A first taste of Non-Linear Beam Dynamics

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- **Introducing aspects of non-linear dynamics**
  - ❑ Effects of **nonlinear perturbations**
    - ■ Resonances, tune shifts, dynamic aperture
  - ❑ **Mathematical tools** for modelling nonlinear dynamics
    - ■ Power series (Taylor) maps and symplectic maps
  - ❑ **Analysis** methods
    - ■ Normal forms, frequency map analysis

- **Illustrate methods and tools for a simple example of an accelerator**
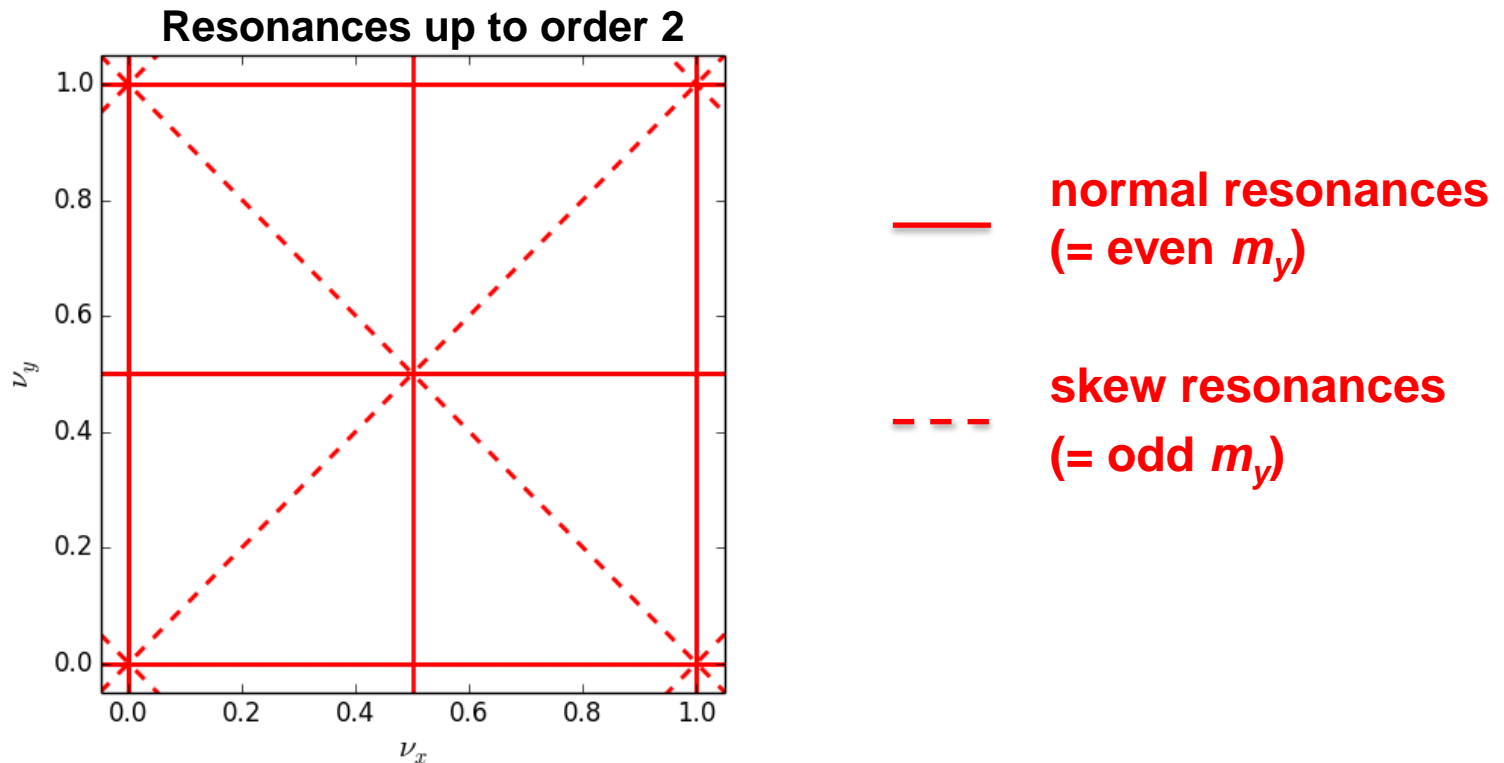 - ❑ **Storage ring**

- Define the **resonance conditions** for periodic lattices
- Explain significance of **symplectic maps**, and describe some of the challenges in calculating and applying symplectic maps
- Outline some of the **analysis methods** that can be used to characterise nonlinear beam dynamics in periodic beamlines.

Resonances

■ If we include vertical as well as horizontal motion, then we find that **resonances** occur when the tunes satisfy

$$m_x \nu_x + m_y \nu_y = \ell$$

where m_x, m_y and l are integers; resonance is of **order $|m_x| + |m_y|$**

Resonances up to order 2

—— **normal resonances (= even m_y)**

- - - **skew resonances (= odd m_y)**

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Resonances up to order 3

—— **normal resonances (= even m_y)**

– – – **skew resonances (= odd m_y)**

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- If we include vertical as well as horizontal motion, then we find that **resonances** occur when the tunes satisfy

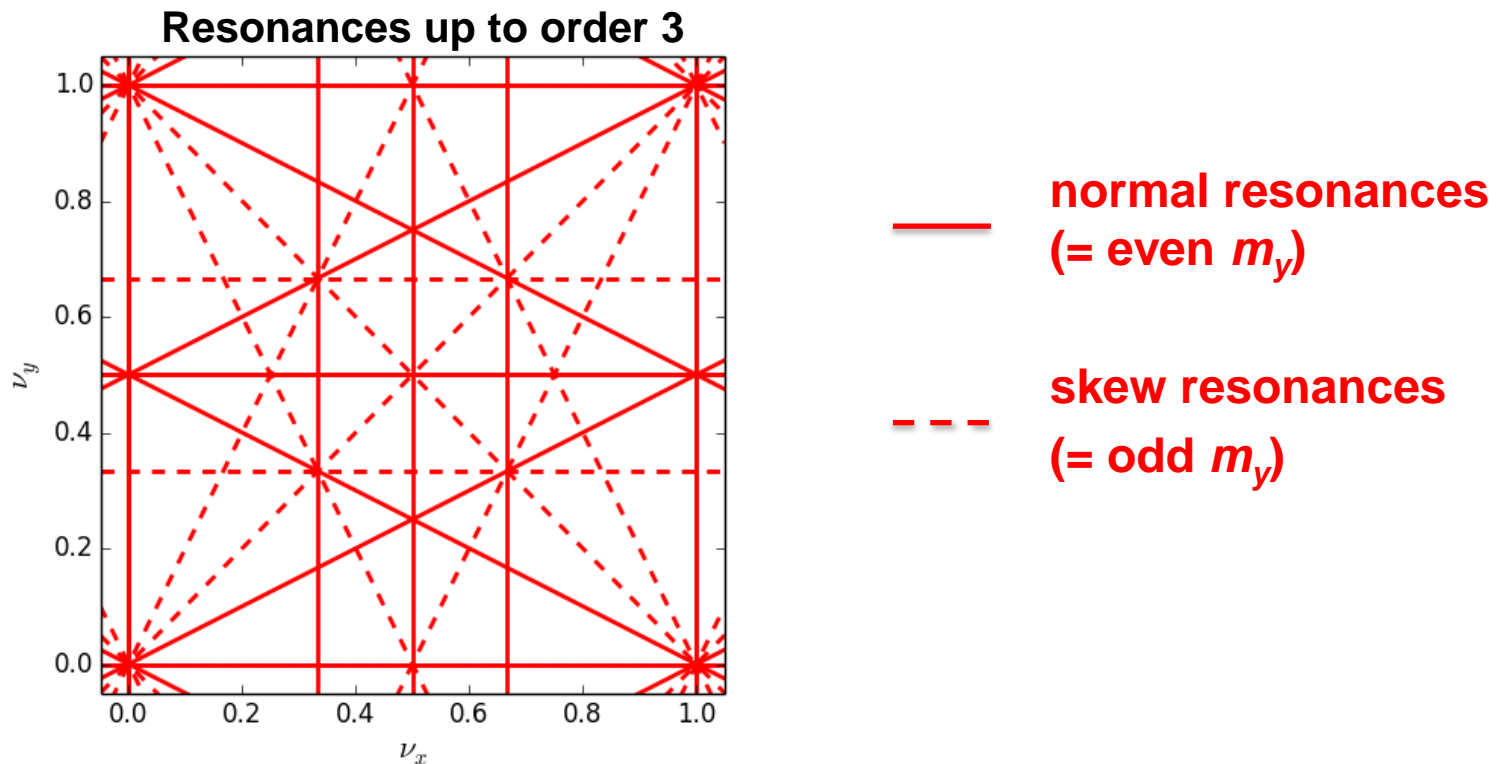$$m_x \nu_x + m_y \nu_y = \ell$$

where m_x, m_y and l are integers; resonance is of **order $|m_x| + |m_y|$**

Resonances up to order 4

normal resonances (= even m_y)

skew resonances (= odd m_y)

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■ If we include vertical as well as horizontal motion, then we find that **resonances** occur when the tunes satisfy
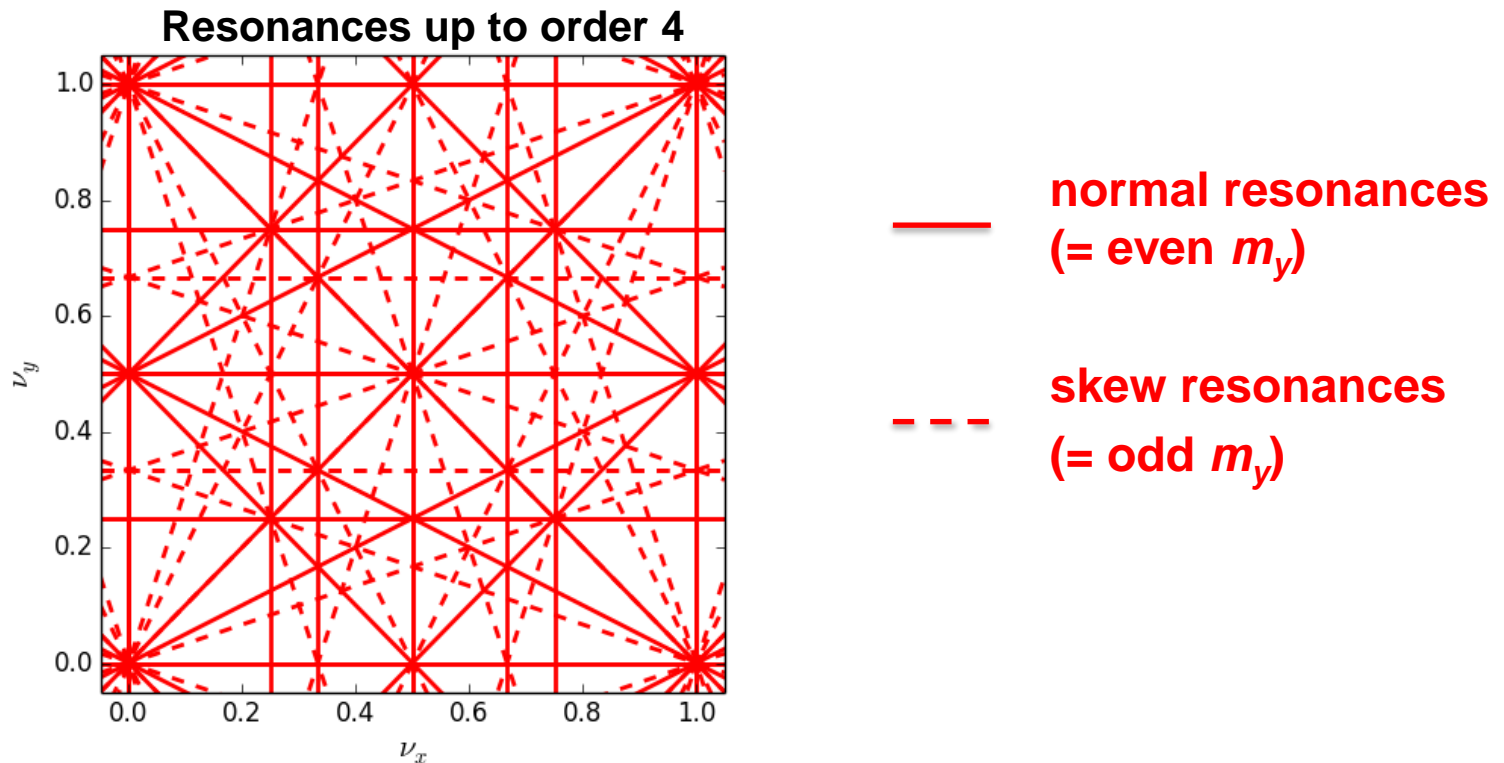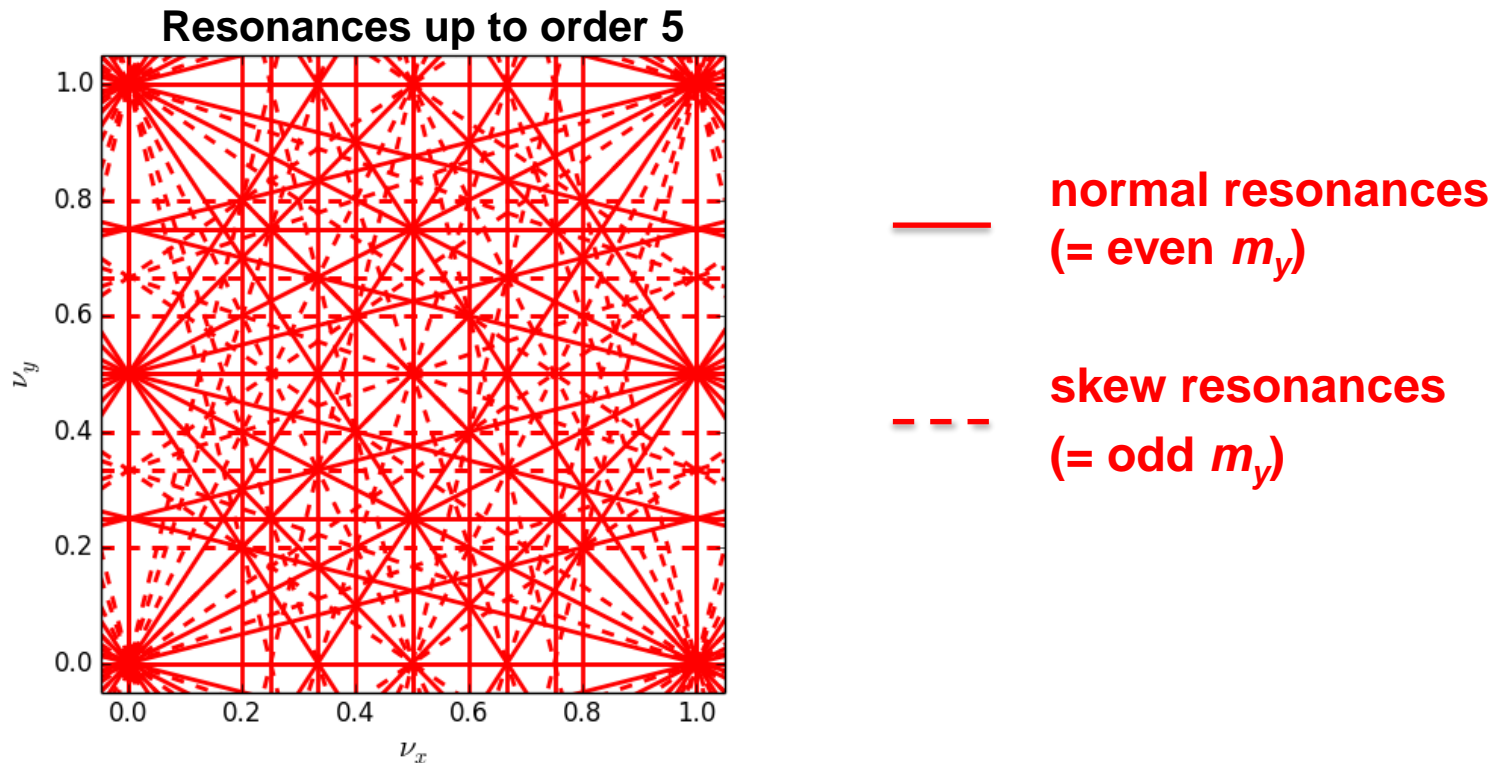
$$m_x \nu_x + m_y \nu_y = \ell$$

where m_x, m_y and l are integers; resonance is of **order $|m_x| + |m_y|$**

Resonances up to order 5

____ **normal resonances (= even m_y)**

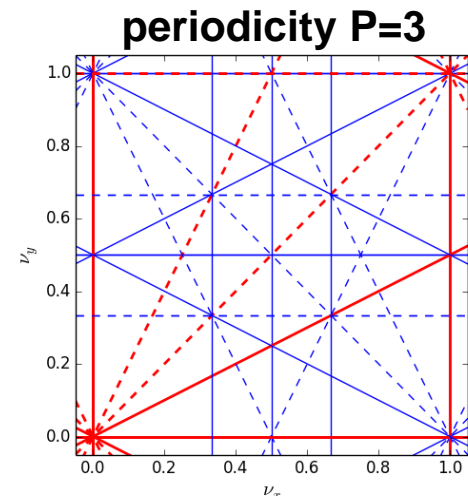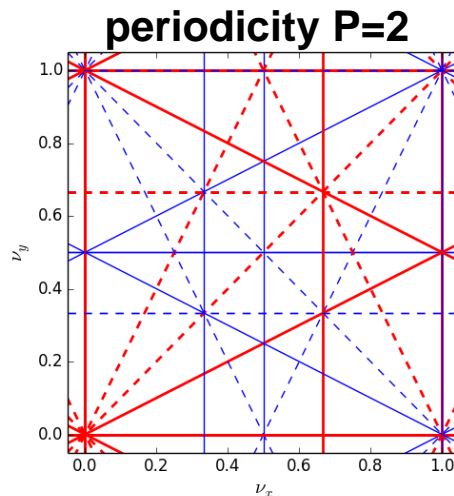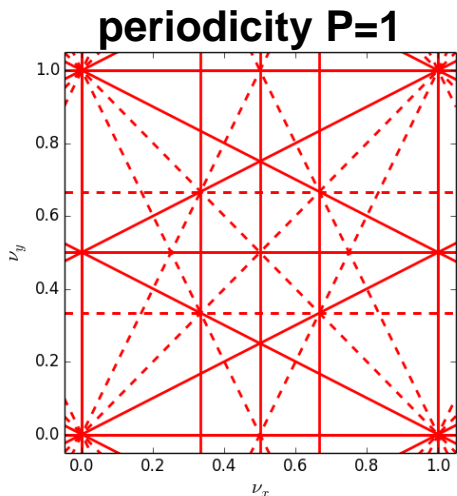- - - **skew resonances (= odd m_y)**

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- **Resonances** are associated with **chaotic motion** for particles in storage rings

- However, the number of **resonance lines** in tune space is **infinite**: any point in tune space will be close to a resonance of some order

- This observation raises two questions:
  - How do we know what the real effect of any given resonance line will be?
  - How can we design a storage ring to minimise the adverse effects of resonances?

- By imposing a **periodicity *P*** in the lattice (i.e. building a machine from *P* identical cells), the resonance condition becomes

$$m_x \frac{\nu_x}{P} + m_y \frac{\nu_y}{P} = l \qquad \rightarrow \qquad \boxed{m_x \nu_x + m_y \nu_y = P\ell}$$

  - … the resonance condition needs to be satisfied by each cell, as conceptually there is no difference between passing one cell P turns or passing a lattice consisting of P identical cells only once

- Resonances for which *l* is integer → systematic

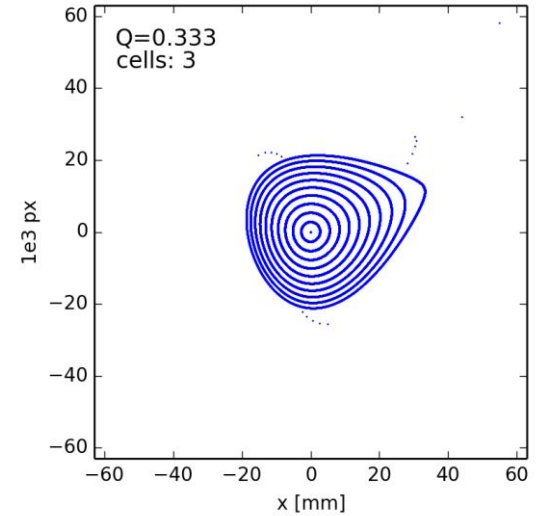- If *l* is NOT integer the resonance cancels → non-systematic
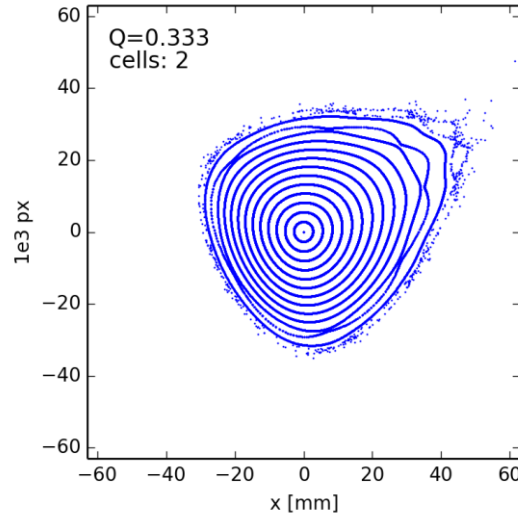
periodicity P=1   **periodicity P=2**   **periodicity P=3**
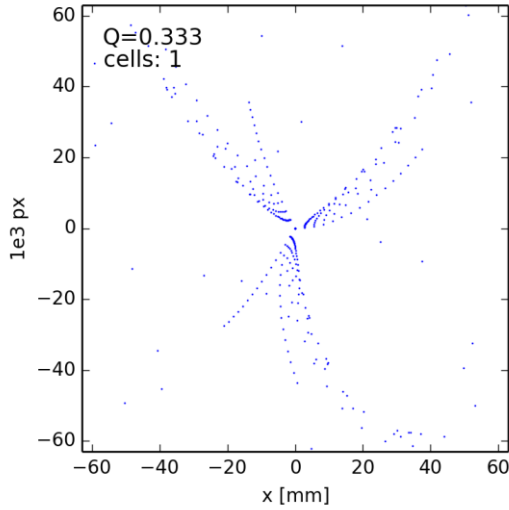

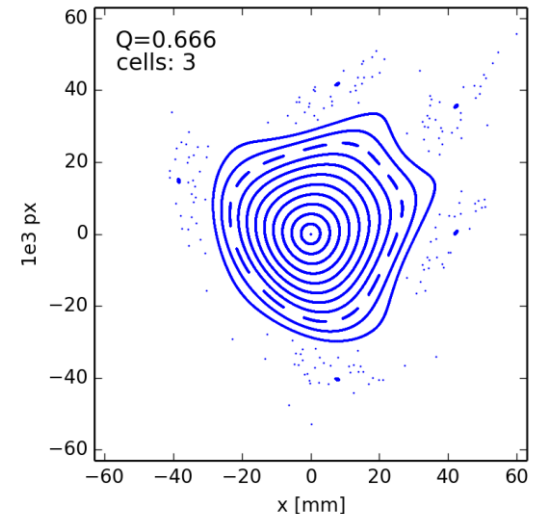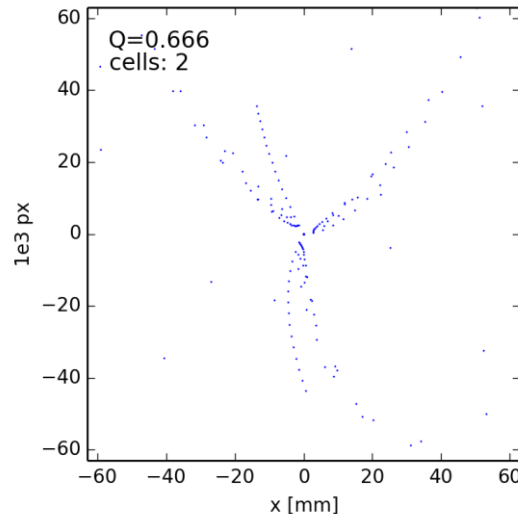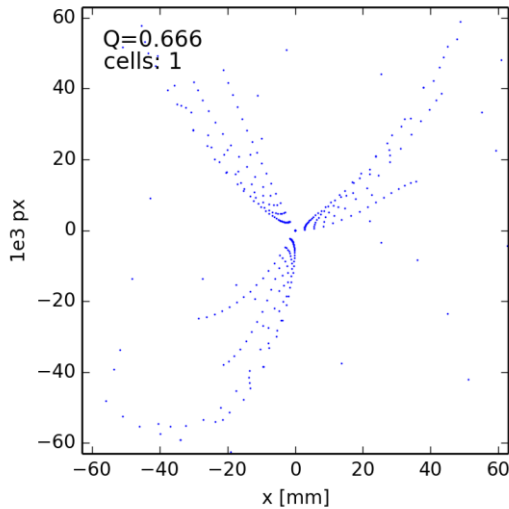solid lines: normal resonances   dashed lines: skew resonances

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$$m_x \nu_x + m_y \nu_y = P\ell$$

■ 3 Qx = 1 resonance

■ 3 Qx = 2 resonance

Advanced Light Source, design lattice periodicity: 12

Measurement of beam loss as function of tune

$72 = 6*12$

$3\nu_x = 43$

$5\nu_x = 72$

Uncorrected Optics
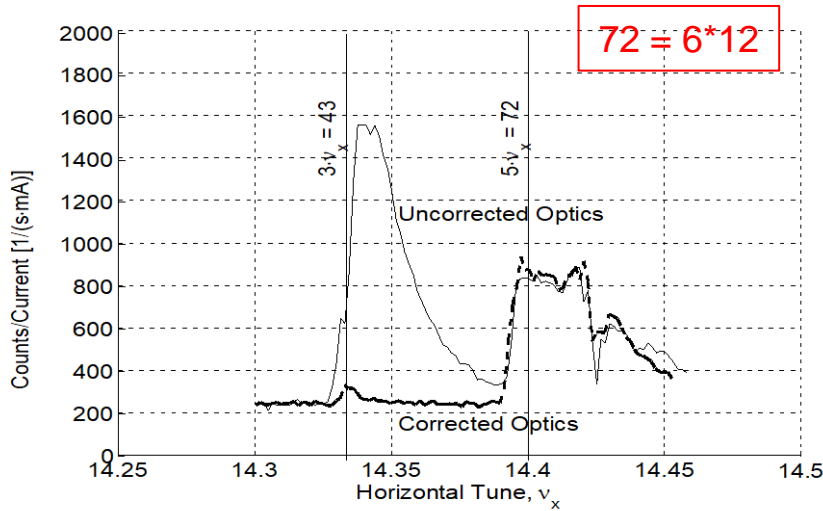
Corrected Optics

Beta beating

Before optics correction: ~30%

After optics correction: <1%

Synchrotron light beam spot

Uncorrected optics   **Corrected optics**

Simulated phase space

D. Robin, C. Steier, J. Safranek, W. Decking, "Enhanced performance of the ALS through periodicity restoration of the lattice", proc. EPAC 2000.

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- CERN SPS (hadron machine) has design lattice periodicity of 6
- **Sextupole resonances can be clearly identified although they should be suppressed by lattice periodicity**
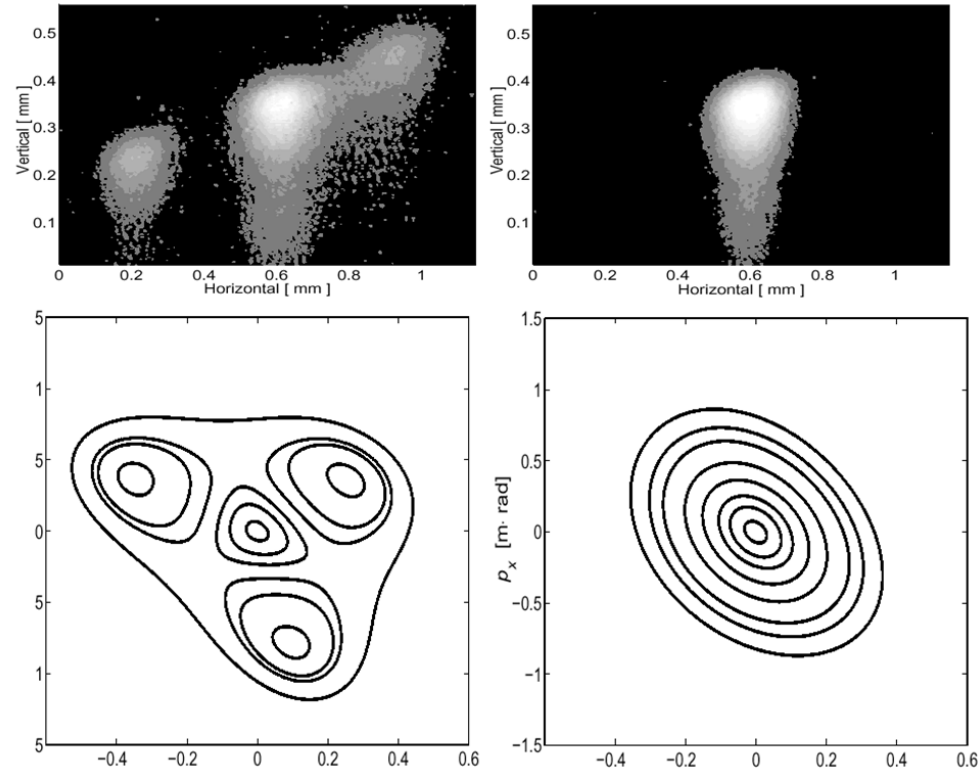- Unfortunately SPS has no individual quadrupoles to restore optics functions distortions (and typical beta beating is only around 10-20%)

Measured losses during tune scan
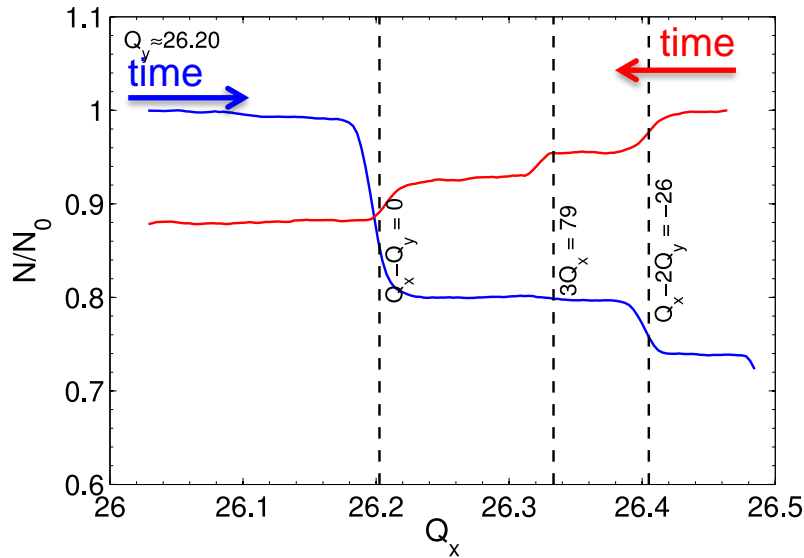
Measured loss rate in 2D scan
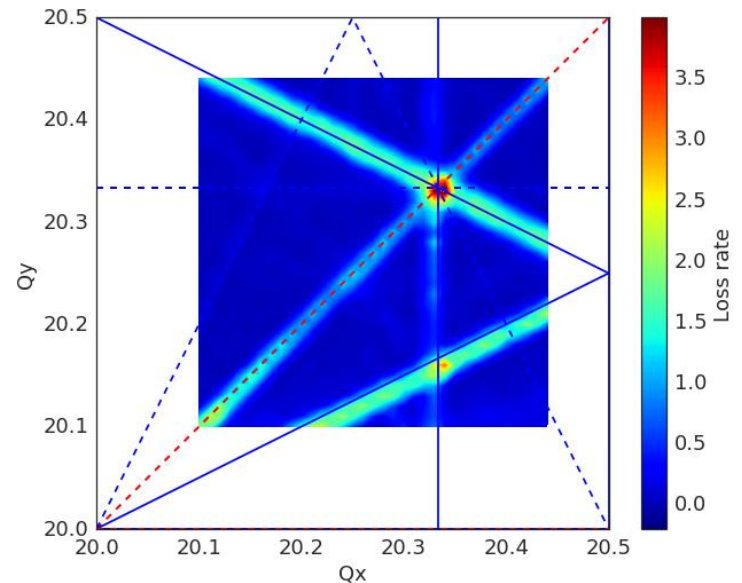
Non-linear map representation

■ For any dynamical variable x_j the **Taylor map** up to **3rd order** can be written as

$$x_j^{\text{new}} = \sum_{k=1}^{6} R_{jk} x_k + \sum_{k=1}^{6} \sum_{l=1}^{6} T_{jkl} x_k x_l + \sum_{k=1}^{6} \sum_{l=1}^{6} \sum_{m=1}^{6} U_{jklm} x_k x_l x_m$$

■ **Taylor series** provide a convenient way of systematically representing **transfer maps** for **beamline components**, or sections of beamline

■ The **main drawback** of Taylor series is that in general, transfer maps can only be represented exactly by series with an **infinite number** of **terms**

■ In practice, we have to **truncate** a **Taylor map** at some order, and we then lose certain desirable properties of the map

■ In particular, a **truncated map** will be usually **non-symplectic**

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- Consider two sets of canonical variables $\vec{x}_i$, $\vec{x}_f$, which represent the evolution of the system between two points in phase space

- A **map** $\mathcal{M} : \vec{x}_i \mapsto \vec{x}_f$ describes the transformation from one set to the other

- This map is **symplectic,** i.e. it **conserves phase space volumes**, if
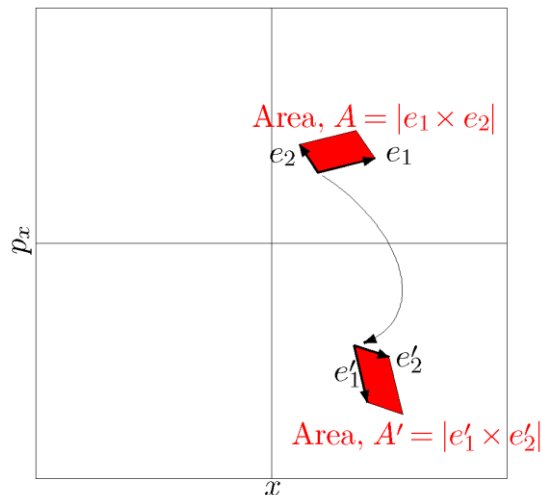
$$J^T S J = S$$

symplecticity condition

$$J_{mn} \equiv \frac{\partial x_{m,f}}{\partial x_{n,i}}$$

Jacobian matrix of the map

$$S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

antisymmetric matrix with block diagonals

Area, $A = |e_1 \times e_2|$

e_2 e_1

e_1' e_2'

Area, $A' = |e_1' \times e_2'|$

… this is Liouville's theorem, and is a property of charged particles moving in electromagnetic fields, in the absence of radiation

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■ The effect of **losing symplecticity** becomes apparent if we compare phase space portraits constructed using symplectic (below, left) and non-symplectic (below, right) transfer maps.

2nd order implicit

2nd order power series in x
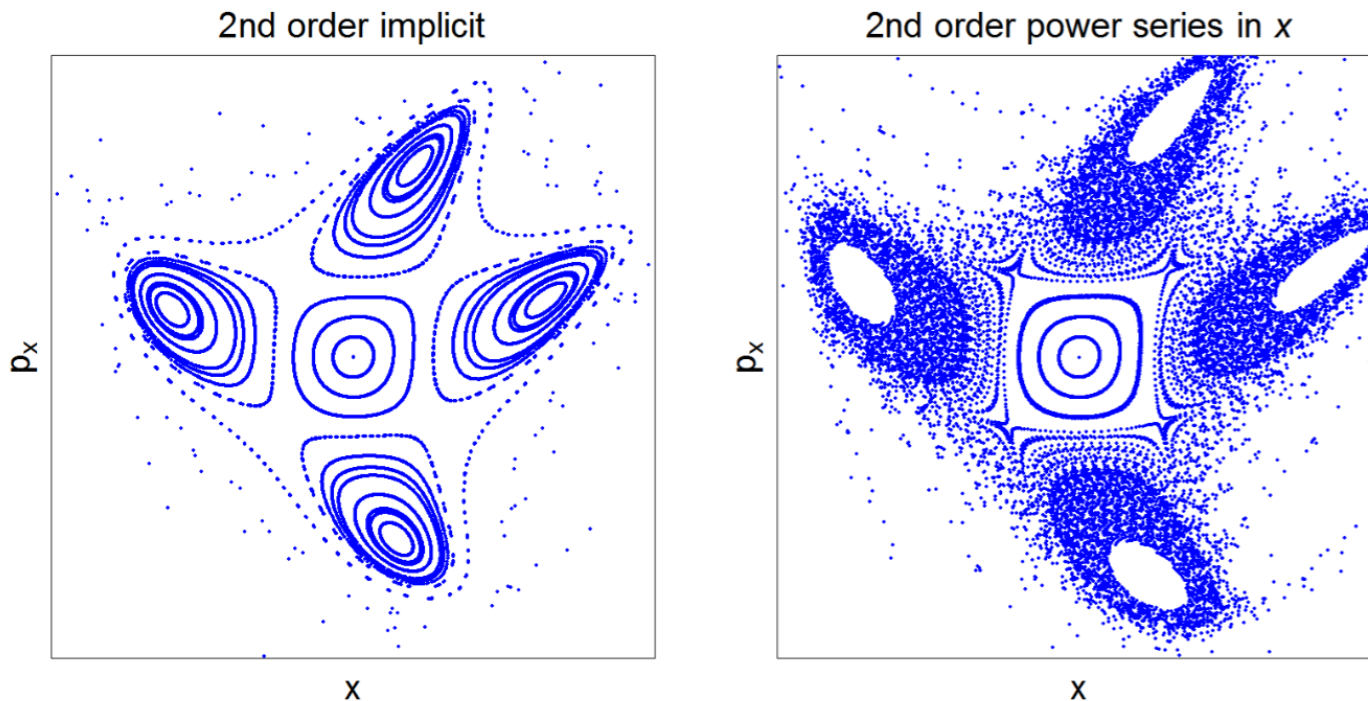
■ Modelling a storage ring using non-symplectic maps can lead to an inaccurate estimate of the dynamic aperture and the beam lifetime

- There are a number of ways of representing transfer maps to ensure symplecticity. These include:

  - Taylor maps can be specially constructed to retain symplecticity with a certain (finite) number of terms. Taylor maps are explicit: once the coefficients have been calculated, the map can be applied simply by substitution of values for the dynamical variables.

  - Mixed-variable generating functions provide an implicit representation: each application of the map requires further solution of equations

  - Lie transformations provide a finite representation for infinite Taylor series, and are useful for analytical studies

- Symplectic Taylor maps with a finite number of terms can be constructed for multipole magnets of any order using the "kick" approximation.

- Consider a sextupole with equations of motion:

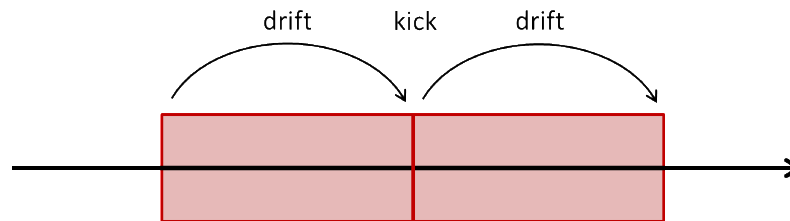$$\frac{dx}{ds} = p_x, \qquad \frac{dp_x}{ds} = -\frac{1}{2}k_2 x^2$$

- Exact solutions using some elementary functions do not exist

- By splitting integration into three steps, it is possible to write an **explicit** and **symplectic** approximate solution

$$0 \le s < L/2 : \qquad x_1 = x_0 + \frac{L}{2}p_{x0}, \qquad p_{x1} = p_{x0},$$

$$s = L/2 : \qquad x_2 = x_1, \qquad p_{x2} = p_{x1} - \frac{1}{2}k_2 L x_1^2,$$

$$L/2 < s \le L : \qquad x_3 = x_2 + \frac{L}{2}p_{x2}, \qquad p_{x3} = p_{x2}$$

- This is an example of a symplectic integrator known as a **"drift–kick–drift"** approximation

Analytical methods for nonlinear dynamics

- There are two approaches widely used in accelerator physics: **perturbation theory** and **normal form** analysis
  - In both these techniques, the goal is to construct a **quantity** that is **invariant** under application of the single-turn transfer map. In both cases the mathematics is unfortunately complicated and fairly cumbersome …
- In the case of a **single sextupole** in a storage ring, we find from **normal form analysis** the following expression for the betatron action as a function of the betatron phase (angle variable):

$$J_x \approx I_0 - \frac{k2L}{8}\left(2\beta_x I_0\right)^{3/2}\frac{\cos(3\mu_x/2 + 2\phi_x) + \cos(\mu_x/2)}{\sin(3\mu_x/2)} + O(I_0^2)$$
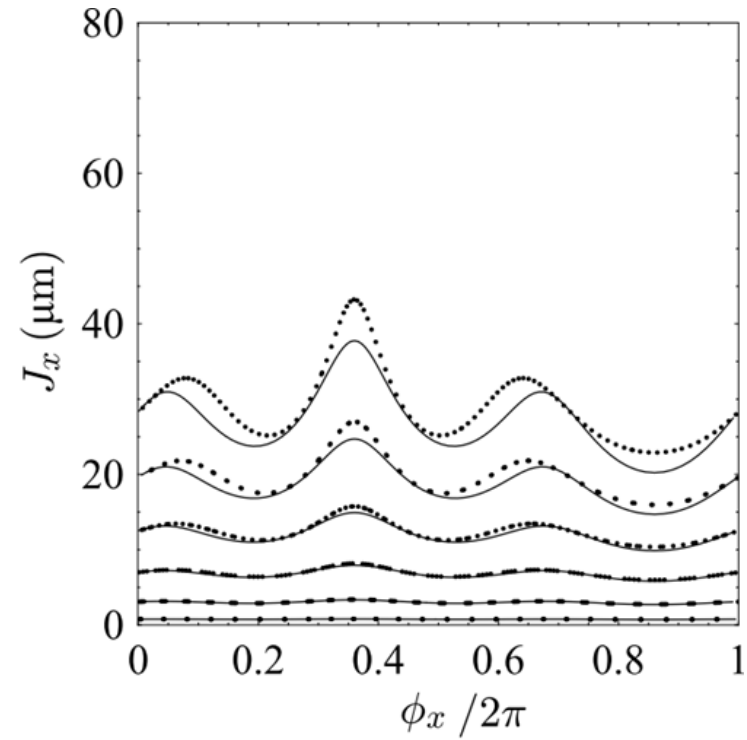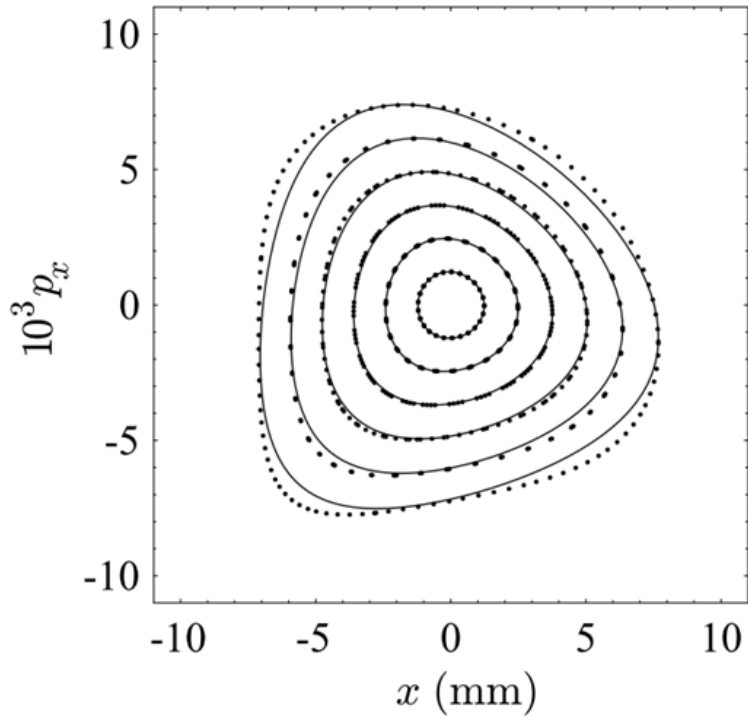
where I_0 is a constant (an invariant of the motion), ϕ_x is the angle variable, and μ_x is the phase advance per cell. The second term becomes **very large** when μ_x is close to **third integer**

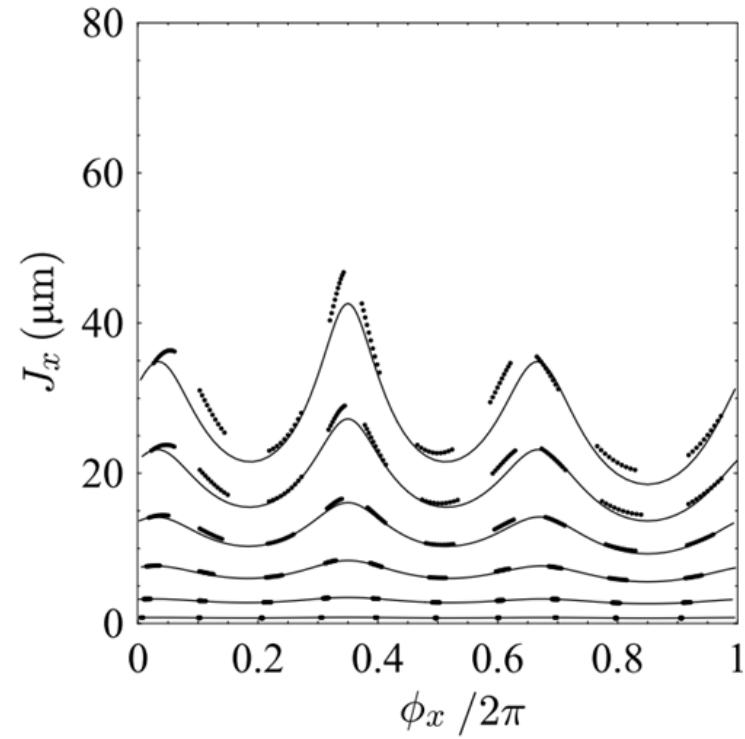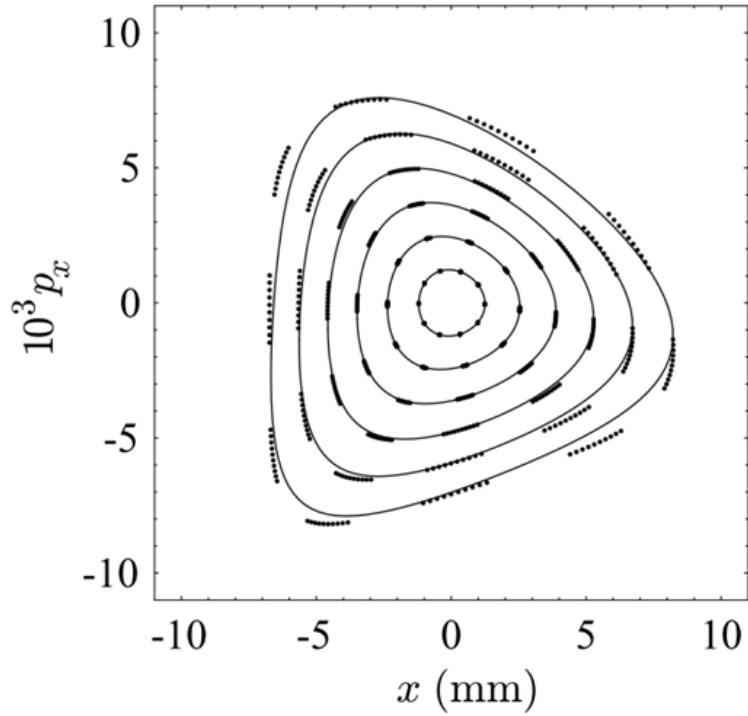- The cartesian variables can be expressed in terms of the action–angle variables:

$$x = \sqrt{2\beta_x J_x}\cos\phi_x,$$

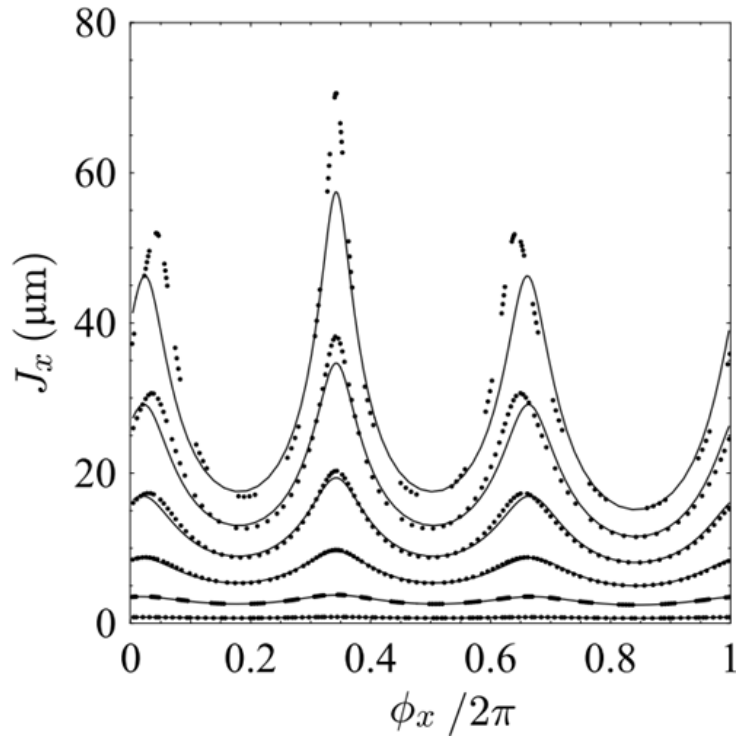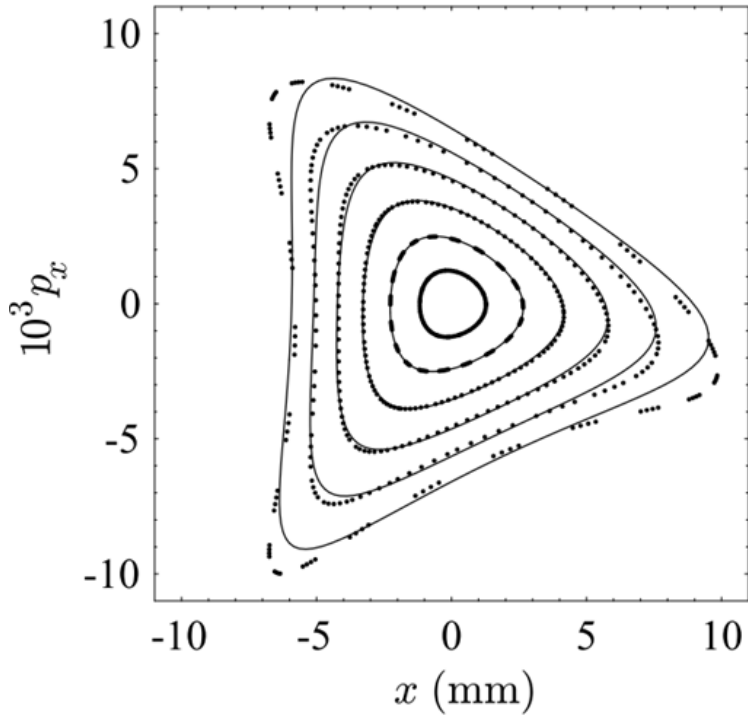$$p_x = -\sqrt{2J_x/\beta_x}(\sin\phi_x + \alpha_x\cos\phi_x)$$

$$\mu_x = 0.28 \times 2\pi$$

$$\mu_x = 0.30 \times 2\pi$$

$$\mu_x = 0.315 \times 2\pi$$

- Close inspection of the plots on the previous slides reveals another effect, in addition to the obvious distortion of the phase space ellipses: the **phase advance** per turn (i.e. the tune) **varies** with increasing **betatron amplitude**

- Normal form analysis (and perturbation theory) can be used to obtain **estimates** for the **tune shift** with amplitude

- In the case of a sextupole, the **tune shift** is **higher-order** in the **sextupole strength**

- An octupole, however, does have a **tune shift** with amplitude in **first-order** of the octupole strength, given by:

$$\nu_x = \nu_{x0} + \frac{k_3 L \beta_x^2}{16\pi} J_x + O(J_x^2)$$

■ The tune shift with amplitude becomes obvious if we track a small number of turns (30) in a lattice with a **single octupole**.

$$\mu_x = 0.330 \times 2\pi \qquad \mu_x = 0.336 \times 2\pi$$
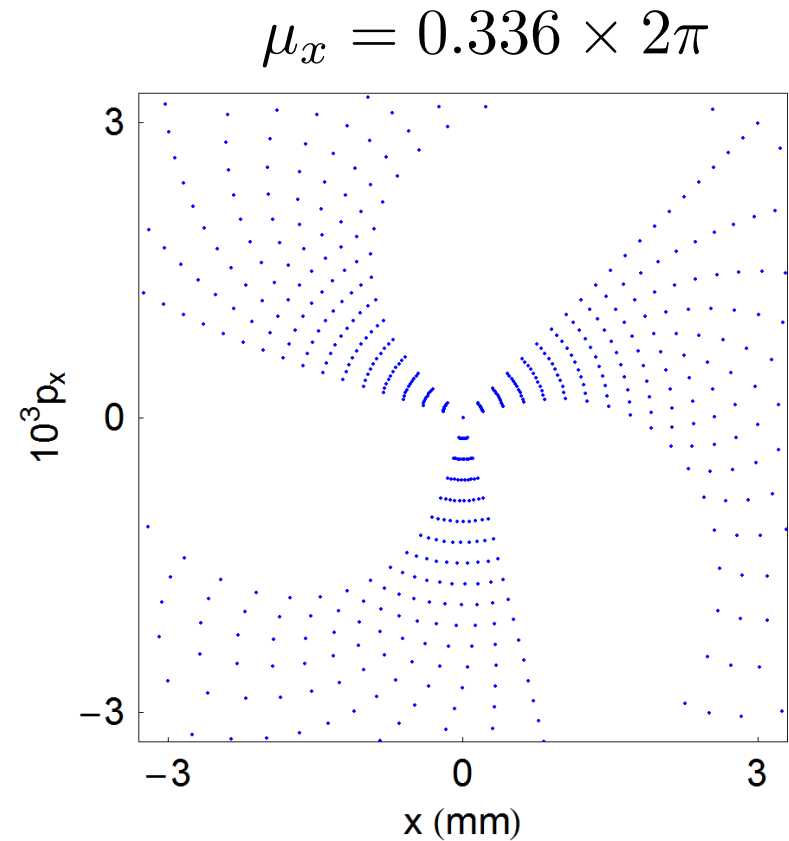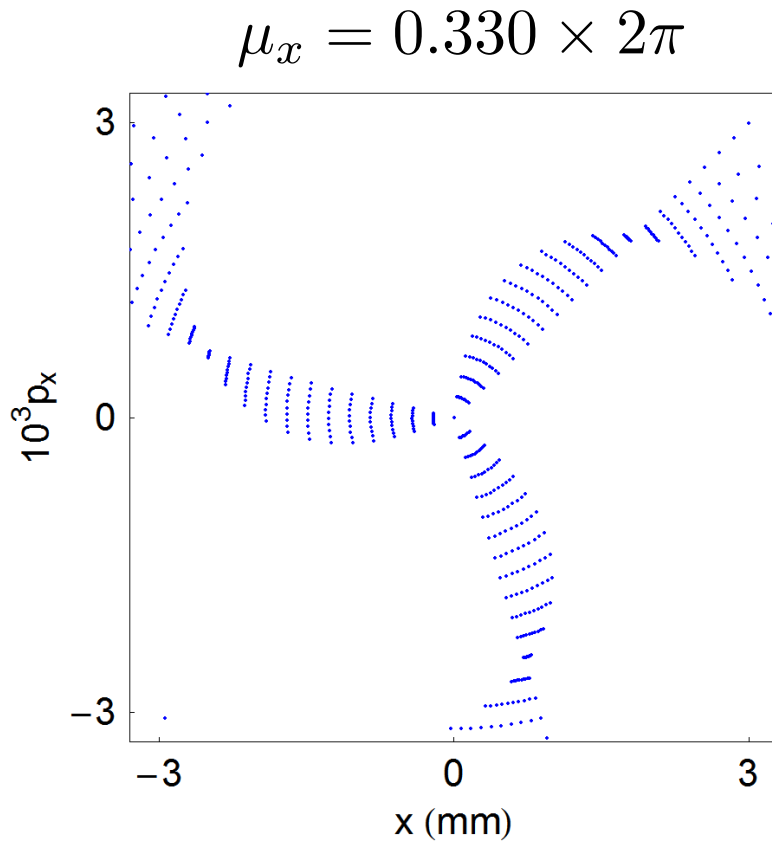
- Simulation of simple storage ring with a **single octupole** close to 4th order resonance
- **Detuning with amplitude** (linear in action)
- Particles in the stable islands have a tune **locked** to the **resonance**
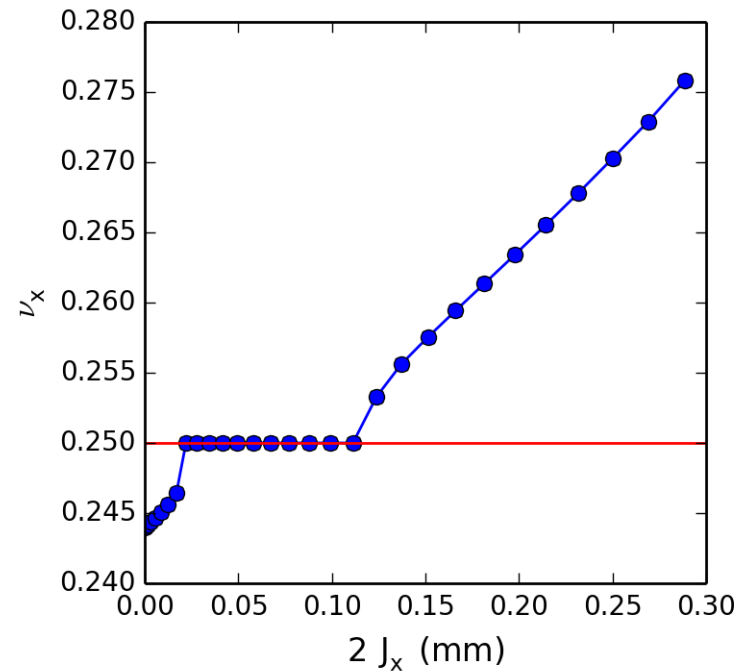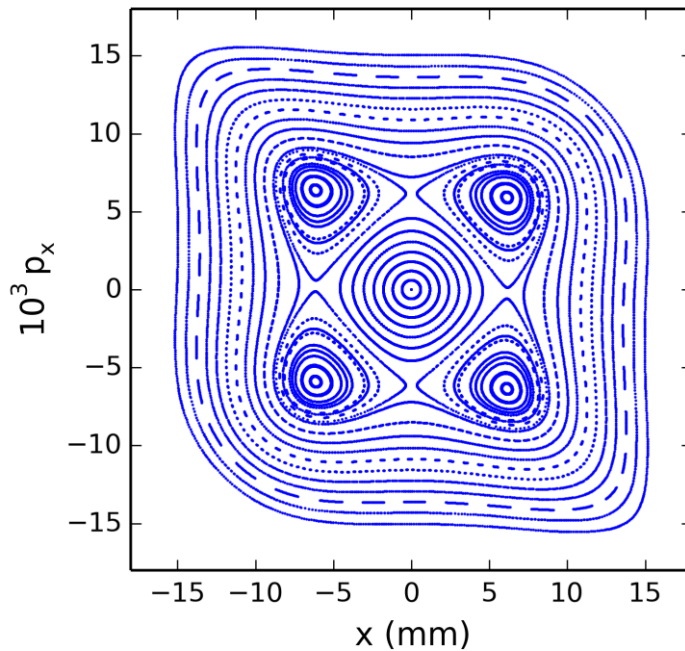
- Simulation of simple storage ring with a **sextupole** and an **octupole** close to **3rd order resonance**

- The **amplitude detuning** induced by the octupole can create **stable islands** even for the **3rd order resonance** (recall the phase-space plot for the case of a single sextupole)
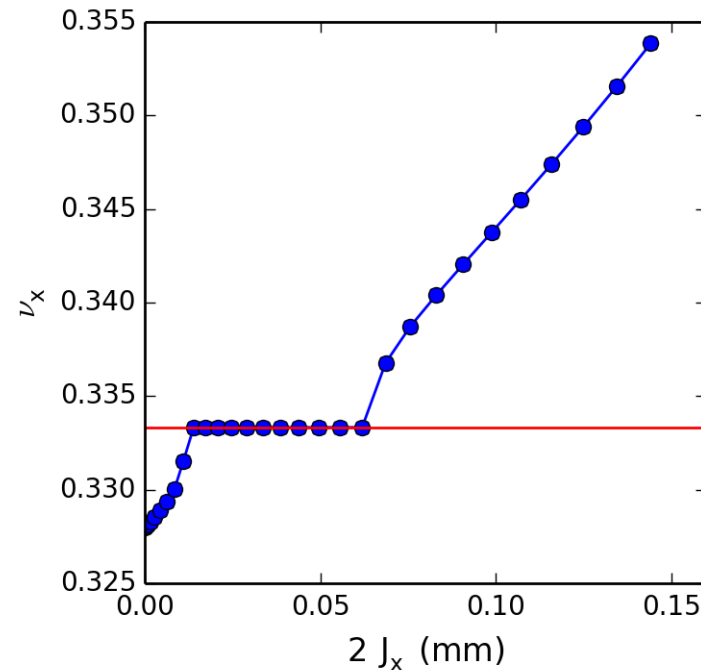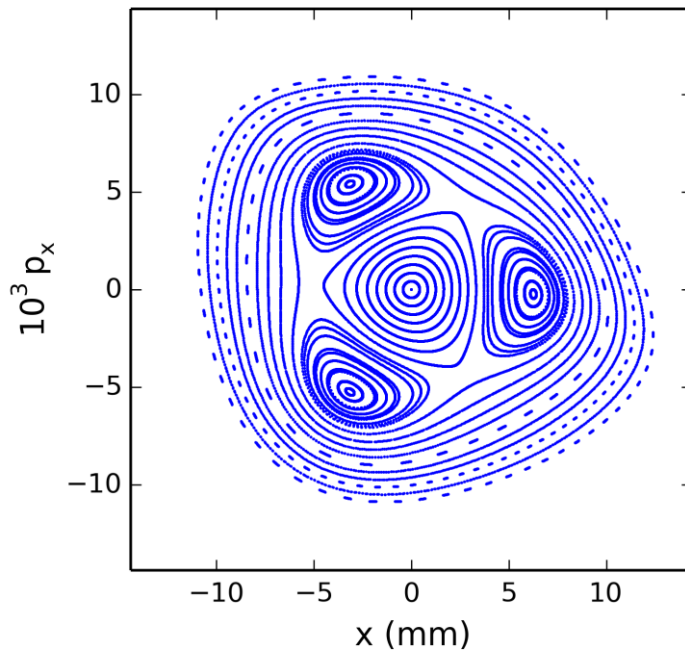
- Perturbation theory and normal form analysis depend on the existence of **constants of motion in the presence of nonlinear perturbations**
- The fact that **constants of motion can exist in the presence of nonlinear perturbations is a consequence of the Kolmogorov–Arnold–Moser (KAM) theorem.**
  - The KAM theorem expresses the general conditions for the existence of constants of motion in nonlinear Hamiltonian systems
- **Resonances do not necessarily result in loss of stable motion**
  - **Resonances will typically** tend to **drive** the amplitudes of **particles with a particular tune to large amplitudes**
  - **For sufficiently large tune-shift with amplitude, there can be a stable region at amplitudes larger than that at which the resonance occurs** (see the example of the octupole in the previous slide)

- The **overlap of two resonances** is associated with a transition from regular to **chaotic motion**
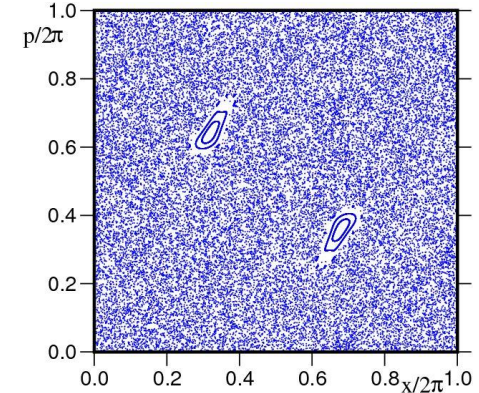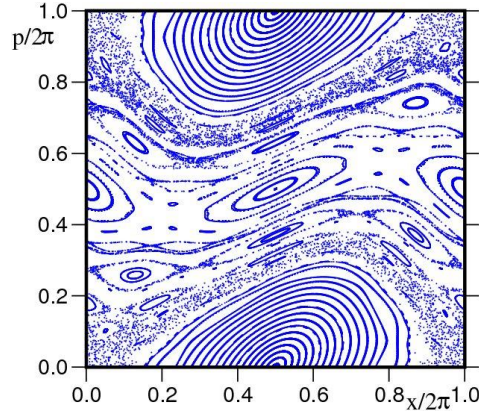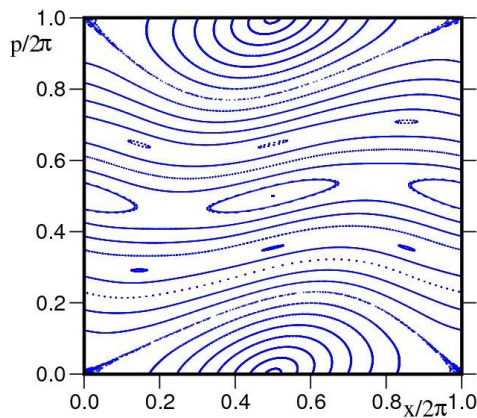

- The **Chirikov criterion** describes the parameter range over which the particle motion becomes chaotic
  - ❑ It is a geometric criterion of the resonance overlap based on the size of resonance islands
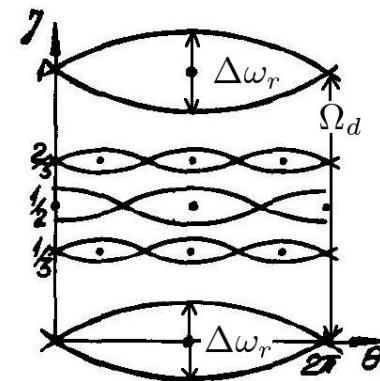
Numerical methods: Dynamic aperture (DA) and Frequency map analysis (FMA)

■ The most direct way to evaluate the nonlinear dynamics performance of a ring is the computation of **Dynamic Aperture** (short: DA), which is the **boundary of the stable region in co-ordinate space**

■ Need a **symplectic tracking code** to follow particle trajectories (a lot of initial conditions) for a number of turns until particles start getting lost → this boundary defines the **Dynamic aperture**

■ Dynamic aperture plots show the maximum initial values of stable trajectories in x-y coordinate space
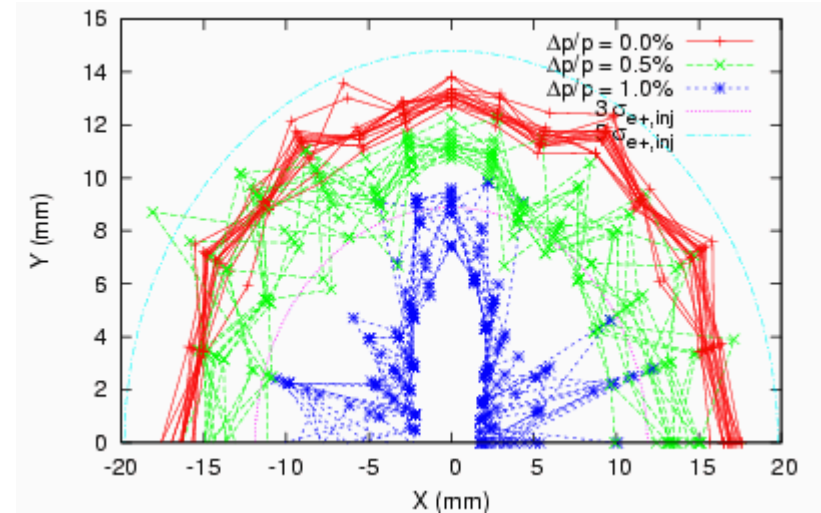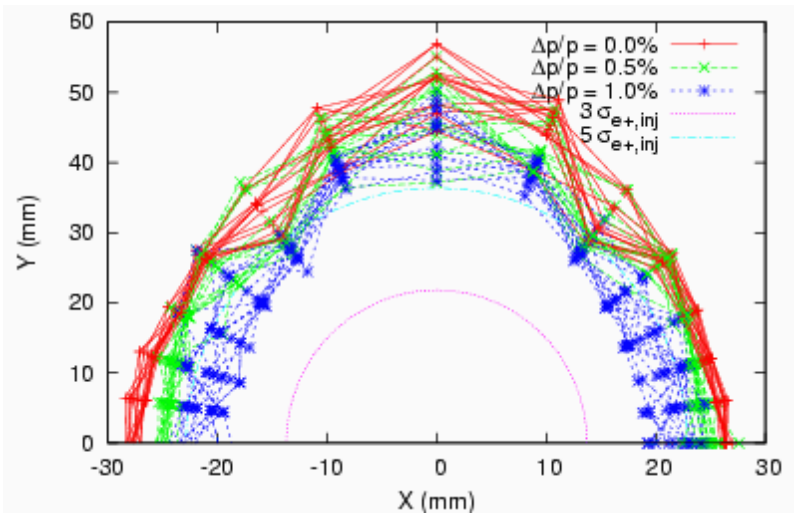
DA simulations for CLIC damping rings

■ **LHC design was based on a large campaign of systematic DA simulations (including margin for stability)**

❑ The goal is to allow significant margin in the design – the measured dynamic aperture is often smaller than the predicted dynamic aperture

DA inferred from measured loss data ⊢⊣
Simulations: $|C^-|=2\times10^{-3}$ ▮ (green)
$|C^-|=4\times10^{-3}$ ▮ (blue)
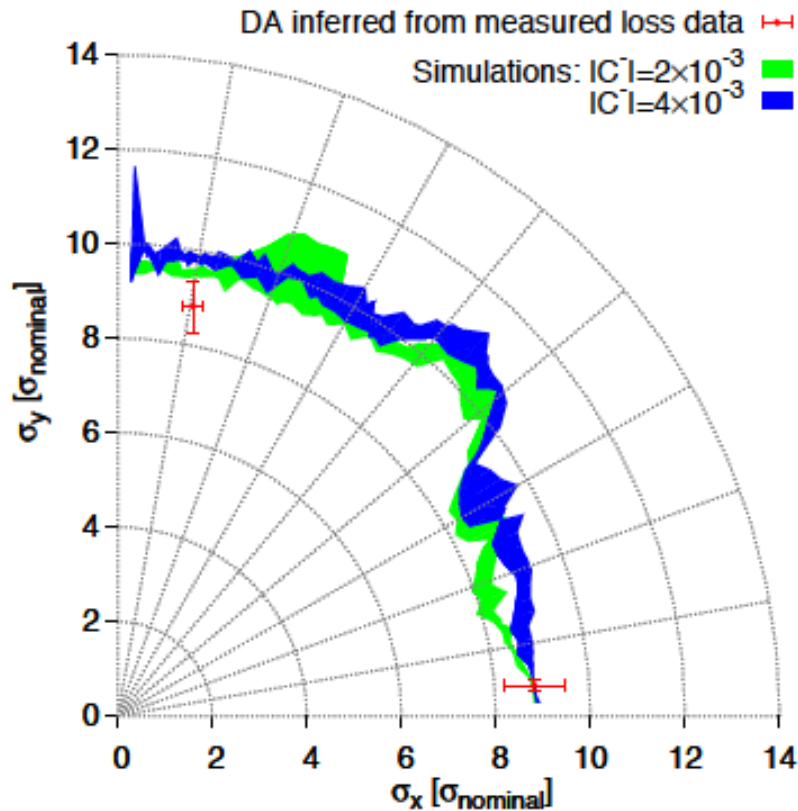
❑ A few years after LHC started operating, a measurement of the DA was performed (kicking the beam to large amplitudes)

❑ Very good agreement between tracking simulations and measurements in the machine

E.Mclean, PhD thesis, 2014

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J. Laskar, "Frequency map analysis and particle accelerators", PAC 2003
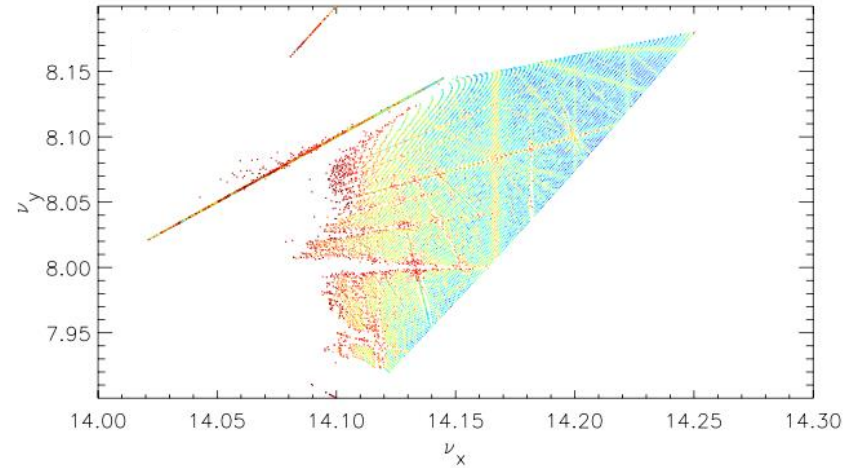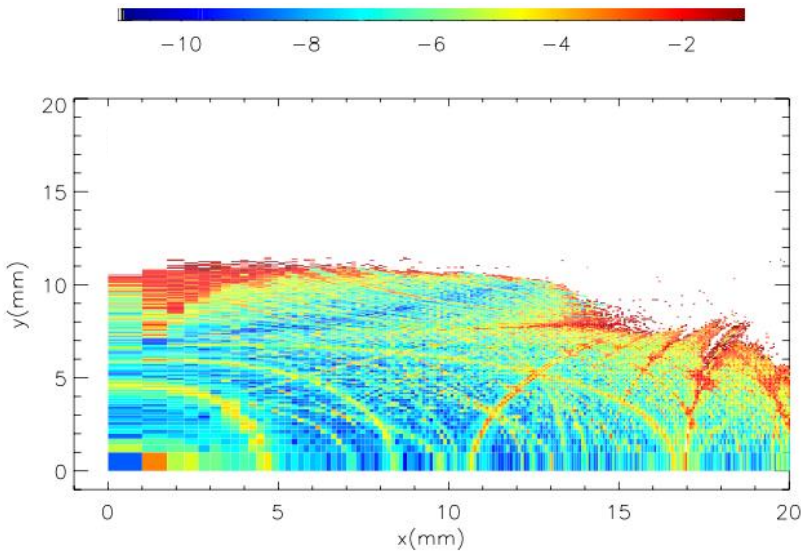
- ■ Numerically integrate the phase space trajectories through the lattice for sufficient number of turns (i.e. perform particle tracking)

- ■ Compute ν_x and ν_y after sufficient number of turns through advanced Fourier methods, and **plot the tune variation by color code in configuration space x-y (left) and tune space ν_x-ν_y (right)**
  - ❑ **regular motion** corresponds to small tune diffusion)
  - ❑ **Chaotic motion** associated with large tune diffusion, occurs at the dynamic aperture of the machine
  - ❑ **Resonances** appear as curves in initial condition space

- **Frequency analysis of turn-by-turn data** of beam oscillations produced by a fast kicker magnet and recorded on a Beam Position Monitor

- **Reproduced in tracking simulations using the non-linear model of the ALS storage ring**

D. Robin, C. Steier, J. Laskar, and L. Nadolski, PRL 2000

experiment

simulation

A first taste of Non-linear Beam Dynamics, CERN Accelerator School, September/October 2023

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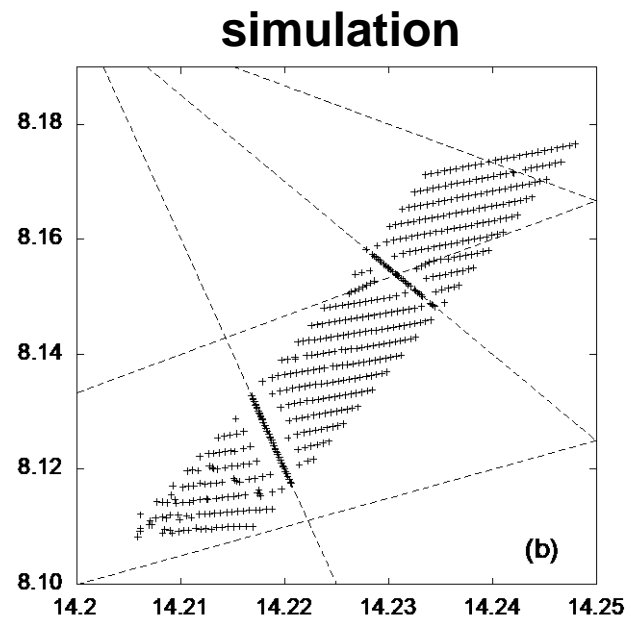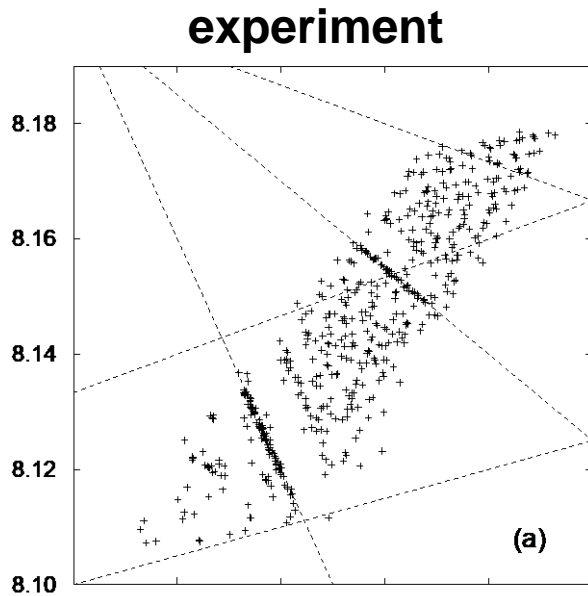
Conclusions and Summary

- **Nonlinear dynamics** appear in a wide variety of accelerator systems, including **single-pass** systems (such as bunch compressors) and **multi-turn** systems (such as storage rings)

- It is possible to model nonlinear dynamics in a given component or section of beamline by representing the **transfer map** as a **power series**

- **Conservation** of **phase space volumes** is an important feature of the beam dynamics in many systems. To conserve phase space volumes, transfer maps must be **symplectic**

- In general, (**truncated**) **power series** maps are **not symplectic**

- To construct a **symplectic transfer map**, the equations of motion in a given accelerator component must be solved using a **symplectic integrator** (e.g. the "drift–kick–drift" approximation for a multipole magnet)

- **Common features** of nonlinear dynamics in accelerators include **phase space distortion**, **tune shifts with amplitude**, **resonances**, and **chaotic** particle trajectories at large amplitudes (**dynamic aperture** limits)

- Analytical methods such as **perturbation theory** and **normal form** analysis can be used to estimate the impact of nonlinear perturbations in terms of quantities such as **resonance strengths** and **tune shifts with amplitude**

- **Frequency map analysis** provides a useful numerical tool for characterising tune shifts and resonance strengths from tracking data

- This can give some **insight** into **limitations** on the **dynamic aperture**