{L}_{-1}^{-1})^2)\langle \Phi_{\iota_1}(z)\ldots\Phi_{\iota_n}(z)\Phi(z)\rangle.$$

The differential operator $D = \mathcal{L}_{-2}^{(z)} + a_-(\mathcal{L}_{-1}^{-1})^2$ thus satisfies (4). Recalling that $a_- = -\frac{3}{2(2h_-+1)}$ and $\Delta = h_-$, using explicit formulas for $\mathcal{L}_{-k}$ we can write

$$D = -\frac{3}{2\,(2h_{1,2}+1)}\left(\sum_{i=1}^{n}\frac{\partial}{\partial z_i}\right)^2 - \sum_{i=1}^{n}\left(\frac{1}{z_i-z}\frac{\partial}{\partial z_i} - \frac{\Delta_{\iota_i}}{(z_i-z)^2}\right)$$

(c) Differentiating the translation invariance equation w.r.t. λ and evaluating at $\lambda = 0$ yields

$$\left[\sum_{i=1}^{n}\frac{\partial}{\partial z_i} + \frac{\partial}{\partial z}\right]F_{\iota_1,\ldots,\iota_n}(z_1,\ldots,z_n,z) = 0,$$

hence when acting on $F_{\iota_1,\ldots,\iota_n}$, we have

$$\sum_{i=1}^{n}\frac{\partial}{\partial z_i} = -\frac{\partial}{\partial z}.$$

Doing this substitution to the equation in previous parts gives (5).

(d) Using the parametrization we get the following second order PDE:

$$\left[\frac{\kappa}{2}\frac{\partial^2}{\partial z^2} - \sum_{i=1}^{n}\left(\frac{1}{z_i-z}\frac{\partial}{\partial z_i} - \frac{2\,\Delta_{\iota_i}}{(z_i-z)^2}\right)\right]F_{\iota_1,\ldots,\iota_n,\iota}(z_1,\ldots,z_n,z) = 0,$$

and this finishes the exercise. Note that this is the equation for the drift terms in the drivers of interacting SLEs.