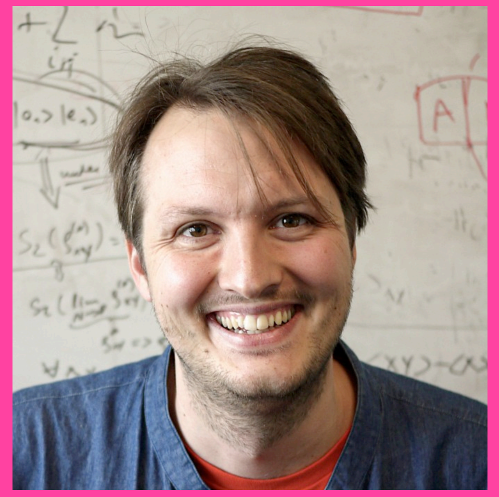
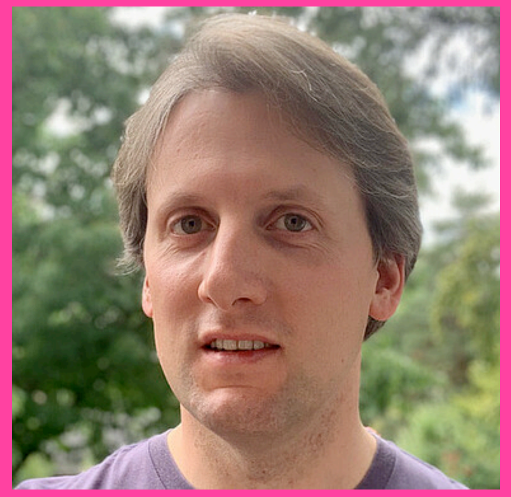
Embezzlement of entanglement, quantum fields, and the classification of von Neumann algebras

Lauritz van Luijk

Joint project with Alex Stottmeister, Reinhard F. Werner and Henrik Wilming

Leibniz Universität Hannover





SRS

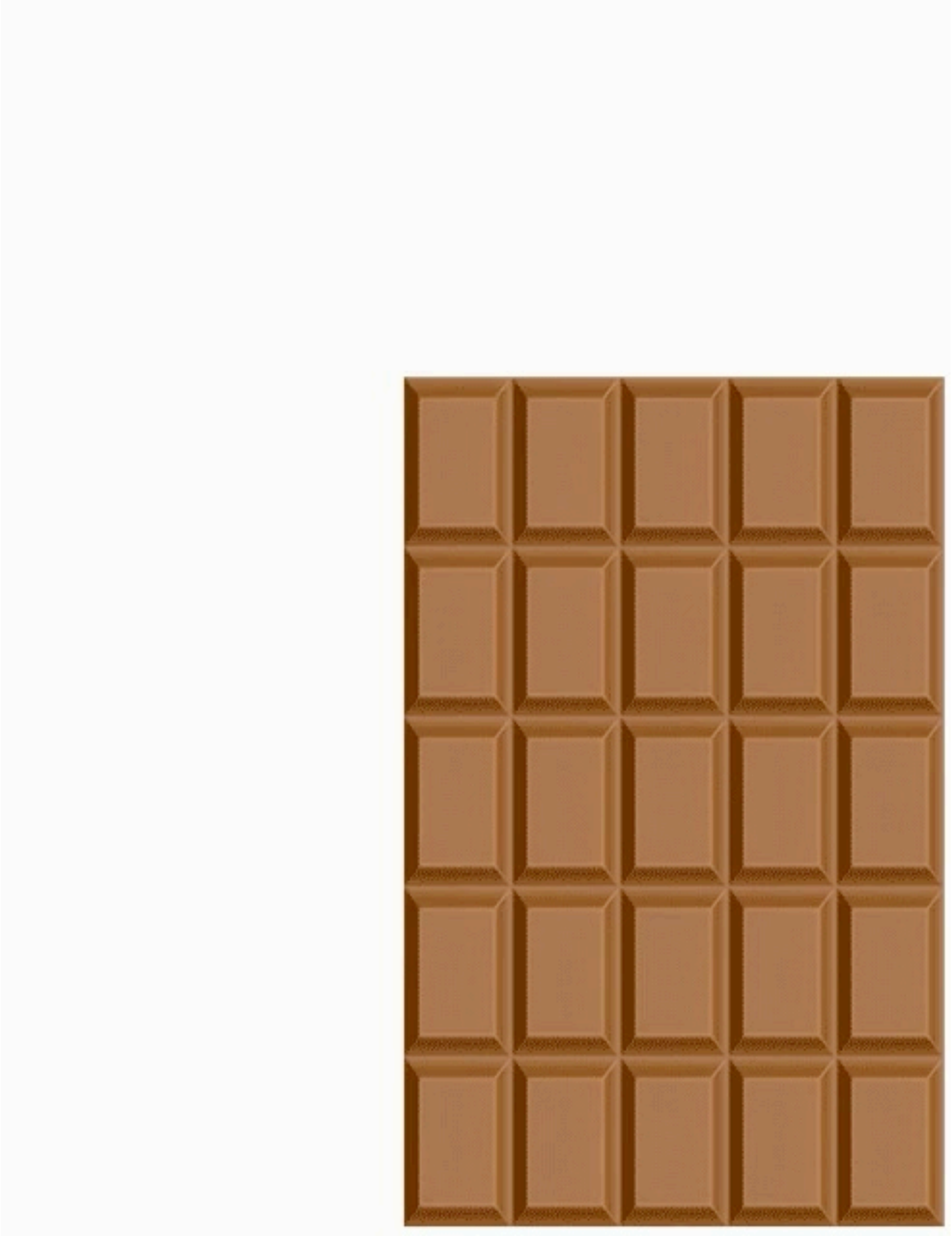
SwissMAP Research Station in Les Diablerets

Embezzlement *Noun*. /ɪmˈbez.əl.mənt/

The crime of secretly taking money that is in your care or that belongs to an organization or business you work for.

Cambridge Dictionary

Embezzlement of chocolate?



user585825, How many whole pieces can be taken out in this way? (Infinite chocolate bar problem), URL (version: 2018-08-20): https://math.stackexchange.com/q/2889188

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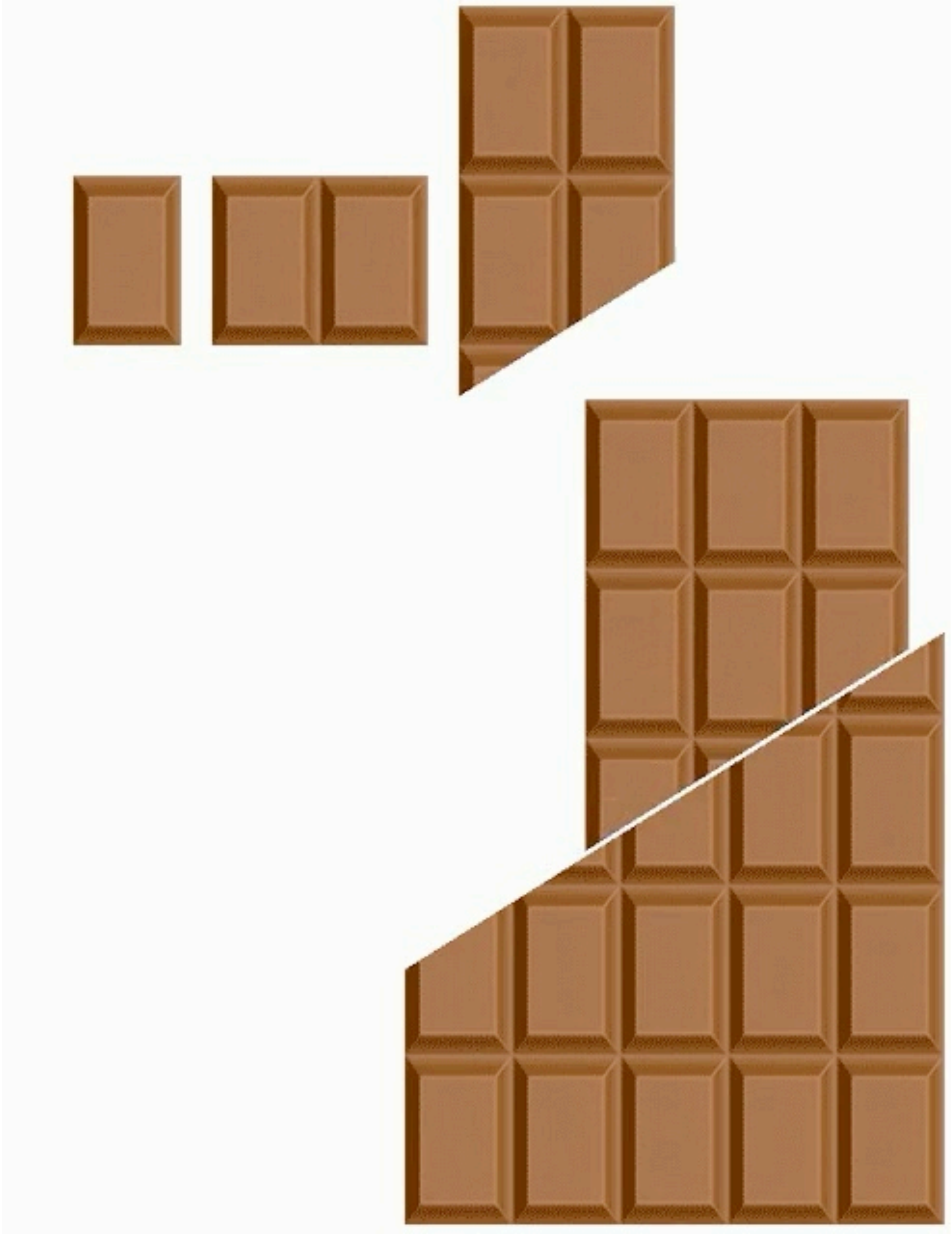

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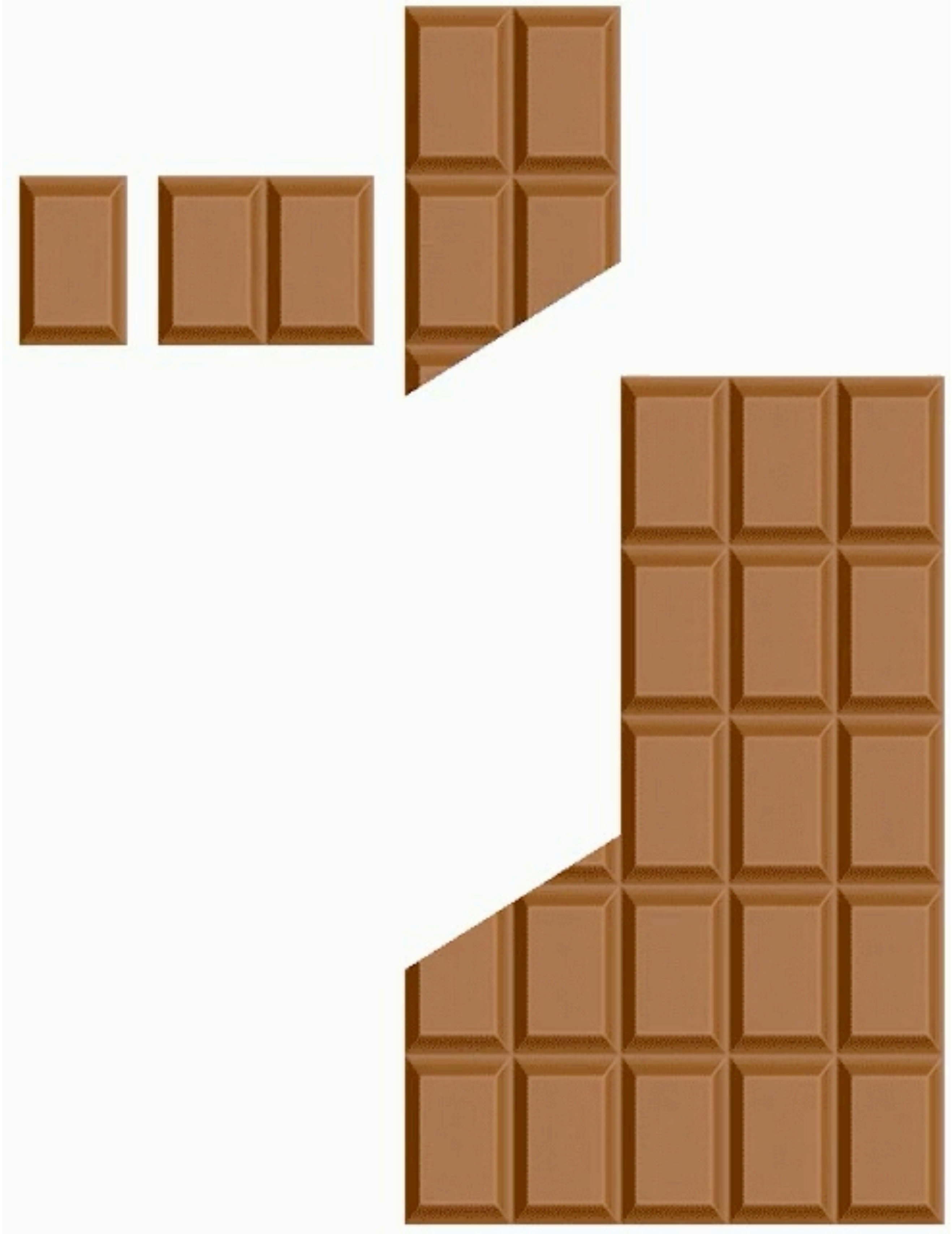

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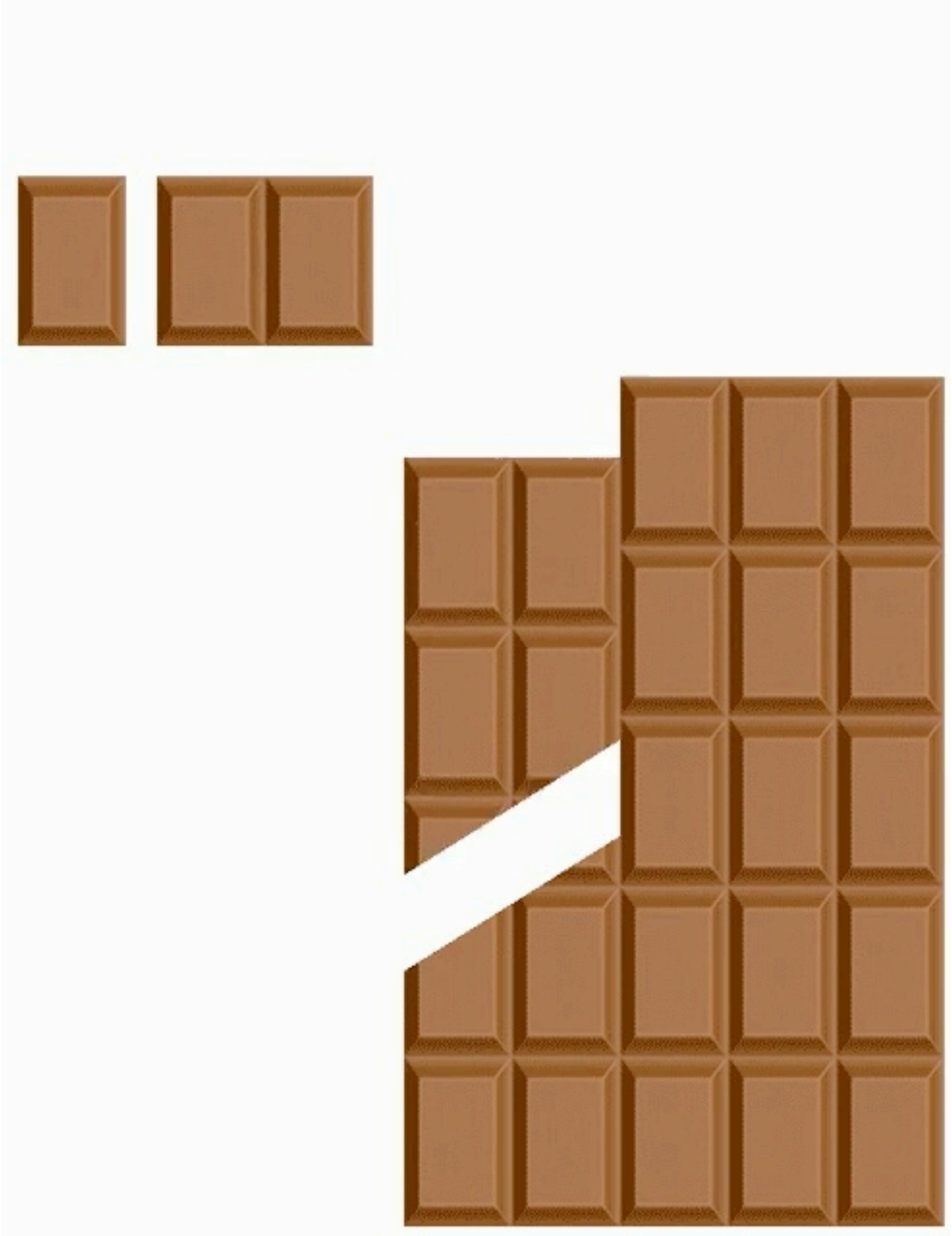

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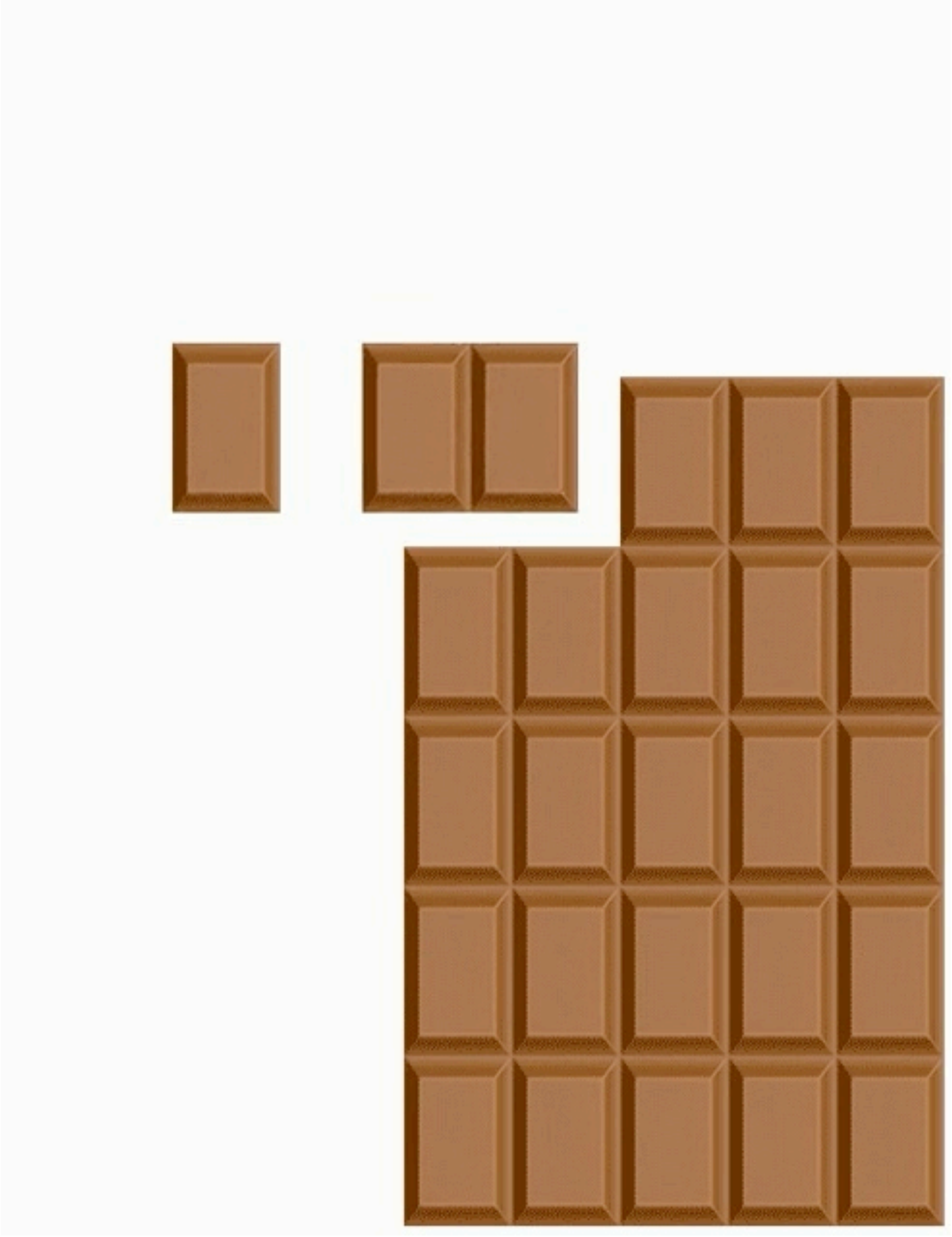

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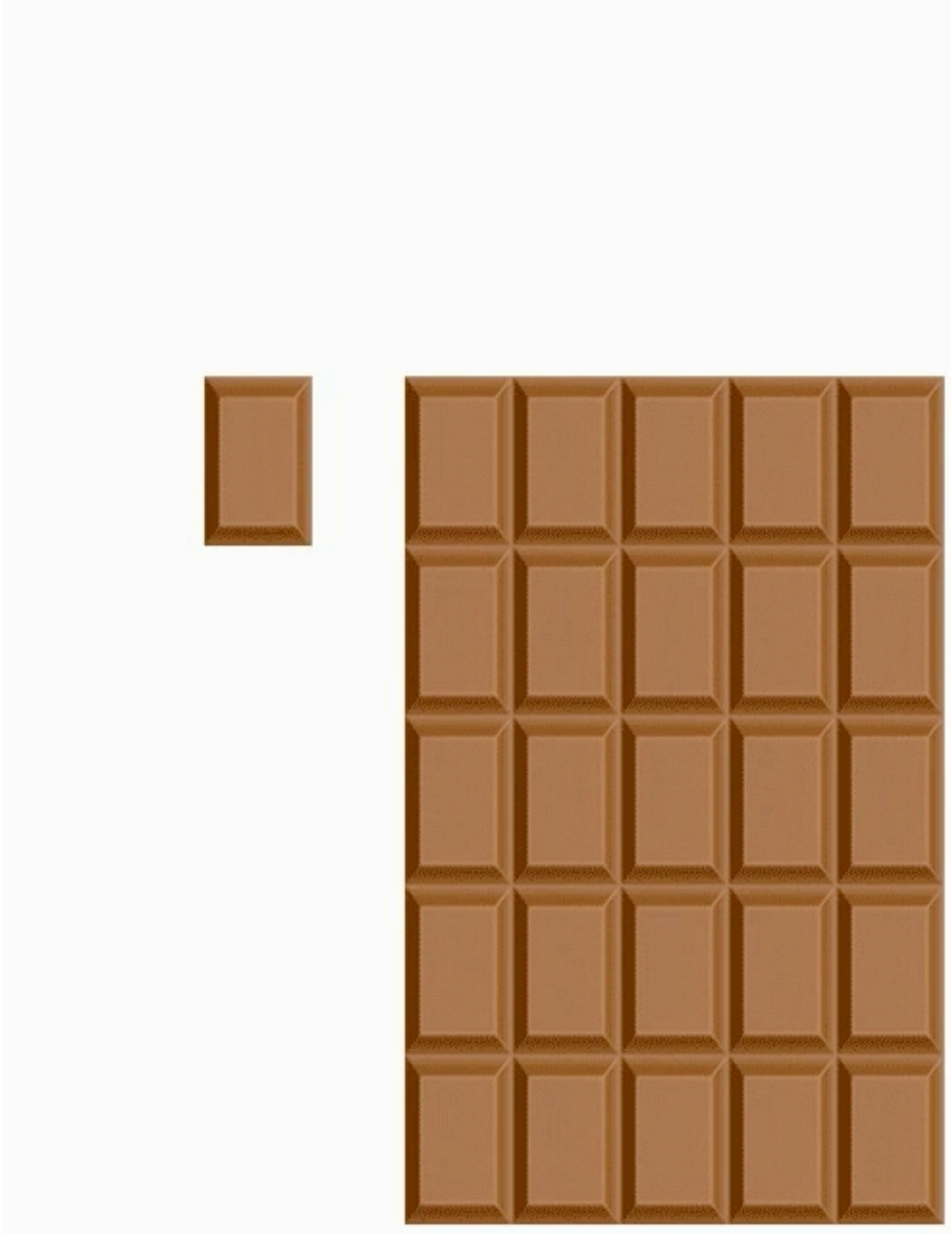

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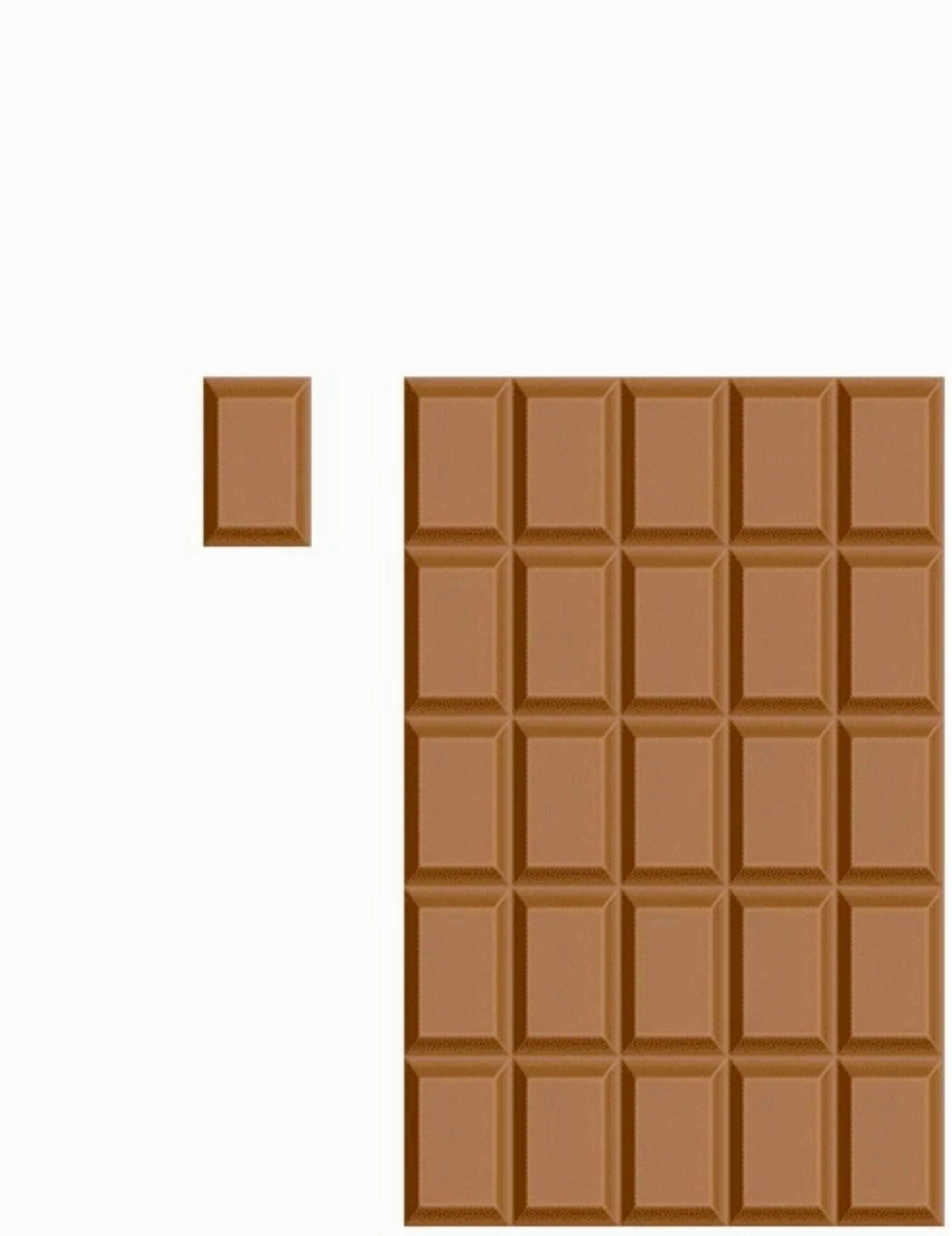

Problem: Easy to detect.

Successful embezzlement requires that it is hard to detect.

Should be difficult to measure the amount of the resource that one wants to embezzle.

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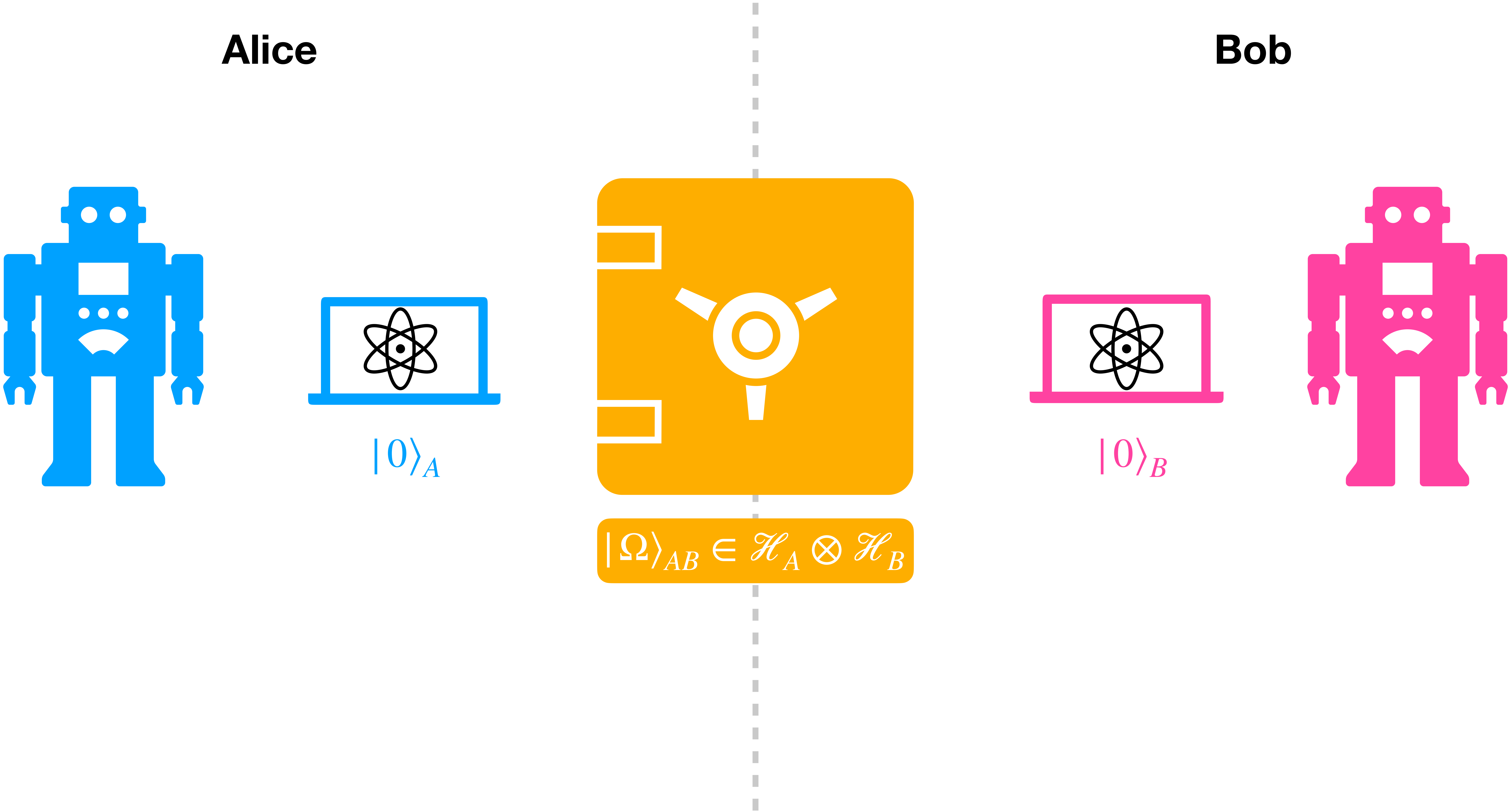
Embezzlement of entanglement

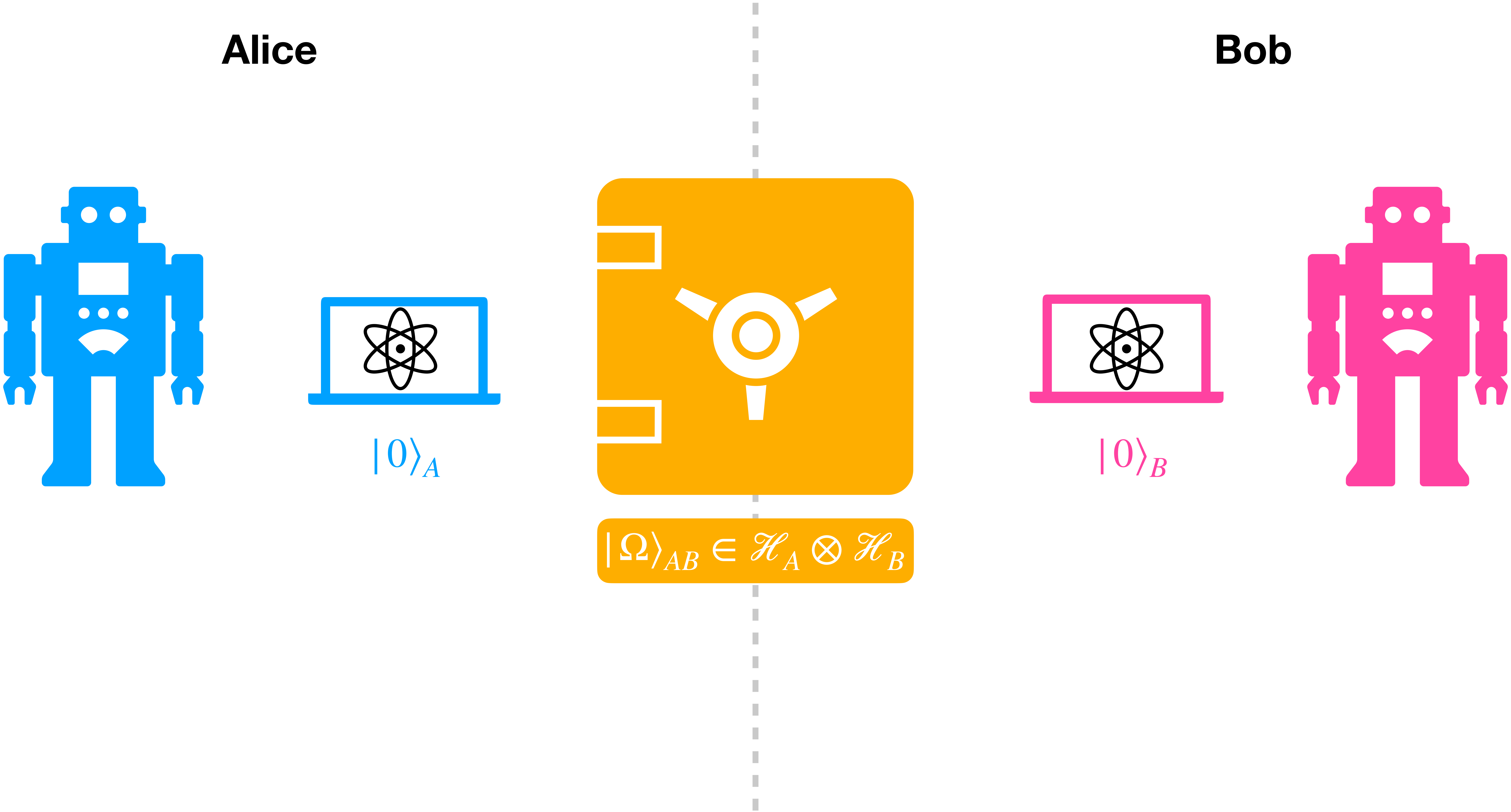
The crime of secretly taking entanglement that is in your care or that belongs to an organization or business you work for.


Alice
Bob
$|0\rangle_A$
$|0\rangle_B$
$|\Omega\rangle_{AB} \in \mathscr{H}_A \otimes \mathscr{H}_B$

Embezzlement of entanglement

The crime of secretly taking entanglement that is in your care or that belongs to an organization or business you work for.

To make sure that no record of the crime exists, this should be done without classical communication!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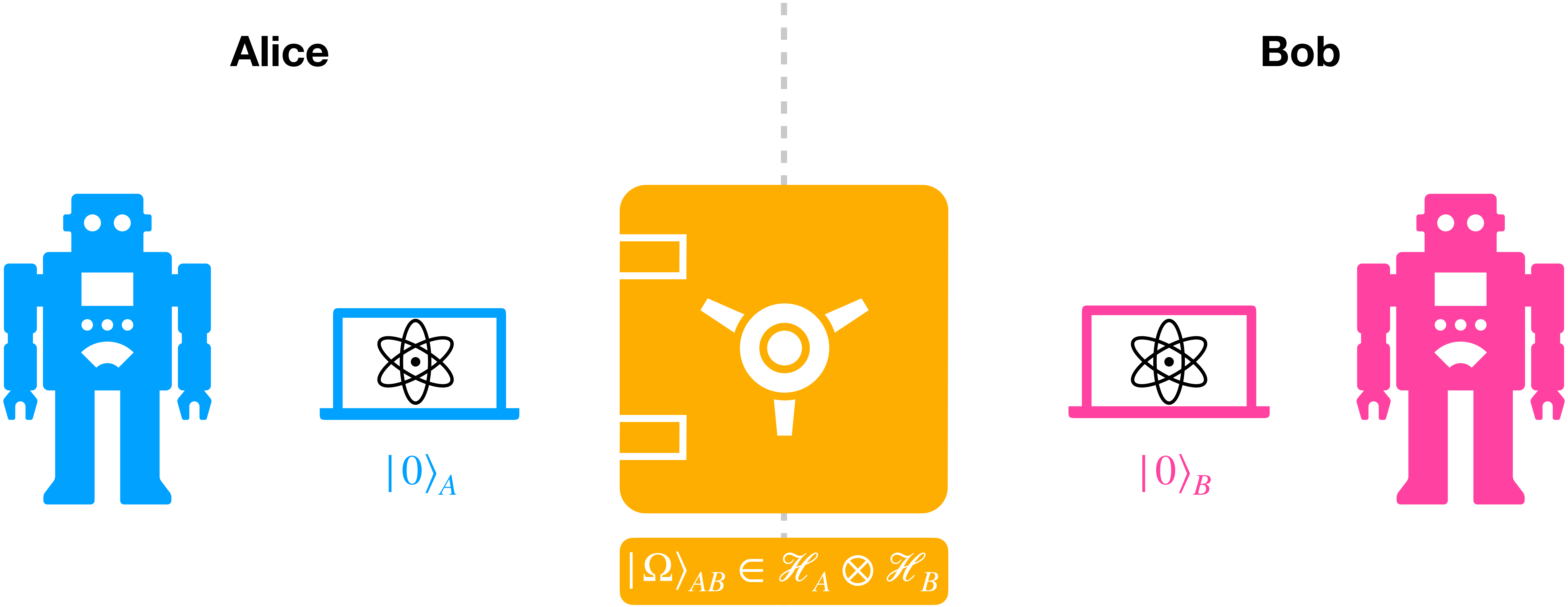
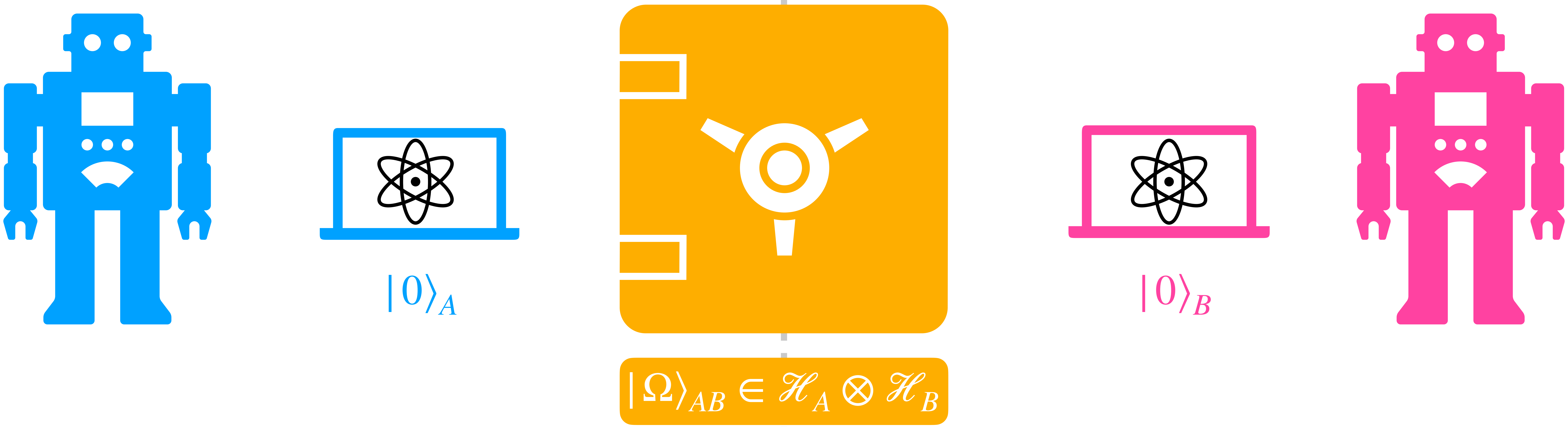


Ideally: $U_A U_B(|\Omega\rangle_{AB} \otimes |0\rangle_A |0\rangle_B) = |\Omega\rangle_{AB} \otimes |\Psi\rangle_{AB}$

Alice **Bob**



Ideally: $U_A U_B(|\Omega\rangle_{AB} \otimes |0\rangle_A |0\rangle_B) = |\Omega\rangle_{AB} \otimes |\Psi\rangle_{AB}$ **Impossible!**

Richard Cleve, Li Liu, and Vern I. Paulsen, “Perfect Embezzlement of Entanglement”, *Journal of Mathematical Physics* 58, no. 1 (2017)

The van Dam — Hayden family

The family of states

$$|\Omega_n\rangle_{AB} := \frac{1}{\sqrt{H_n}} \sum_{j=1}^{n} \frac{1}{\sqrt{j}} |j\rangle_A |j\rangle_B$$

fulfills

$$\inf_{U_A, U_B} \| U_A U_B (|\Omega_n\rangle_{AB} \otimes |0\rangle_A |0\rangle_B) - |\Omega_n\rangle_{AB} \otimes |\Psi\rangle_{AB} \| \leq 2 \frac{\log(d)}{\log(n)}$$

Schmidt rank d

Embezzling family

W. van Dam and P. Hayden, "Universal Entanglement Transformations without Communication", *Physical Review A* 67, no. 6 (2003): 060302

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$$\lambda_j \sim \frac{1}{j}$$

D. Leung and B. Wang, "Characteristics of Universal Embezzling Families", *Physical Review A* 90, no. 4 (2014): 042331

Elia Zanoni, Thomas Theurer, and Gilad Gour, "Complete Characterization of Entanglement Embezzlement", (2023), arXiv.2303.17749.

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- The normalization factor H_n diverges with n.
  Cannot simply take the limit!

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Elia Zanoni, Thomas Theurer, and Gilad Gour, "Complete Characterization of Entanglement Embezzlement", (2023), arXiv.2303.17749.

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*Might depend on quantum gravity

$\otimes$ $\longrightarrow$ $\mathscr{M}$

Tensor product framework

Commuting operator framework

See also: L. van Luijk, R. Schwonnek, AS, R. F. Werner, “The Schmidt rank for the commuting operator framework”, arXiv:2307.11619

Setting: Commuting operator framework

	Commuting Operator	Tensor Product
Hilbert space	$\mathscr{H}_{AB}$	$\mathscr{H}_A \otimes \mathscr{H}_B$
Quantum States	$\lvert\Omega\rangle \in \mathscr{H}_{AB}$	$\lvert\Omega\rangle \in \mathscr{H}_A \otimes \mathscr{H}_B$
Operators of Alice	$\mathscr{M}_A \subseteq \mathscr{B}(\mathscr{H}_{AB})$	$\mathscr{B}(\mathscr{H}_A) \otimes 1$
Operators of Bob	$\mathscr{M}_B \subseteq \mathscr{B}(\mathscr{H}_{AB})$	$1 \otimes \mathscr{B}(\mathscr{H}_B)$
Haag Duality	$\mathscr{M}_A = \mathscr{M}_B'$	automatic
No classical d.o.f.	$\mathscr{M}_A \cap \mathscr{M}_A' = \mathbb{C}1$	automatic

$$\mathscr{M}' := \{a \in \mathscr{B}(\mathscr{H}) \mid [a, \mathscr{M}] = 0\}$$

Commutant

- All Hilbert spaces separable
- All algebras von Neumann algebras:

$$\mathscr{M} = \mathscr{M}''$$

(closed in weak operator topology)

- Factor:

$$\mathscr{M} \cap \mathscr{M}' = \mathbb{C}1$$

Bipartite systems and embezzling states

Bipartite system: A triple $(\mathscr{H}, \mathscr{M}, \mathscr{M}')$.

Embezzling state: $|\Omega\rangle_{AB} \in \mathscr{H}$ such that for all $|\Psi\rangle_{AB} \in \mathbb{C}^n \otimes \mathbb{C}^n$ and all $\varepsilon > 0$ exist local unitaries such that

$$\| U_A U_B \big(|\Omega\rangle_{AB} \otimes |0\rangle_A |0\rangle_B \big) - |\Omega\rangle_{AB} \otimes |\Psi\rangle_{AB} \| < \varepsilon$$

$U_A \in \mathscr{M} \otimes M_n \otimes 1, \quad U_B \in \mathscr{M}' \otimes 1 \otimes M_n$

Matrix algebras: $M_n = M_{n \times n}(\mathbb{C}) \cong \mathscr{B}(\mathbb{C}^n)$

Exact embezzlement

Is exact embezzlement (with $\varepsilon = 0$) possible in commuting operator framework?

Theorem.

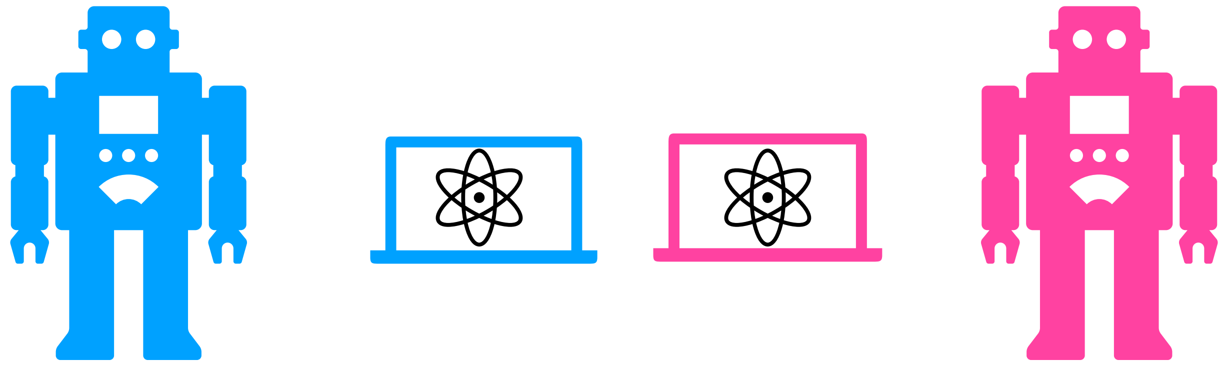
Exact embezzling states exist, but only if $\mathscr{H}$ is **non-separable**.

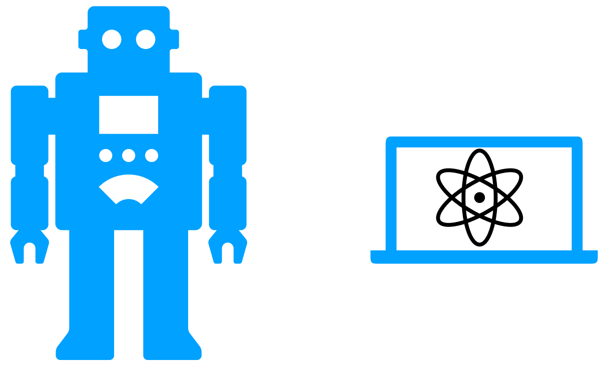
From bipartite to monopartite: Tensor product framework

Pure state entanglement theory

Nielsen’s theorem

Majorization theory on reduced state





Purification

$$|\Psi\rangle_{AB} \in \mathscr{H}_A \otimes \mathscr{H}_B$$

$$\psi_A \in \mathscr{S}(\mathscr{H}_A)$$

M. A. Nielsen, “Conditions for a Class of Entanglement Transformations”, *Physical Review Letters* 83, (1999)

From bipartite to monopartite

Monopartite embezzling state: State ω on $\mathscr{M}$ such that for all $\varepsilon > 0$ and all states ψ on M_n there exists a unitary such that

$$\|U(\omega \otimes \langle 0| \cdot |0\rangle)U^* - \omega \otimes \psi\| < \varepsilon$$

$U \in \mathscr{M} \otimes M_n$

Standard bipartite system: Every marginal ω on $\mathscr{M}$ has purification:

$$\omega(a) = \langle \Omega_\omega | a | \Omega_\omega \rangle \quad \forall a \in \mathscr{M}$$

Same is true for $\mathscr{M}'$.

From bipartite to monopartite II

Result: On a standard bipartite system, entanglement embezzlement is equivalent to monopartite embezzlement:

$$\| U_A U_B (|\Omega\rangle_{AB} \otimes |0\rangle_A |0\rangle_B) - |\Omega\rangle_{AB} \otimes |\Psi\rangle_{AB} \| < \varepsilon$$

$$\Updownarrow$$

$$\| U_A (\omega \otimes \langle 0 | \cdot | 0 \rangle) U_A^* - \omega \otimes \psi_A \| < \varepsilon'$$

Quantifying embezzlement

Quantifying embezzlement

Quality of a state to act as an embezzler in worst case:

$$\kappa(\omega) = \sup_{n\in\mathbb{N}} \sup_{\rho,\sigma} \inf_{U} ||U(\omega \otimes \rho)U^* - \omega \otimes \sigma||$$

Minimize error over unitaries

Maximize error over all n-dim. density matrices

Maximize over all finite dimensions

Quantifying embezzlement

Quality of a state to act as an embezzler in worst case:

$$\kappa(\omega) = \sup_{n\in\mathbb{N}} \sup_{\rho,\sigma} \inf_{U} \| U(\omega \otimes \rho)U^* - \omega \otimes \sigma \|$$

Best worst-case error:

Worst worst-case error:

$$\kappa_{\min} = \inf_{\omega} \kappa(\omega) \quad \leq \quad \kappa(\omega) \quad \leq \quad \kappa_{\max} = \sup_{\omega} \kappa(\omega)$$

$\kappa_{\min}$ and $\kappa_{\max}$ are algebraic invariants of $\mathscr{M}$

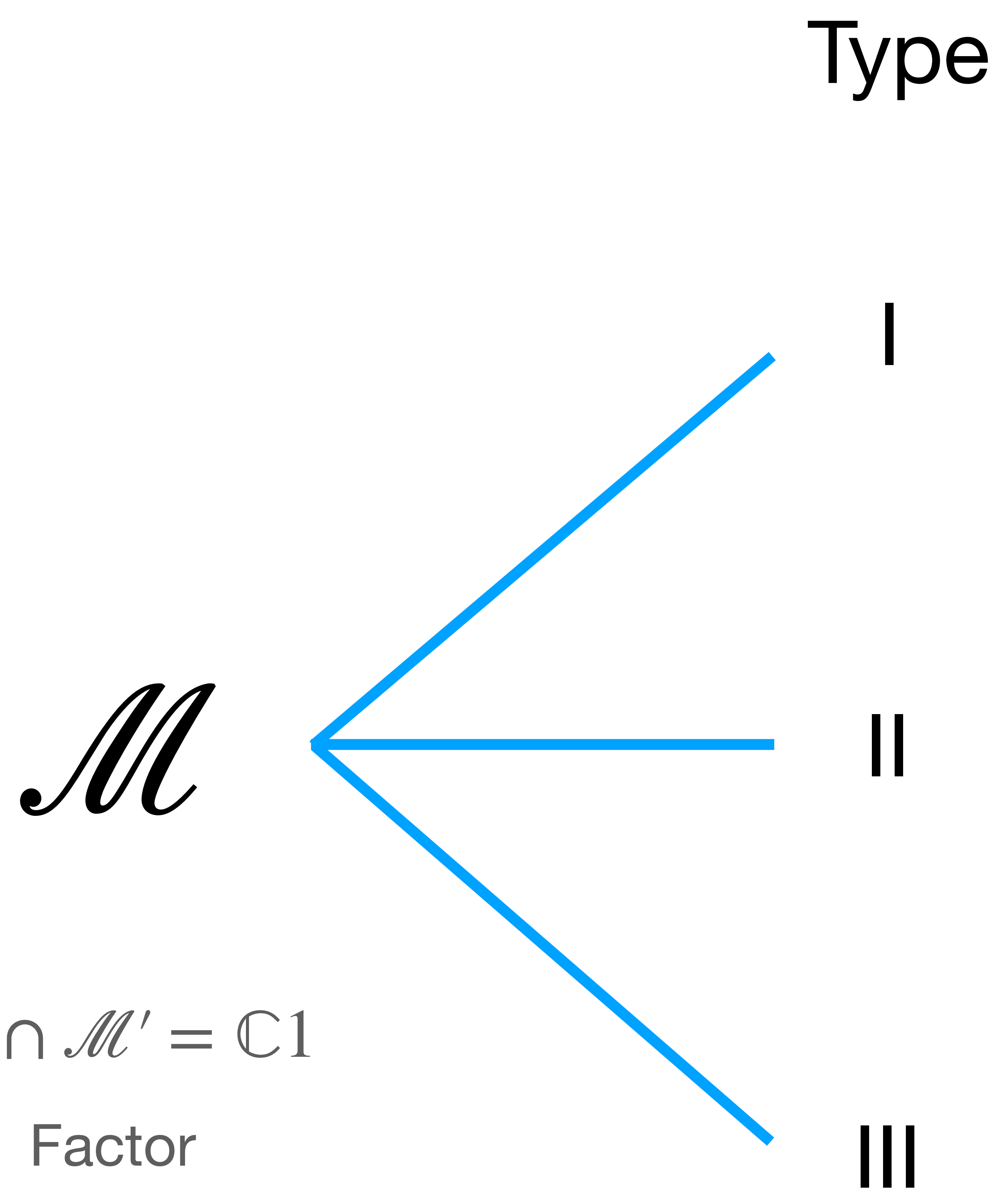
Types of Factors

$$\mathscr{M}$$

$\mathscr{M} \cap \mathscr{M}' = \mathbb{C}1$

Factor

Types of Factors



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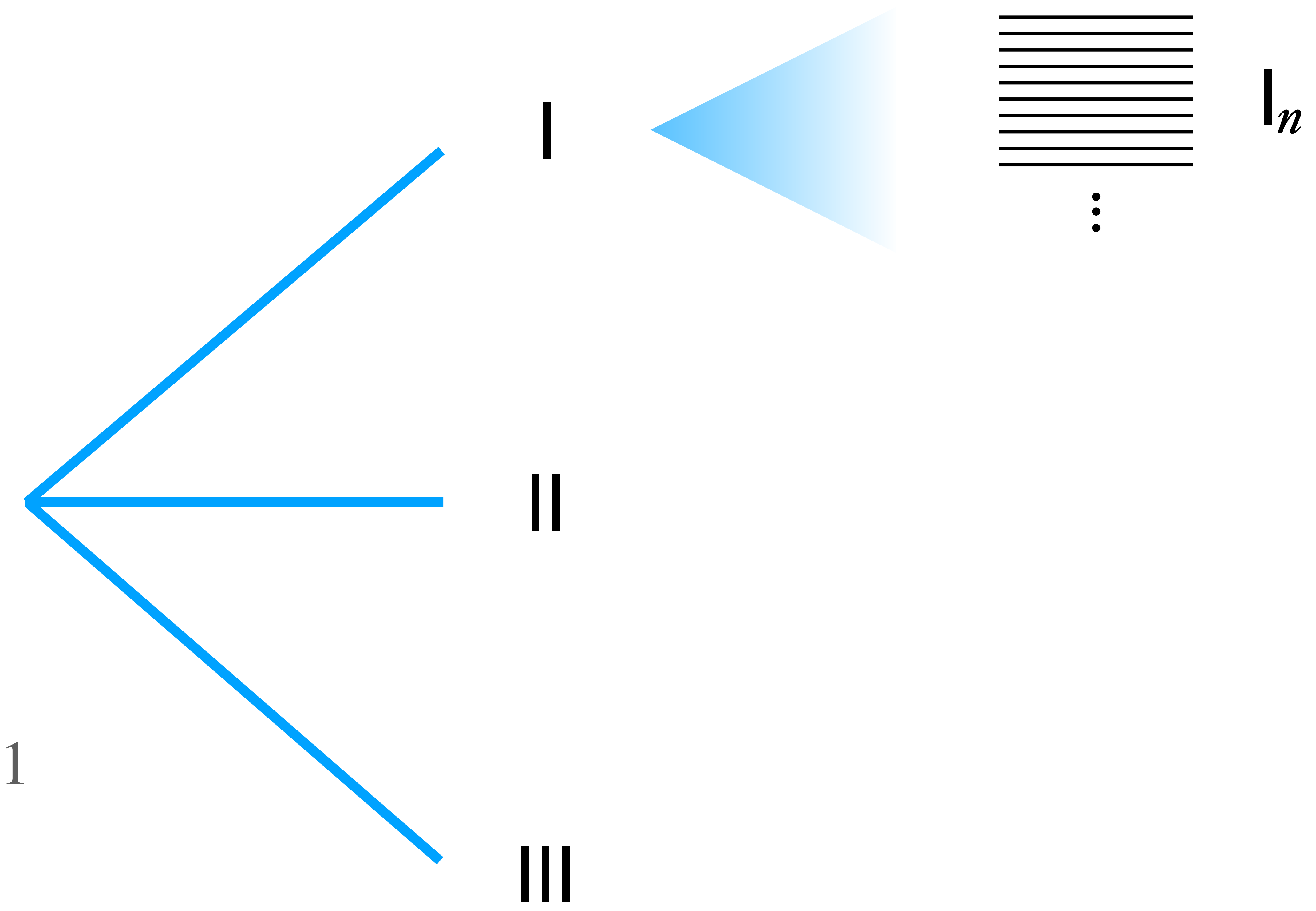
Factor

Types of Factors

Type | Subtype

$\mathscr{M}$



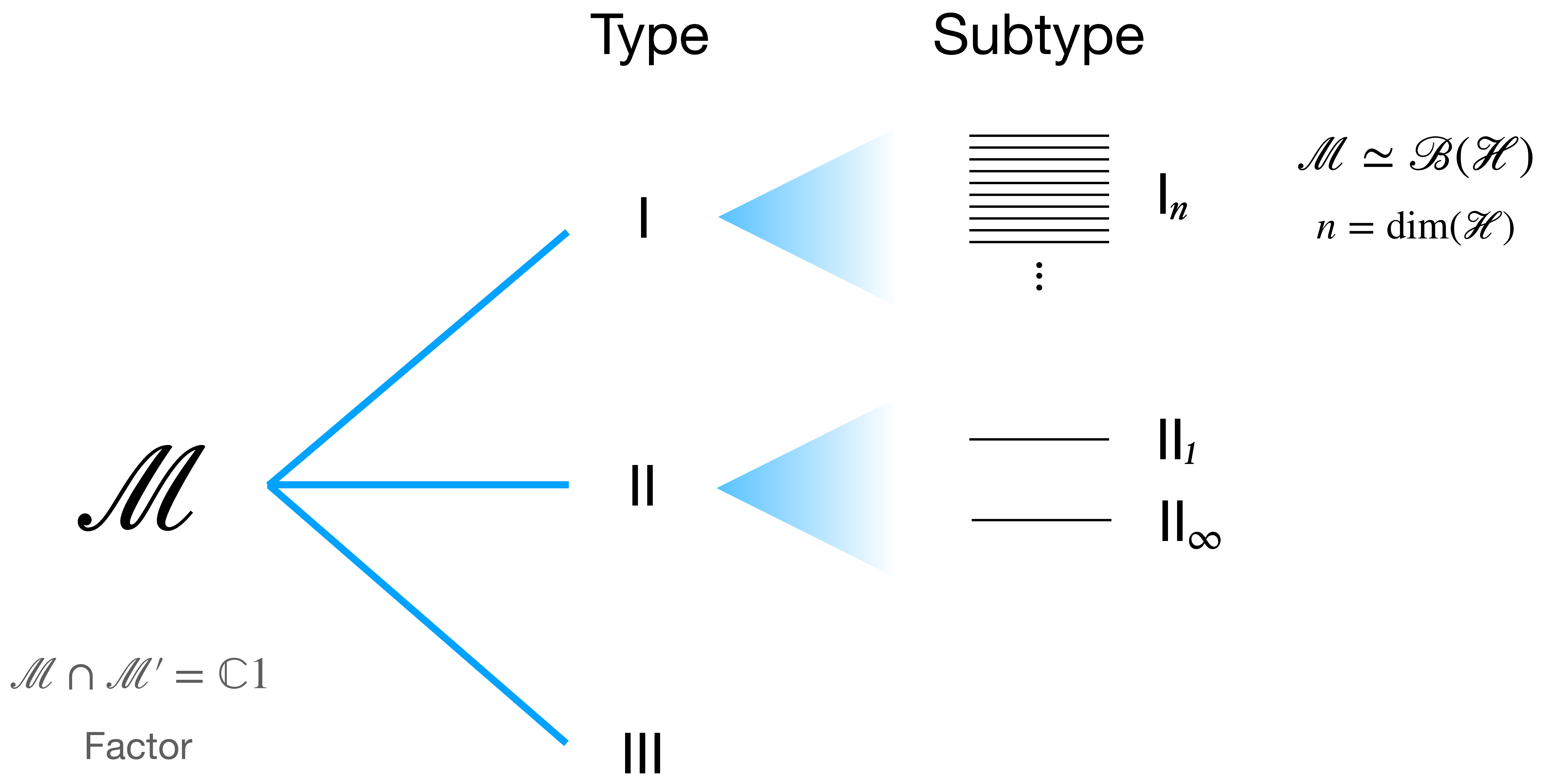
$$\mathscr{M} \simeq \mathscr{B}(\mathscr{H})$$

$$n = \dim(\mathscr{H})$$

$$\mathscr{M} \cap \mathscr{M}' = \mathbb{C}1$$

Factor

Types of Factors



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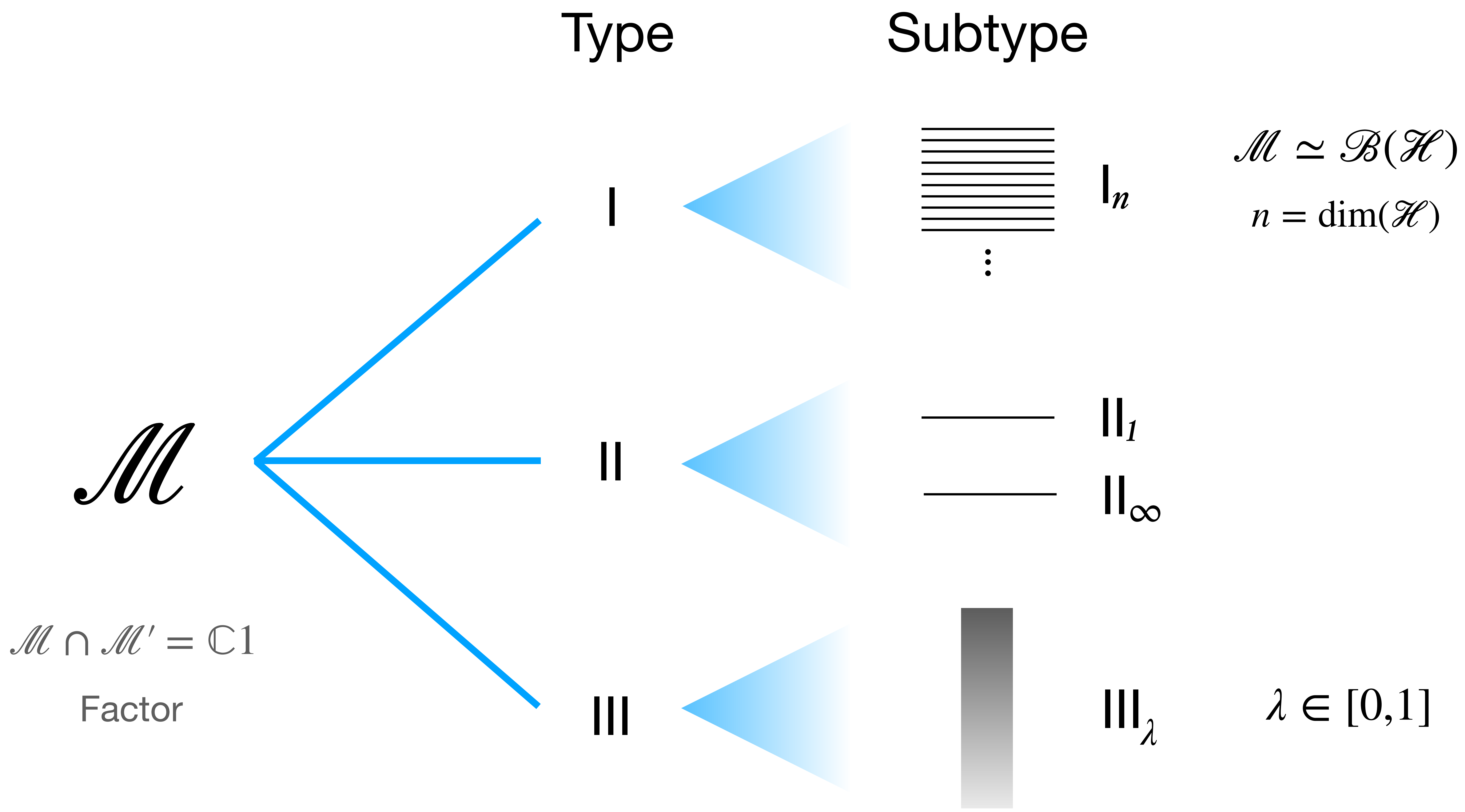

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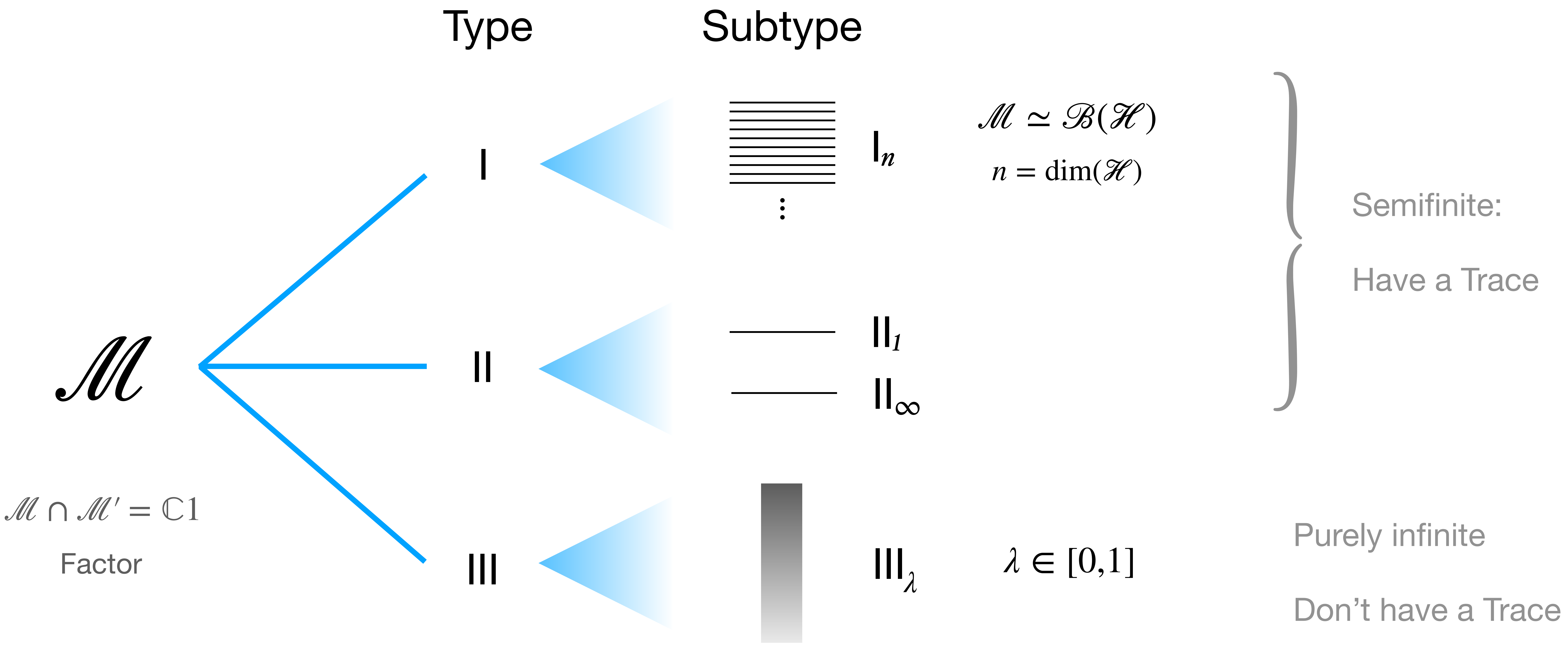

Main results

Type	I	II	III		
Subtype	*	*	$\lambda = 0$	$0 < \lambda < 1$	$\lambda = 1$
κ_{min}	2	2	$\in [0,2]$	**0**	**0**
κ_{max}	2	2	2	$2\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}$	**0**

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No embezzling states

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No embezzling states

Some embezzling states

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No embezzling states (Types I and II)

Some embezzling states ($0 < \lambda < 1$)

All states embezzling ($\lambda = 1$)

Main results

Type	I	II	III		
Subtype	*	*	$\lambda = 0$	$0 < \lambda < 1$	$\lambda = 1$
$\kappa_{\min}$	2	2	$\in [0,2]$	**0**	**0**
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No embezzling states

All states embezzling

Some embezzling states

Sometimes embezzling states, sometimes not

Main results

Type	I	II	III		
Subtype	*	*	$\lambda = 0$	$0 < \lambda < 1$	$\lambda = 1$
κ_{min}	2	2	$\in [0,2]$	**0**	**0**
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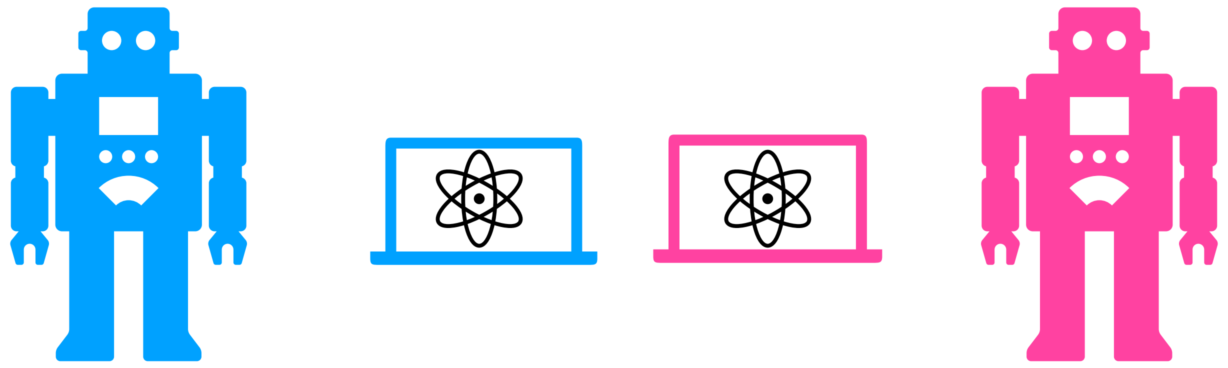
Within type III, the embezzlement quantifiers tell us the subtype!

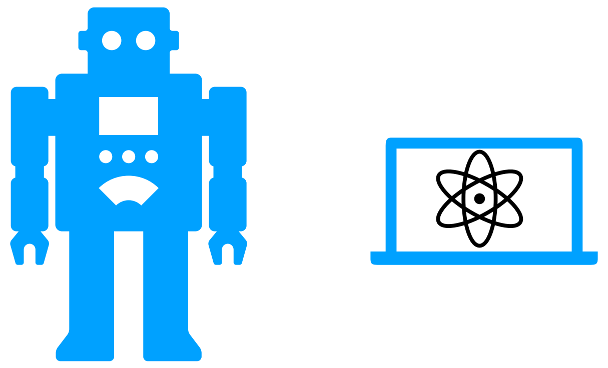
Embezzlement reveals Connes' classification.

From bipartite to monopartite: Commuting operator framework

Embezzling state

Monopartite
embezzling state





Purification

$|\Omega\rangle \in \mathscr{H}$

ω on $\mathscr{M}$

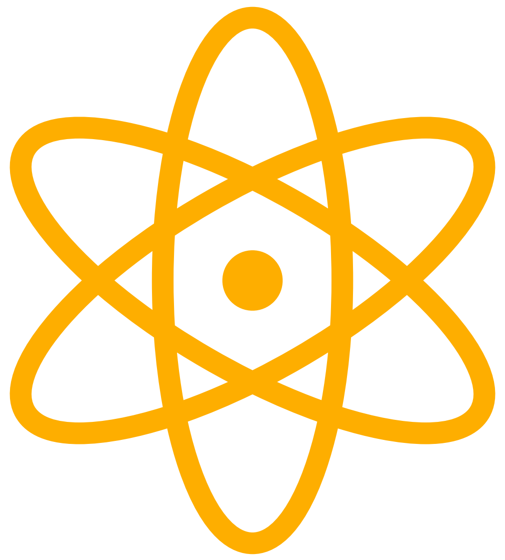
How to replace majorization theory?

Fundamental ingredient

$(\mathscr{M}, \omega)$ ⟶ **Flow of Weights** ⟶ $\big((X, \sigma_s), P_\omega\big)$

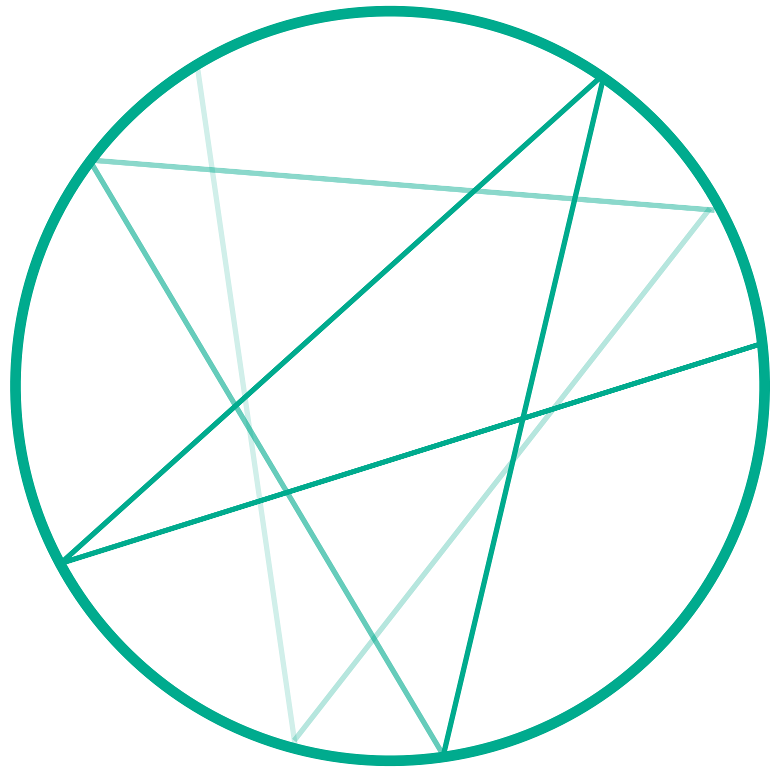
von Neumann Algebra,
state

classical dynamical system,
probability measure



Alain Connes and Masamichi Takesaki, “The Flow of Weights on Factors of Type III”, *Tohoku Mathematical Journal* 29, no. 4 (1977): 473–575

Uffe Haagerup and Erling Størmer, “Equivalence of Normal States on von Neumann Algebras and the Flow of Weights”, *Advances in Mathematics* 83, no. 2 (1990): 180–262



Fundamental ingredient

$(\mathscr{M}, \omega) \longrightarrow$ *Flow of Weights* $\longrightarrow \big((X, \sigma_s), P_\omega\big)$

von Neumann Algebra,
state

classical dynamical system,
probability measure

Embezzlement $\longleftrightarrow$ Invariance under σ_s

Spectral states

ω on $\mathscr{M}$ $\longrightarrow$ P_ω state (= prob. distr.) on X

Theorem. Distance of unitary orbits = distance of spectral states:

$$\inf_{U\in\mathscr{U}(\mathscr{M})} \|U\omega U^* - \varphi\| = \|P_\omega - P_\varphi\|$$

U. Haagerup and E. Størmer. "Equivalence of normal states on von Neumann algebras and the flow of weights", *Advances in Mathematics* 83.2 (1990)

- P_ω contains full spectral information
- The set $\{P_\omega : \omega\}$ can be characterized precisely

Spectral states for $\mathscr{M} = M_n$

Flow of weights: $X = (0,\infty), \ \sigma_s(t) = e^s t$

Spectral states for $\mathscr{M} = M_n$

Flow of weights: $X = (0,\infty), \ \sigma_s(t) = e^s t$

Density matrix $\longrightarrow$ Probability density

$$\rho = \sum_i p_i P_i \qquad\longrightarrow\qquad dP_\rho(t) = D_\rho(t)\,dt$$

Spectral states for $\mathscr{M} = M_n$

Flow of weights: $X = (0,\infty)$, $\sigma_s(t) = e^s t$

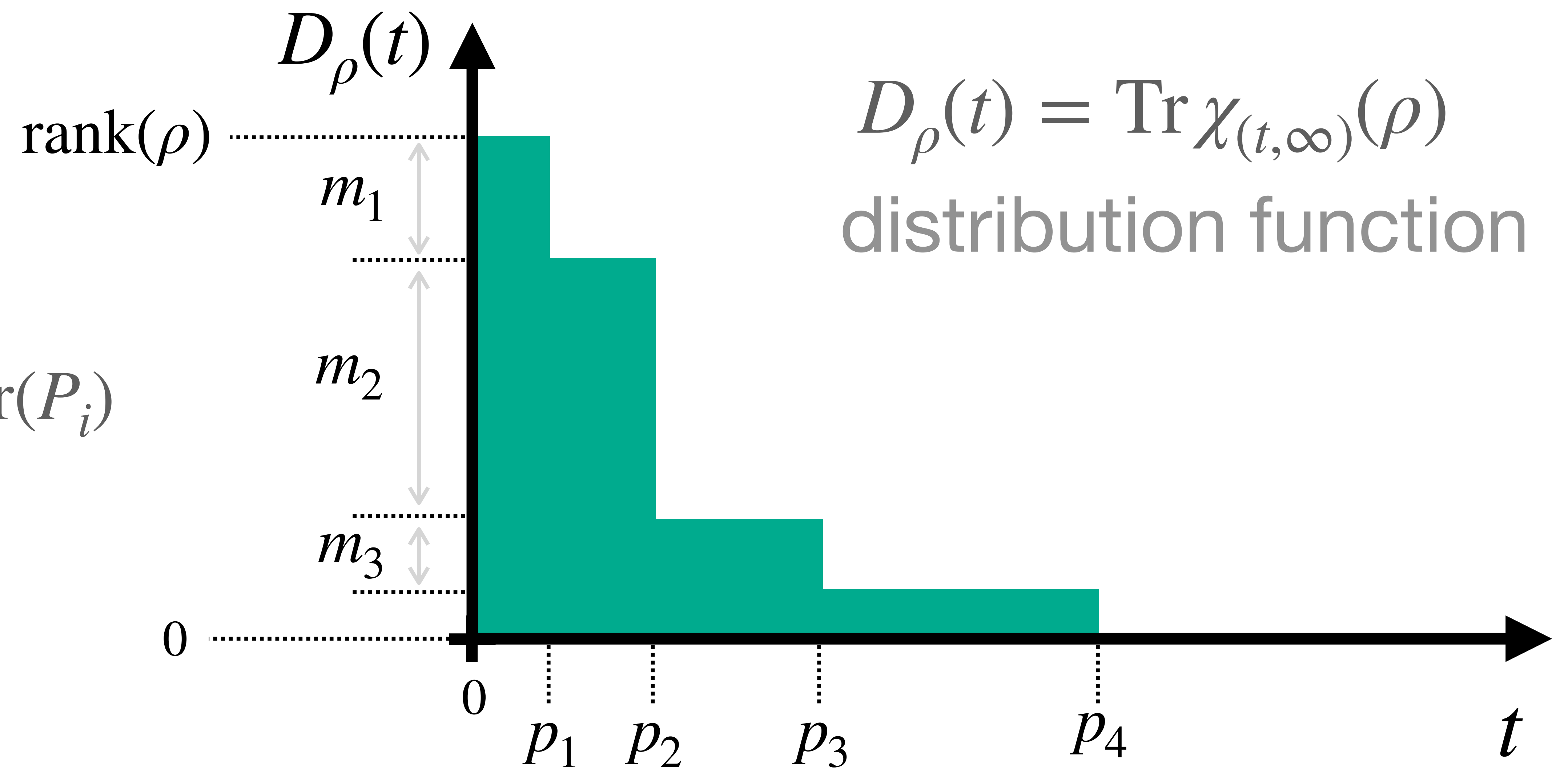
Density matrix

$$\rho = \sum_i p_i P_i$$

$\longrightarrow$

Probability density

$$dP_\rho(t) = D_\rho(t)\, dt$$

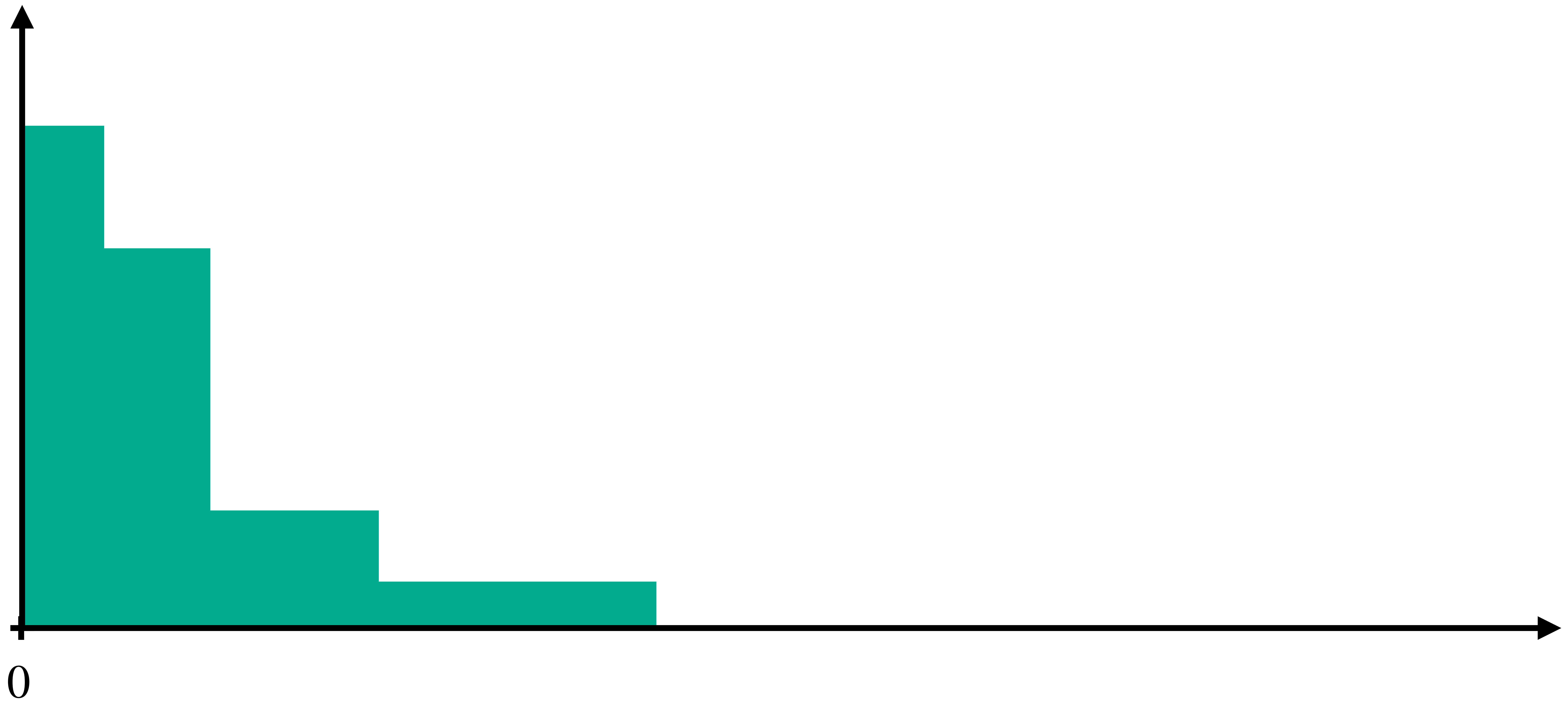


$D_\rho(t) = \mathrm{Tr}\,\chi_{(t,\infty)}(\rho)$

distribution function

Multiplicities $m_i = \mathrm{Tr}(P_i)$

Spectral states for $\mathscr{M} = M_n$ Flow of weights: $X = (0,\infty)$, $\hat{\sigma}_s(t) = e^s \cdot t$



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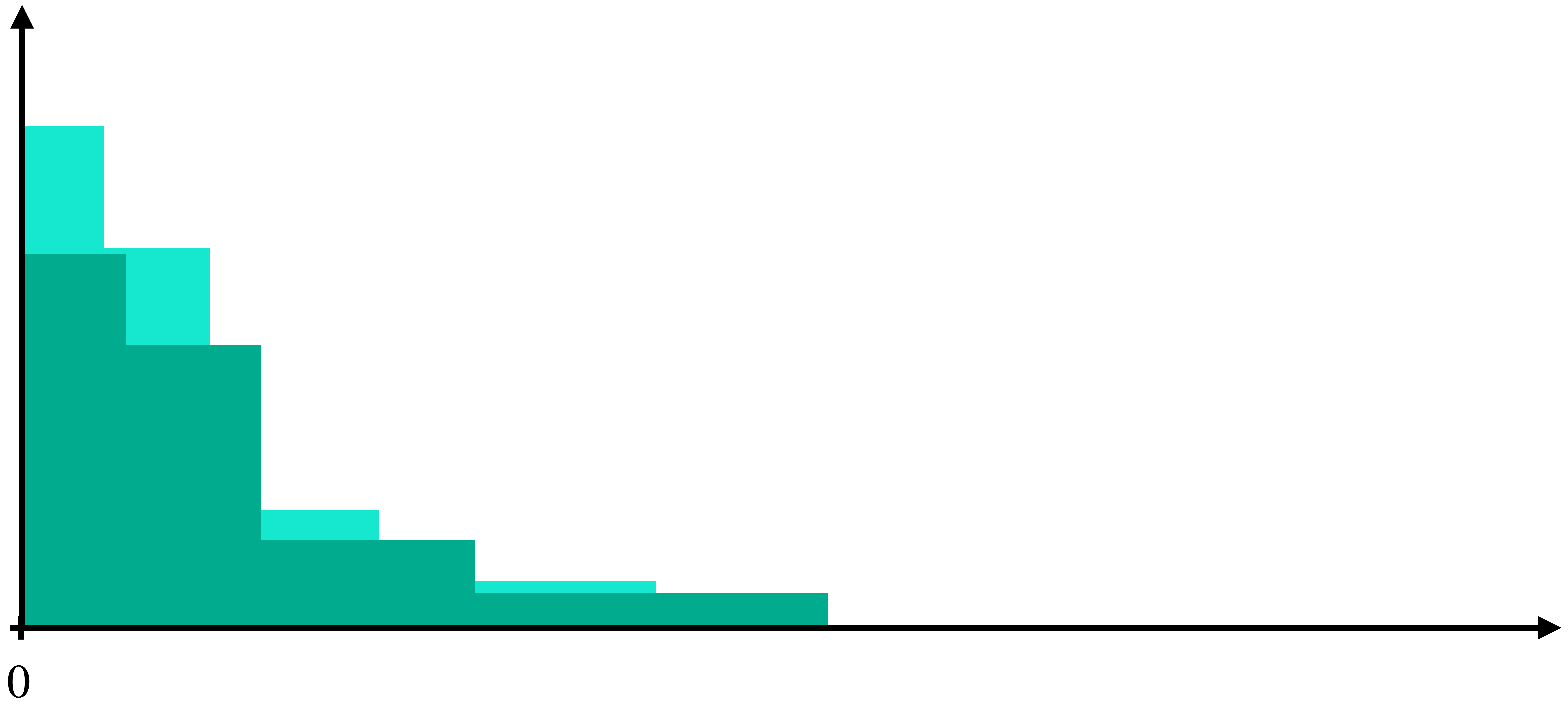

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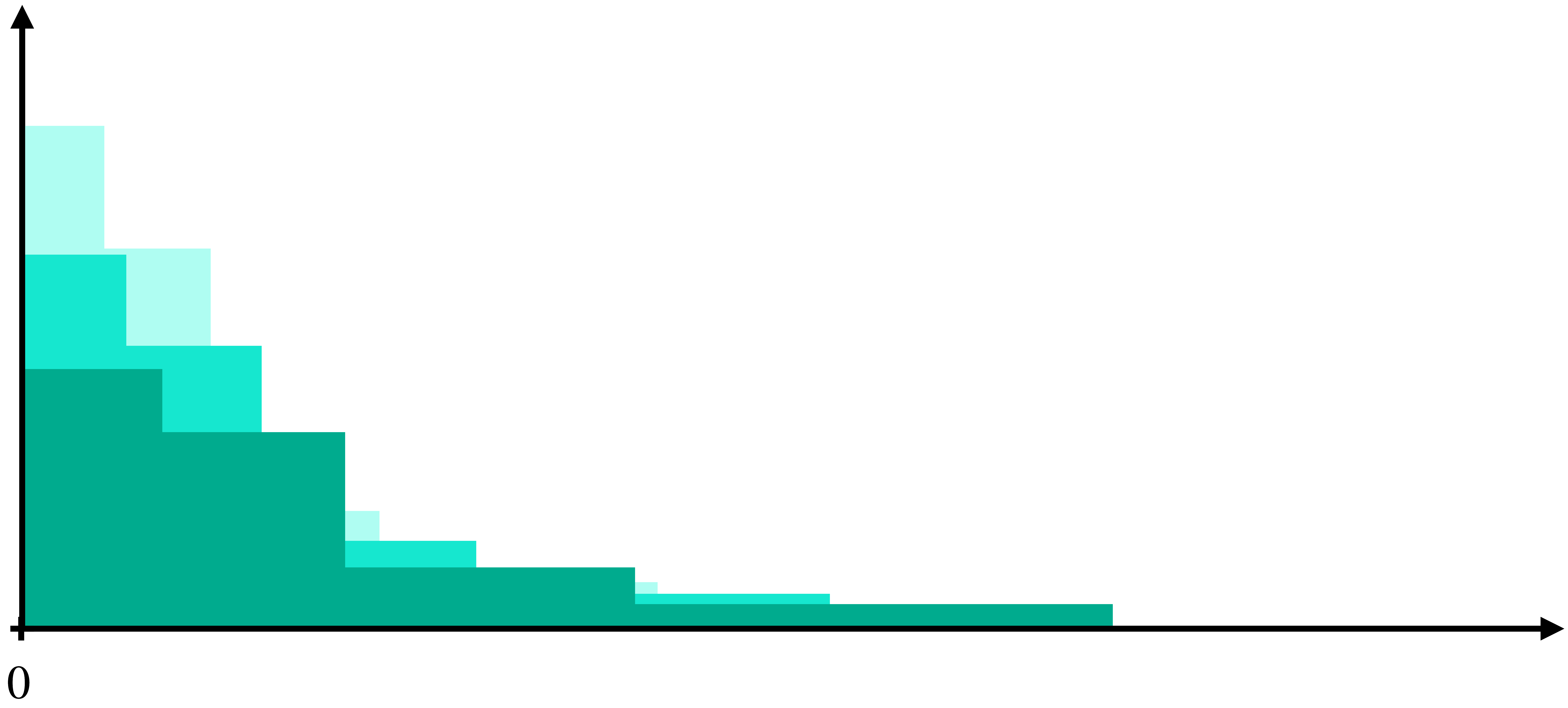

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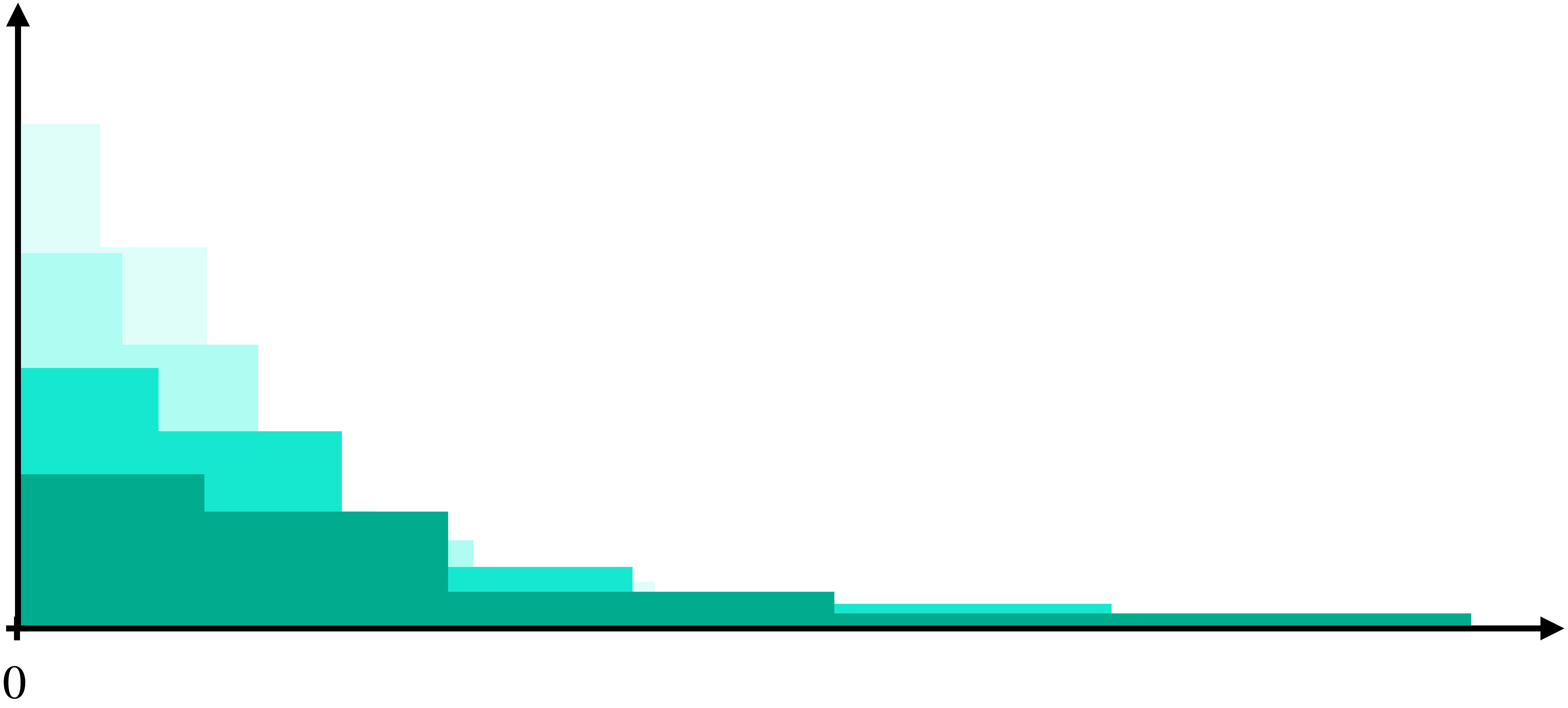

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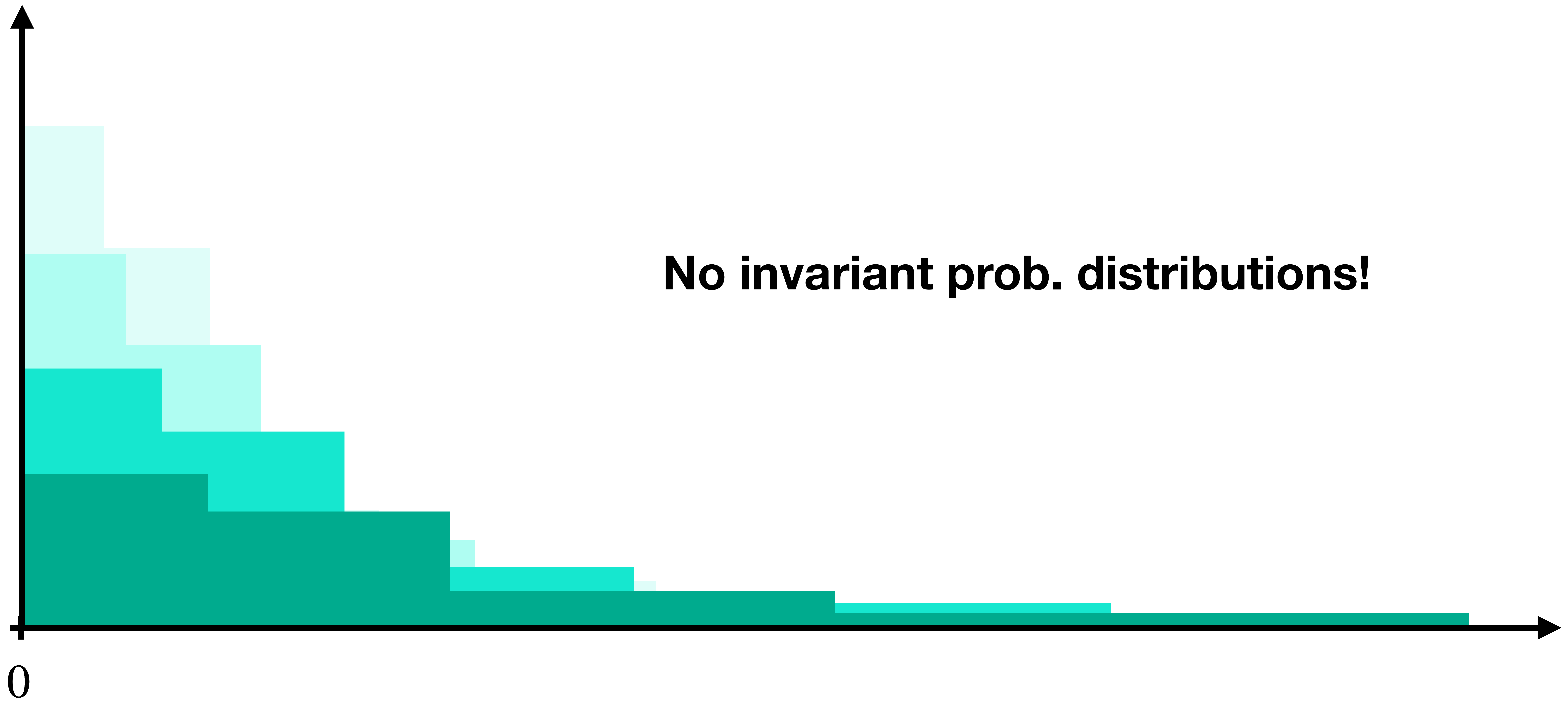

No invariant prob. distributions!

Spectral states and tensor products

$$D_{\omega\otimes\langle 0|\cdot|0\rangle}(t) = D_\omega(t) \qquad\qquad D_{\omega\otimes\frac{1}{n}\mathrm{Tr}}(t) = nD_\omega(tn)$$

Spectral states and tensor products

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Proof sketch: If ω is embezzling, we need for any n:

$$D_\omega(t) = D_{\omega\otimes\langle 0|\cdot|0\rangle}(t) = D_{\omega\otimes\frac{1}{n}\mathrm{Tr}}(t) = nD_\omega(nt)$$

This implies

$$D_\omega\left(\frac{m}{n}t\right) = \frac{n}{m}D_\omega(t) \xrightarrow{\text{Right cont.}} D_\omega(t) = \frac{1}{t}D_\omega(1) \qquad \blacksquare$$

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$D_\omega(t)$ is integrable, but $1/t$ is not.

Spectral states and tensor products

$$D_{\omega\otimes\langle 0|\cdot|0\rangle}(t) = D_\omega(t) \qquad D_{\omega\otimes\frac{1}{n}\mathrm{Tr}}(t) = nD_\omega(tn) = e^{\log n}D_\omega(e^{\log n}t)$$

Tensor product

Ergodic flow

Theorem. If ω is embezzling, then $\lambda_\omega(t) \propto \frac{1}{t}$.

Corollary. If $\mathscr{M}$ is semifinite, there are no embezzling states.

Proof sketch: Flow of weights is the same as for M_n. $D_\omega(t)$ is integrable, but $1/t$ is not.

The fundamental theorem

$(\mathscr{M}, \omega) \longrightarrow$ **Flow of Weights** $\longrightarrow ((X, \hat{\sigma}_s), P_\omega)$

von Neumann Algebra,
state

classical dynamical system,
probability measure

Embezzlement $\longleftrightarrow$ Invariance under $\hat{\sigma}_s$

The fundamental theorem

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von Neumann Algebra, state

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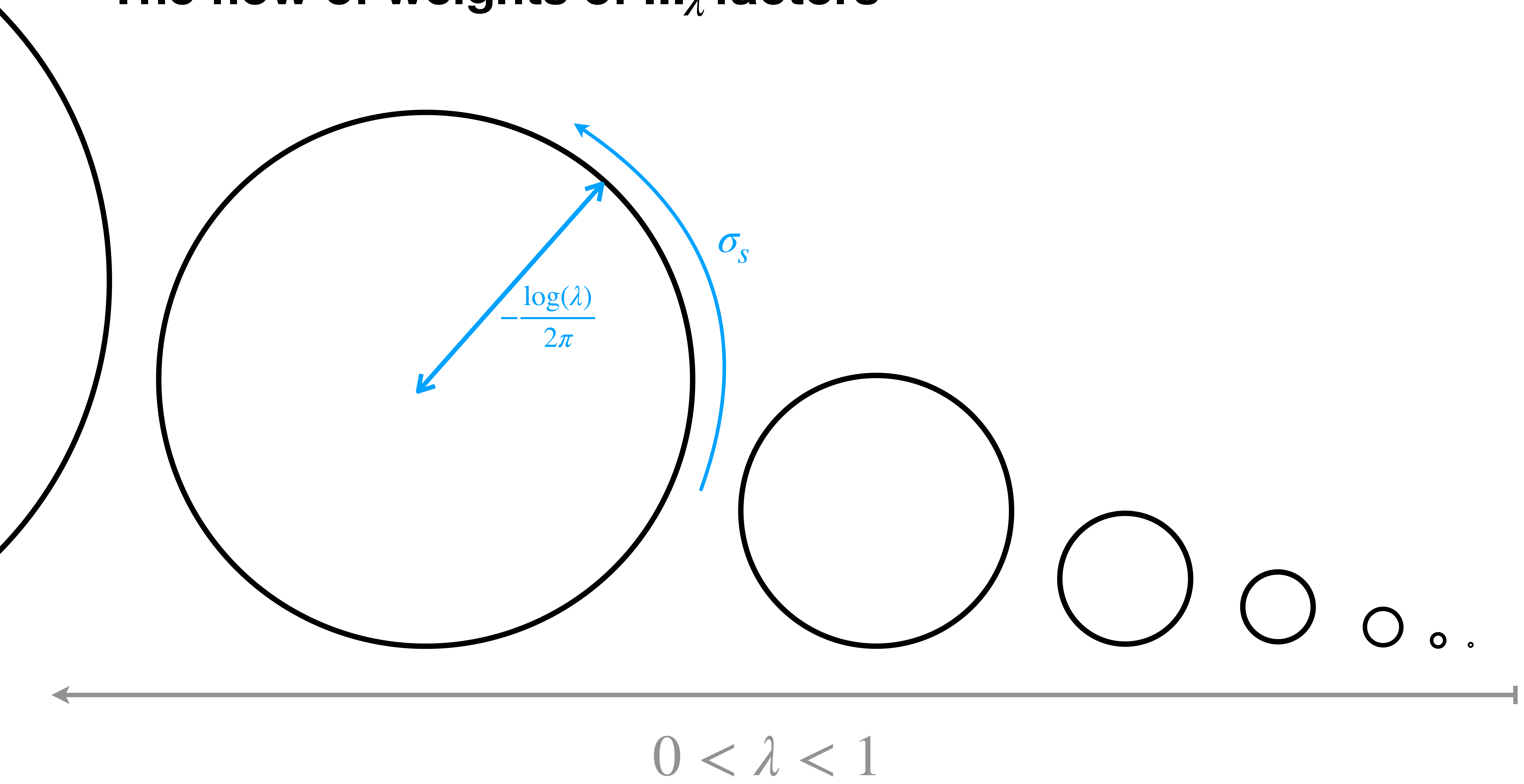
Theorem. $\kappa(\omega) = \sup_{s\in\mathbb{R}} ||\sigma_s(P_\omega) - P_\omega||$

Embezzlement $\longleftrightarrow$ Invariance under $\hat{\sigma}_s$

The flow of weights of III_λ factors



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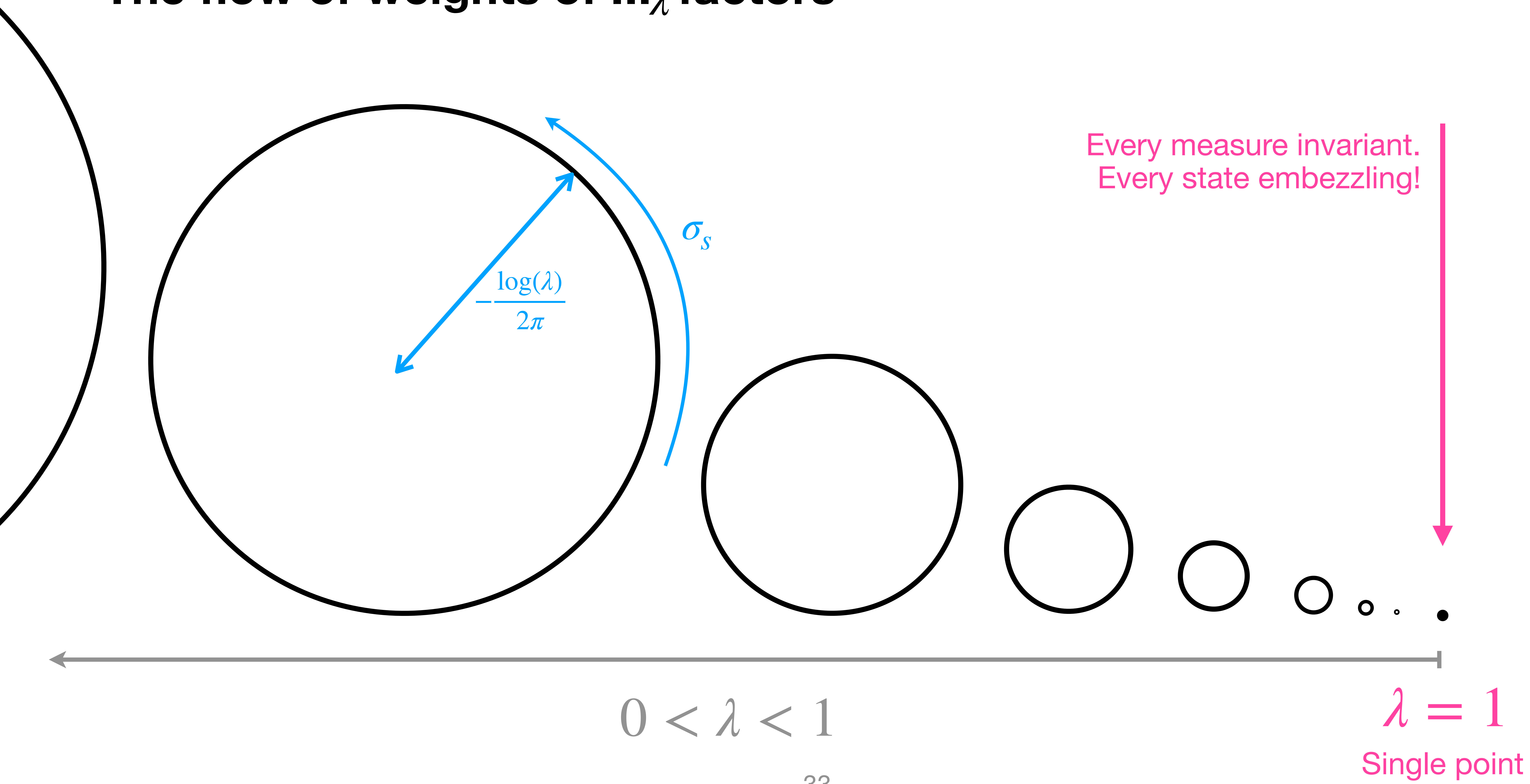

Main results

Type	I	II	III		
Subtype	*	*	$\lambda = 0$	$0 < \lambda < 1$	$\lambda = 1$
$\kappa_{\min}$	2	2	$\in [0,2]$	**0**	**0**
$\kappa_{\max}$	2	2	2	$2\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}$	**0**

Universal embezzlement characterizes type III_1

Universal embezzlement $\Longleftrightarrow$ type III_1 factors

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$\Longleftarrow$

1. A factor is type III_1 if and only if

$$\inf_{u \in \mathcal{M}} \|u\omega_1 u^* - \omega_2\| = 0 \qquad \text{(Connes—Størmer)}$$

2. Type III factors are properly infinite:

$$\mathcal{M} \simeq \mathcal{M} \otimes \mathcal{B}(\mathcal{H})$$

Alain Connes and Erling Størmer, "Homogeneity of the State Space of Factors of Type III1", *Journal of Functional Analysis* 28, no. 2 (1978)

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$\Longrightarrow$

3. Embezzling states have maximal modular spectrum:

$$\mathrm{Sp}\,\Delta_\omega = \mathbb{R}^+$$

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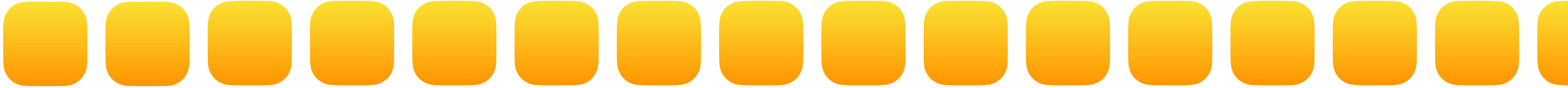
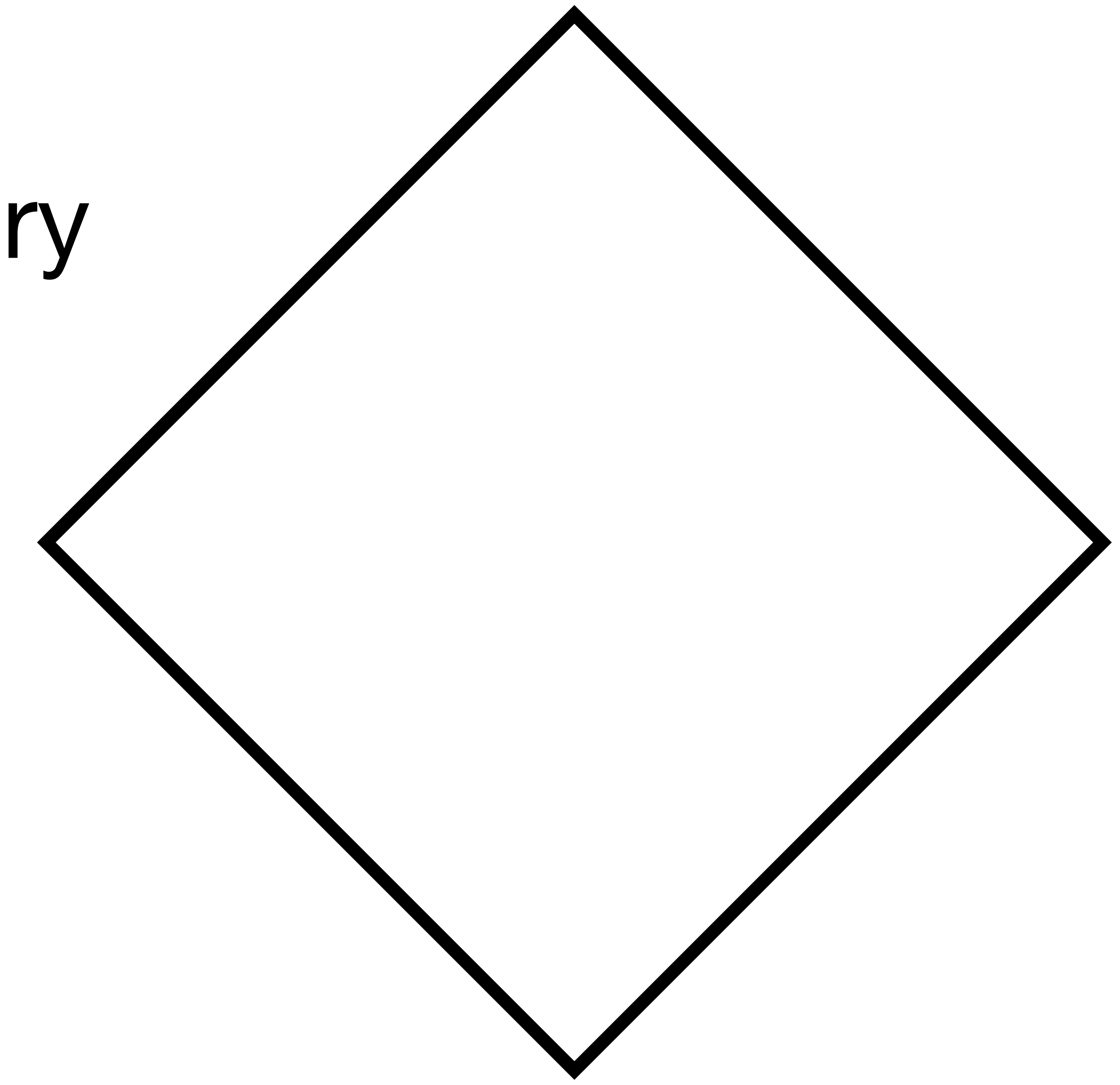
Type III_1 factors and where to find them

Statistical Mechanics: Infinite spin chains



Relativistic Quantum Field Theory



Why universal embezzlers are Type III_1

$$\mathscr{M} \cong \bigotimes_{j=1}^{\infty} (M_2, \rho_j)$$

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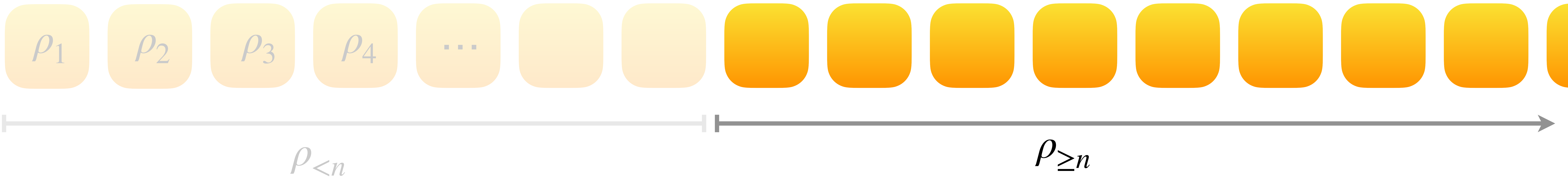
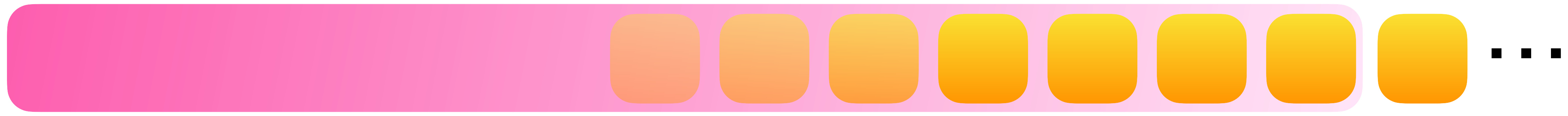


Infinite spin chain isomorphic to original spin chain

↓

Universal embezzler: $\rho_{\geq n}$ must be embezzling.

Why universal embezzlers are Type III_1



ω

σ

Why universal embezzlers are Type III_1

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$$\sigma \approx \sigma_{<n} \otimes \rho_{\geq n}$$

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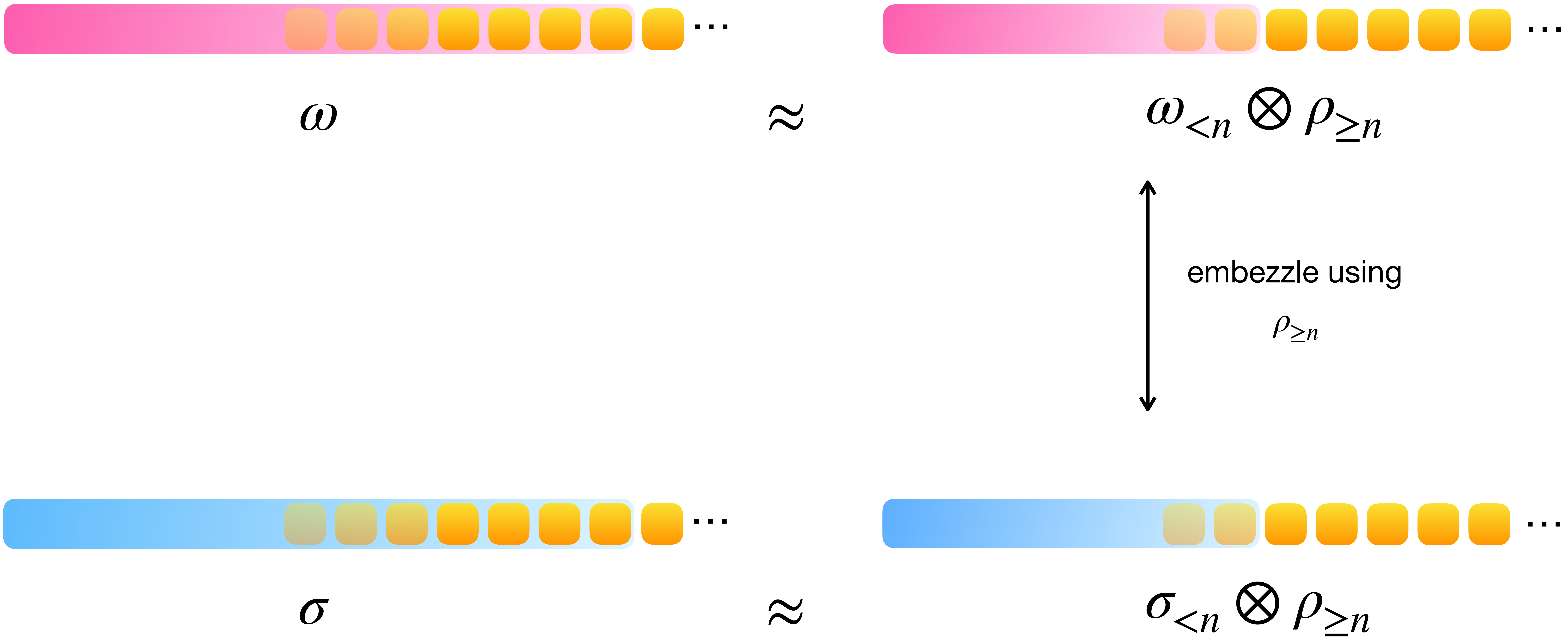

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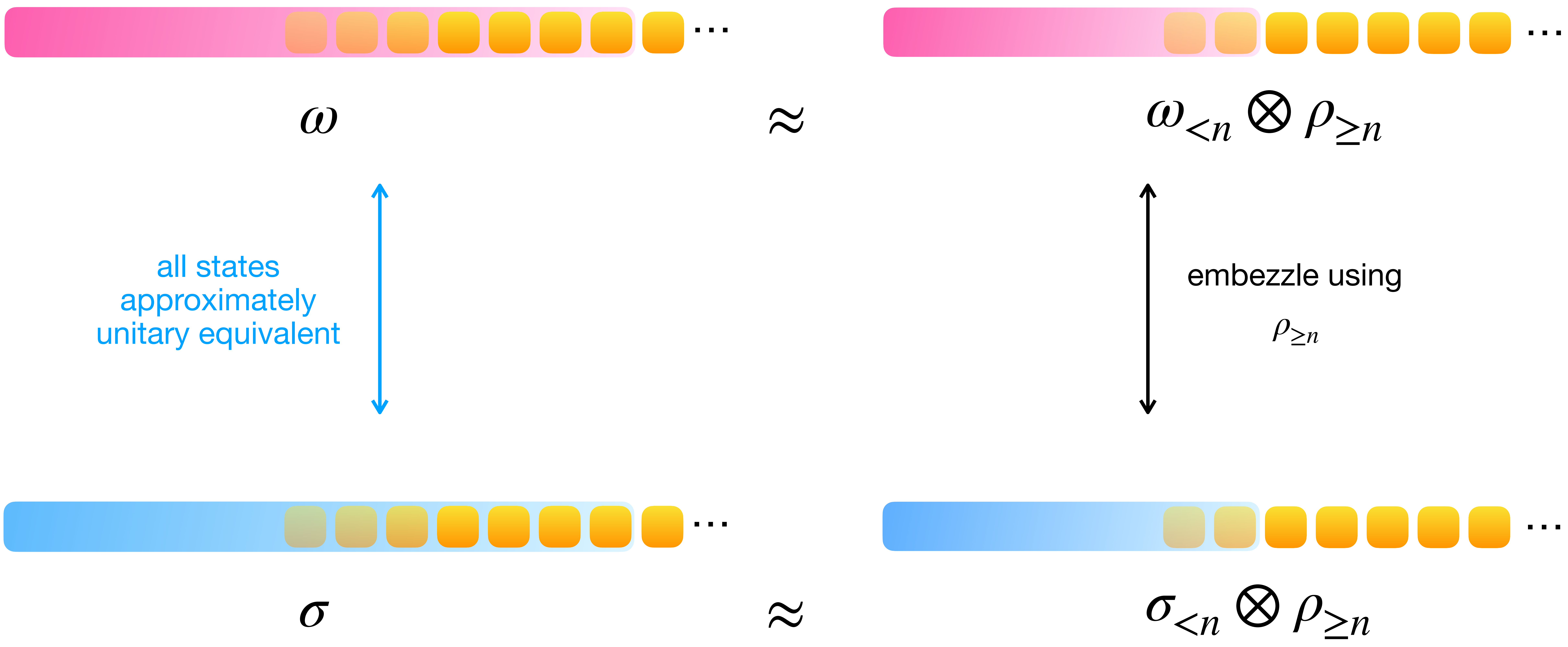
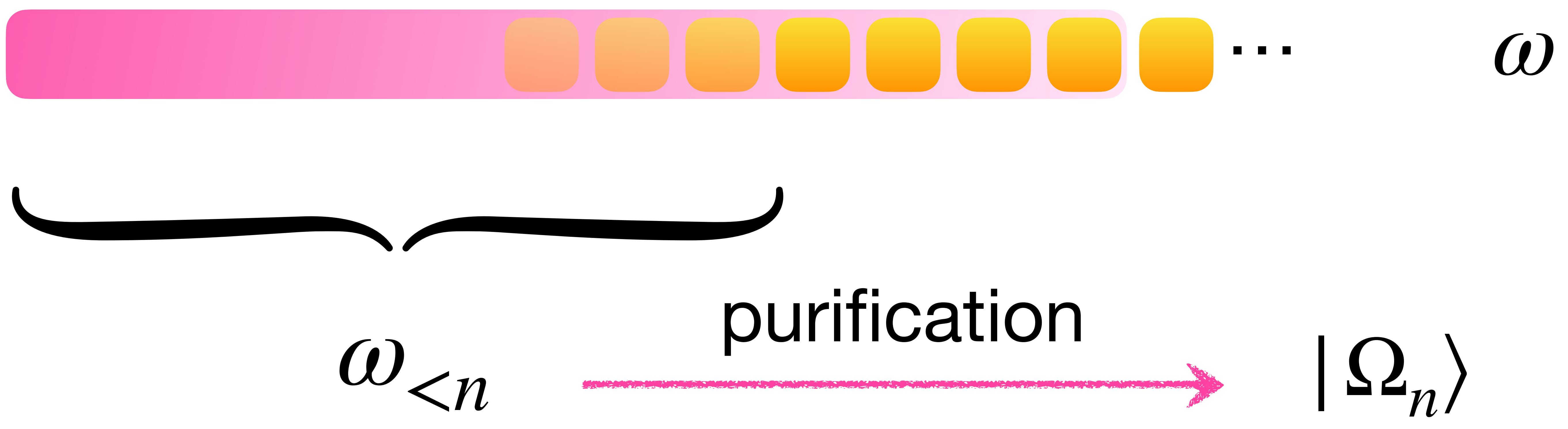


From embezzlers to embezzling families

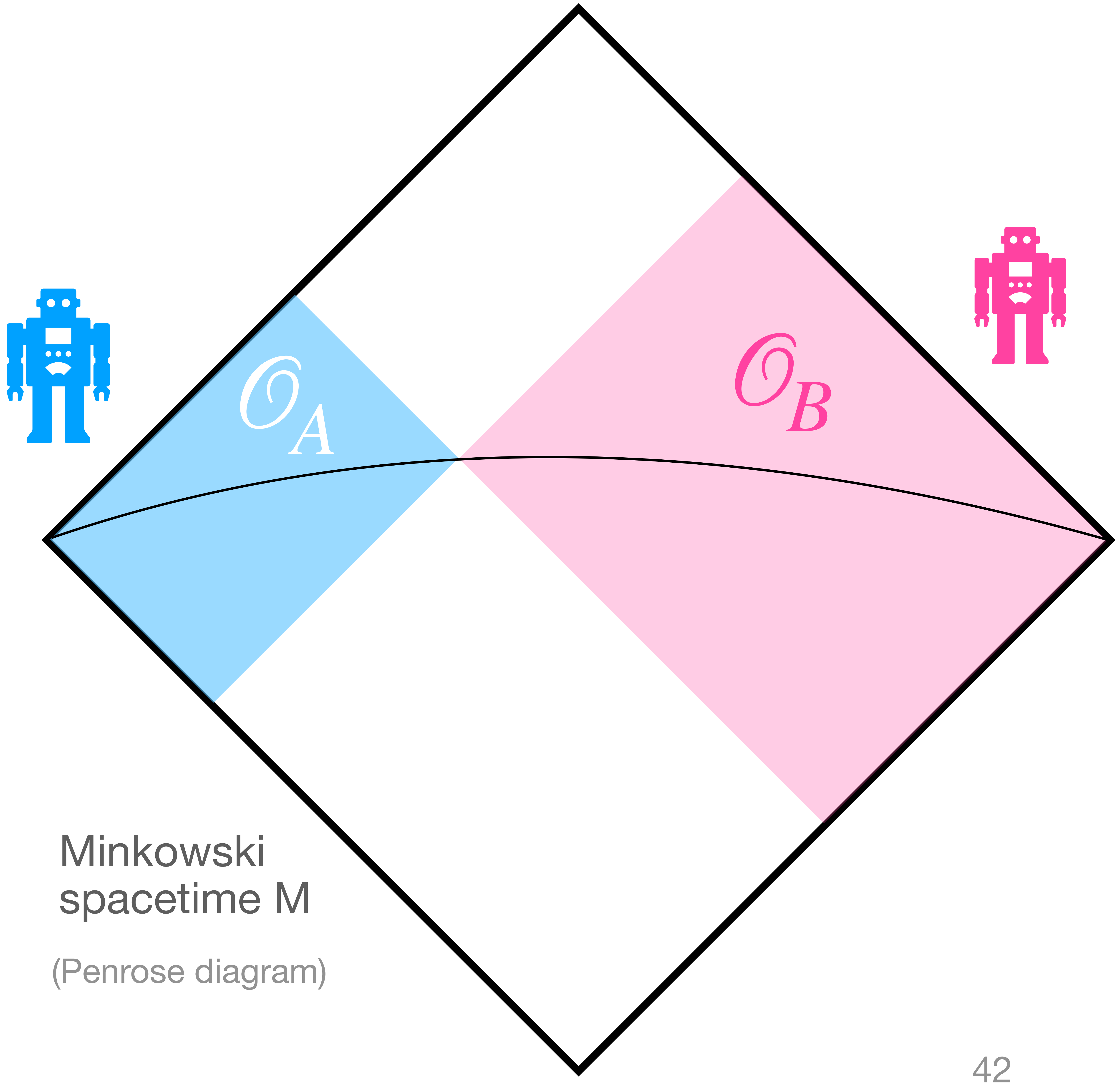


Theorem. Any hyperfinite embezzler induces an embezzling family.

Spin chains are examples of hyperfinite von Neumann algebras.

work in progress

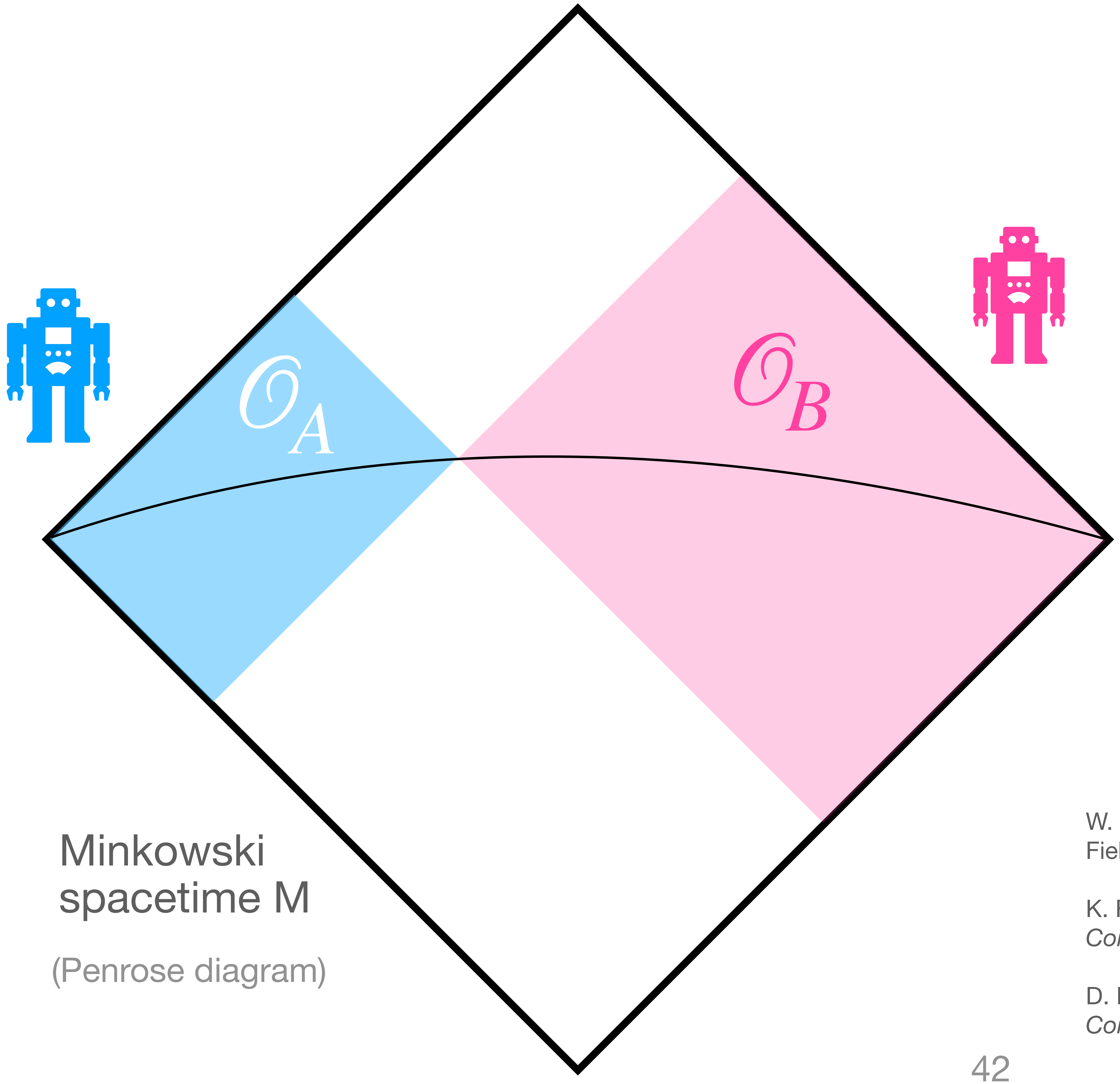
Relativistic QFT



Minkowski spacetime M

(Penrose diagram)

Relativistic QFT



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General result in algebraic QFT:

$$\mathscr{M}(\mathcal{O}_A) = \mathscr{M}(\mathcal{O}_B)' \quad \text{type III}_1$$

W. Driessler, “Comments on Lightlike Translations and Applications in Relativistic Quantum Field Theory”, *Communications in Mathematical Physics* 44, no. 2 (January 1975): 133–41,

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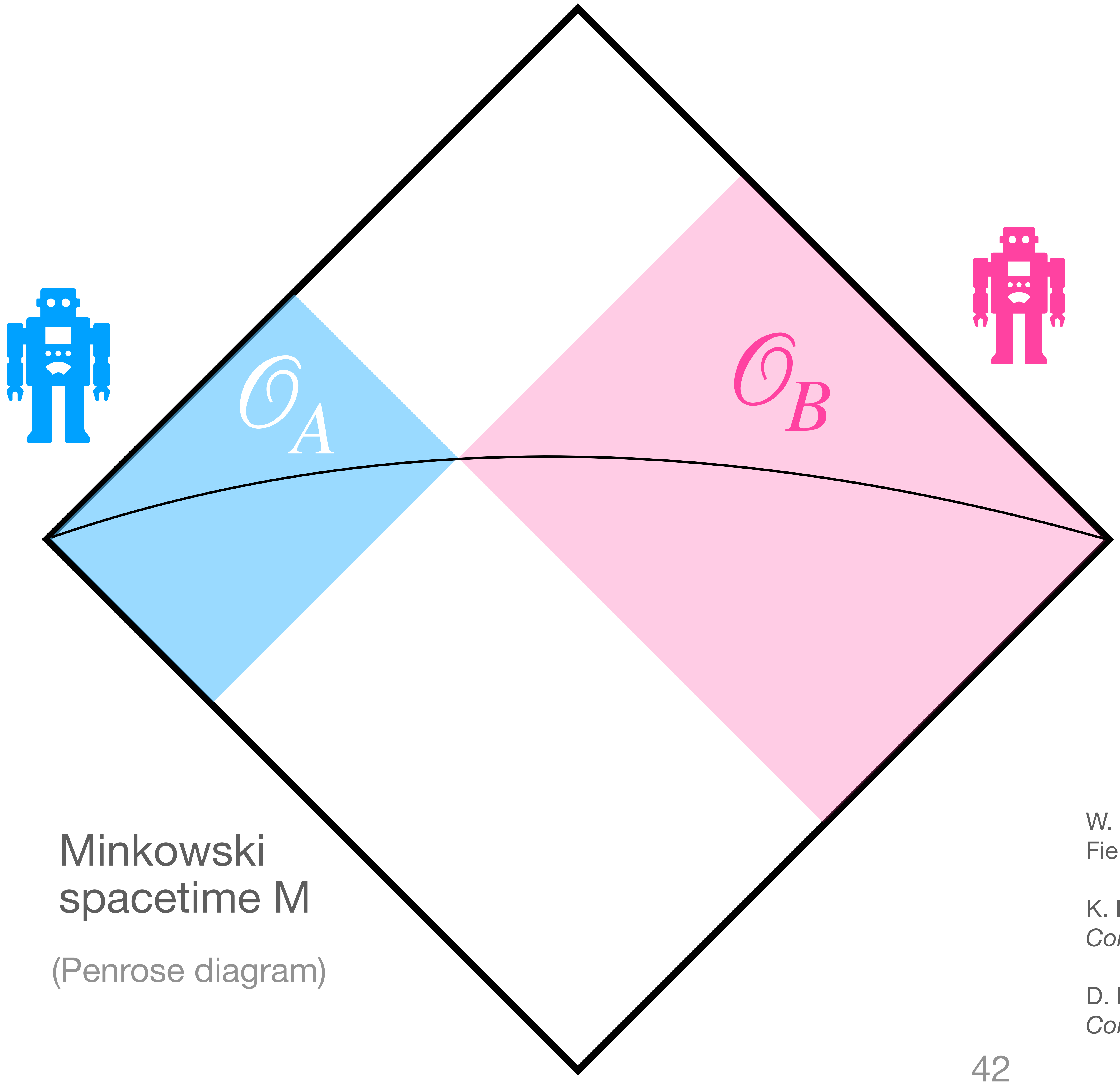

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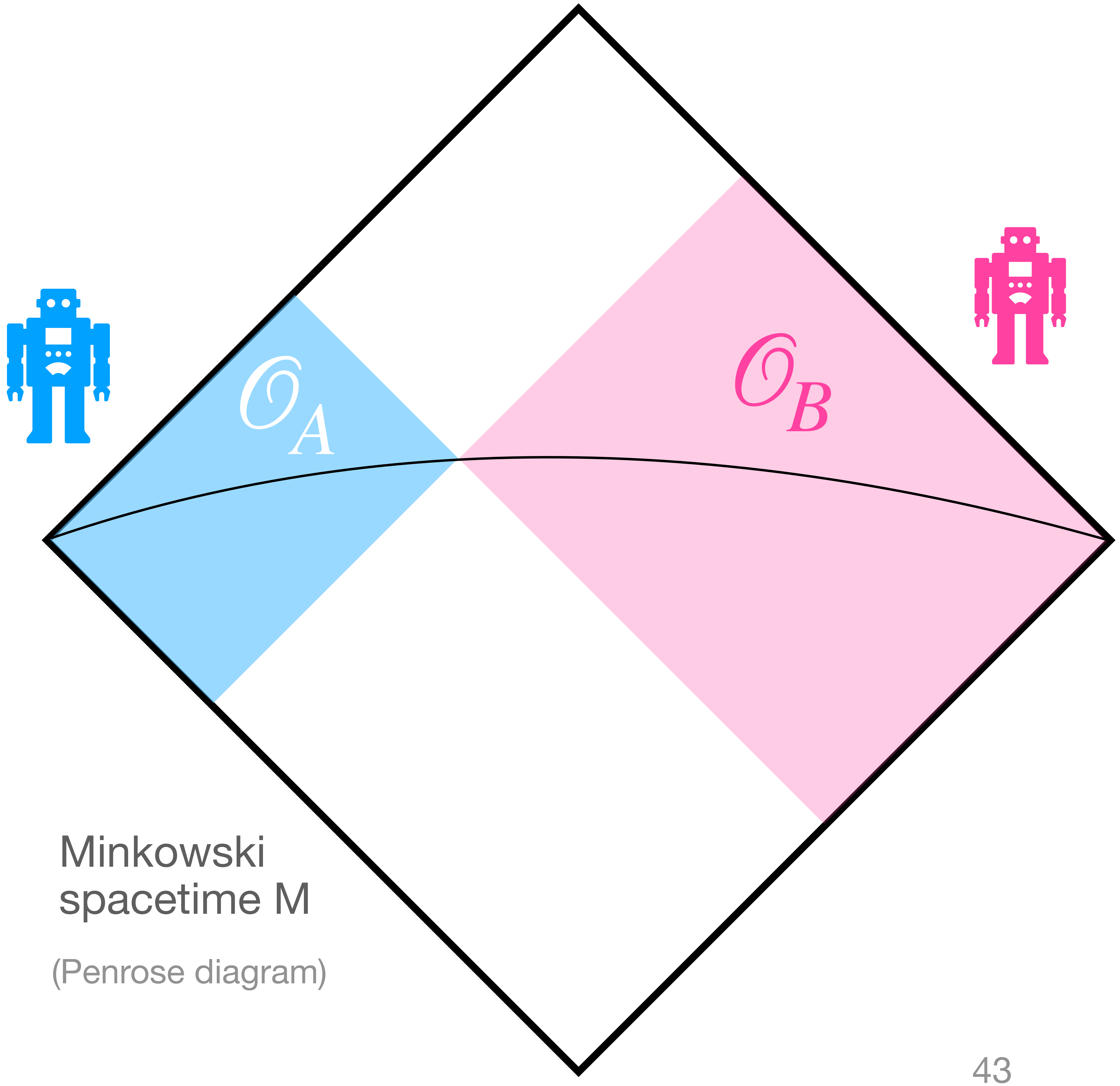

Minkowski spacetime M

(Penrose diagram)

Relativistic quantum fields are universal embezzlers.

- Operational interpretation of diverging vacuum entanglement.
- Explains why they can violate Bell inequalities: Alice and Bob can embezzle Bell states!

Stephen J. Summers and Reinhard Werner, "The Vacuum Violates Bell's Inequalities", *Physics Letters A* 110, no. 5 (1985): 257–59

Bell inequalities

CHSH coefficient in a bipartite system:

$$\beta(|\Omega\rangle, \mathscr{M}, \mathscr{M}') = \sup_{\substack{a_\pm \in \mathscr{M}, b_\pm \in \mathscr{M}' \\ -1 \le a_\pm, b_\pm \le 1}} \langle \Omega | (a_+ b_+ + a_+ b_- + a_- b_+ - a_- b_-) | \Omega \rangle$$

Theorem. In a standard bipartite system, every state fulfills

$$\beta\big(|\Omega\rangle, \mathscr{M}, \mathscr{M}'\big) \ge 2\sqrt{2} - 8\sqrt{\kappa(\omega)}$$

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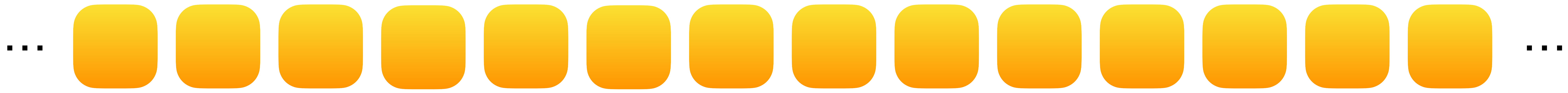
ω embezzling $\implies$ $\beta(|\Omega\rangle) = 2\sqrt{2}$

The critical XY model is embezzling

$$H = -\frac{1}{2}\sum (X_n X_{n+1} + Y_n Y_{n+1})$$



$$\mathscr{A}_\Lambda = \bigotimes_{n\in\Lambda} M_2(\mathbb{C}), \quad \Lambda \subset \mathbb{Z}$$

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Alice:

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If $(\mathscr{H}, \pi, |\Omega\rangle)$ = GNS rep. of the groundstate $\omega : \mathscr{A}_\mathbb{Z} \to \mathbb{C}$, then

$$\left(\mathscr{H}, \mathscr{M}_A := \overline{\pi(\mathscr{A}_L)}^w, \mathscr{M}_B := \overline{\pi(\mathscr{A}_R)}^w\right)$$

is a standard bipartite system of **type III$_1$**. Hence, $|\Omega\rangle$ is embezzling.

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$\forall\, |\Psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n, \varepsilon > 0 \;\; \exists N > 0$, unitaries $U_L \in \mathscr{A}_{[-N,0]} \otimes M_n, U_R \in \mathscr{A}_{[1,N]} \otimes M_n$,

such that

$$\left\| U_L U_R \big(\omega \otimes \langle 00| \cdot |00\rangle\big) U_L^* U_R^* - \omega \otimes \langle\Psi| \cdot |\Psi\rangle \right\| < \varepsilon$$

1. Is there a quantum state that can embezzle states of arbitrary Schmidt rank to arbitrary precision?
   Embezzling state.
2. Are there systems where all states are embezzling states?
   Universal embezzler.
3. Do universal embezzlers exist in nature? QFT.

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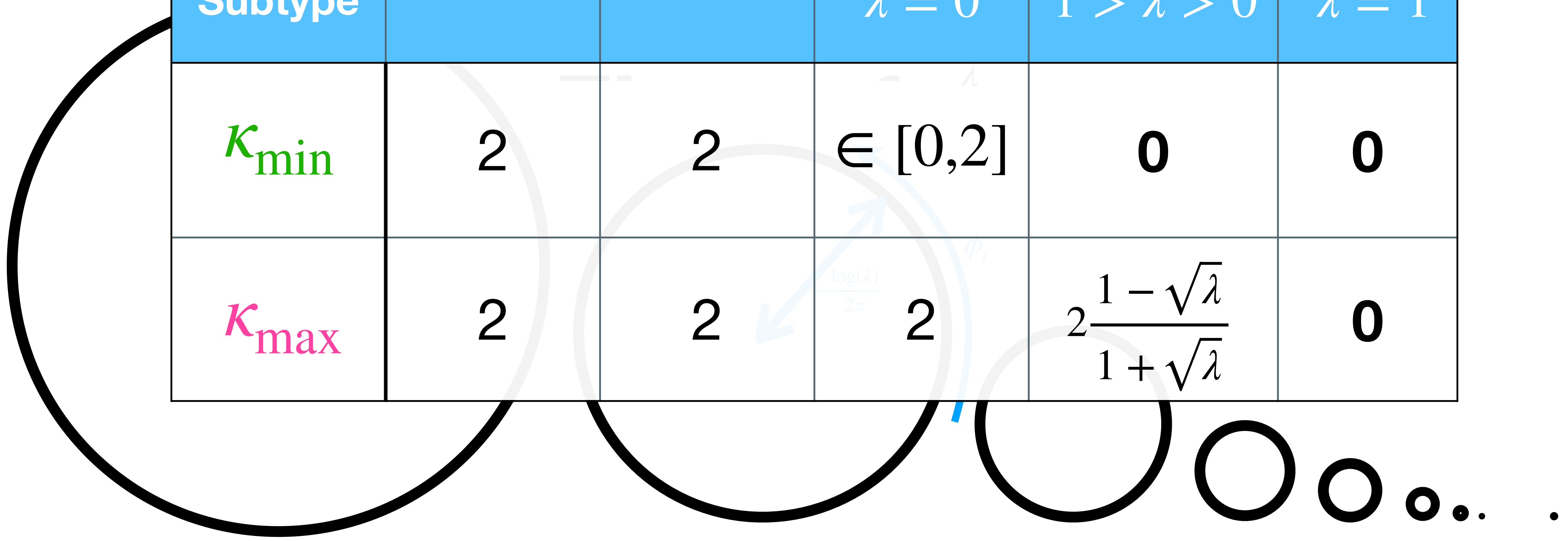

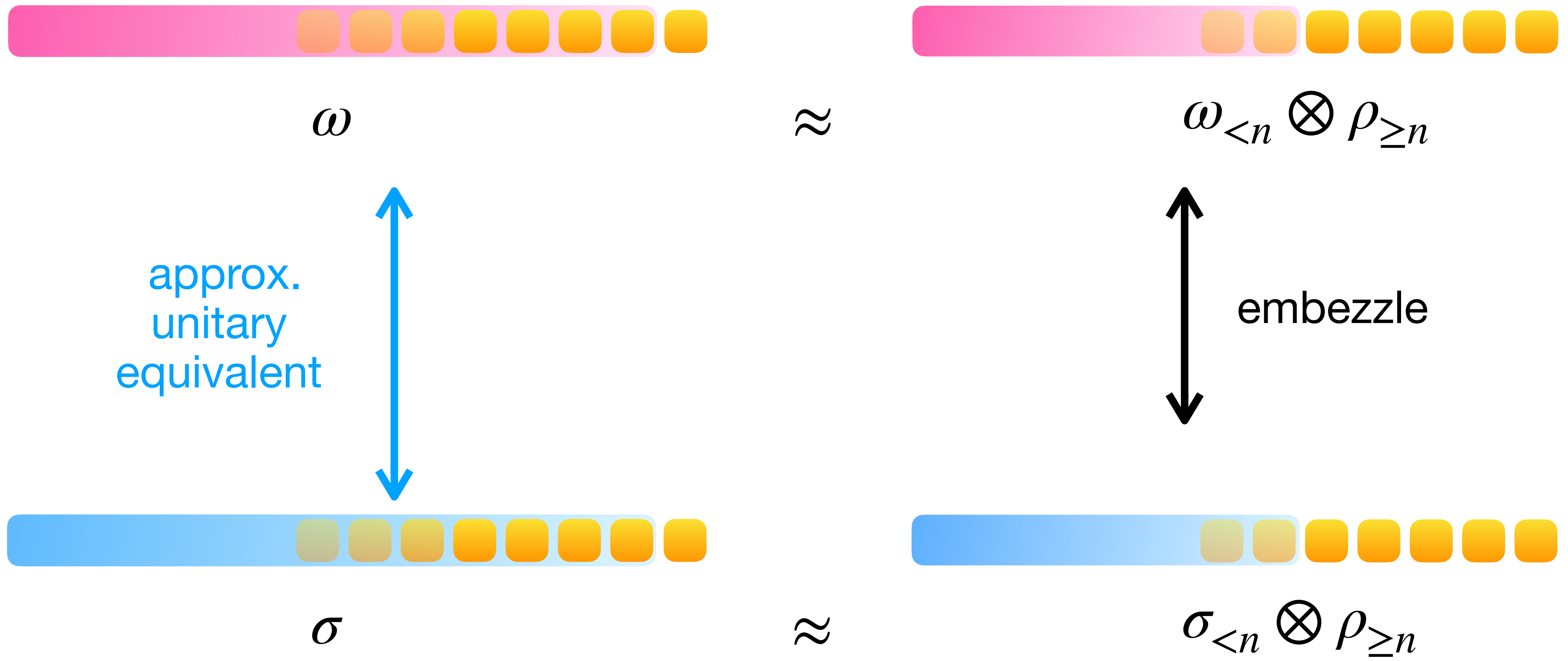

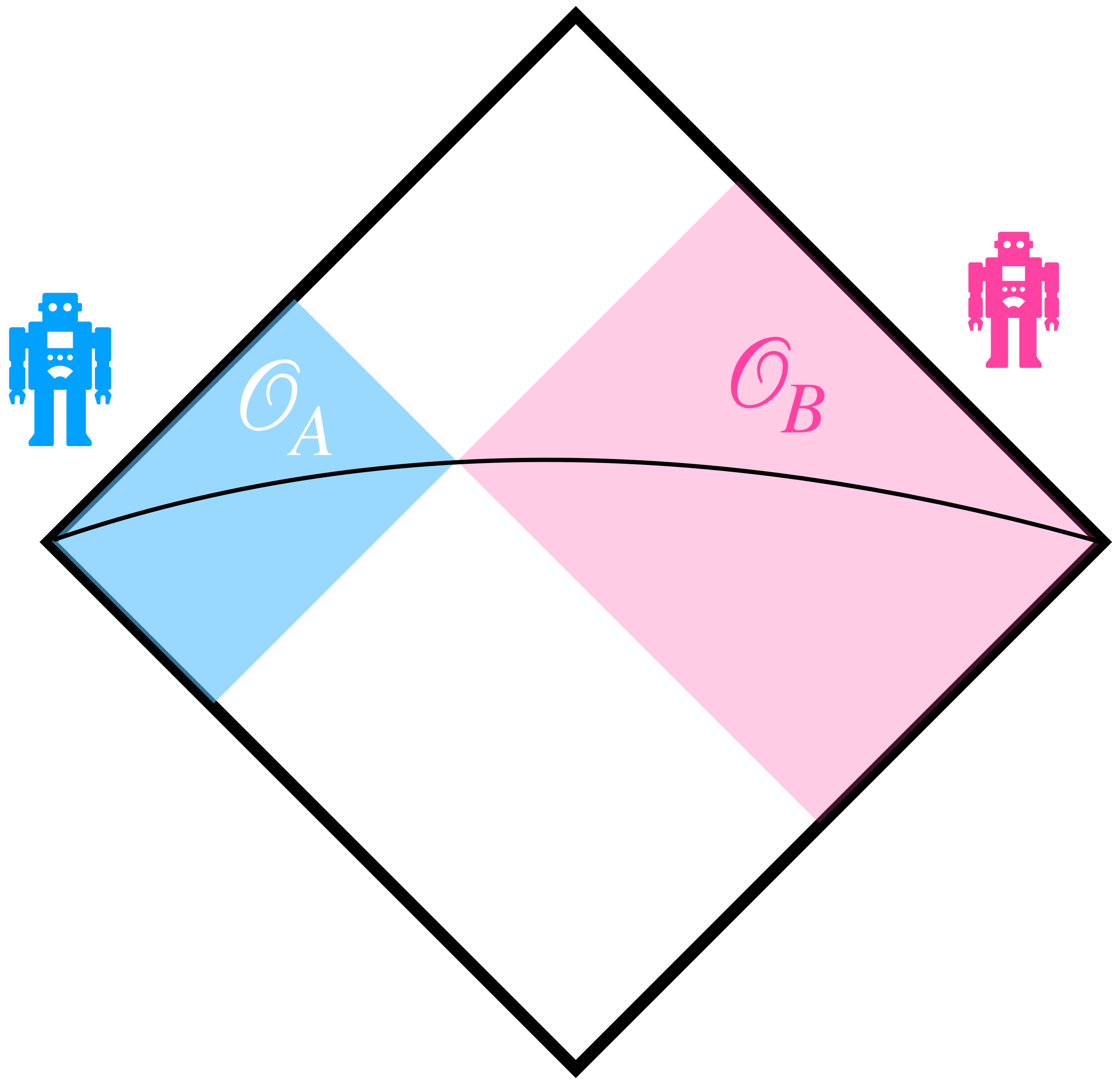

Some open problems

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Three answers

1. Is there a quantum state that can embezzle states of arbitrary Schmidt rank to arbitrary precision? Embezzling state. ✓

2. Are there systems where all states are embezzling states? Universal embezzler. ✓

3. Do universal embezzlers exist in nature? QFT. ✓*

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Chandrasekaran, Longo, Penington, and Witten, "*An algebra of observables for de Sitter space*", JHEP (2023)

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Thank you for your attention!

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