1 Exercise

1) **Information erasure:** Consider a system in a state ρ to be converted to some fixed pure state $|\psi\rangle$, by introducing an external system B involving N qubits, $|0\rangle\langle 0|^{\otimes N}$, and applying to the composite system a unital channel. During this process, B is simultaneously converted from a pure state $|0\rangle$ to a maximally mixed flat state $(\mathbb{1}/2)^{\otimes N}$. Find the relation between the entropy of $\Omega = \rho \otimes |0\rangle\langle 0|^{\otimes N}$ and the one of $\Upsilon = |\psi\rangle\langle\psi| \otimes (\mathbb{1}/2)^{\otimes N}$. Can you derive an upper bound on $S(\rho)$?

2) **Different geometries:** Use conformal maps to find the entanglement entropy of a subsystem of length ℓ within a finite system of length L with periodic boundary conditions, in its ground state. You can repeat a similar analysis for a semi-infinite line, $[0, \infty)$, and the subsystem A is the finite interval $[0, \ell)$. What is the coefficient in front of the logarithm with respect to the one with periodic boundary conditions? Is there an interpretation?

3) **Charged moments:** We want to compute the charged moments $Z_n(\alpha) = \mathrm{Tr}(\rho_A^n e^{i\alpha Q_A})$, where Q_A is the charge restricted to the subsystem $A = [u, v]$ on an infinite line. You can think of $Z_n(\alpha)$ as the partition function on a Riemann surface in the presence of a charge flux that you can put in only one sheet (e.g. the first one). You can introduce a composite twist field $\mathcal{T}_{n,\alpha} = \mathcal{T}_n \mathcal{V}_\alpha$, where $\mathcal{V}_\alpha$ is a primary operator generating the magnetic flux, with dimension h_α. Following the same steps we used for the standard twist field, find the scaling dimension of $\mathcal{T}_{n,\alpha}$ and prove that $Z_n(\alpha) = c_{n,\alpha}|v-u|^{-\frac{c}{6}(n-1/n)-\frac{2}{n}(h_\alpha+\bar{h}_\alpha)}$.

4) **Local quench:** Suppose we physically cut a system at the boundaries between two subsystems A and B. In this state the two subsystems are unentangled. Let us join up the pieces at time $-t$. What is the physical cut in the density matrix and how does it change with respect to the one of the global quantum quench? What is the entanglement entropy between the two parts in which the system was divided before the quench? The two half-chains are joined together at the point $z_1 = (0, i\tau)$ and you can use the one-point function of the twist field in the upper-half-plane plane. What is the asymptotic expression for $t \gg \epsilon$? Are there any free dynamical parameters in the result you find? *The z-plane with two slits can be mapped into the half-plane $Re[w] > 0$ by $w = z/\epsilon + \sqrt{(z/\epsilon)^2 + 1}$, where ϵ is a cutoff similar to τ_0. For the last question, remember that $S_A(t = 0) = 0$.*

5) **Negativity with boundaries:** Consider two adjacent intervals, the first one starting from the boundary, i.e. $A_1 = [0, \ell_1]$ and $A_2 = [\ell_1, \ell_1 + \ell_2]$ where $B = [\ell_1 + \ell_2, \infty)$ is the remainder. For simplicity, place the spatial coordinate along the imaginary direction in the complex plane, while the imaginary time is on the real direction. By images method, what is the expression of $\mathrm{Tr}(\rho_A^{T_2})^n$. Can you draw some general conclusion for $\ell_1 = \ell_2$?

6) **Entanglement Hamiltonian:** For simple one dimensional geometries, the entanglement Hamiltonian (i.e. the logarithm of the reduced density matrix) may be written explicitly in a local form using the physical energy density T_{00}

$$H_A = \int_A dx \beta(x) T_{00}(x)$$

This form of the entanglement Hamiltonian implies that ρ_A represents an ensemble with the physical energy density T_{00} in local thermal equilibrium with local temperature $\beta(x)$. Its entanglement entropy is therefore just the thermal entropy, obtained by integrating the thermal entropy density over the region A. If $A = x > 0$ is the half-line, $\beta(x) = 2\pi x$. By using a proper conformal mapping, compute the entanglement for one single interval $A = [u, v]$.

7) **Entanglement in higher dimensions:** Consider a free massive scalar theory in d Euclidean dimensions for a free massive scalar field

$$S = \int d^d x [\partial_\mu \varphi(x) \partial^\mu \varphi(x) + m^2 \varphi(x)\varphi(x)].$$

We denote the space coordinates by x_i, $i = 1, \cdots, d-1$, and the Euclidean time by x_0. Let A and $\bar{A}$ be regions with $x_1 > 0$ and $x_1 \leq 0$, respectively. The entangling surface Σ is chosen to be a $(d-2)$-dimensional hyperplane at $x_1 = 0$: $\Sigma = \{(x_0, x_i) | x_0 = x_1 = 0\}$. Let us introduce the metric of the spacetime

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\tau^2 + \sum_{i=2}^{d-1} \mathrm{d}x_i^2, \tag{1}$$

where we have used the polar coordinates for the plane parametrised by (x_0, x_1). The metric of the n-fold cover $\mathcal{M}_n$ of the original spacetime is (1) with $r \geq 0$ and $0 \leq \tau \leq 2\pi n$. Thus, $\mathcal{M}_n = \mathcal{C}_{n,\alpha} \times \mathbb{R}^{d-2}$, where $\mathcal{C}_n$ is the two-dimensional cone parametrized by (r, τ).

The $\mathrm{Tr}\rho_A^n$ on $\mathcal{M}_n$ is given by

$$\begin{aligned} \ln \mathrm{Tr}\rho_A^n &= -\frac{1}{2} \ln \det(-\nabla^2 + m^2) = -\frac{1}{2} \mathrm{tr} \ln(-\nabla^2 + m^2) \\ &= \frac{1}{2} \int_{\epsilon^2}^{\infty} \frac{ds}{s} \mathrm{tr} \left[e^{-s(-\nabla^2 + m^2)} - e^{-s} \right], \end{aligned} \tag{2}$$

where the parameter $\epsilon^2 \ll 1$ is introduced as a regulator for the UV divergences. Because of the direct product structure of $\mathcal{M}_n$, the Laplacian decomposes into the sum of those on $\mathcal{C}_n$ and $\mathbb{R}^{d-2}$: $\nabla^2 = \nabla^2_{\mathcal{C}_n} + \nabla^2_{\mathbb{R}^{d-2}}$. The rotational symmetry of the cone $\mathcal{C}_n$ allows the Fourier decomposition of the real fields by the modes $\exp(i\tau a)$, where $a = \frac{l}{n}$, with integer l. Therefore, the eigenfunctions $\varphi_{k,a}(r, \tau)$ of the Laplacian are parametrized by (k, a) satisfying

$$\begin{aligned} \nabla^2_{\mathcal{C}_n} \varphi_{k,a}(r, \tau) &= -k^2 \varphi_{k,a}(r, \tau), k \in \mathbb{R}^+, \\ \varphi_{k,a}(r, \tau) &= \sqrt{\frac{k}{2\pi n}} e^{i\tau a} J_{|a|}(kr), \end{aligned} \tag{3}$$

where J_a is the Bessel function of the first kind. The eigenfunctions form an orthonormal basis on the cone $\mathcal{C}_n$, namely

$$\int_{\mathcal{C}_n} \mathrm{d}^2 x \varphi_{k,a}(x) \varphi^*_{k',a'}(x) = \delta_{na,na'} \delta(k - k').$$

The orthonormal basis of the eigenfunctions of the Laplacian on $\mathbb{R}^{d-2}$ is spanned by the plane waves, $\varphi_{\mathbf{k}^\perp}(y) = \exp(i\mathbf{k}_\perp \cdot \mathbf{y})/(2\pi)^{(d-2)/2}$, with eigenvalues $-k_\perp^2$. Exploiting these two sets of eigenfunctions, compute the trace of the kernel in Eq. (2). Using Eqs. (2) and the result above, you can compute $\ln \mathrm{Tr}\rho_A^n$ for a massive scalar field. As a consistency check, focus your attention on $d = 2$: for a semi-infinite line, you should obtain obtain

$$\ln \mathrm{Tr}\rho_A^n = \left(-\frac{1}{24} \left(n - \frac{1}{n} \right) \right) (-\mathrm{Ei}(-m^2 \epsilon^2)), \tag{4}$$

where

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} dt \frac{e^{-t}}{t}. \tag{5}$$

By expanding around $\epsilon = 0$, prove that

$$S_1 = \frac{1}{6}(-\ln(m\epsilon) - \frac{\gamma_E}{2}), \tag{6}$$

where γ_E is the Euler-Mascheroni constant.