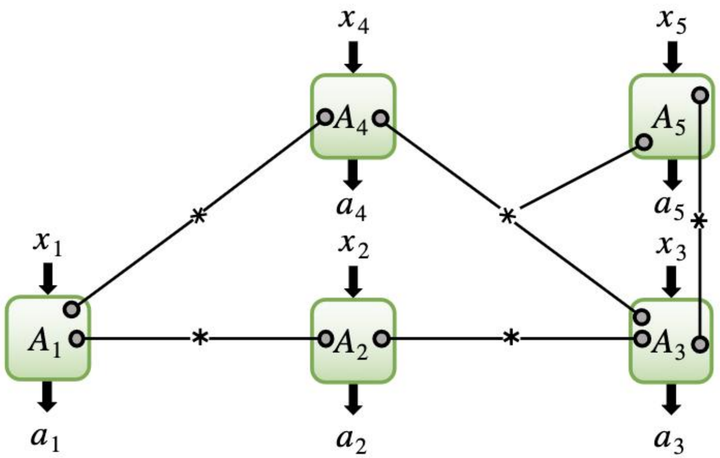
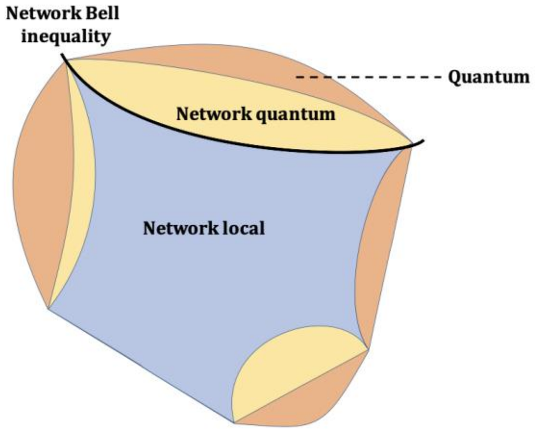
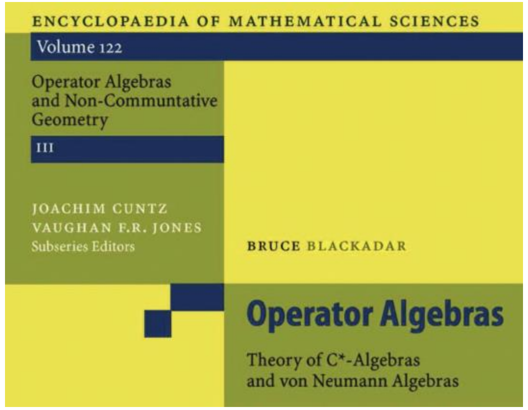
Network correlations

and

polynomial optimization over states







David Gross, University of Cologne

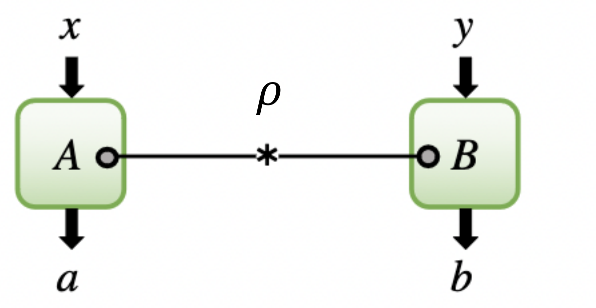
Joint work with Laurens Ligthart and Mariami Gachechiladze

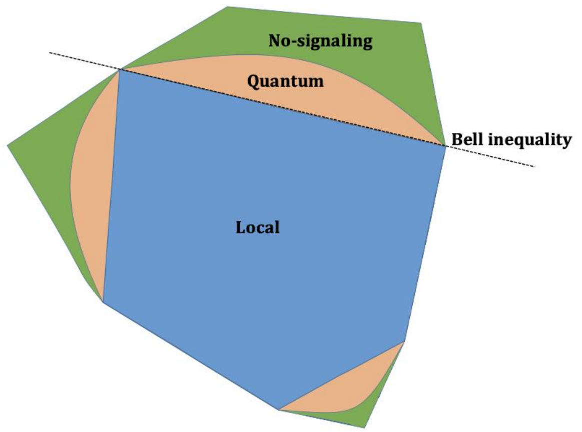
arXiv:2110.14659 (CMP), arXiv:2212.11299 (JMP)

Recap: Bi-partite quantum correlations

Tensor product vs commuting models, MIP*=RE, SDP hierarchies, GNS construction…

Bipartite correlations, tensor product model





Scenario:

- Two parties A, B,
- with finite local settings x, y and finite outcomes a, b.

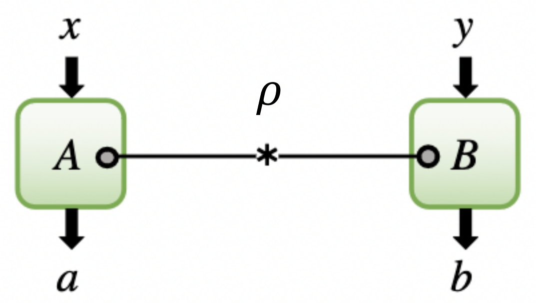
A distribution $p(a, b|x, y)$ has *tensor-product model* if ∃

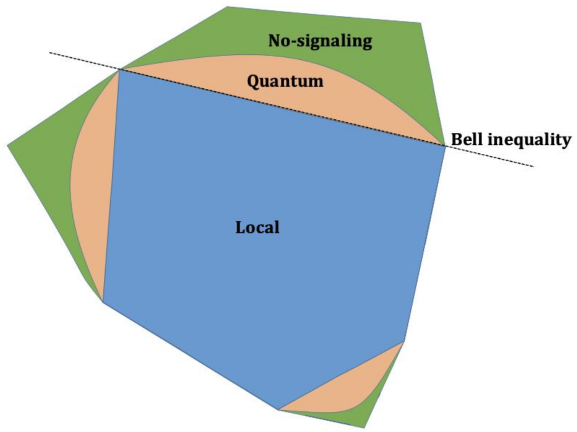
- Hilbert spaces H_A, H_B,
- A joint state ρ on $H_A \otimes H_B$
- POVMs $\{A_{\alpha|x}\}$, $\{B_{\beta|y}\}$, such that

$$p(\alpha, \beta|x, y) = \rho\left(A_{\alpha|x} \otimes B_{\beta|y}\right).$$

*Most figures stolen from Tavakoli et al., *Bell nonlocality in networks*.

Bipartite correlations, commuting model





Scenario:

- Two parties A, B,
- with finite local settings x, y and finite outcomes a, b.

A distribution $p(a, b|x, y)$ has *commuting operator model* if ∃

- A Hilbert space H,
- A joint state ρ on H,
- Mutually commuting POVMs $\{A_{\alpha|x}\}, \{B_{\beta|y}\}$

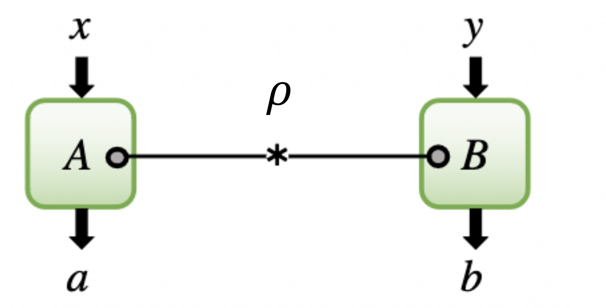
$$[A_{\alpha|x}, B_{\beta|y}] = 0$$

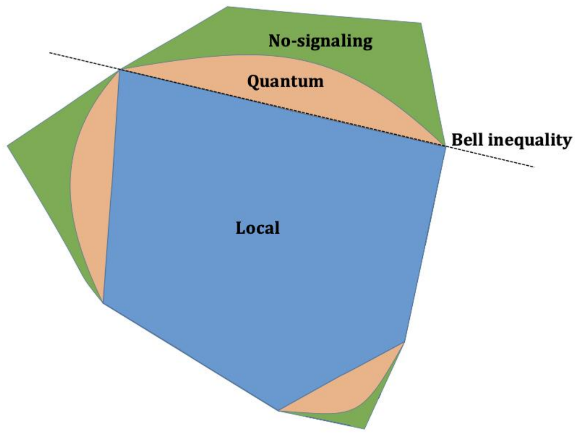
such that

$$p(\alpha, \beta|x, y) = \rho\left(A_{\alpha|x} B_{\beta|y}\right).$$

*Most figures stolen from Tavakoli et al., *Bell nonlocality in networks*.

Tensor product vs commuting op models 1/2





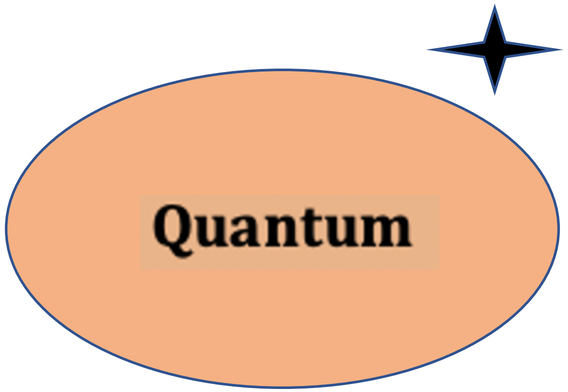
- TP model common in quantum information
- CO model common relativistic QFT

- Agree in finite dimensions, so QI convention fine
- Since 2000: CO known to be more general

- Both are reasonable mathematical models.

[Ji et al., MIP^*=RE, and all that]

Weak membership problem



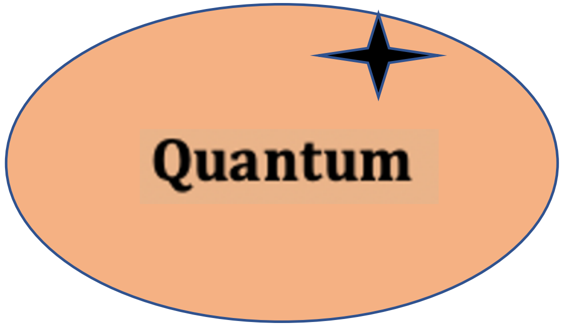
Given $p(a, b|x, y)$, error bound ϵ.

- promised not to be on boundary
- Can one algorithmically decide whether it has a quantum model, "up to ϵ"?

- Emphasis on decidability, not computational complexity.
- Strategy: Find tight inner / outer approximations.

[Ji et al., MIP^*=RE, and all that]

Inner approximations

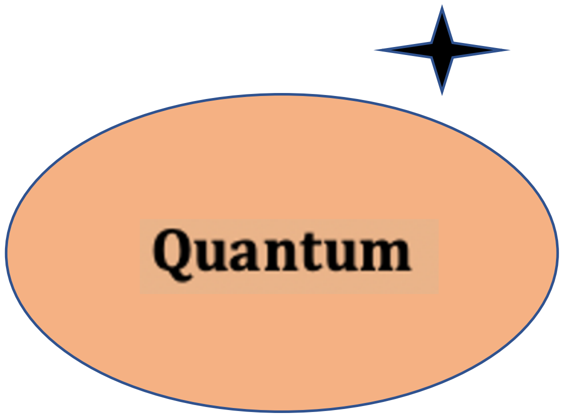


- Correlations with finite-dim model are dense
- For every finite dimension, there is a *dense enumerable dense set* of models
- $\Longrightarrow$ Their union over all dims is still dense and enumerable

- $\Longrightarrow$ Every member can be identified in finitely many steps

[Ji et al., MIP^*=RE, and all that]

Outer approximations 1/4



- Consider algebra $\mathcal{F}$ with symbolic *generators* $a_{\alpha|x}, b_{\beta|y}$
- and *relations*

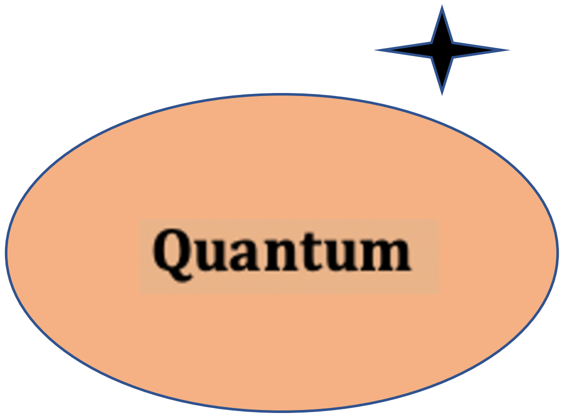
$$[a_{\alpha|x}, b_{\beta|y}] = 0,\ a_{\alpha|x}^2 = a_{\alpha|x},\ \sum_\alpha a_{\alpha|x} = I, \ldots$$

- (i.e. equiv.-classes of finite non-comm polynomials in symbols)
- This is an abstract *-algebra, the *free *-algebra given the generators and relations*.

Idea: A state implementing $p(a,b|x,y)$ induces a state on $\mathcal{F}$. But there's an enumerable set of conditions for such a state existing.

[Navascues, Pironio, Acin]

Outer approximations 2/4



A *state* on a *-algebra is a linear functional satisfying

- $\rho(1) = 1$ [Normalization]
- $\rho(x^* x) \geq 0$ [Positivity]

- Suffices to check positivity on words in generators
- $\Longrightarrow$ Enumerable set of conditions (one SDP per degree)

- A concrete choice of commuting POVMs $A_{\alpha|x}, B_{\beta|y}$ gives a representation

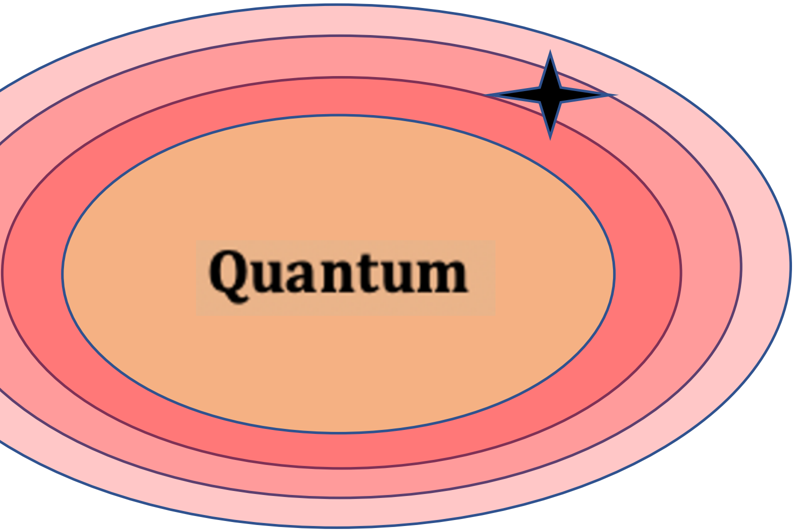
$$\pi: a_{\alpha|x} \longmapsto A_{\alpha|x}, b_{\beta|y} \longmapsto B_{\beta|y}$$

- …and a concrete state gives rise to an abstract state

$$\rho(x) = \rho(\pi(x))$$

[Navascues, Pironio, Acin]

Outer approximations 3/4



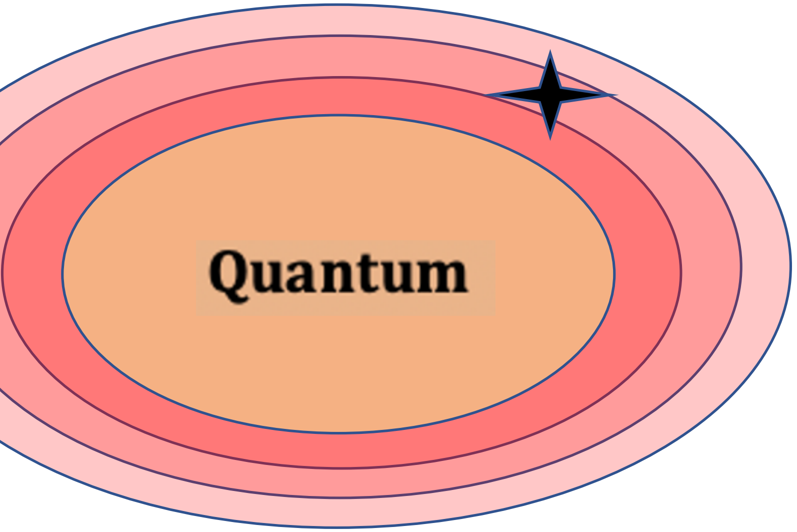
Given $p(a,b|x,y)$, at level $n \geq 2$ of the hierarchy:

- $\mathcal{F}_n$:= subspace of free algebra spanned by words of degree *n*
- Accept if there exist a linear functional ρ_n on $\mathcal{F}_n$,
- that is positive on $\mathcal{F}_n$ and such that

$$p(\alpha,\beta|x,y) = \rho_n\left(a_{\alpha|x} b_{\beta|y}\right).$$

[Navascues, Pironio, Acin]

Outer approximations 4/4



Converse via GNS construction.

- On the free algebra, define

$$\|x\| := \sup\{\|\pi(x)\| \mid \pi \text{ is a representation of } \mathcal{F}\}$$

- Completion gives *universal C*-algebra* given gens and relations
- From all ρ_n, get state ρ on C* algebra realizing $p(a, b|x, y)$
- GNS construction gives **commuting operator** quantum model.

- (No obvious relations that are only realizable on tensor prods)

[Navascues, Pironio, Acin]

Optimization

Abstract point of view on hierarchy:

- ”Semi-definite programming over state space of universal C*-algebra”:

Input

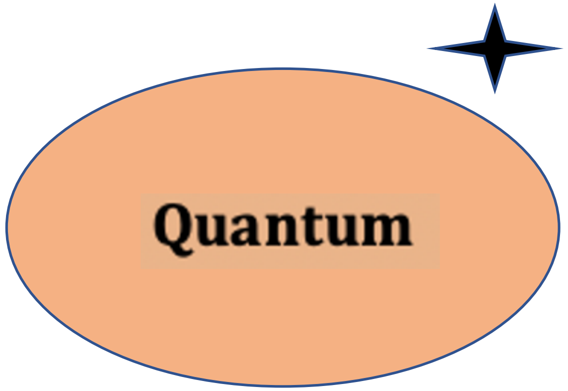
- Generators $\{g_i\}$
- Relations $\{a_j \geq 0\}$ for a_j in the free *-algebra
- Linear functions $\{l_k\}$ on the free *-algebra

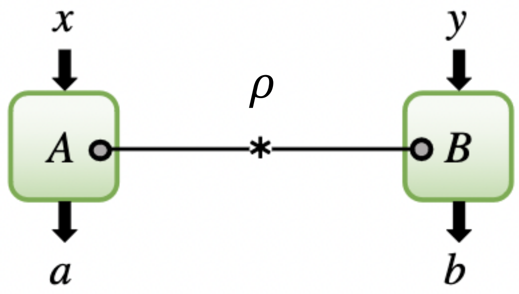
Output

$$\begin{aligned} &\min l_0(\rho) \\ &\text{s.t. } l_k(\rho) = 0 \quad \forall k \geq 1 \end{aligned}$$

With minimum over states of the of the universal C^*-algebra.

Weak membership problem: Summary





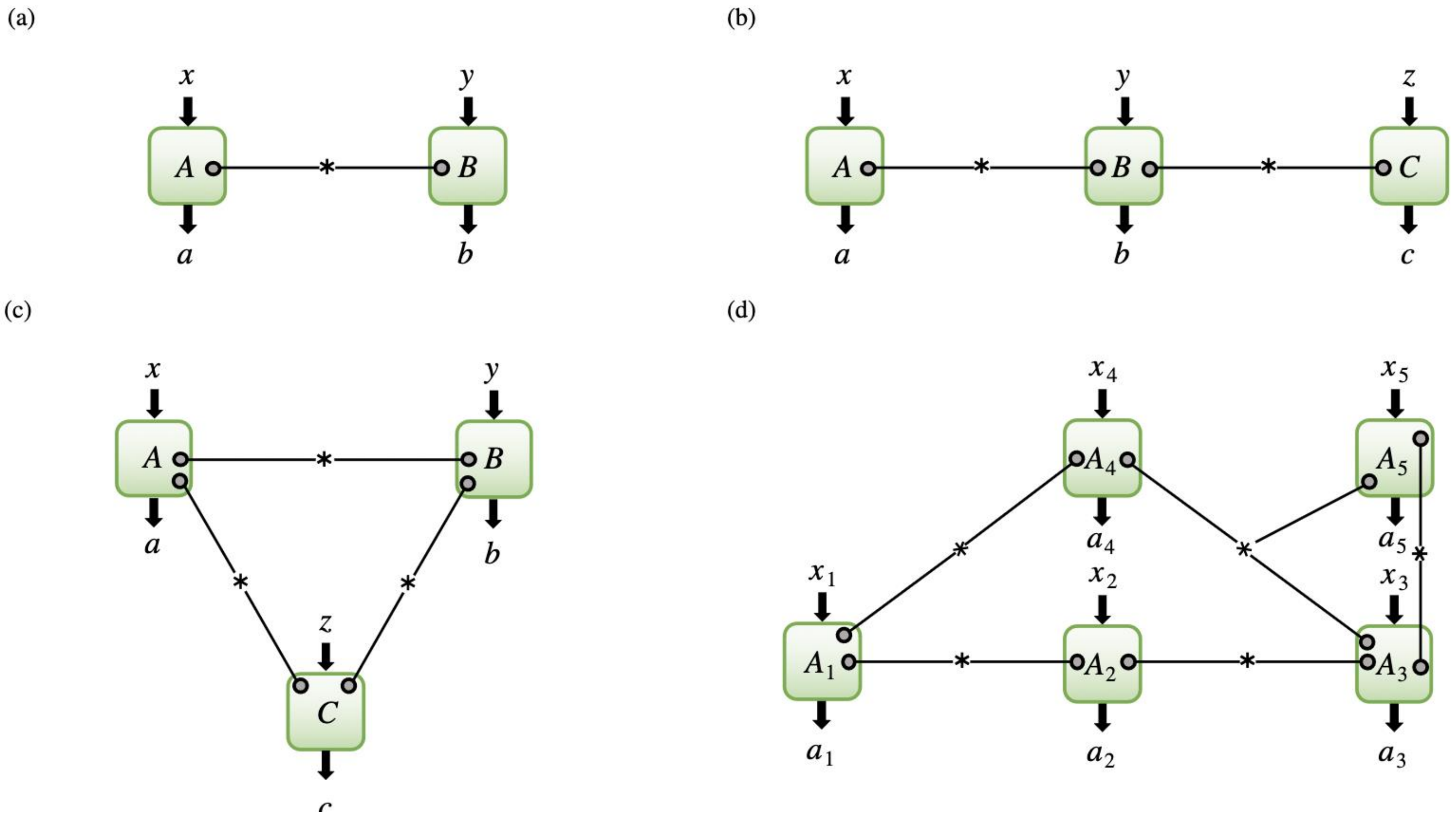
Given $p(a,b|x,y)$, error bound ϵ.

- If there is a co model, can find it (by enumerating dense set of models)
- If there is no co model, can witness that (by converging hierarchy of SDP relaxations)

- [MIP*=RE] $\Longrightarrow$ TP model is algorithmetically undecidable

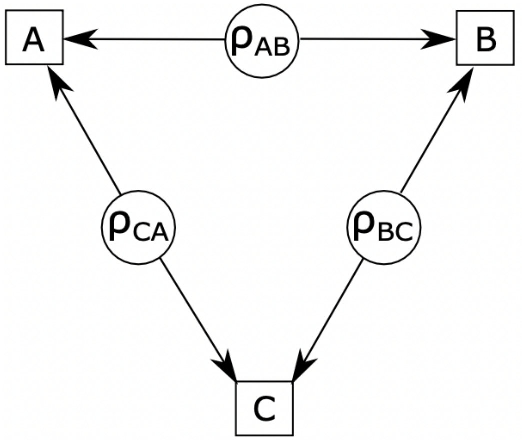
- So we have done what we could.

Now make everything more complicated.



Can we solve the membership problem for those?

Two solutions

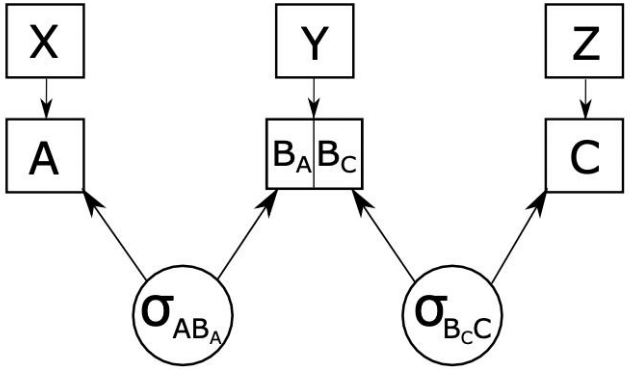


In general: A hack.

[arXiv:2110.14659]

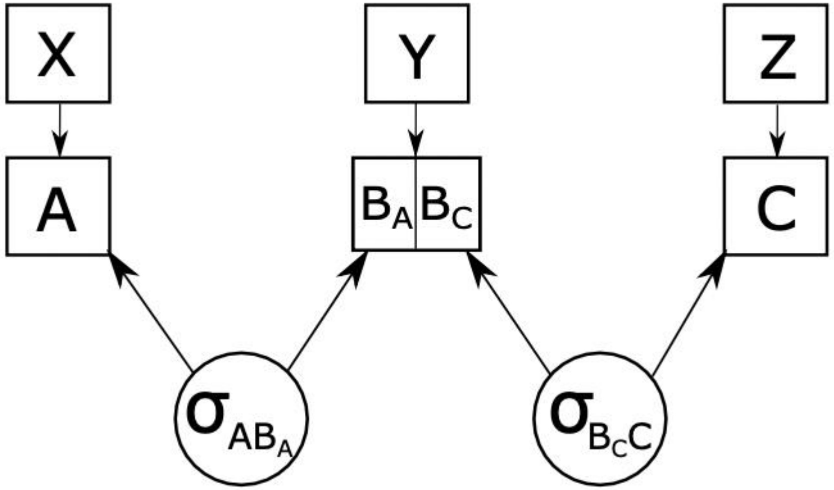
For bi-linear scenario: No hack.

[arXiv:2212.11299]

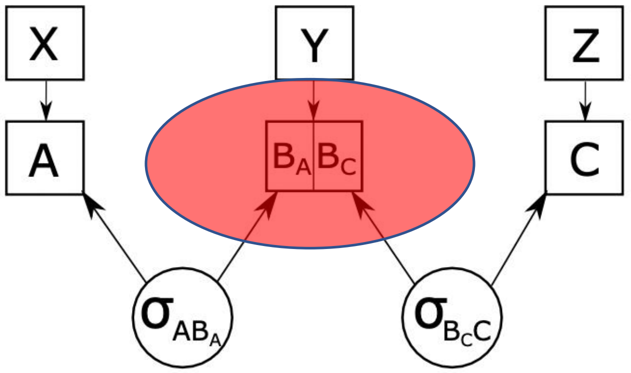


Bi-local scenario

- For now:
  Work in commuting operator model



Quantum bilocal scenario



$p(\alpha, \beta, \gamma | x, y, z)$ is *compatible* if $\exists$

- Commuting operator algebras $\mathcal{A}, \mathcal{B}_A, \mathcal{B}_C, \mathcal{C}$
- A joint state ρ that factorizes

$$\rho(a\, b_A\, b_C\, c) = \rho(a\, b_A)\rho(b_C\, c)$$

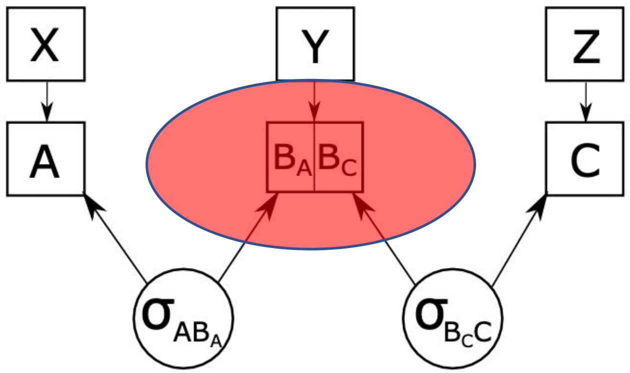
- POVMs

$$\{A_{\alpha|x}\} \subset \mathcal{A}, \{B_{\beta|y}\} \subset \mathcal{B}_A\, \mathcal{B}_C, \{C_{\gamma|z}\} \subset \mathcal{C}$$

such that,

$$p(\alpha, \beta, \gamma | x, y, z) = \rho\left(A_{\alpha|x} B_{\beta|y}\, C_{\gamma|z}\right).$$

Challenges



Consider the independence constraint

$$\rho(a\ b_A\ b_C\ c) = \rho(a\ b_A)\rho(b_C\ c)$$

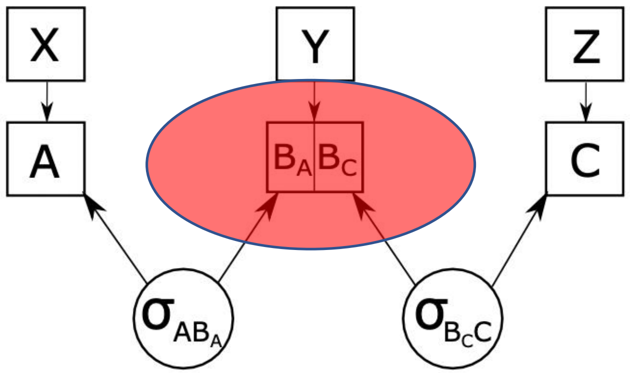
Challenge 1:

- It is non-linear in the state

Challenge 2:

- It is phrased in terms of the *internal locality structure* of B
- We have no natural ansatz for $\mathcal{B}_A\ \mathcal{B}_C$.

Challenges



Ignore Challenge 2 for now and relax:

$$\rho(a\ b_A\ b_C\ c) = \rho(a\ b_A)\rho(b_C\ c) \implies \rho(ac) = \rho(a)\rho(c)$$

Necessary condition for a quantum model:

$\exists$ state ρ of the universal algebra generated by POVMs, s.t.

$$p(\alpha, \beta, \gamma | x, y, z) = \rho\left(A_{\alpha|x} B_{\beta|y}\ C_{\gamma|z}\right)$$
$$\rho(ac) = \rho(a)\rho(c)$$

Special case of “state polynomial optimization problem for universal C*-algebras”.

Polynomial optimization

State polynomial optimization for universal C-algebras*:

Input

- Generators $\{g_i\}$
- Relations $\{a_j \geq 0\}$ for a_j nc-polynomial in generators.
- Polynomials p_k in states evaluated at words in generators

Output

$$\begin{aligned} &\min p_0(\rho) \\ &\text{s.t. } p_k(\rho) = 0 \qquad \forall k \geq 1 \end{aligned}$$

With minimum over states of the of the universal C^*-algebra.

[Ligthart, DG 22; Ligthart, Gachechiladze, DG 20; Klep, Magron, Volcic, Wang 23]

Polynomial optimization

$$\min p_0(\rho)$$
$$\text{s.t. } p_k(\rho) = 0 \quad \forall k \geq 1$$

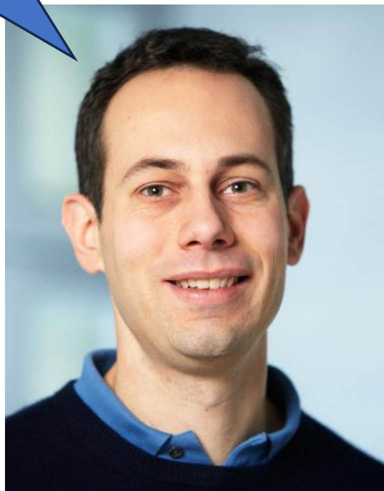
Result:

- There is a monotonously convergent SDP hierarchy that solves the state-polynomial optimization problem.

Idea:

- Replace polynomials by their polarizations (=linearizations over many independent copies)
- Relax independence (non-linear) to symmetry (linear)
- Use suitable de Finetti Theorem

”Symmetry implies independence”



[Ligthart, DG 22; Ligthart, Gachechiladze, DG 20; Klep, Magron, Volcic, Wang 23]

Proof sketch 1/3: Polarization

$$\min p_0(\rho)$$
$$\text{s.t. } p_k(\rho) = 0 \quad \forall k \geq 1$$

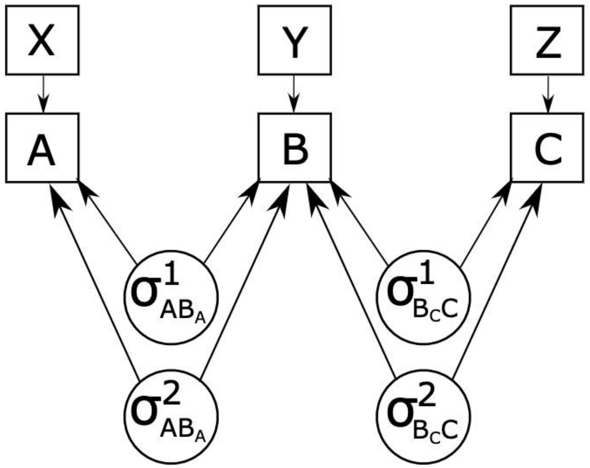
Polarization

- Consider $\Pi = \rho^{\otimes 2}$. Then:

$$\rho(ac) - \rho(a)\rho(c) = 0 \quad \Leftrightarrow \quad \Pi\left(a^{(1)}c^{(1)} - a^{(1)}c^{(2)}\right) = 0$$

- That's a *linear* constraint on a symmetric *product* state
- Linear expression is the *polarization* of the polynomial

- Have to get rid of non-linear "productness" constraint.



[Ligthart, DG 22; C.f. (Quantum) Inflation technique, Wolfe, Spekkens, Fritz, Navascues, Pozas-K., Grinberg, Rosset, Acin, …]

$$\min\ p_0(\rho) \quad \text{s.t.}\ \ p_k(\rho) = 0 \quad \forall k \geq 1$$

Proof sketch 2/3: de Finetti

de Finetti*

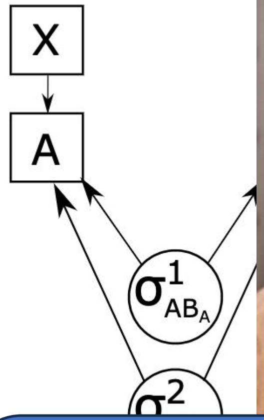
- Separable states = convex combinations of products
- For ∞-ly many copies, can enforce separability linearly:

$$\Pi \text{ invariant under } S_\infty \quad \Leftrightarrow \quad \Pi = \int \rho^{\otimes\infty}\, \mathrm{d}\mu(\rho).$$

- Remaining problem:

$$\Pi = \int \rho^{\otimes\infty}\, \mathrm{d}\mu(\rho), \quad \Pi\left(a^{(1)}c^{(1)} - a^{(1)}c^{(2)}\right) = 0$$

- …does *not* imply that every product fulfills constraint.



*None was published for commuting model, but Werner-Raggio's can be adapted

[Ligthart, DG 22; Ligthart, Gachechiladze, DG 20]

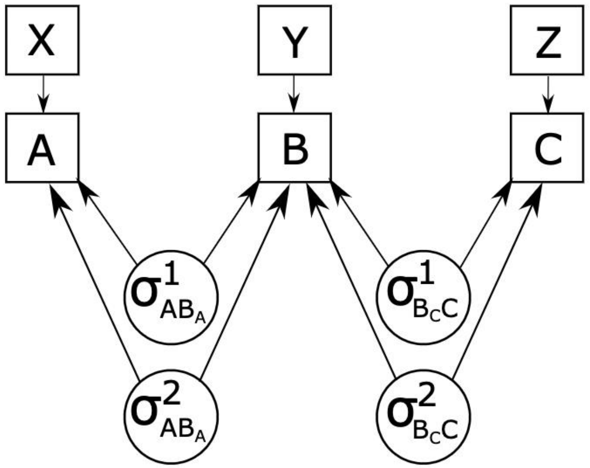
Proof sketch 3/3: Polarization

$$\min \; p_0(\rho) \quad \text{s.t.} \;\; p_k(\rho) = 0 \quad \forall k \geq 1$$

Polarization, once more!

- Take polarization of $p = (\rho(ac) - \rho(a)\rho(c))^2$

$$y := a^{(1)}c^{(1)}a^{(2)}c^{(2)} - 2a^{(1)}c^{(1)}a^{(2)}c^{(3)} + a^{(1)}c^{(2)}a^{(3)}c^{(4)}$$



- Because the polynomial is non-negative

$$\Pi = \int \rho^{\otimes\infty} \, \mathrm{d}\mu(\rho), \qquad \Pi(y) = 0$$

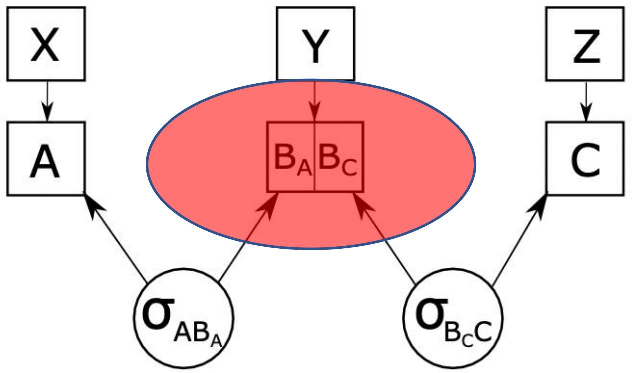
does imply that every product state satisfies

$$\rho(ac) - \rho(a)\rho(c) = 0$$

…can turn that into algorithm for state-polynomial optimization.

[Ligthart, DG 22]

Back to Bi-local scenario



Can now solve:

∃ state ρ of the universal algebra generated by POVMs, s.t.

$$p(\alpha,\beta,\gamma|x,y,z) = \rho\left(A_{\alpha|x}B_{\beta|y}\,C_{\gamma|z}\right)$$
$$\rho(ac) = \rho(a)\rho(c)$$

But: This was only a relaxation of the actual constraint

$$\rho(a\,b_A\,b_C\,c) = \rho(a\,b_A)\rho(b_C\,c)$$

Turns out: In the bi-local scenario (and so far only there!) can construct a quantum model from a solution of the relaxation.

Proof idea 1/2: Recovering $\mathcal{B}_A, \mathcal{B}_C$

Assume:

- There's a textbook quantum model given by

$$\left|\psi_{AB_A}\right\rangle \otimes \left|\psi_{B_C C}\right\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{\mathcal{B}_\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}_\mathcal{C}} \otimes \mathcal{H}_\mathcal{C}$$

- with $\left|\psi_{AB_A}\right\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{\mathcal{B}_\mathcal{A}}$ full Schmidt rank.

Then $\mathcal{B}_A$ is the commutant of$\mathcal{A}$ in $B(\mathcal{H}_A \otimes \mathcal{H}_{\mathcal{B}_\mathcal{A}})$.

Proof idea 2/2: GNS for product states

- Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be the universal algebras generated by POVMs
- Let ρ be a state such that

$$\rho(a\, c) = \rho(a)\rho(c)$$

Then:

- GNS representation for $\mathcal{A}, \mathcal{C}$ factorizes

$$\mathcal{H}_\rho = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{C}, \qquad |\rho\rangle = |\alpha\rangle \otimes |\gamma\rangle$$

- Define

$$\mathcal{B}_\mathcal{A} = \pi(\mathcal{A})', \qquad \mathcal{B}_\mathcal{C} = \pi(\mathcal{C})'$$

- There's a channel $\Lambda\colon\ \mathcal{B} \to \mathcal{B}_\mathcal{A} \otimes \mathcal{B}_\mathcal{C}$ such that

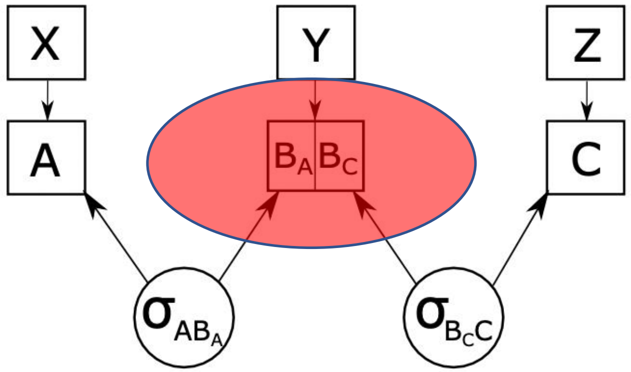
$$\rho(\mathrm{abc}) \;=\; \langle\alpha\gamma|\, \pi(\mathrm{a})\, \Lambda(\mathrm{b})\, \pi(\mathrm{c})|\alpha\gamma\rangle$$

Blackboard

Bi-local summary

Thm.: A set of correlations $p(\alpha, \beta, \gamma | x, y, z)$ has a quantum bilocal model if and only if the state-polynomial problem below is feasible.

This can be checked using a monotonously convergent SDP hierarchy.

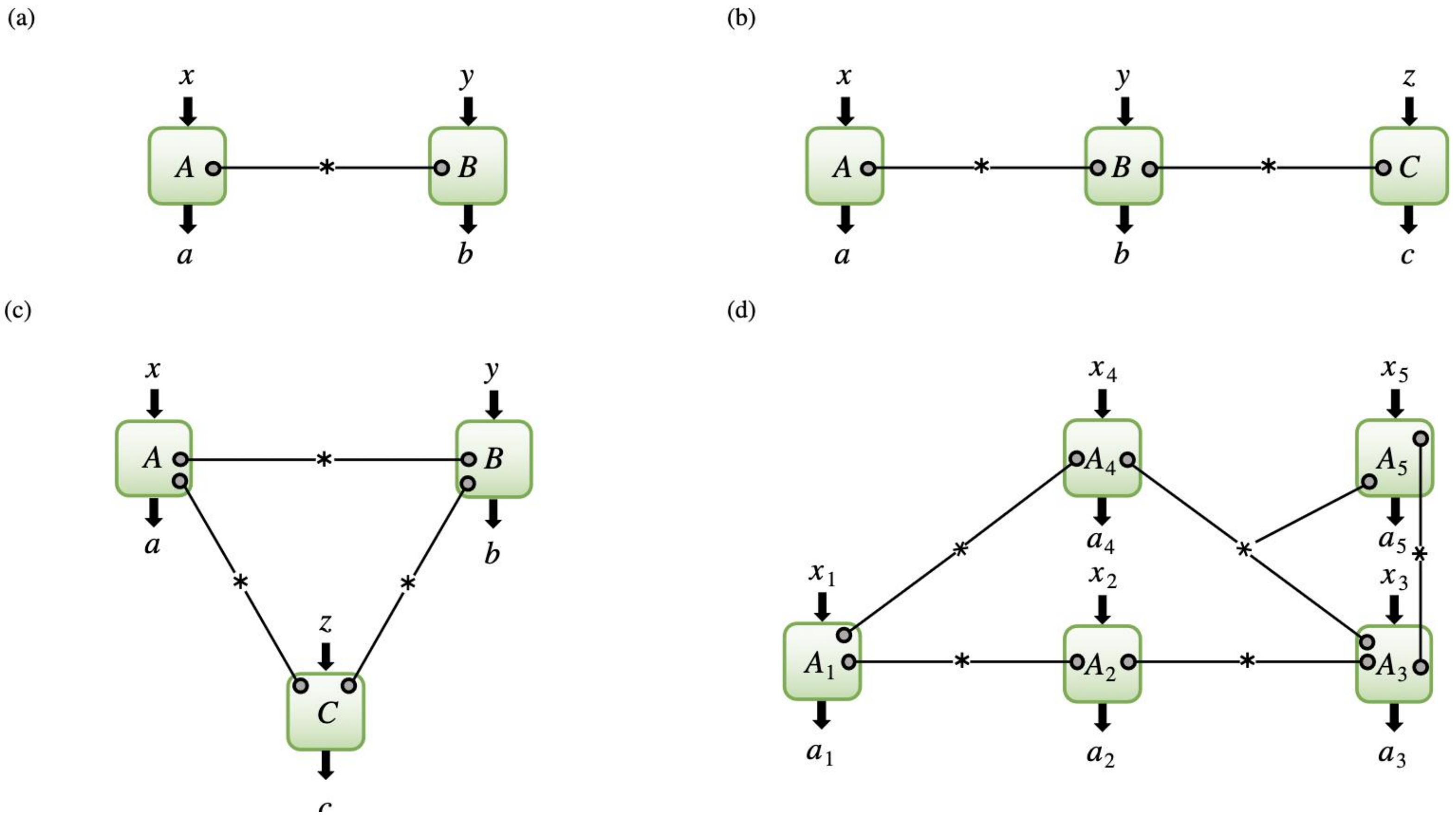


∃ state ρ of the universal algebra generated by POVMs, s.t.

$$p(\alpha, \beta, \gamma | x, y, z) = \rho\left(A_{\alpha|x} B_{\beta|y}\, C_{\gamma|z}\right)$$
$$\rho(ac) = \rho(a)\rho(c)$$

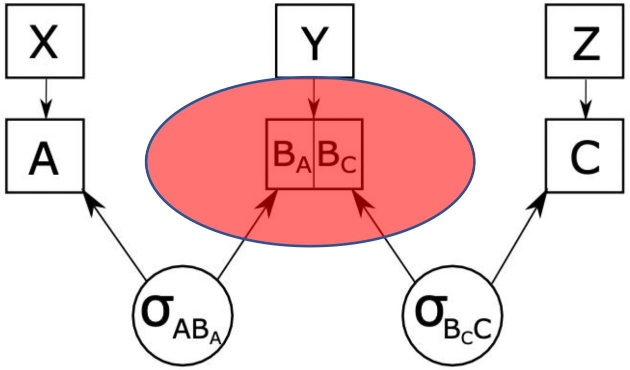
[Ligthart, DG 22]

Now make everything more complicated.



Can we solve the membership problem for those?

Recall “2nd Challenge”



Difficulty:

- Factorization constraint

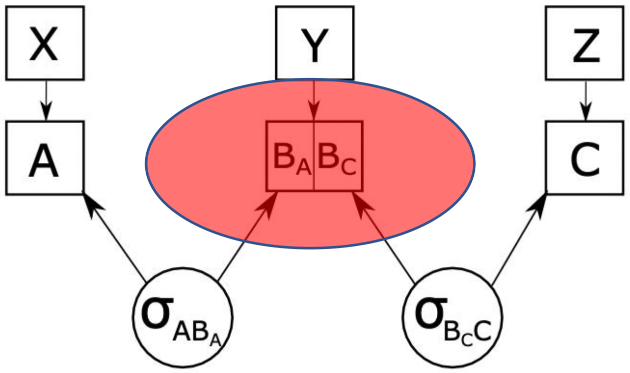
$$\rho(a\, b_A\, b_C\, c) = \rho(a\, b_A)\rho(b_C\, c) \qquad (🏒)$$

involves operators b_a, b_c *that need not lie in algebra generated by the observables*.

- “Unclear how observables lies relative to the locality structure that defines causal structure.”

- Could impose (🏒) *if we knew which operators to impose it on.*

Recall “2nd Challenge”



Difficulty:

- Factorization constraint

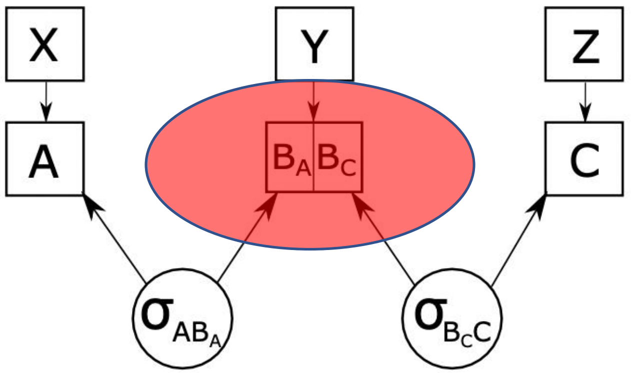
$$\rho(a\, b_A\, b_C\, c) = \rho(a\, b_A)\rho(b_C\, c) \qquad (🥌)$$

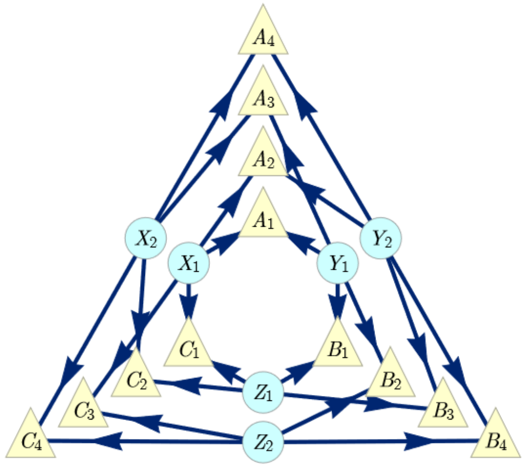
involves operators b_a, b_c *that need not lie in algebra generated by the observables*.

- “Unclear how observables lies relative to the locality structure that defines causal structure.”

- Could impose (🥌) *if we knew which operators to impose it on.*

A hack





"Top-down" doesn't work:

- Don't know how to write $\mathcal{B}_A, \mathcal{B}_C$ in terms of observables.

"Bottom-up" does!

- Introduce generators for $\mathcal{B}_A, \mathcal{B}_C$ and expand observables

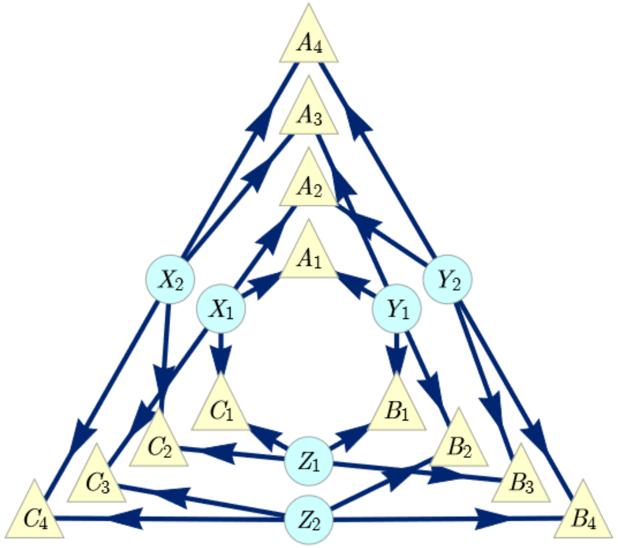
$$B_{\beta|y} = \sum_{i=1}^{r} b_a^{(i)} \cdot b_c^{(i)}$$

- Requires explicit upper bound on the Schmidt rank of POVM elements.

Network summary

Thm.: A set of correlations has a network quantum model **of bounded operator Schmidt rank** if and only if the associated state-polynomial problem is feasible.

This can be checked using a monotonously convergent SDP hierarchy.



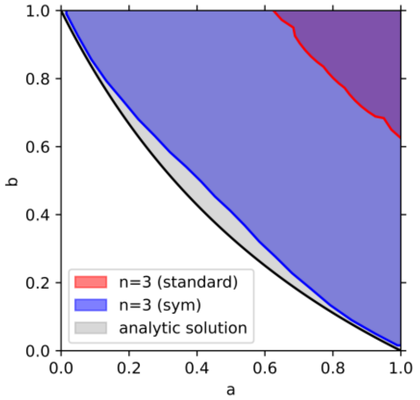
The convergent SDP hierarchy is a variant of the *quantum inflation hierarchy* of Wolfe, Spekkens, Fritz, Navascues, Pozas-K., Grinberg, Rosset, Acin.

[Ligthart, Gachechiladze, DG 20]

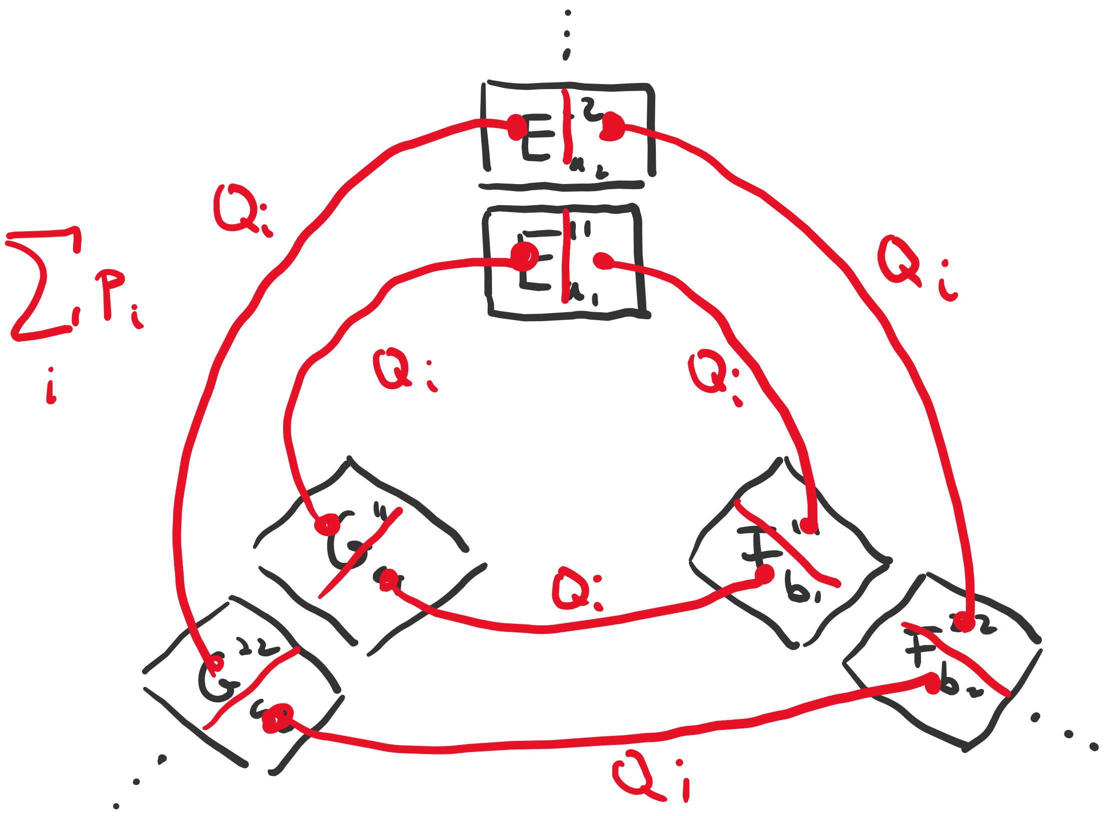
Outlook, Miscellanea, and Curiosities

- Recall: In bilocal scenario, GNS representation for $\mathcal{A}, \mathcal{C}$ factorizes
$$\mathcal{H}_\rho = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{C}$$
- *Can* turn this into algorithm for deciding membership with bi-partite tensor product model…
- …for states that can be prepared using “entanglement swapping with bounded probability”

- Cute GPT generalization with applications to non-negative matrix rank. [with Martin Plávala].



Thank you!



David Gross, University of Cologne

Joint work with Laurens Ligthart and Mariami Gachechiladze