1. Exercises Day 2

An SLE(κ)-curve γ on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ can be described in terms of mapping out functions, which are (properly normalized) conformal maps $g_t : H_t \to \mathbb{H}$, where H_t is the unbounded connected component of $\mathbb{H} \setminus \gamma[0,t]$. With the so-called capacity parameterization, g_t satisfies the Loewner differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t},$$

where the driving function is $W_t = \sqrt{\kappa} B_t$, and B is the standard Brownian motion. The domain Markov property together with conformal invariance can be used to show that for every $s \geq 0$, the curve $\gamma_t^s := g_s(\gamma_{s+t}) - W_s$ is also an SLE(κ) curve in $\mathbb{H}$ independent of $\gamma[0,s]$.
The sets $K_t := \overline{\mathbb{H} \setminus H_t}$ are called hulls associated to γ. A swallowing time T_z of $z \in \overline{\mathbb{H}}$ is the first time instance $\gamma[0,t]$ hits z or disconnects z from ∞ in $\overline{\mathbb{H}}$:

$$T_z = \inf\{t \geq 0 : z \in K_t\}.$$

Exercise 1. *This exercise is continuation of Exercise 1 from day one. Let $\kappa \in (4,8)$ and $x \in [0,1]$.*

(a) Let γ be an SLE(κ) curve starting from 0. Show that $P(T_x \leq T_1) = 1$.

(b) Consider the martingale $M_t = P\left(T_x < T_1 \mid \gamma[0,t]\right)$. Using properties of SLE(κ), argue that

$$M_t = \begin{cases} \mathbf{1}\left(T_x < T_1\right) & \text{if } t \geq T_x \wedge T_1, \\ F\left(\frac{g_t(x) - W_t}{g_t(1) - W_t}\right) & \text{if } t < T_x \wedge T_1, \end{cases}$$

where $F(x) = P(T_x < T_1)$.

A stochastic process X_t satisfying the following stochastic differential equation (SDE)

$$dX_t = \mu_t dt + \sigma_t dB_t$$

is a local martingale if and only if the finite variation part μ_t is zero: $\mu_t \equiv 0$. If $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuously twice differentiable function, then by Ito's formula the process $f(t, X_t)$ satisfies the following SDE:

$$df(t, X_t) = \left(\partial_1 + \mu_t \partial_2 + \frac{\sigma_t^2}{2}\partial_2^2\right) f(t, X_t) dt + \frac{\sigma_t}{2}\partial_2 f(t, X_t) dB_t. \tag{1}$$

Exercise 2.

(a) Under the assumption $F \in C^2([0,1])$, apply Equation (1) on $F(\frac{g_t(x) - W_t}{g_t(1) - W_t})$ to conclude that for M_t to be a local martingale, F should satisfy the differential equation

$$F'(x)\left(2\left(x^{-1} - x\right) + \kappa\left(x - 1\right)\right) + \frac{\kappa}{2} F''(x)\left(x - 1\right)^2 = 0, \qquad x \in (0,1). \tag{2}$$

(b) Solve for F, and then (use the optional stopping theorem ($\mathbb{E}[M_{T_x}] = \mathbb{E}[M_0]$) to) deduce that

$$P(T_x < T_1) = \frac{\int_x^1 (1-u)^{\frac{8}{\kappa} - 2} u^{-\frac{4}{\kappa}} du}{\int_0^1 (1-u)^{\frac{8}{\kappa} - 2} u^{-\frac{4}{\kappa}} du}.$$

(c) Show that $P(T_{1/2} < T_1) = \frac{1}{2}$ if and only if $\kappa = 6$.

(d) Let $\varphi : \Omega \to \mathbb{H}$ be the conformal map from the rhombus Ω to the upper half plane $\mathbb{H}$ such that when extended continuously on the boundary, φ satisfies*

$$\varphi(p_1) = 0, \qquad \varphi(p_3) = 1, \qquad \varphi(p_4) = \infty.$$

Show that $\varphi(p_2) = \frac{1}{2}$.

(e) *Conclude that the only possible conformally invariant scaling limit for the critical percolation interface from Exercise* 1 *is* SLE(6). HINT: *How does an LR-crossing crossing from Exercise* 1 *relate to* $P(T_x < T_1)$*?*

The Verma module $M_{c,h}$ of weight h and central charge c is the unique (up to isomorphism) highest weight module satisfying the following universality property: if V is another (non-zero) highest weight module of weight h and central charge c, then (unless $V \cong M_{c,h}$) there exists a singular vector $w \in M_{c,h}$ such that we have an isomorphism

$$V \cong M_{c,h}/(\mathfrak{Vir}\, w)$$

sending a highest weight vector of V to a highest weight vector of $M_{c,h}$.

Physics postulate. Suppose we have a family $(\Phi_\iota(z))_{\iota\in I}$ of primary fields with conformal weights Δ_ι. Physics arguments show that each $\Phi_\iota(z)$ is a highest weight vector of some irreducible Verma module V_{c,Δ_ι}. The Virasoro action on $\Phi_\iota(z)$ manifests in the level of correlation functions as

$$\langle \Phi_{\iota_1}(z_1)\ldots\Phi_{\iota_n}(z_n) L_{-k}^m \Phi_\iota(z)\rangle = (L_{-k}^{(z)})^m \langle \Phi_{\iota_1}(z_1)\ldots\Phi_{\iota_n}(z_n)\Phi_\iota(z)\rangle, \tag{3}$$

for $\iota_1,\ldots,\iota_n \in I,$ and $k,m\in\mathbb{Z}_{\geq 0}$, where $\mathcal{L}_{-k}^{(z)}$ is a first order differential operator given by

$$\mathcal{L}_{-k}^{(z)} = \sum_{i=1}^n \left(\frac{(k-1)}{(z_i-z)^k}\Delta_{\iota_i} - \frac{1}{(z_i-z)^{k-1}}\frac{\partial}{\partial z_i}\right).$$

Exercise 3. *This is continuation of Exercises 2 and 3 from day 1. Let* $(\Phi_\iota(z))_{\iota\in I}$ *be a family of primary fields with central charge c and conformal weights* Δ_ι*. Denote by* V_{c,Δ_ι} *the* $\mathfrak{Vir}$*-module generated by* Φ_ι*. Assume* $\Phi(z)$ *is a primary field from the collection with conformal weight* $\Delta = h_-$*, where* h_- *is as in Exercise 3.*

(a) *Use Exercise 2 to show that the lowest level at which the Verma module* $M_{c,\Delta}$ *has singular vectors is* 2*. Conclude that* $V_{c,\Delta} \cong M_{c,\Delta}/(\mathfrak{Vir}\, w_{c,\Delta})$*, where* $w_{c,\Delta}$ *is the* 2*-level singular vector from Exercise 3.*

(b) *Assuming that the correlation function*

$$F_{\iota_1,\ldots,\iota_n}(z_1,\ldots,z_n,z) = \langle \Phi_{\iota_1}(z_1)\ldots\Phi_{\iota_n}(z_n)\Phi(z)\rangle$$

satisfies Equations (3), find a differential operator D expressed in terms of $\mathcal{L}_{-k}^{(z)}$ *such that the following differential equation is satisfied:*

$$DF_{\iota_1,\ldots,\iota_n}(z_1,\ldots,z_n,z) = 0. \tag{4}$$

Hint: note that $\langle\cdot\rangle$ *is a linear operator.*

(c) *Assuming translation invariance, i.e.*

$$F_{\iota_1,\ldots,\iota_n}(z_1+\lambda,\ldots,z_n+\lambda,z+\lambda) = F_{\iota_1,\ldots,\iota_n}(z_1,\ldots,z_n,z),$$

show that equation (4) becomes the following BPZ equation:

$$\left[-\frac{3}{2\,(2\Delta+1)}\frac{\partial^2}{\partial z^2} - \sum_{i=1}^n\left(\frac{1}{z_i-z}\frac{\partial}{\partial z_i} - \frac{\Delta_{\iota_i}}{(z_i-z)^2}\right)\right] F_{\iota_1,\ldots,\iota_n}(z_1,\ldots,z_n,z) = 0. \tag{5}$$

(d) *Consider the parameterization* $c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$ *from Exercise 3(c). Assuming* $\Delta_{\iota_i} = h_- = \frac{6-\kappa}{2\kappa}$ *for every i, write down the BPZ-equation (5) in terms of* κ*.*