Quantum expanders – Random constructions & Applications

Based on joint works with:
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Plan

Classical expanders

G a d-regular graph on n vertices (d edges at each vertex).
A its (normalized) adjacency matrix, i.e. the $n \times n$ matrix s.t. $A_{kl} = e(k,l)/d$ for all $1 \leqslant k,l \leqslant n$.
number of edges between vertices k and l ↵

$\lambda_1(A), \ldots, \lambda_n(A)$ eigenvalues of A, ordered s.t. $|\lambda_1(A)| \geqslant \cdots \geqslant |\lambda_n(A)|$.

G regular $\Longrightarrow \lambda_1(A) = 1$ with associated eigenvector the uniform probability $u = (1/n, \ldots, 1/n)$.
The *spectral expansion parameter* of G is $\lambda(G) := |\lambda_2(A)|$.

Observation: $\lambda(G) = |\lambda_1(A - J)|$, where J is the adjacency matrix of the *complete graph* on n vertices, i.e. the matrix whose entries are all equal to $1/n$.
$\longrightarrow \lambda(G)$ is a distance measure between G and the complete graph.

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Definition [Classical expander]

A d-regular graph G on n vertices is an *expander* if it is sparse (i.e. $d \ll n$) and spectrally expanding (i.e. $\lambda(G) \ll 1$).

$\longrightarrow$ G is both 'economical' and 'resembling' the complete graph.
For instance, a random walk supported on G converges fast to equilibrium.
Indeed, for any probability p on $\{1,\ldots,n\}$, $\forall\, q \in \mathbf{N}$, $\|A^q p - u\|_1 \leqslant \sqrt{n}\|A^q p - u\|_2 \leqslant \sqrt{n}\lambda(G)^q$.
exponential convergence, at rate $|\log \lambda(G)|$ ↵

Quantum analogue of the transition matrix associated to a regular graph

Classical - Quantum correspondence:

- $p \in \mathbf{R}^n$ probability vector $\longleftrightarrow$ $\rho \in \mathcal{M}_n(\mathbf{C})$ density operator (PSD and trace 1 operator).
- $A : \mathbf{R}^n \to \mathbf{R}^n$ transition matrix $\longleftrightarrow$ $\Phi : \mathcal{M}_n(\mathbf{C}) \to \mathcal{M}_n(\mathbf{C})$ quantum channel (CPTP map).
- G regular: A leaves u invariant $\longleftrightarrow$ Φ unital: Φ leaves I/n invariant.

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- G regular: A leaves u invariant $\longleftrightarrow$ Φ unital: Φ leaves I/n invariant.

Question: What is the analogue of the degree in the quantum setting?

Answer: The Kraus rank.
[Recall: Given a CP map Φ on $\mathcal{M}_n(\mathbf{C})$, its *Kraus representation* is:

$$\Phi : X \in \mathcal{M}_n(\mathbf{C}) \mapsto \sum_{i=1}^{d} K_i X K_i^* \in \mathcal{M}_n(\mathbf{C}), \text{ where } K_1, \dots, K_d \in \mathcal{M}_n(\mathbf{C}). \quad (\star)$$

↳ Kraus operators of Φ

The minimal d s.t. Φ can be written as $(\star)$ is the *Kraus rank* of Φ (it is always at most n^2).]

Indeed, the degree and the Kraus rank both quantify the 1-iteration spreading:

- G a d-regular graph: If $|\mathrm{supp}(p)| = 1$, then $|\mathrm{supp}(Ap)| \leqslant d$.
- Φ a Kraus rank d unital quantum channel: If $\mathrm{rank}(\rho) = 1$, then $\mathrm{rank}(\Phi(\rho)) \leqslant d$.

Quantum expanders

Φ a Kraus rank d unital quantum channel on $\mathcal{M}_n(\mathbf{C})$.
$\lambda_1(\Phi),\ldots,\lambda_{n^2}(\Phi)$ eigenvalues of Φ, ordered s.t. $|\lambda_1(\Phi)| \geqslant \cdots \geqslant |\lambda_{n^2}(\Phi)|$.

Φ unital $\Longrightarrow \lambda_1(\Phi) = 1$ with associated eigenstate the maximally mixed state I/n.
The *spectral expansion parameter* of Φ is $\lambda(\Phi) := |\lambda_2(\Phi)|$.

Observation: $\lambda(\Phi) = |\lambda_1(\Phi - \Pi)|$, where Π is the *fully depolarizing channel* on $\mathcal{M}_n(\mathbf{C})$, i.e. $\Pi : X \in \mathcal{M}_n(\mathbf{C}) \mapsto \mathrm{Tr}(X)\, I/n \in \mathcal{M}_n(\mathbf{C})$.
$\longrightarrow \lambda(\Phi)$ is a distance measure between Φ and the fully depolarizing channel.

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$\longrightarrow$ Φ is both ‘economical’ and ‘resembling’ the fully depolarizing channel.
For instance, the dynamics associated to Φ converges fast to equilibrium.
Indeed, for any state ρ on $\mathbf{C}^n$, $\forall\, q \in \mathbf{N}$, $\|\Phi^q(\rho) - I/n\|_1 \leqslant \sqrt{n}\,\|\Phi^q(\rho) - I/n\|_2 \leqslant \sqrt{n}\lambda(\Phi)^q$.
exponential convergence, at rate $|\log \lambda(\Phi)|$ ↲

Plan

Constructions of optimal classical expanders

Fact: For any d-regular graph G on n vertices, $\lambda(G) \geqslant 2\sqrt{d-1}/d - o_n(1)$.
$\longrightarrow$ G is called a *Ramanujan graph* if it is an optimal expander, i.e. $\lambda(G) \leqslant 2\sqrt{d-1}/d$.

Question: Do Ramanujan graphs exist?

1. Explicit constructions of exactly Ramanujan graphs only for $d = p^m + 1$, p prime.
2. Random constructions of almost Ramanujan graphs for all d.
3. Existence of exactly Ramanujan graphs for all d.

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In fact, for large n, almost all regular graphs are almost Ramanujan:

Theorem [Uniform random regular graph (Friedman, Bordenave)]

Fix $d \in \mathbf{N}$. Let G be uniformly distributed on the set of d-regular graphs on n vertices.
Then, for all $\varepsilon > 0$, $\mathbf{P}\left(\lambda(G) \leqslant \frac{2\sqrt{d-1}}{d} + \varepsilon\right) = 1 - o_n(1)$.

Remarks: (permutation model)
• First proven for a simpler model of random regular graphs: for d even, pick $\sigma_1, \ldots, \sigma_{d/2} \in \mathcal{S}_n$ independent uniformly distributed and let G have edges $\{(k, \sigma_i(k)), (k, \sigma_i^{-1}(k))\}_{1 \leqslant k \leqslant n, 1 \leqslant i \leqslant d/2}$.
• Result remains true for d_n growing with n, up to a constant multiplicative factor:
$\mathbf{P}\left(\lambda(G) \leqslant C/\sqrt{d_n} + \varepsilon\right) = 1 - o_n(1)$.

Constructions of optimal quantum expanders

Fact: For any Kraus rank d unital quantum channel Φ on $\mathcal{M}_n(\mathbf{C})$, $\lambda(\Phi) \geqslant c/\sqrt{d} - o_n(1)$.
$\longrightarrow$ Φ is considered an optimal expander if $\lambda(\Phi) \leqslant C/\sqrt{d}$.

Question: Do optimal quantum expanders exist?
First attempts at exhibiting explicit constructions (inspired by classical ones): not optimal.
$\longrightarrow$ What about random constructions?

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Question: How to sample a unital quantum channel randomly?

Idea: Pick random Kraus operators $K_1, \ldots, K_d \in \mathcal{M}_n(\mathbf{C})$, under the constraint $\begin{cases} \sum_{i=1}^d K_i^* K_i = I \\ \sum_{i=1}^d K_i K_i^* = I \end{cases}$.

Let $\Phi : X \in \mathcal{M}_n(\mathbf{C}) \mapsto \sum_{i=1}^d K_i X K_i^* \in \mathcal{M}_n(\mathbf{C})$ be the associated random unital quantum channel.

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Theorem [Independent paired Haar unitaries as Kraus operators (Hastings, Pisier)]

Fix $d \in \mathbf{N}$ even. Pick $U_1, \ldots, U_{d/2} \in \mathcal{M}_n(\mathbf{C})$ independent Haar unitaries. Let $K_i = U_i/\sqrt{d}$, $1 \leqslant i \leqslant d/2$. The random CP map Φ associated to the K_i's, K_i^*'s is TP and unital by construction.
Then, for all $\varepsilon > 0$, $\mathbf{P}\left(\lambda(\Phi) \leqslant \frac{2\sqrt{d-1}}{d} + \varepsilon \right) = 1 - o_n(1)$.

Remarks:

- Optimal constant for self-adjoint Kraus rank d unital quantum channels on $\mathcal{M}_n(\mathbf{C})$.
- Same result, up to a constant multiplicative factor, for d independent unitary Kraus operators.

More random examples of optimal quantum expanders

Question: Can the previous result be extended to other (non self-adjoint) random models? And to a regime where d is not fixed but grows with n?

Difficulty: Imposing that Φ is both TP and unital is very constraining.
However, the definition of expander can be generalized to 'close to unital' quantum channels, whose fixed point ρ_* has a large entropy: $S(\rho_*) \geqslant \alpha S(I/n) = \alpha \log n$, for some $0 < \alpha < 1$.
[Note: We now have $\lambda(\Phi) = |\lambda_1(\Phi - \Pi_{\rho^*})|$, where $\Pi_{\rho_*} : X \in \mathcal{M}_n(\mathbf{C}) \mapsto \mathrm{Tr}(X)\,\rho_* \in \mathcal{M}_n(\mathbf{C})$.]

Classical analogy: Relaxation of the exact regularity condition, e.g. to look at Erdős-Rényi graphs.

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Theorem [Independent Gaussians as Kraus operators (Lancien/Pérez-García)]

Pick $G_1, \ldots, G_d \in \mathcal{M}_n(\mathbf{C})$ independent Gaussian matrices. Let $\tilde{K}_i = G_i/\sqrt{d}$, $1 \leqslant i \leqslant d$.
↳ i.i.d. Gaussian entries (mean 0 and variance $1/n$)
The random CP map $\tilde{\Phi}$ associated to the $\tilde{K}_i$'s is not TP but almost: $\mathbf{P}\left(\Sigma := \sum_{i=1}^d \tilde{K}_i^* \tilde{K}_i \simeq I\right) \simeq 1$.
With $K_i = \tilde{K}_i \Sigma^{-1/2}$, $1 \leqslant i \leqslant d$, the random CP map Φ associated to the K_i's is TP by construction.
Then, $\mathbf{P}\left(S(\rho_*) \geqslant \log n - \frac{C'}{\sqrt{d}} \text{ and } \lambda(\Phi) \leqslant \frac{C}{\sqrt{d}} \right) \geqslant 1 - e^{-cn}$, for $C, C', c > 0$ constants.

Remark: Other model that was proven to be a.s. an optimal expander as n grows (for d fixed): blocks of a Haar isometry $V : \mathbf{C}^n \hookrightarrow \mathbf{C}^n \otimes \mathbf{C}^d$ as Kraus operators (González-Guillén/Junge/Nechita).

How much can the previous examples be generalized?

Theorem [Independent general random matrices as Kraus operators (Lancien/Youssef)]

1. Let $A \in \mathcal{M}_n(\mathbf{R})$ be a doubly stochastic matrix s.t. $|\lambda_2(A)| \leqslant \frac{C}{\sqrt{d}}$, with $d \geqslant (\log n)^4$.
   E.g. A the adjacency matrix of a d-regular graph G on n vertices s.t. $\lambda(G) \leqslant \frac{C}{\sqrt{d}}$.
2. Let $W \in \mathcal{M}_n(\mathbf{C})$ be a random matrix with independent centered entries, s.t.
   $\forall\, 1 \leqslant k, l \leqslant n$, $\mathbf{E}|W_{kl}|^2 = A_{kl}$ and $\left(\mathbf{E}|W_{kl}|^{2p}\right)^{1/p} \leqslant C' p^{\beta} A_{kl},\ p \in \mathbf{N}$.
   [$\beta = 0$: bounded entries. $\beta = 1$: sub-Gaussian entries. $\beta = 2$: sub-exponential entries.]
3. Pick $W_1, \ldots, W_d \in \mathcal{M}_n(\mathbf{C})$ independent copies of W. Let $K_i = \frac{W_i}{\sqrt{d}}$, $1 \leqslant i \leqslant d$, and Φ be the random CP map with the K_i's as Kraus operators.

Then, Φ is on average TP and unital, and s.t. $\mathbf{E}\lambda(\Phi) \leqslant \dfrac{C''}{\sqrt{d}}$.

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Interest: Constructing a random optimal quantum expander from any optimal classical expander.
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$\longrightarrow$ Optimal quantum expanders can be obtained from random Kraus operators which are sparse and whose entries have any distribution following the moments' growth assumption.

Remark: With more assumptions, one could compute variances and show that conclusions hold not only on average but also with positive / high probability.

Proof idea to show that $\mathbf{E}\,\lambda(\Phi) \leqslant C/\sqrt{d}$

Goal: In all cases, we want to upper bound $\mathbf{E}|\lambda_2(\Phi)| = \mathbf{E}|\lambda_1(\Phi - \Pi_{\rho^*})|$.
First step: Upper bound $\mathbf{E}|\lambda_1(\Phi - \mathbf{E}(\Phi))|$ (and then show that $\mathbf{E}(\Phi)$ is close to Π_{ρ^*}).

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- Observation 1: $|\lambda_1(\Psi)| \leqslant s_1(\Psi) = \|\Psi\|_\infty$.
- Observation 2: $\|\Psi\|_\infty = \|M_\Psi\|_\infty$, where for $\Psi : X \mapsto \sum_{i=1}^d K_i X L_i^*$, $M_\Psi = \sum_{i=1}^d K_i \otimes \bar{L}_i$.

[Identification $\Psi : \mathcal{M}_n(\mathbf{C}) \to \mathcal{M}_n(\mathbf{C}) \equiv M_\Psi : \mathbf{C}^n \otimes \mathbf{C}^n \to \mathbf{C}^n \otimes \mathbf{C}^n$ preserves the operator norm.]
$\longrightarrow$ We want to upper bound $\mathbf{E}\| \underbrace{M_\Phi - \mathbf{E}(M_\Phi)}_{=:X} \|_\infty$, where $M_\Phi = \sum_{i=1}^d K_i \otimes \bar{K}_i$ with the K_i's random.

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↱ Haar unitaries, Gaussians, blocks of Haar isometry

- For concrete models, this can be done by a moments' method:

By Jensen's inequality, we have: $\forall\, p \in \mathbf{N},\ \mathbf{E}\|X\|_\infty \leqslant \mathbf{E}\|X\|_p \leqslant (\mathbf{E}\,\mathrm{Tr}|X|^p)^{1/p}$.
The term on the r.h.s. can be estimated and provides a good upper bound for $p \simeq n^\gamma$.

↳ by Weingarten or Wick calculus

- For the general case, we use recent results on estimating the operator norm of random matrices with dependencies and non-homogeneity (Bandeira/Boedihardjo/van Handel, Brailovskaya/van Handel):

Setting $X = \sum_{i=1}^d Z_i$, with $Z_i := K_i \otimes \bar{K}_i - \mathbf{E}(K_i \otimes \bar{K}_i)$, $1 \leqslant i \leqslant d$, we have for $p \simeq \log n$,

$$\mathbf{E}\|X\|_\infty \lesssim \|\mathbf{E}(XX^*)\|_\infty^{1/2} + \|\mathbf{E}(X^*X)\|_\infty^{1/2} + (\log n)^{3/2}\|\mathbf{Cov}(X)\|_\infty^{1/2} + (\log n)^2 \left(\sum_{i=1}^d \mathbf{E}\,\mathrm{Tr}|Z_i|^p \right)^{1/p}.$$

Plan

Model of random 'physically relevant' states of many-body quantum systems

Useful subset of many-body quantum states: *matrix product states (MPS)*.

- They admit an *efficient description*: number of parameters that scales linearly rather than exponentially with the number of subsytems.
- They are *good approximations of several 'physically relevant' states*, such as ground states of gapped local Hamiltonians on 1D systems (Hastings, Landau/Vazirani/Vidick).

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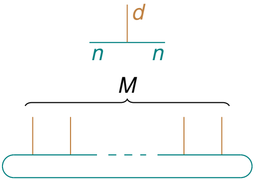
Model of random translation-invariant (TI) MPS:

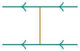
Pick $K_1, \ldots, K_d \in \mathcal{M}_n(\mathbf{C})$ at random.
Corresponding random TI MPS:
(M sites, *physical dimension* d, *bond dimension* n)

$$|\chi\rangle = \sum_{i_1,\ldots,i_M=1}^{d} \mathrm{Tr}\left(K_{i_1} \cdots K_{i_M}\right) |i_1\rangle \otimes \cdots \otimes |i_M\rangle \in (\mathbf{C}^d)^{\otimes M}$$



Associated random *transfer (super) operator*:

$$\Phi_\chi : X \in \mathcal{M}_n(\mathbf{C}) \mapsto \sum_{i=1}^{d} K_i X K_i^* \in \mathcal{M}_n(\mathbf{C})$$

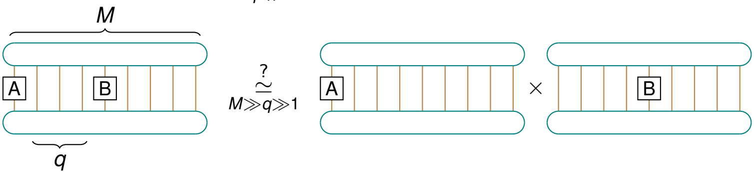


[Note: Φ_χ is the CP map version of the matrix $T_\chi = \sum_{i=1}^{d} K_i \otimes \bar{K}_i \in \mathcal{M}_n(\mathbf{C}) \otimes \mathcal{M}_n(\mathbf{C})$.]

Correlations between the 1-site observables A, B separated by q sites in the MPS $|\chi\rangle \in (\mathbf{C}^d)^{\otimes M}$:

$$\gamma_\chi(A,B,q) := \left| \left\langle A \otimes I^{\otimes q} \otimes B \otimes I^{\otimes (M-q-2)} \right\rangle_\chi - \left\langle A \otimes I^{\otimes (M-1)} \right\rangle_\chi \left\langle I^{\otimes (q+1)} \otimes B \otimes I^{\otimes (M-q-2)} \right\rangle_\chi \right|.$$

Question: Do we have $\gamma_\chi(A,B,q) \underset{q \ll M \to \infty}{\longrightarrow} 0$? And if so, at which speed?

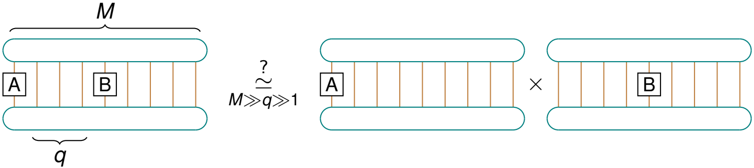


Correlations in a TI MPS and spectrum of its transfer operator

Correlations between the 1-site observables A, B separated by q sites in the MPS $|\chi\rangle \in (\mathbf{C}^d)^{\otimes M}$:

$$\gamma_\chi(A,B,q) := \left| \left\langle A \otimes I^{\otimes q} \otimes B \otimes I^{\otimes (M-q-2)} \right\rangle_\chi - \left\langle A \otimes I^{\otimes (M-1)} \right\rangle_\chi \left\langle I^{\otimes (q+1)} \otimes B \otimes I^{\otimes (M-q-2)} \right\rangle_\chi \right|.$$

Question: Do we have $\gamma_\chi(A,B,q) \underset{q \ll M \to \infty}{\longrightarrow} 0$? And if so, at which speed?



Fact: Let $\lambda_1(\Phi_\chi), \ldots, \lambda_{n^2}(\Phi_\chi)$ be the eigenvalues of the transfer operator Φ_χ, ordered s.t. $|\lambda_1(\Phi_\chi)| \geqslant \cdots \geqslant |\lambda_{n^2}(\Phi_\chi)|$. Setting $\varepsilon(\chi) = |\lambda_2(\Phi_\chi)|/|\lambda_1(\Phi_\chi)|$, we have

$$\gamma_\chi(A,B,q) \leqslant C(\chi)\varepsilon(\chi)^q \|A\|_\infty \|B\|_\infty .$$

$\longrightarrow$ If $|\lambda_2(\Phi_\chi)| < |\lambda_1(\Phi_\chi)|$, correlations between two 1-site observables decay exponentially with the distance separating the two sites, at a rate $\tau(\chi) = |\log \varepsilon(\chi)|$.
Correlation length in the MPS χ: $\xi(\chi) := 1/\tau(\chi) = 1/|\log \varepsilon(\chi)|$.

Conclusion: Estimating $\xi(\chi)$ boils down to estimating $|\lambda_1(\Phi_\chi)|$ and $|\lambda_2(\Phi_\chi)|$.

Decay of correlations in random TI MPS

Examples of distribution for $K_1, \ldots, K_d \in \mathcal{M}_n(\mathbf{C})$:

1. $K_i = U_i/\sqrt{d}$, $1 \leqslant i \leqslant d$, where the U_i's are i.i.d. Haar unitaries.
2. $K_i = V_i$, $1 \leqslant i \leqslant d$, where $V = \sum_{i=1}^{d} V_i \otimes |i\rangle$ is a Haar isometry.
3. $K_i = G_i/\sqrt{d}$, $1 \leqslant i \leqslant d$, where the G_i's are i.i.d. Gaussians with mean 0 and variance $1/n$.
   (More generally: $K_i = W_i/\sqrt{d}$, $1 \leqslant i \leqslant d$, where the W_i's are i.i.d. matrices with independent centered entries having variance profile a doubly stochastic matrix A s.t. $|\lambda_2(A)| \leqslant C/\sqrt{d}$.)

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Theorem [Correlation length of a random TI MPS (Lancien/Pérez-García)]

Let $|\chi\rangle \in (\mathbf{C}^d)^{\otimes M}$ be a random TI MPS, with associated $K_1,\dots,K_d \in \mathcal{M}_n(\mathbf{C})$ sampled according to one of the models above.
For large n, its correlation length is typically upper bounded by $2/\log d$.

One-slide summary of “Implications for decay of correlations in random MPS”

Matrix product states (MPS) form a subset of *many-body quantum states*.

They are particularly useful because:

- They admit an *efficient description*: number of parameters that scales linearly rather than exponentially with the number of subsytems.
- They are *good approximations of several ‘physically relevant’ states*, such as ground states of gapped local Hamiltonians on 1D systems (Hastings, Landau/Vazirani/Vidick).

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with the distance separating the sites

between observables measured on distinct sites

Main result: Random MPS typically have correlations that decay exponentially fast, with a *small correlation length* (Lancien/Pérez-García).

Proof strategy: Observe that the correlation length is given by $1/|\log\lambda(\Phi)|$ for Φ a random quantum channel associated to the random MPS (its so-called *transfer operator*).

Some perspectives

- What about *explicit constructions* of optimal quantum expanders?
  Important for applications (cryptography, error correction, condensed matter physics, etc)

  Previously known constructions required a large amount of randomness.
  First step towards *derandomization*: sparse matrices with ± 1 entries as Kraus operators.
  Other direction: unitary Kraus operators sampled according to a 'simple' measure that 'resembles' the uniform one, e.g. an approximate t-design (work in progress).

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- What about identifying the *full spectral distribution* of random quantum channels?

  Known: for a random Kraus rank d quantum channel $\Phi : \mathcal{M}_n(\mathbf{C}) \to \mathcal{M}_n(\mathbf{C})$, the eigenvalues of $\Phi - \Pi_{\rho_*}$ are typically inside a disc of radius $C/\sqrt{d}$ for large n.
  But what is the exact radius and are they uniformly distributed inside this disc?
  Answer in the self-adjoint case: asymptotically (as $n, d \to \infty$) the spectrum of $\sqrt{d}(\Phi - \Pi_{\rho_*})$ follows a semi-circular distribution (Lancien/Oliveira Santos/Youssef).

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- Do the results about the typical spectral gap of random quantum channels remain true when we impose *extra symmetries* on the model?

- What about looking at other, related, notions of expansions, such as *geometric* ones (Bannink/Briët/Labib/Maassen) or *linear-algebraic* ones (Li/Qiao/Wigderson/Wigderson/Zhang)?

References

- **T. Bannink, J. Briët, F. Labib, H. Maassen.** Quasirandom quantum channels. 2020.
- **J. Friedman.** A proof of Alon's second eigenvalue conjecture. 2008.
- **A. Bandeira, M. Boedihardjo, R. van Handel.** Matrix concentration inequalities and free probability. 2021.
- **C. Bordenave.** A new proof of Friedman's second eigenvalue theorem and its extension to random lifts. 2020.
- **T. Brailovskaya, R. van Handel.** Universality and sharp matrix concentration inequalities. 2022.
- **C. González-Guillén, M. Junge, I. Nechita.** On the spectral gap of random quantum channels. 2018.
- **M.B. Hastings.** Solving gapped Hamiltonians locally. 2006.
- **M.B. Hastings.** Random unitaries give quantum expanders. 2007.
- **C. Lancien, P. Oliveira Santos, P. Youssef.** Limiting spectral distribution of random self-adjoint quantum channels. 2024.
- **C. Lancien, D. Pérez-García.** Correlation length in random MPS and PEPS. 2021.
- **C. Lancien, P. Youssef.** A note on quantum expanders. 2023.
- **Z. Landau, U. Vazirani, T. Vidick.** A polynomial-time algorithm for the ground state of 1D gapped local Hamiltonians. 2015.
- **Y. Li, Y. Qiao, A. Wigderson, Y. Wigderson, C. Zhang.** On linear-algebraic notions of expansion. 2022.
- **G. Pisier.** Quantum expanders and geometry of operator spaces. 2014.

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