1 Exercise

The massless Dirac field theory is described by the action

$$\mathcal{S}_{\mathrm{D}} = \int \mathrm{d}x_0 \mathrm{d}x_1 \bar{\psi}\gamma^\mu \partial_\mu \psi, \tag{1}$$

where $\bar{\psi} = \psi^\dagger \gamma^0$. The γ^μ matrices can be represented in terms of the Pauli matrices as $\gamma^0 = \sigma_1$ and $\gamma^1 = \sigma_2$.

The partition function corresponding to $\mathrm{Tr}\,[\rho_A^n]$ has a single fermionic field defined on a Riemann surface made of n different sheets, and can be mapped to an equivalent one in which one deals with a n-component field, which is instead defined on a single sheet:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}, \tag{2}$$

where ψ_j is the Dirac field of the j-th copy of the system. The transformation among the different copy can be encoded in the twist matrix:

$$T_a = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ (-1)^{n+1} & & 0 & \end{pmatrix}. \tag{3}$$

Can you find an appropriate change of basis which diagonalizes this matrix? Write down the boundary conditions that the decoupled fields will satisfy.
What is the total partition function?
Suppose that A is composed of p intervals

$$A = \bigcup_i [u_i, v_i]. \tag{4}$$

Once you write down the total partition function in terms of the single-copy partition function, how can you solve the problem of multivaluedness of the fields?
Hint 1): Absorb the phase into a pure gauge transformation

$$\psi_k(x) = e^{i\int_{x_0}^{x} \mathrm{d}y^\mu A_\mu^k} \psi_k^G(x), \tag{5}$$

Hint 2): Require a null phase condition around the branch points for ψ_k^G + Stokes theorem.
The Dirac Euclidean lagrangian in a gauge field is

$$\mathcal{L}_k = \bar{\psi}_k^G \gamma^\mu (\partial_\mu - iA_\mu^k)\psi_k^G. \tag{6}$$

Can you write down the partition function as an expectation value (involving the gauge field) on the massless, ungauged theory?
Does the formula simplify by using famous one-dimensional techniques? Hint: use $j_k^\mu = \epsilon^{\mu\nu}\partial_\nu \phi_k/\sqrt{\pi}$, and vertex operators $V_a(y) = e^{-ia\phi_k(y)}$.
Focus on the case $p = 2$, with $A_1 = [u_1, v_1], A_2 = [u_2, v_2]$ and write down the final formula in terms of the cross-ratio

$$\ell_1 = |v_1 - u_1|, \qquad \ell_2 = |v_2 - u_2|, \qquad d = |u_2 - v_1|, \qquad x = \frac{\ell_1 \ell_2}{(d+\ell_1)(d+\ell_2)}. \tag{7}$$

The action 1 is $U(1)$ invariant. Can you write down the charged moments associated to the charge $Q_{\mathrm{D}} = \int \mathrm{d}x_1 \psi^\dagger \psi$? Hint: The twist matrix should be modified such that

$$T_\alpha \mathcal{O}_j(e^{2\pi i}(z-u)) = e^{i\alpha/n}\mathcal{O}_{j+1}(z-u). \tag{8}$$

Modify the boundary conditions on the gauge field accordingly and finally prove that

$$Z_{k,n}^{A_1:A_2}(\{\alpha_j\}) = \langle V_{\frac{k}{n}+\frac{\alpha_1}{2\pi n}}(u_1) V_{-\frac{k}{n}+\frac{\alpha_2}{2\pi n}}(v_1) V_{\frac{k}{n}+\frac{\alpha_3}{2\pi n}}(u_2) V_{-\frac{k}{n}+\frac{\alpha_4}{2\pi n}}(v_2)\rangle. \tag{9}$$