ALPS exercises in Quantum Computing
Digital quantum simulation tutorial, II. Schwinger model hopping propagator via shears

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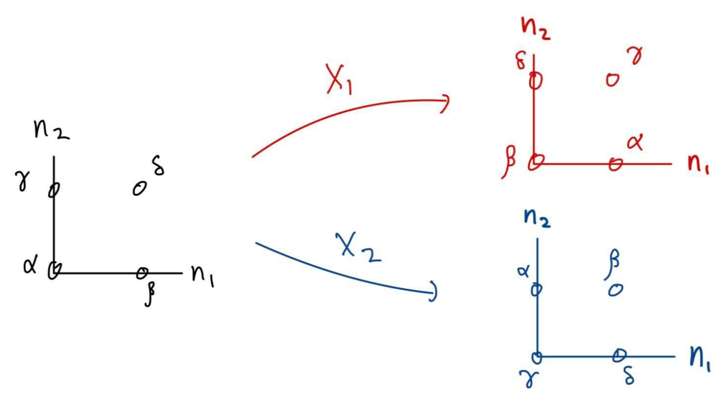
May 16, 2024

In this tutorial, we learn how to effect time evolution for the hopping term of a link in the Schwinger model. We will take the Hamiltonian to be $H = x\,\sigma_1^- \sigma_2^+ U + \text{H.c.}$, where 1 and 2 refer to the fermions at the left and right sides of the link, and $U = \sum_{E=E_{min}}^{E_{max}-1} |E+1\rangle\langle E|$ is the truncated link operator. Note that, in the computational basis, H can be described as off-diagonal with respect to all three variables. The steps below will show how to diagonalize H such that its time evolution is "easy" to circuitize.

Shearing approach to the hopping term

1. Consider first the toy Hamiltonian $H' = x\,\sigma_1^- \sigma_2^+ + \text{H.c.}$ and recall $\sigma^+ = |0\rangle\langle 1|$, $\sigma^- = |1\rangle\langle 0|$. Note how H' is off-diagonal with respect to both matter qubits in the computational basis.

   - Draw points for the four computational basis states in the $n_1 \times n_2$ plane, and make from them a mathematical graph showing what states are mixed by the application of H'. (An edge joining points $|a\rangle$ and $|b\rangle$ means that $\langle a| H' |b\rangle \neq 0$.)
   - Some gates, when applied to the basis states, have a very simple geometric action on the graph. For example, the single-qubit gates X_1 or X_2 correspond to reflections as follows:

   Come up with a two-qubit gate V' which, when applied to the basis states, has the effect of realigning the edge(s) in the graph of H' parallel to the n_1 axis. Using V', explicitly evaluate the transformed Hamiltonian, $V'HV'^\dagger$. How would you characterize $V'HV'^\dagger$ and its associated graph in terms of being diagonal vs. off-diagonal?

2. Now consider another toy Hamiltonian, $H'' = x\,\sigma_1^- U + \text{H.c.}$, which is off-diagonal on one matter qubit and on the bosonic E register.

   - Draw points for the computational basis states in the $n_1 \times E$ plane, and make from them a mathematical graph showing what states are mixed by the application of H''.
   - Let $\lambda^+ = U + |E_{min}\rangle\langle E_{max}|$ and $\lambda^- = U^\dagger + |E_{max}\rangle\langle E_{min}|$ denote cyclic incrementers on the electric field. Use λ^- to devise a gate V'' that, when applied to all qubits, aligns the graph of H'' parallel to the n_1 axis. You should find that V'' commutes with the gate V' found previously.

3. If all was done as expected, you should have found two unitary transformations V' and V'', which when applied to the original H, transform it as

$$\begin{aligned} V''V'H(V''V')^\dagger &= x\,\sigma_1^- \;|1\rangle\langle 1|_2 \,(1 - |E_{max}\rangle\langle E_{max}|) + \text{H.c.} \\ &= x\,X_1 \;|1\rangle\langle 1|_2 \,(1 - |E_{max}\rangle\langle E_{max}|). \end{aligned}$$

   In reality, the choice of V' and V'' is not unique and your projection operators could vary. Regardless, $V''V'H(V''V')^\dagger$ should be off-diagonal on the 1-mode qubit alone and fully diagonalized by a Hadamard gate on the same qubit.

   - Separating out the two terms $x\,Z_1\,|1\rangle\langle 1|_2$ and $-x\,Z_1\,|1\rangle\langle 1|_2\,|E_{max}\rangle\langle E_{max}|$ of the fully diagonalized Hamiltonian (they commute), how would you simulate each one? (What controls are needed?)