Stochastic quantum thermodynamics of clocks

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1. Thermodynamics of clocks.
2. Quantum clocks & and measurement noise.
3. Experiment using a superconducting circuit qubit.
4. Quantum thermodynamic uncertainty relations.

Thermodynamics of clocks.

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See GJM *The thermodynamics of clocks*, Contemporary Physics, 2020.

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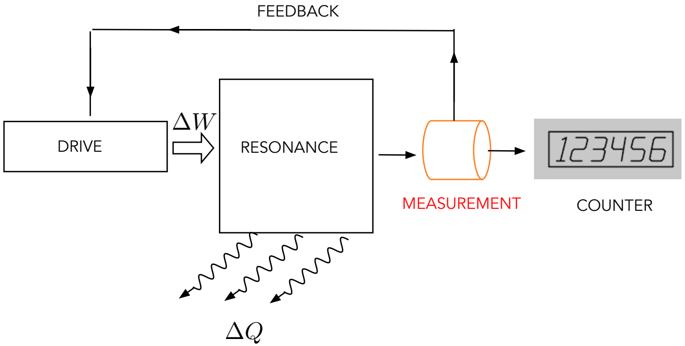
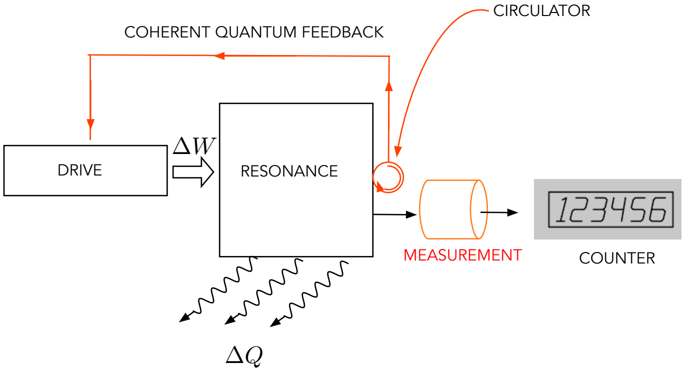
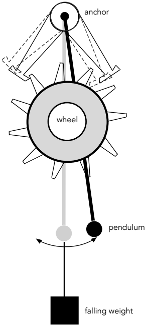


Thermodynamics of quantum clocks.



Work driven clocks and limit cycles .



Pendulum clock: a driven non linear oscillator in a non equilibrium steady state.

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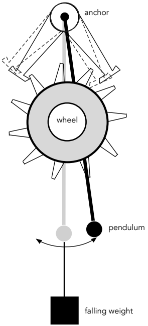

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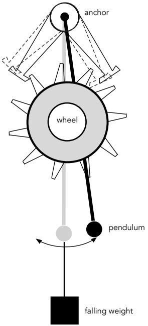

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Equations of motion:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - \Gamma y + K(x, y) + \overbrace{\eta(t)}^{\text{thermal noise}} \end{aligned}$$

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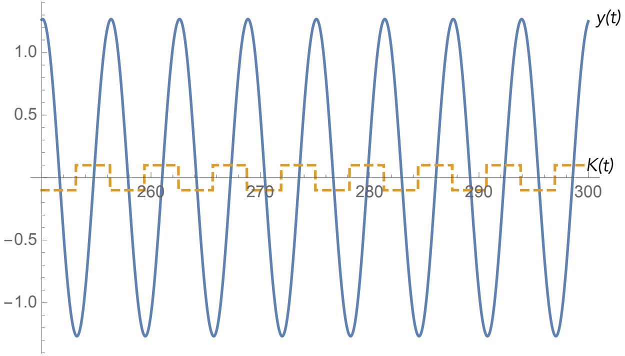
Adiabatic elimination of the ratchet dynamics to define the 'kick function'

$$K(x,y) = -\mu\,\mathrm{sign}(\sin\psi_0\, x - \cos\psi_0\, y)$$

ψ_0 is fixed by the design of the escapement and μ has units of frequency.

Work driven clocks and limit cycles .

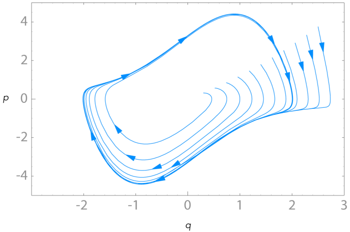
On the limit cycle (with no noise):



Angular momentum (solid) and the impulse function (dashed) versus time for the kicked pendulum with $\sin\psi_0 = 0.1$ $\Gamma = 0.1$ $\mu = 0.1$.

Phase reduction.

periodic clocks exhibit self sustained oscillations known as *limit cycles* or *relaxation oscillations*. A limit cycle is a *one-dimensional attractor*.



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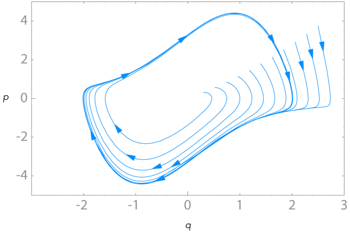

Phase function $\theta(t) = \Theta(q(t), p(t))$, with $0 \leq \theta(t) < 2\pi$ determines clock period.

see Sacré & Sepulchre, IEEE CONTROL SYSTEMS MAGAZINE, APRIL 2014

Phase reduction and noise.

Normal form for Hopf bifurcation:

$$\begin{aligned}\dot{x} &= [\mu x - \omega y - (x^2 + y^2)y]dt \\ \dot{y} &= [\omega x + \mu y - (x + y^2)x]dt + \sigma dW(t)\end{aligned}$$

μ is the amplification rate.

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limit cycle: $x(t) = \sqrt{\mu}\cos(\theta(t)), \quad y(t) = \sqrt{\mu}\sin(\theta(t))$.

$$d\theta(t) = \omega + \frac{\sigma}{\mu} dW(t)$$

the larger the limit cycle the slower the phase noise.

See Aminzare et al. IEEE 58th Conference on Decision and Control (CDC), 2019.

Phase reduction and noise.

First passage time distribution: probability for time T taken for phase to change by 2π... the period is a random variable.

Phase reduction and noise.

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Wald or Inverse Gaussian distribution of periods.

$$W(T,\alpha,\lambda) = \sqrt{\frac{\lambda}{2\pi}} T^{-3/2} \exp\left[-\frac{\lambda}{2\alpha^2 T}(T-\alpha)^2\right] \qquad t \geq 0.$$

where α, λ are positive real parameters (λ is called the spread parameter).

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$$\begin{aligned}\overline{T} &= \alpha\\ \overline{\Delta T^2} &= \frac{\alpha^3}{\lambda}\end{aligned}$$

Phase reduction and noise.

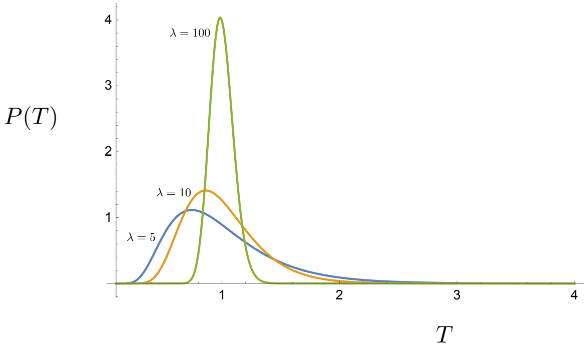
Normal form limit cycle:

$$\overline{T} = \frac{2\pi}{\omega}$$
$$\overline{\Delta T^2} = \frac{2\pi\sigma^2}{\omega^3\mu}$$

fluctuations in period get smaller as the limit cycle gets bigger.

More work, more heat dissipated, better the clock (See Erker et al. 2017).

Noise and fluctuations in period.



Mean period $\bar{T} = 1$ and three different values of the spread parameter, λ. Increasing λ is decreasing noise.

Quantum clocks & and measurement noise.

Example: the laser.

A laser is a quantum clock, if you add a counter.

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Dissipative *nonlinear* oscillator (Wiseman,PRA, 60, 4083).

$$\dot{\rho} = G n_s \mathcal{D}[a^\dagger] \overbrace{\left(\mathcal{A}[a^\dagger] + n_s\right)^{-1}}^{\text{gain sat.}} \rho + \kappa \mathcal{D}[a]\rho$$

$$\mathcal{D}[a]\rho = a\rho a^\dagger - \frac{1}{2}(a^\dagger a\rho + \rho a^\dagger a) \quad \mathcal{A}[a]\rho = \frac{1}{2}(a^\dagger a\rho + \rho a^\dagger a)$$

G is the small signal gain, n_s saturation photon number and κ is the cavity decay rate.

Equivalent classical model

Dynamics of the average field $\alpha(t) = \mathrm{tr}(a\rho(t))$

$$\dot{\alpha} = -\frac{\kappa\alpha}{2}\left(1 - \frac{Gn_s}{\kappa(|\alpha|^2 + n_s)}\right)$$

Well above threshold this is similar to the van der Pol oscillator in a rotating frame.

There are two fixed points $\alpha_0 = 0$ and $|\alpha_0|^2 = Gn_s/\kappa$ for $G >> \kappa$. The second solution is the above threshold limit-cycle solution.

Quantum model

What to measure?

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Photon counting ? .... a Poisson distribution of counts. No oscillatory clock signal.

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Photon counting ? .... a Poisson distribution of counts. No oscillatory clock signal.

No point in measuring output intensity.

Quantum model: measurement noise

Measure field amplitude by heterodyne detection to get a clock signal.

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Measure field amplitude by heterodyne detection to get a clock signal.

measured current:

$$J_{het}(t)dt = \kappa\langle a(t)\rangle_c dt + \sqrt{\frac{\kappa}{\eta}}dW(t)$$

where $0 < \eta \leq 1$ is the photo-detector quantum efficiency, $dW(t)$ is a complex valued Weiner process and $\langle a(t)\rangle_c$ is the mean field amplitude conditioned on the entire history of the observed current.

Quantum model: measurement noise

Conditional quantum dynamics:

$$d\rho_c = \mathcal{L}\rho_c dt + \sqrt{\frac{\eta}{\kappa}}\mathcal{H}[a]\rho_c dW^*$$

dW^*, complex Weiner increment, $\mathcal{H}[A]$ is defined by

$$\mathcal{H}[\mathcal{A}]\rho = A\rho + \rho A - \mathrm{tr}[A\rho + \rho A]\rho$$

is nonlinear...as expected.

Wiseman and Milburn "Quantum Measurement and Control, CUP, 2011

Quantum model: measurement noise

Well above threshold (on the classical limit cycle)

$$\dot{\rho}_c = \Gamma\mathcal{D}[a^\dagger a] + \sqrt{\eta}\mathcal{H}[a]\rho_c dW^*$$

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$$\dot{\rho}_c = \Gamma\mathcal{D}[a^\dagger a] + \sqrt{\eta}\mathcal{H}[a]\rho_c dW^*$$

Writing $\rho_c(t)$ as a mixture of coherent states as

$$\rho_c(t) = \int d^2\alpha P_c(\alpha, t)|\alpha\rangle\langle\alpha|$$

define $\alpha = re^{i\phi}$

$$d\phi_c = \sqrt{\Gamma}dW$$

non-thermal phase diffusion rate $\Gamma = \kappa/2\bar{n}$.

Quantum model: measurement noise

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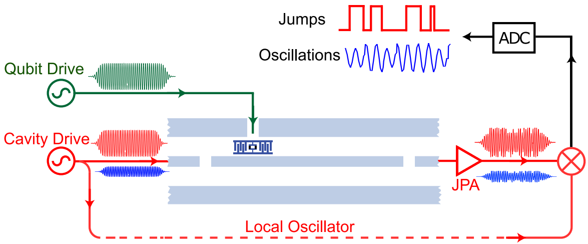
non-thermal phase diffusion rate $\Gamma = \kappa/2\bar{n}$.
Energy dissipated by cavity loss is $\propto \kappa\bar{n}$

Experiment using a superconducting circuit qubit.

Quantum model: measurement noise

Simple model:



Transmon qubit dispersively coupled to a superconducting microwave cavity.
Cavity and qubit are driven coherently.
The output field is subject to a homodyne measurement.

Quantum model: measurement noise

$$d\rho_c = \overbrace{-iE[a+a^\dagger,\rho_c]dt}^{\text{cavity drive}} - \overbrace{i\Omega[\sigma_x,\rho_c]dt}^{\text{qubit drive}} - i\chi[a^\dagger a\sigma_z,\rho_c]dt + \gamma\mathcal{D}[\sigma_-]\rho_c + \kappa\mathcal{D}[a]\rho_c dt + \sqrt{\eta\kappa}\mathcal{H}[a]\rho_c dW(t).$$

χ: dispersive coupling.
γ: qubit amplitude damping rate, κ: cavity damping rate,
η photodetector, efficiency

$\mathcal{D}[A]$ and $\mathcal{H}[A]$ are superoperators:
$\mathcal{D}[A]\rho = A\rho A^\dagger - \frac{1}{2}(A^\dagger A\rho + \rho A^\dagger A)$ and
$\mathcal{H}[A]\rho = A\rho + \rho A^\dagger - \mathrm{tr}(A\rho + \rho A^\dagger)\rho$.

Quantum model: measurement noise

Adiabatically eliminate the cavity field.
conditional master equation for the reduced state of the qubit (ρ_σ) only:

$$d\rho_\sigma = -i[H_\sigma, \rho_\sigma]dt + \gamma\mathcal{D}[\sigma_-]\rho_\sigma dt + \Gamma\mathcal{D}[\sigma_z]\rho_\sigma dt - \sqrt{\Gamma}\mathcal{H}[\sigma_z]\rho_\sigma dW(t).$$

Here $H_\sigma = \Omega\sigma_x + \Delta\sigma_z$

$$\frac{\Gamma}{\kappa} = 4\left(\frac{\chi}{\kappa}\right)^2 n_0 \quad \text{measurement dephasing rate}$$

$$\frac{\Delta}{\kappa} = \left(\frac{\chi}{\kappa}\right) n_0 \quad \text{effective Stark shift}$$

$n = |\alpha_0|^2 = 4|E|^2/\kappa$

Γ can be tuned by varying E.

Quantum model: measurement noise

Bloch sphere conditional dynamics.

$$\begin{aligned} dX &= -2\Delta Ydt - \gamma_2 Xdt \\ dY &= 2\Delta Xdt - 2\Omega Zdt - \gamma_2 Ydt \\ dZ &= 2\Omega Ydt - \gamma(1+Z)dt - 2\sqrt{\Gamma}(1-Z^2)dW \end{aligned}$$

where $\gamma_2 = \gamma/2 + 2\Gamma$ is the transverse decay rate of the conditional polarization.

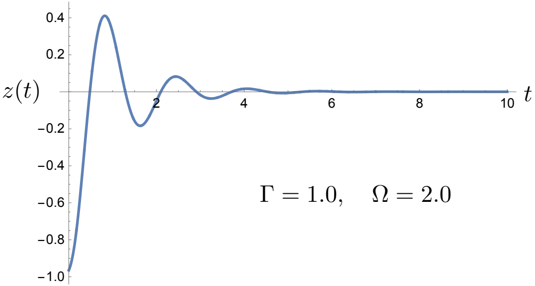
Spontaneous emission rate, $\gamma << 1$, so $\gamma_2 \approx 2\Gamma$.

The unconditional dynamics is obtained by averaging over the noise. $\overline{dW} = 0$.

Quantum model: measurement noise

Unconditonal dynamics:

underdamped : $\Omega > \Gamma/2$.



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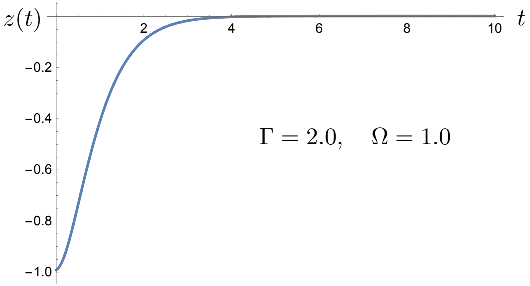

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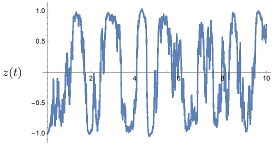
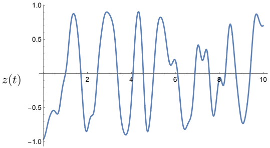
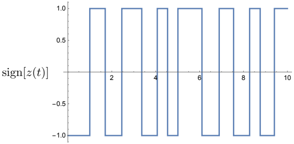




low pass filter



clock signal

Quantum model: measurement noise

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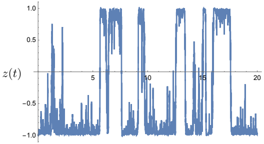
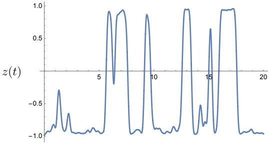


$\Gamma = 5, \quad \Omega = 2$



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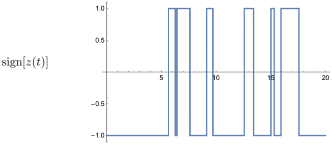

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Quantum model: measurement noise

Conditonal dynamics: under-damped : $\Omega > \Gamma/2$.

Coherent with phase noise.

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Quantum-jump regime ... a non periodic clock.

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Coherent with phase noise.

Conditonal dynamics: over-damped : $\Omega < \Gamma/2$.

Quantum-jump regime ... a non periodic clock.

In both cases, the noise is quantum not thermal.

It arises from measurement itself.

Quantum model: measurement noise

Quality measure:

$$N_{osc} = \frac{(\mathrm{E}[T])^2}{\mathrm{Var}[T]}$$

Simulate quantum stochastic master equation:

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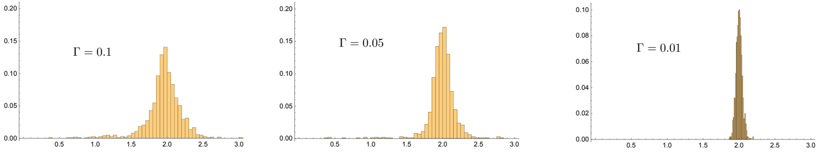

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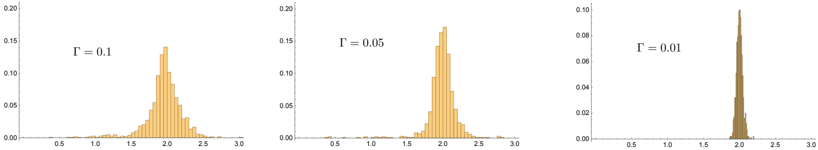


Decreasing measurement strength makes a better clock.

Quantum model: measurement noise

Linearise noise on unconditional steady state:

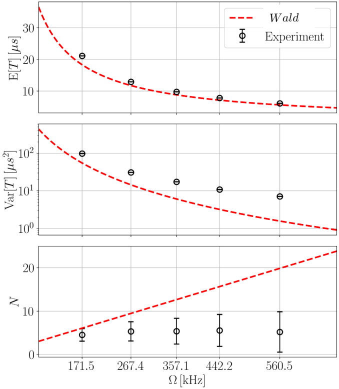
$$P(T) = \sqrt{\frac{\pi}{\Gamma T^3}} \exp\left[-\frac{(\pi - \Omega T)^2}{\Gamma T}\right] .$$

$$\mathrm{E}[T] = \frac{\pi}{\Omega} , \quad \mathrm{Var}[T] = \frac{\pi\Gamma}{2\Omega^3}$$

Define

$$N_{osc} = \frac{(\mathrm{E}[T])^2}{\mathrm{Var}[T]} = 2\pi\left(\frac{\Omega}{\Gamma}\right) .$$

Experiment.


Wald
Experiment
E[T] [μs]
30
20
10
Var[T] [μs²]
10²
10¹
10⁰
N
20
10
0
171.5
267.4
357.1
442.2
560.5
Ω [kHz]

Quantum model: measurement noise

Simulations N_{osc}:

Γ	variance	numerical	Wald
0.1	0.070	56	98.7
0.05	0.033	101	197
0.01	0.039	987	1024
0.001	0.002	1861	1974

better match for weak measurement.

Quantum thermodynamic uncertainty relations.

Quantum model: measurement noise

Quantum parameter estimation for weak continuous measurement.

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Period estimation is a First Passage Time problem.

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Kewming, et al. *First Passage Times for Continuous Quantum Measurement Currents*, arXiv:2308.07810

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Quantum thermodynamic uncertainty relations.

Quantum kinetic uncertainty relations (KUR).

$$\frac{\mathrm{Var}[T]}{(\partial_\theta \mathrm{E}_\theta[T]|_{\theta=0})^2} \geq \frac{1}{I_Q(0)}\,.$$

θ, estimated parameter,
$I_Q(0) = I_Q(\theta = 0)$ is the quantum Fisher information (QFI).

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$I_Q(0) = I_Q(\theta = 0)$ is the quantum Fisher information (QFI).

$$I_Q(0) = \mathrm{E}[T](\overbrace{\mathcal{N}}^{classsical} + \overbrace{\mathcal{Q})}^{quantum}\,,$$

where

$$\mathcal{N} = \Gamma\,, \qquad \mathcal{Q} = \frac{4\Omega^2}{\Gamma}\,,$$

See Van Vu et al. *Thermodynamics of Precision in Markovian Open Quantum Dynamics*, Phys. Rev. Lett. (2022).

Quantum kinetic uncertainty relations (KUR).

Why does the Fisher information have a quantum contribution?

Quantum kinetic uncertainty relations (KUR).

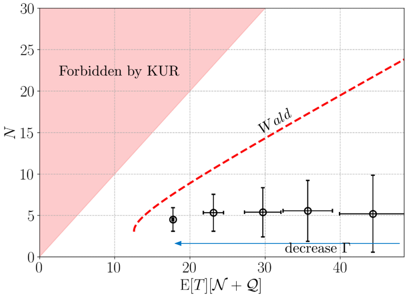
Why does the Fisher information have a quantum contribution?

Quantum probabilities are given by the Born rule

$$P(x|\theta) = |\psi_\theta(x)|^2$$

Need amplitude and phase to determine statistical distance. See Braunstein, Caves, GJM Annals of physics (1996).

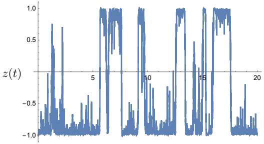
Experiment.



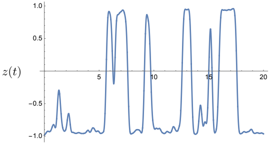
Forbidden by KUR
Wald
decrease Γ
N
$\mathrm{E}[T][\mathcal{N}+\mathcal{Q}]$
0
5
10
15
20
25
30
0
10
20
30
40

Quantum jump regime.

Conditonal dynamics: over-damped : $\Omega < \Gamma/2$.



$\Gamma = 5, \quad \Omega = 2$



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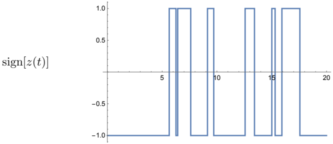


clock signal

Quantum jump regime.

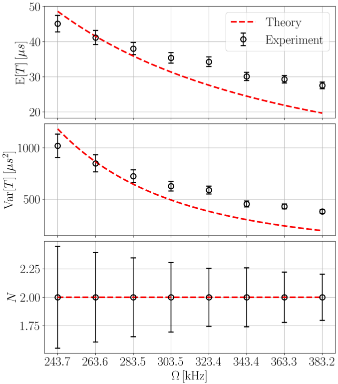
FPT distribution:

$$P(T) = \mu^2 T e^{-\mu T} \,,$$

$$\mu = \Gamma\Omega^2/(\Gamma^2 + \Delta^2)$$

$$\mathrm{E}[T] = \frac{2}{\mu}, \quad \mathrm{Var}[T] = \frac{2}{\mu^2}, \quad N_{\mathrm{jump}} = 2 \,.$$

Quantum jump regime: experiment



Conclusions.

- All clocks are dissipative systems pushed away from thermal equilibrium by work or measurement.
- At low temperature, weak continuous measurement plays a special role to extract a clock signal.
- At low temperature, quantum noise (spontaneous emission, tunnelling, measurement back action) limits clock accuracy.
- Use coherent quantum feedback.
- Engineer quantum noise for better clocks

  (eg Wiseman, et al. *The Heisenberg limit for laser coherence*, Nat. Phys. (2021).

Acknowledgements.

Arkady Fedorov, UQ

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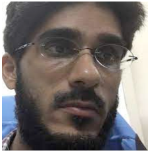
Michael Kewming (TCD)



Adil Gangat, NTT (USA)

Pete Evans (UQ)