

A first taste of Non-Linear Beam Dynamics

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- Introducing aspects of non-linear dynamics
 - Effects of **nonlinear perturbations**
 - Resonances, tune shifts, dynamic aperture
 - **Mathematical tools** for modelling nonlinear dynamics
 - Power series (Taylor) maps and symplectic maps
 - **Analysis** methods
 - Normal forms, frequency map analysis

- Illustrate methods and tools for a simple example of an accelerator
 - **Storage ring**

- Provide an introduction to some of the **key concepts** of nonlinear dynamics in particle accelerators
- Describe some of the **sources** of nonlinearities
- Outline some of the **tools** used for modelling
- Explain the significance and potential **impact** of nonlinear dynamics in some accelerator systems

From Linear to Non-linear

- Particle motion through linear components such as drifts, dipoles and quadrupoles can be represented by **linear transfer maps**
- For example, in a drift space of length L , the **horizontal coordinate** and the (scaled) **horizontal momentum** from initial position 0 to a final position 1 are

$$x_1 = x_0 + Lp_{x0}$$

$$p_{x1} = p_{x0}$$

- Note that the horizontal momentum is

$$p_x = \frac{\gamma m v_x}{P_0} \approx \frac{dx}{ds}$$

where γ is the relativistic factor, m is the rest mass of the particle, v_x is the horizontal velocity, and P_0 is the reference momentum

- **Linear transfer maps** can be written in terms of **matrices** and for example for a drift space of length L

$$\begin{pmatrix} x_1 \\ p_{x1} \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ p_{x0} \end{pmatrix}$$

- In general, a **linear transformation** can be written as

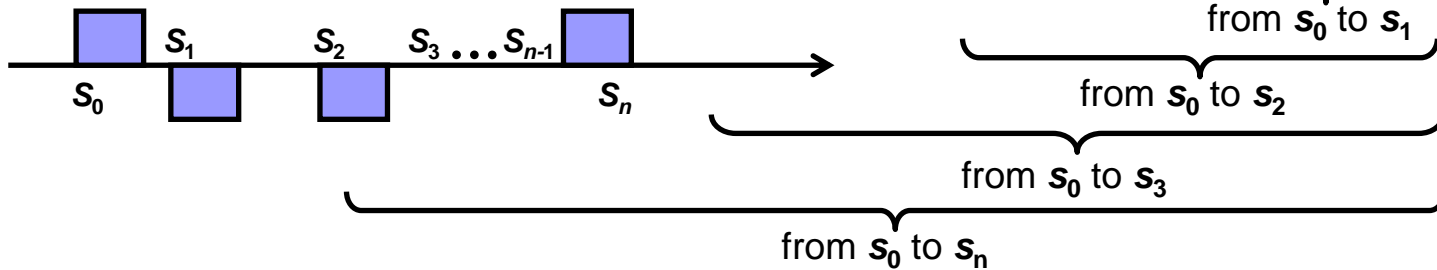
$$\vec{x}_1 = R\vec{x}_0 + \vec{A}$$

where the phase space vectors are $\vec{x} = (x, p_x)$

- The transfer matrix R and the vector \vec{A} are constants, i.e. they do not depend on \vec{x}_0

- The transfer matrix for a section of beamline can be found by **multiplying the transfer matrices** for the accelerator components within that section

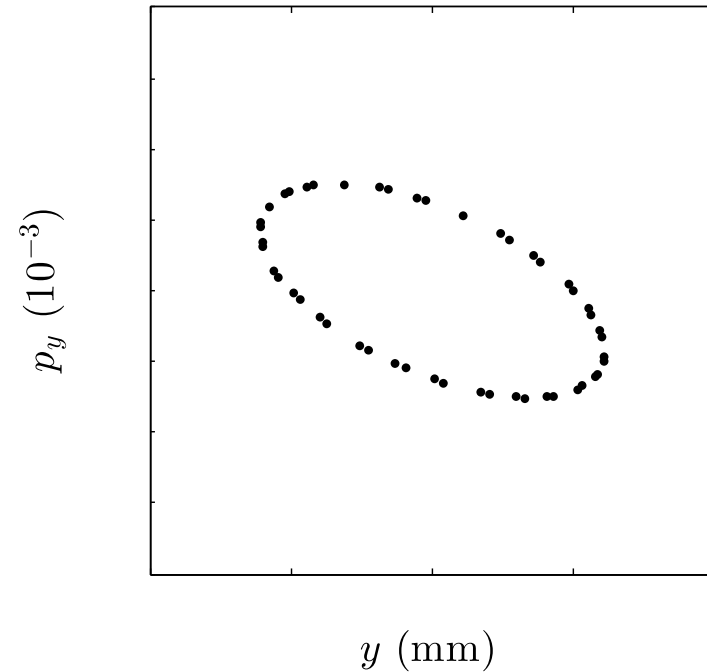
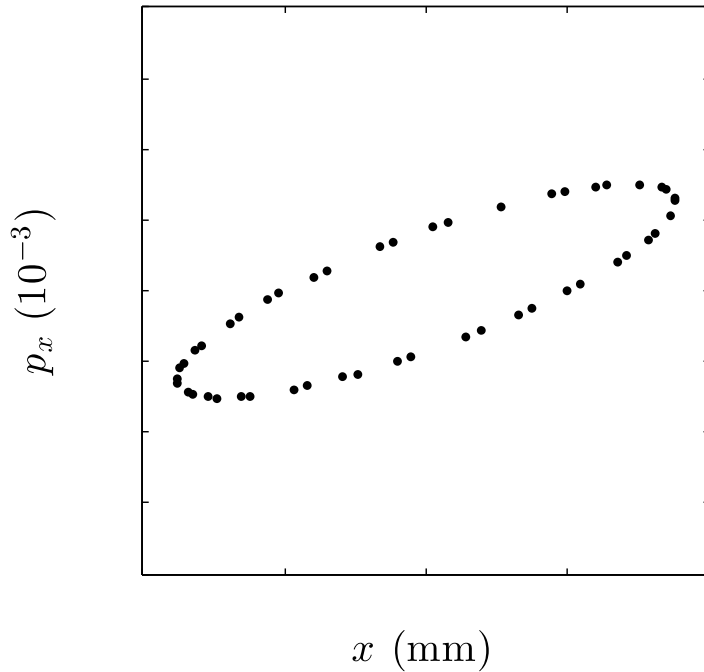
$$R(s_n | s_0) = R(s_n | s_{n-1}) \dots R(s_3 | s_2) \cdot R(s_2 | s_1) \cdot R(s_1 | s_0)$$



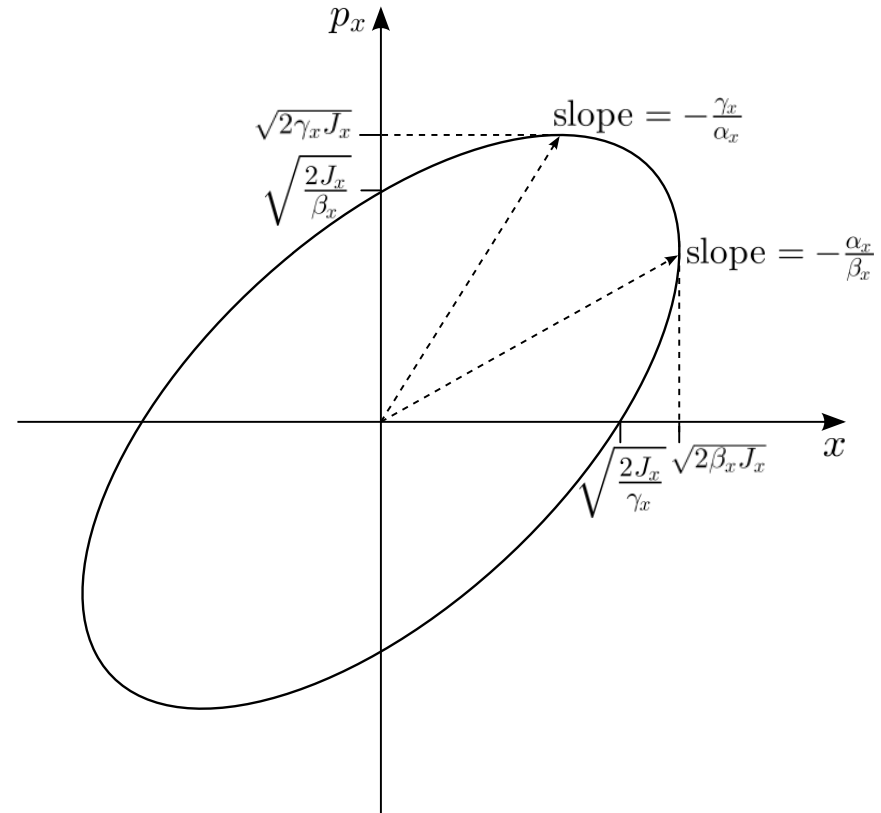
- For **periodic beamlines** (i.e. a beamline constructed from a repeated unit), the transfer matrix for a single period can be parameterised in terms of the **Courant–Snyder parameters** (α, β, γ) and the **phase advance** μ :

$$R = \begin{pmatrix} \cos \mu + \alpha \sin \mu & \beta \sin \mu \\ -\gamma \sin \mu & \cos \mu - \alpha \sin \mu \end{pmatrix}$$

- The characteristics of the particle motion can be represented by a **phase space portrait** showing the co-ordinates and momenta of a particle after an increasing number of passes through full periods of the beamline



- If the transfer map for each period is linear, then the phase space portrait is an **ellipse** with area πJ_x
- The **action** J_x characterises the **amplitude** of the betatron oscillations
- The **shape** of the ellipse is described by the **Courant–Snyder parameters**
- The rate at which particles move around the ellipse (**phase advance** per period) is **independent** of the betatron **action**



- Nonlinearities in particle dynamics can come from a number of different **sources**, e.g.
 - **Stray fields** in drift spaces
 - Higher-order **multipole components** in dipoles and quadrupoles
 - Higher-order **multipole magnets** (sextupoles, octupoles...) used to control various properties of the beam;
 - **Electromagnetic fields** generated by a **bunch** of particles, acting on individual particles within the same or another bunch (space-charge forces, beam-beam effects...)
- The effects of nonlinearities can be quite dramatic
- It is paramount to have some understanding of nonlinear dynamics for **optimising** the **design** and **operation** of many accelerator systems

Non-linear transfer maps and effects of non-linearities

- As example, consider the **vertical field** component in a **sextupole** (i.e. a nonlinear) magnet:

$$\frac{B_y}{B\rho} = \frac{1}{2}k_2x^2$$

with $B\rho$ the beam rigidity and k_2 the normalized sextupole gradient

- In “thin lens” approximation, the **deflection** of a particle passing through the sextupole of length L is

$$\Delta p_x = - \int \frac{B_y}{B\rho} ds = -\frac{1}{2}k_2Lx^2$$

- The thin lens **transfer map** for the sextupole is

$$\begin{aligned} x_1 &= x_0 , \\ p_{x1} &= p_{x0} - \frac{1}{2}k_2Lx^2 \end{aligned}$$

- A **nonlinear transfer map** can be represented as a **power series**

$$x_1 = A_1 + R_{11}x_0 + R_{12}p_{x0} + T_{111}x_0^2 + T_{112}x_0p_{x0} + T_{122}p_{x0}^2 + \dots$$

$$p_{x1} = A_2 + R_{21}x_0 + R_{22}p_{x0} + T_{211}x_0^2 + T_{212}x_0p_{x0} + T_{222}p_{x0}^2 + \dots$$

- The **coefficients** R_{ij} are components of the **transfer matrix** R
- The coefficients of the **higher-order** (nonlinear) terms are conventionally represented by T_{ijk} (2nd order), U_{ijkl} (3rd order) and so on...
- The values of the **indices** correspond to **components** of the phase space vector:

index value	1	2	3	4	5	6
component	x	p_x	y	p_y	z	δ

Example of a periodic system: a simple storage ring

- As example, consider the transverse dynamics in a **simple storage ring**, assuming:
 - ❑ The storage ring is constructed from some number of **identical cells** consisting of dipoles, quadrupoles and sextupoles.
 - ❑ The **phase advance** per cell can be tuned from close to zero, up to about $0.5 \times 2\pi$.
 - ❑ There is **one sextupole per cell**, which is located at a point where the horizontal beta function is 1 m, and the alpha function is zero.
 - ❑ Usually, storage rings will contain (at least) two sextupoles per cell, to correct horizontal and vertical chromaticity. To keep things simple, we will use only one sextupole per cell.

- The **chromaticity**, and hence the sextupole strength, will normally be a **function** of the **phase advance**
- To investigate the nonlinear effects of sextupoles, we shall keep the **sextupole strength** $k_2 L$ **fixed**, and **change** only the **phase advance**
- We can assume that the **map** from **one sextupole** to the **next** is **linear**, and corresponds to a **rotation** in phase space through an angle equal to the phase advance:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \mapsto \begin{pmatrix} \cos \mu_x & \sin \mu_x \\ -\sin \mu_x & \cos \mu_x \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix}$$

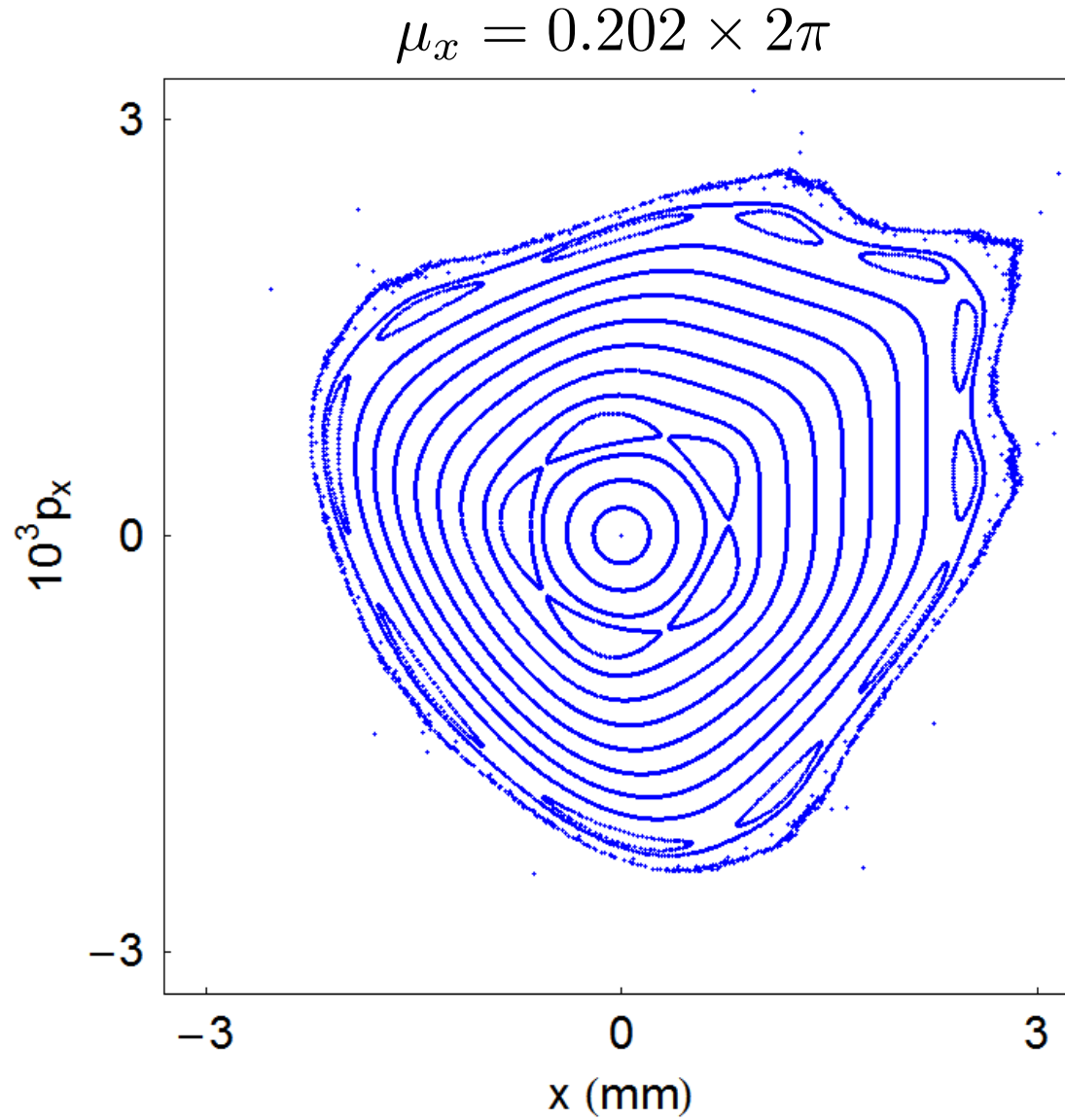
- Again to keep things simple, we shall consider only **horizontal motion**, and assume that the vertical co-ordinate $y = 0$
- In “thin lens” approximation, the **deflection** of a particle passing through the sextupole of length L is

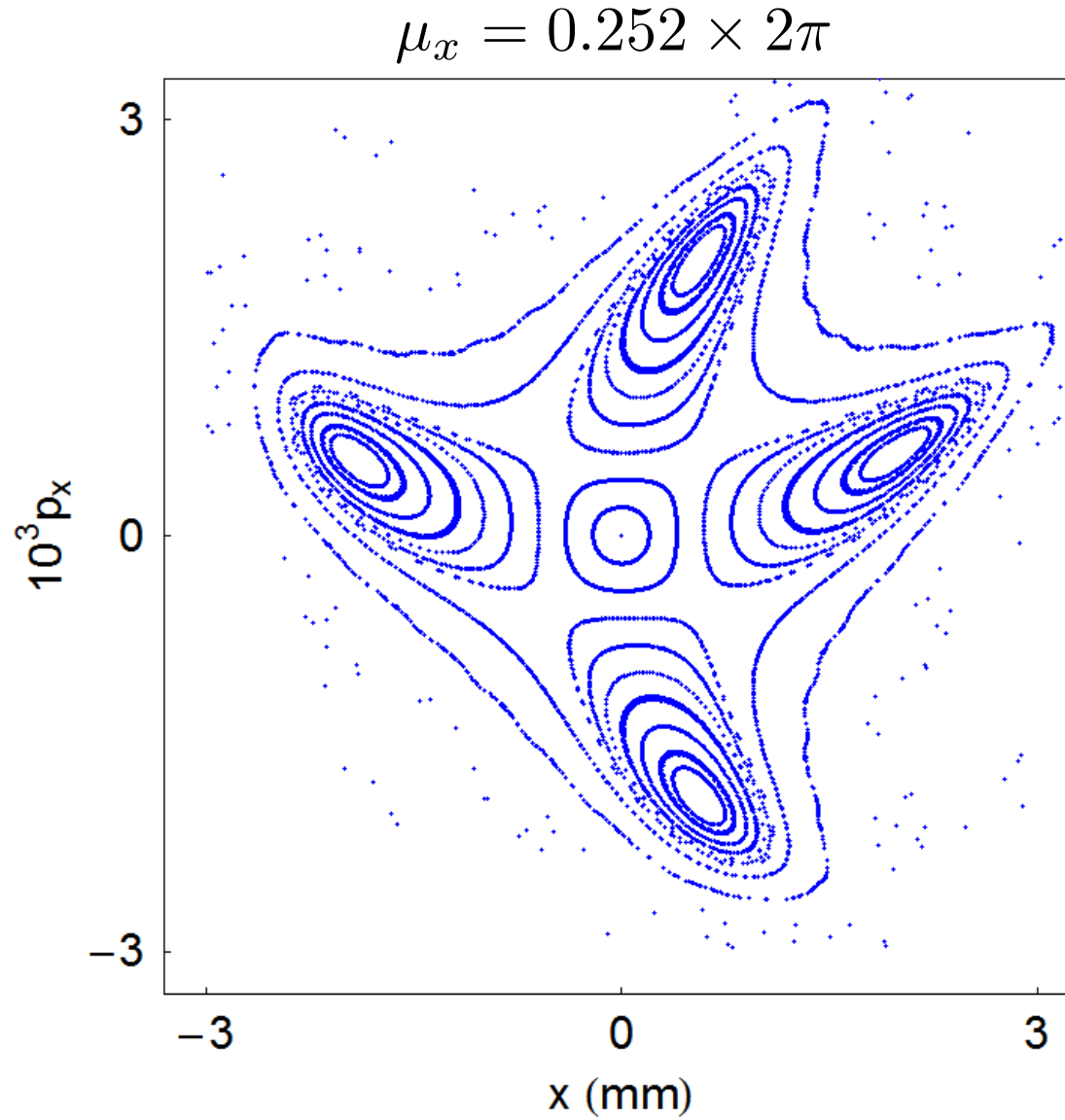
$$\Delta p_x = - \int \frac{B_y}{B_\rho} ds = -\frac{1}{2} k_2 L x^2$$

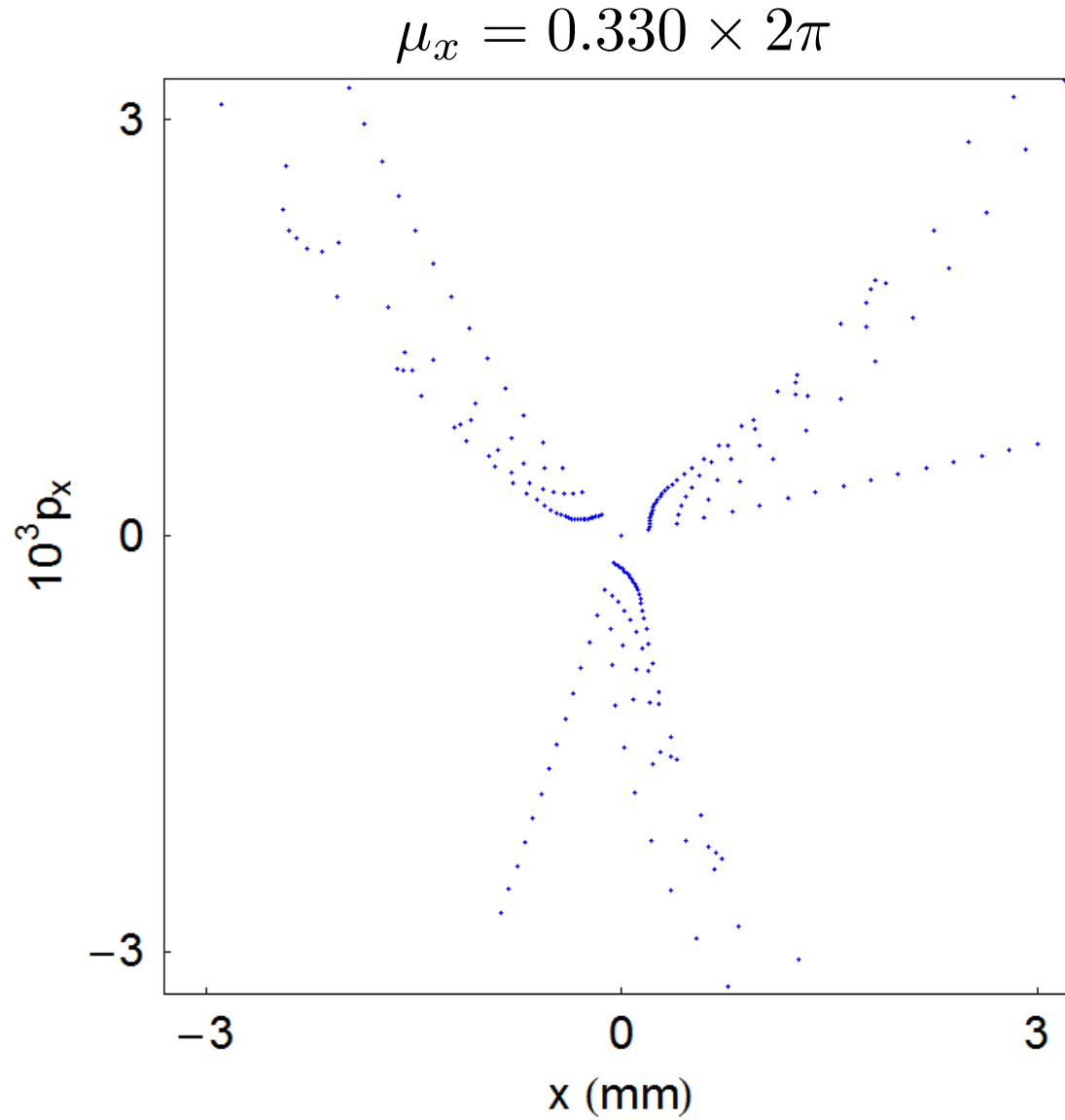
- The map for a particle moving through a short **sextupole** can thus be represented by a “**kick**” in the horizontal momentum:

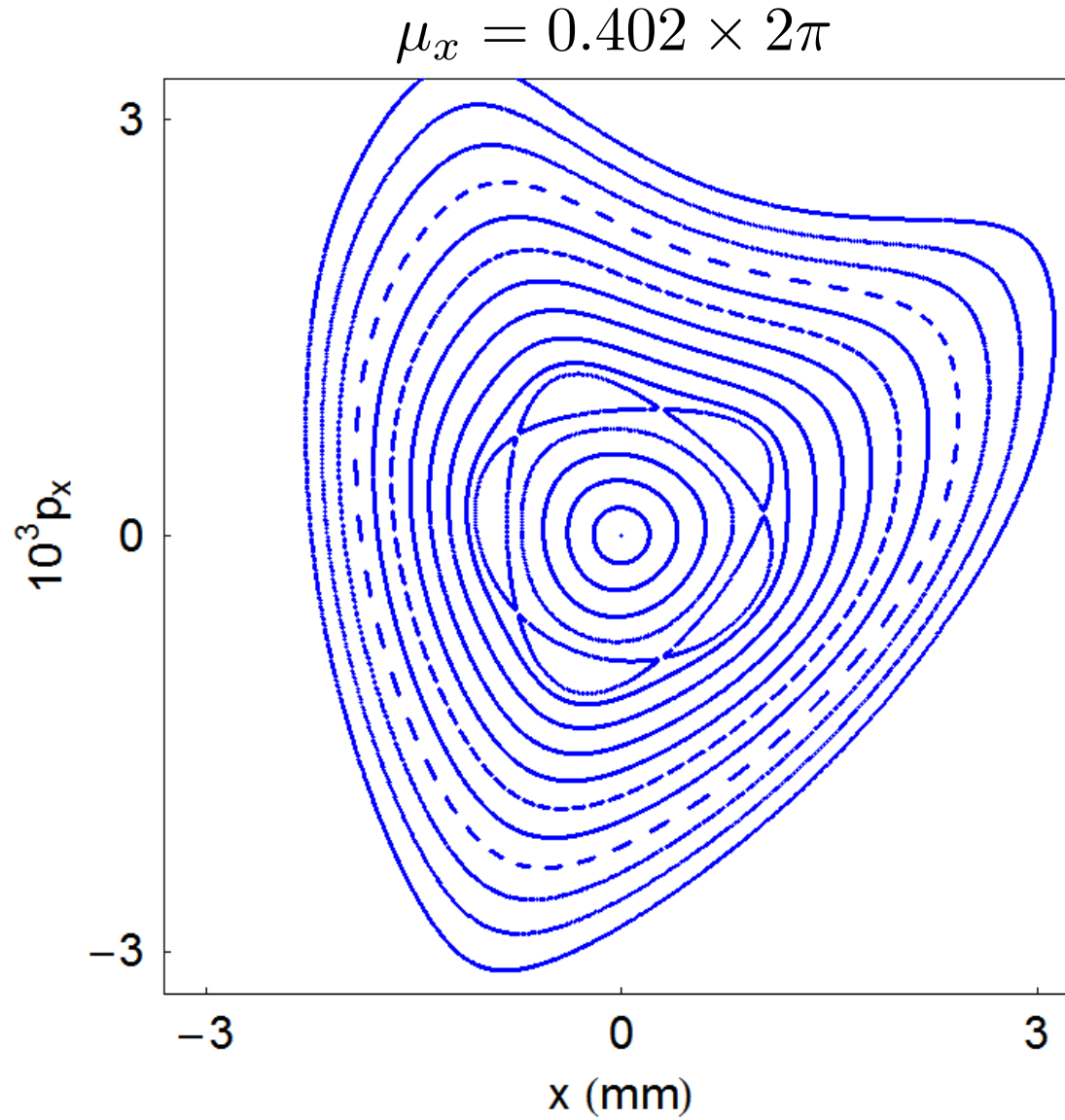
$$\begin{aligned} x &\mapsto x, \\ p_x &\mapsto p_x - \frac{1}{2}k_2 L x^2 \end{aligned}$$

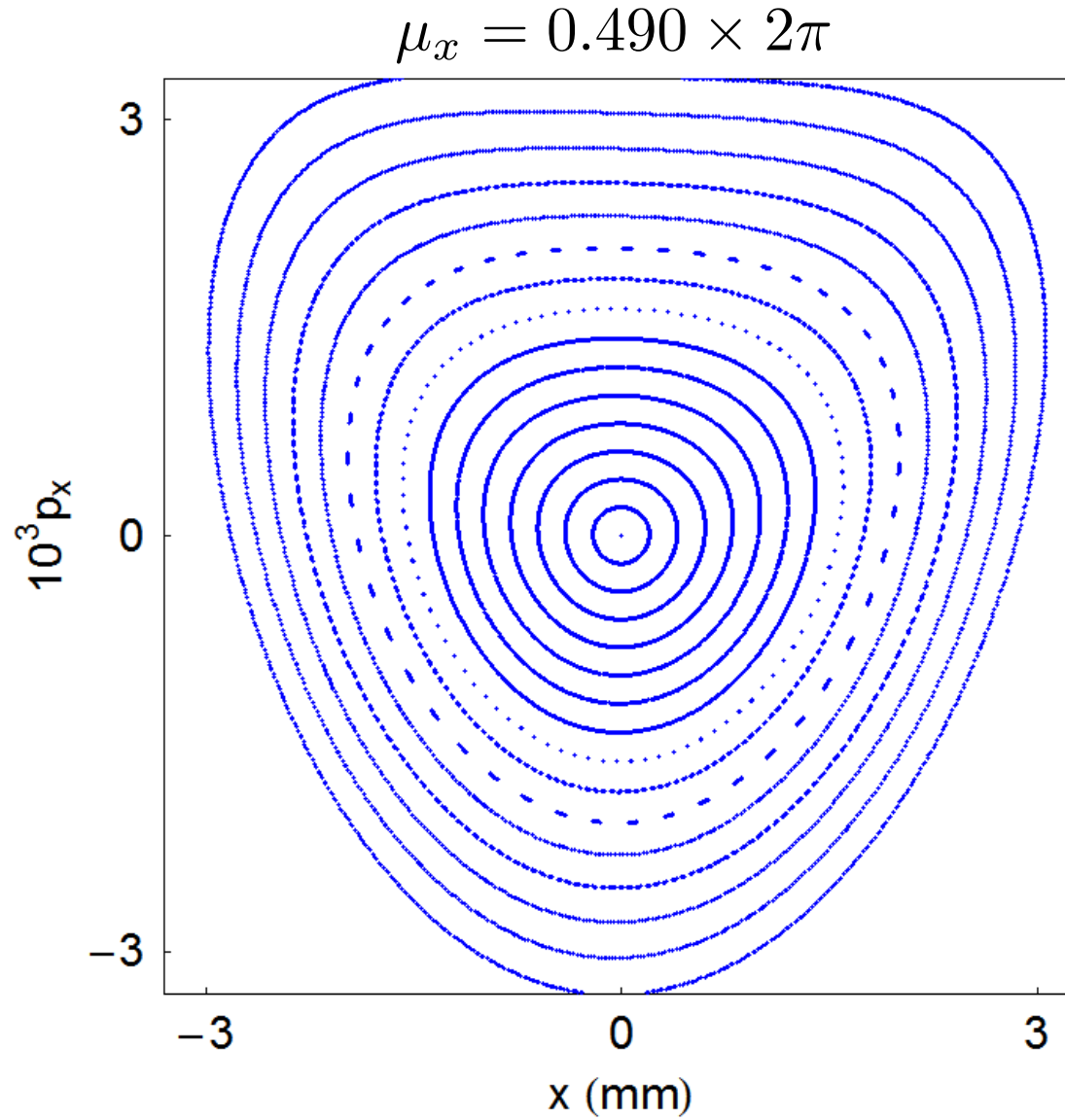
- Let us choose a fixed value $k_2 L = -600 \text{ m}^{-2}$, and look at the effects of the maps for different phase advances.
- For each case, we construct a **phase space portrait** by plotting the values of the dynamical variables after repeated application of the map (rotation + sextupole) for a range of initial conditions.
- First, let us look at the phase space portraits for a range of phase advances from $0.2 \times 2\pi$ to $0.5 \times 2\pi$











- **There are interesting features in these phase space portraits to which it is worth drawing attention:**
 - ❑ For small amplitudes (small x and p_x), **particles trace out closed loops around the origin**: this is what we expect for a linear map
 - ❑ As the **amplitude is increased**, “**islands**” appear in phase space: the phase advance (for the linear map) is often close to m/p where m is an integer and p is the number of islands
 - ❑ Sometimes, a larger number of islands appears at larger amplitude
 - ❑ Usually, there is a **closed curve that divides a region of stable motion from a region of unstable motion**. Outside that curve, the amplitude of particles increases without limit as the map is repeatedly applied
 - ❑ **The area of the stable region depends strongly on the phase advance**: for a phase advance close to $2\pi/3$, it appears that the stable region almost vanishes altogether
 - ❑ As the **phase advance is increased towards π** , the **stable area becomes large**, and distortions from the linear ellipse become small

Effect of phase advance on nonlinear dynamics

- An important observation is that the **effect** of the sextupole in the periodic cell **depends strongly** on the **phase advance** across the cell
- We can start to understand the significance of the phase advance by considering **two special cases**:
 - Phase advance equal to an **integer** times 2π
 - Phase advance equal to a **half integer** times 2π

- Let us consider first a **phase advance** equal to an **integer** times 2π . In that case, the linear part of the map is just the identity

$$x \mapsto x ,$$

$$p_x \mapsto p_x$$

- The **combined effect** of the **linear map** and the **sextupole kick** is:

$$x \mapsto x ,$$

$$p_x \mapsto p_x - \frac{1}{2}k_2 L x^2$$

- Clearly, the **horizontal momentum** will **increase** without limit
- There are **no stable regions** of phase space, apart from $x = 0$

- Now consider what happens if the phase advance of a cell is a **half integer** times 2π , so the linear part of the map is just a rotation through π
- If a **particle** starts at the entrance of a sextupole with $x = x_0$ and $p_x = p_{x0}$, then at the **exit** of that sextupole:

$$\begin{aligned} x_1 &= x_0 , \\ p_{x1} &= p_{x0} - \frac{1}{2}k_2 L x_0^2 \end{aligned}$$

- Then, after passing to the **entrance** of the **next sextupole**, the coordinates will be:

$$\begin{aligned} x_2 &= \cos(\pi)x_1 = -x_1 = -x_0 , \\ p_{x2} &= \cos(\pi)p_{x1} = -p_{x1} = -p_{x0} + \frac{1}{2}k_2 L x_0^2 \end{aligned}$$

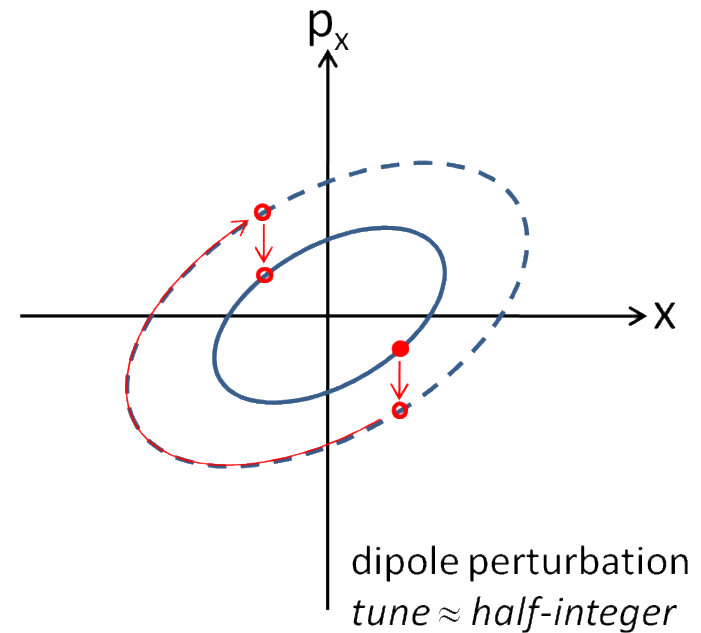
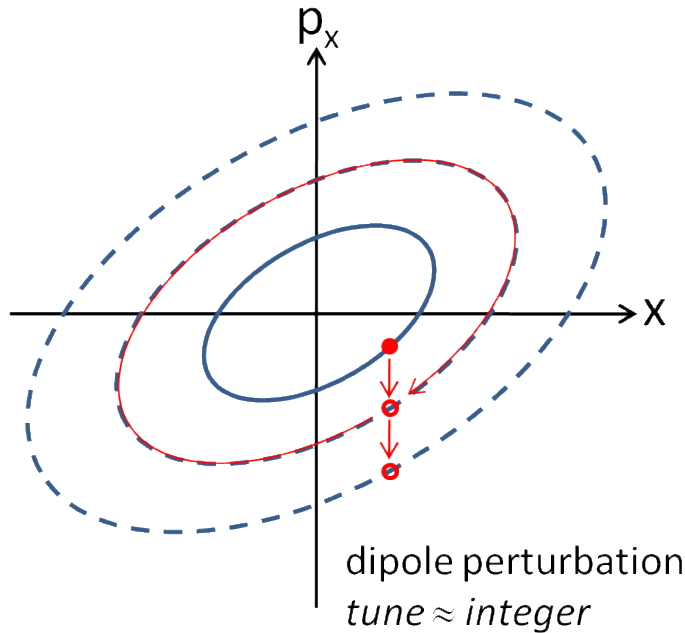
- Finally, after **passing** through the **second sextupole**:

$$x_3 = x_2 = -x_0 ,$$

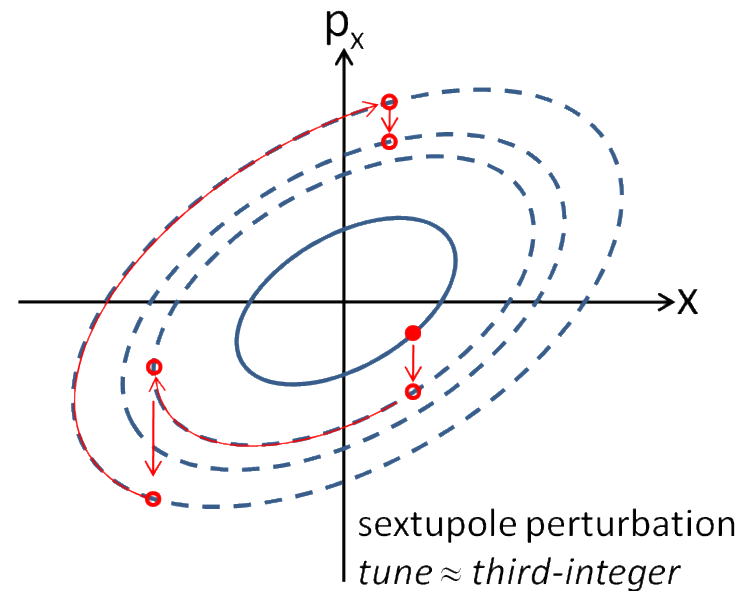
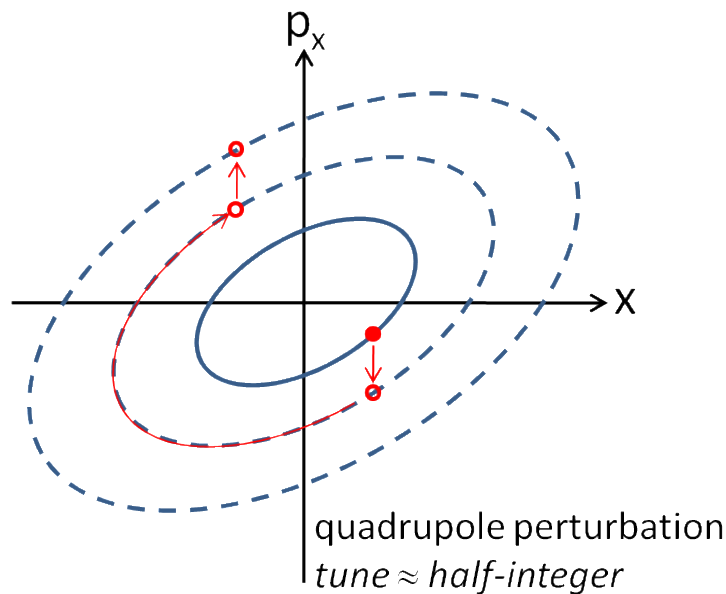
$$p_{x3} = p_{x2} - \frac{1}{2}k_2 L x_2^2 = -p_{x0}$$

- In other words, the **momentum kicks** from the two sextupoles exactly **cancel** each other
- The resulting map is a purely **linear phase space rotation** by π .
- In this situation, we expect the **motion** to be **stable** (and periodic), independent of the amplitude

- The effect of the phase advance on the sextupole “kicks” is **similar** to the effect on **perturbations** arising from **dipole** and **quadrupole errors** in a storage ring
- In the case of **dipole errors**, the **kicks add up** if the phase advance is an **integer**, and **cancel** if the **phase advance** is a **half integer**



- In the case of **quadrupole errors**, the **kicks add up** if the phase advance is a **half integer** times 2π
- **Higher-order multipoles** drive **higher-order resonances** but the effects are less easily illustrated on a phase space diagram



Conclusions and Summary

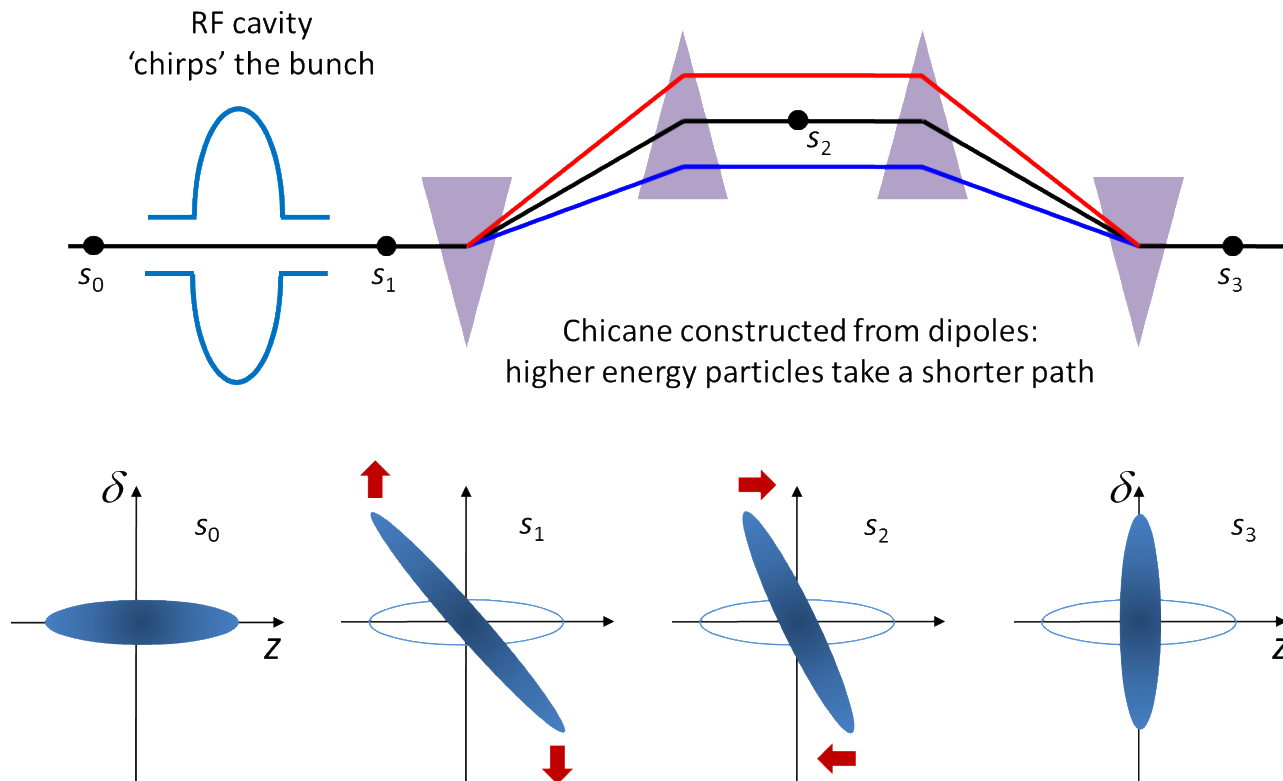
- **Nonlinear effects** can **limit** the **performance** of an accelerator system
- Sometimes the **effects** are **small enough** that they can be **ignored**
- In many cases, a **system designed without** taking account of **nonlinearities** will **not achieve** the specified **performance**
- If we analyse and understand the **nonlinear behaviour** of a system, then, we may be able to devise means of **compensating** any adverse effects

- **Nonlinear effects** can arise from a number of **sources** in accelerators, including stray fields, higher-order multipole components in magnets, space-charge, ...
- The **transfer map** for a nonlinear element (such as a sextupole) may be represented as a **power series** in the initial values of the phase space variables
- The effects of **nonlinearities** in accelerator systems vary widely, depending on the **type of system** in which they occur (e.g. a periodic accelerator)
- In some cases, **nonlinear effects** can limit the **performance** of an accelerator system. In such cases, it is important to take nonlinearities into account in the **design** of the system

BACK UP

Nonlinear effects in a bunch compressor

- A **bunch compressor** reduces the **length** of a bunch, by performing a **rotation** in longitudinal phase space
- Bunch compressors are used, for example, in **free electron lasers** to increase the peak current



Distribution of particles 'rotates' in longitudinal phase space (area is conserved).

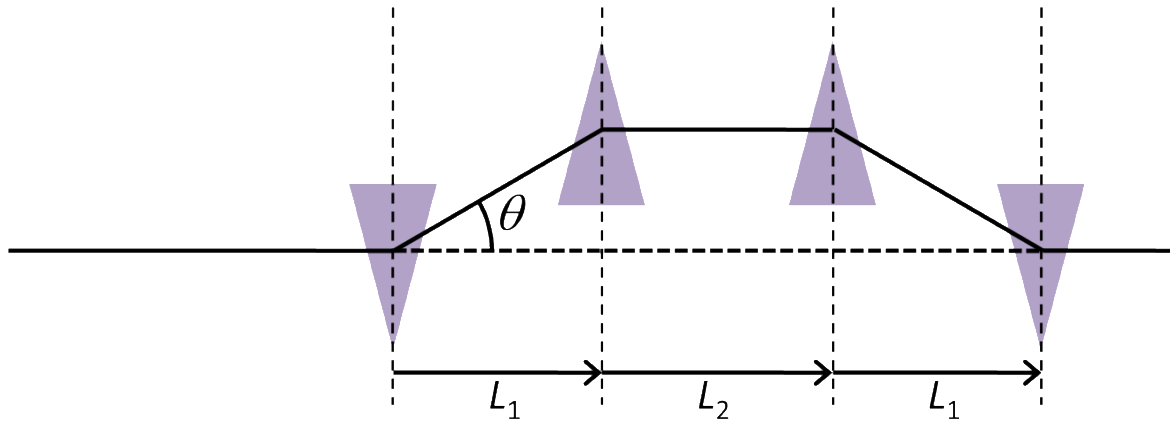
- The **RF cavity** is designed to “chirp” the bunch, i.e. to provide a **change in energy deviation** as a function of **longitudinal position** z within the bunch
- The **energy deviation** δ of a particle with energy E from a reference energy E_0 is defined as:

$$\delta = \frac{E - E_0}{E_0}$$

- The **transfer map** for the **RF cavity** in the bunch compressor with voltage V and frequency $\frac{\omega}{2\pi}$ is:

$$\begin{aligned} z_1 &= z_0 , \\ \delta_1 &= \delta_0 - \frac{eV}{E_0} \sin \left(\frac{\omega z_0}{c} \right) \end{aligned}$$

- Neglecting synchrotron radiation, the chicane does not change the energy of the particles. However, the **path length L** depends on the **energy** of the particle.



- If we assume that the bending angle in a dipole is small:

$$L = \frac{2L_1}{\cos \theta} + L_2$$

- The **bending angle** is a function of the **energy** of the particle:

$$\theta = \frac{\theta_0}{1 + \delta}$$

- The **change** in the co-ordinate z is the **difference** between the **nominal** path length, and the length of the path actually taken by the particle
- Hence, the **chicane transfer map** can be written:

$$\begin{aligned} z_2 &= z_1 + 2L_1 \left(\frac{1}{\cos \theta_0} - \frac{1}{\cos(\theta(\delta_1))} \right), \\ \delta_2 &= \delta_1 \end{aligned}$$

where θ_0 is the nominal bending angle of each dipole in the chicane, and $\theta(\delta)$ is given by

$$\theta(\delta) = \frac{\theta_0}{1 + \delta}$$

- Clearly, the **complete transfer map** for the bunch compressor is **nonlinear**, but how important are the nonlinear terms?

- To understand the effects of the nonlinear part of the map, we will study a **specific example**
- First, we will “**design**” a bunch compressor using only the **linear part** of the map
- The linear part of a transfer map can be obtained by **expanding** the map as a **Taylor series** in the dynamical variables, and keeping only the **first-order** terms
- After finding appropriate values for the various **parameters** using the **linear transfer map**, we shall see how our **design** has to be **modified** to take account of the **nonlinearities**

- To **first order** in the dynamical variables, the **map** for the **RF cavity** can be written:

$$\begin{aligned}
 z_1 &= z_0 , \\
 \delta_1 &= \delta_0 + R_{65} z_0 \quad \text{with} \quad R_{65} = -\frac{eV}{E_0} \frac{\omega}{c}
 \end{aligned}$$

- The **map** for the **chicane** is

$$\begin{aligned}
 z_2 &= z_1 + R_{56} \delta_1 , \\
 \delta_2 &= \delta_1 \quad \text{with} \quad R_{56} = 2L_1 \frac{\theta_0 \sin \theta_0}{\cos^2 \theta_0}
 \end{aligned}$$

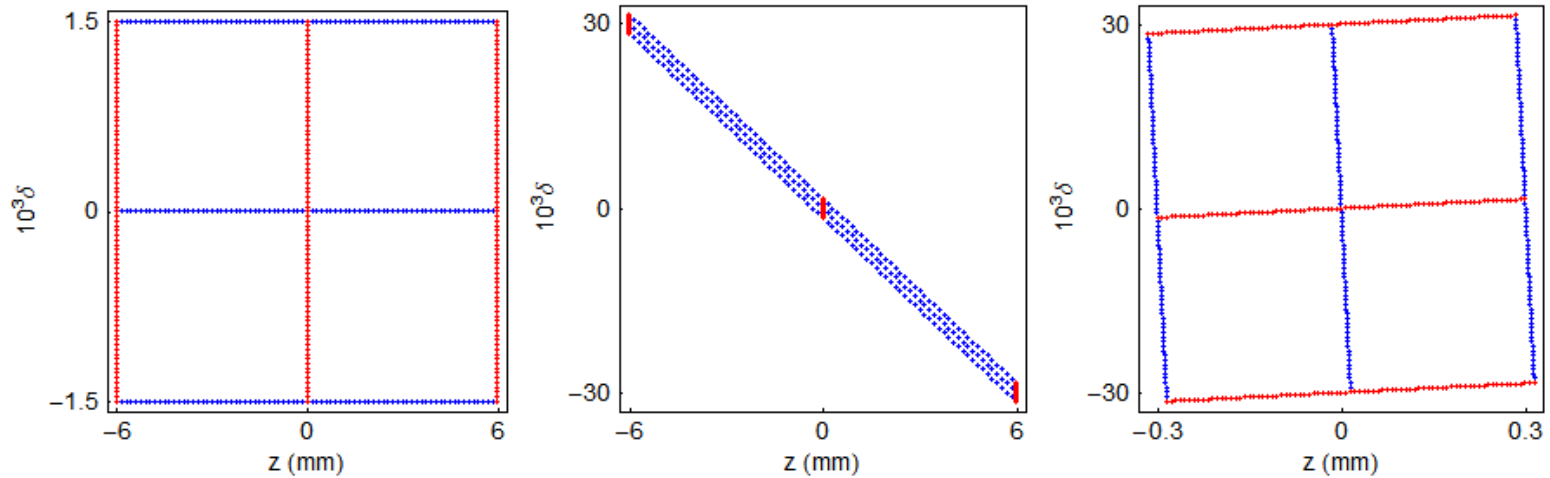
- As a specific example, consider a bunch compressor for the International Linear Collider (ILC)

Initial rms bunch length	$\sqrt{\langle z_0^2 \rangle}$	6 mm
Initial rms energy spread	$\sqrt{\langle \delta_0^2 \rangle}$	0.15%
Final rms bunch length	$\sqrt{\langle z_2^2 \rangle}$	0.3 mm

- **Two constraints** determine the values of R_{65} and R_{56}
 - The **bunch length** should be **reduced** by a **factor 20**
 - There should be **no “chirp”** on the bunch at the exit of the bunch compressor
- With these constraints, we find (see Appendix):

$$R_{65} = -4.9937 \text{ m}^{-1} \qquad R_{56} = 0.19975 \text{ m}$$

- We can illustrate the effect of the linearised bunch compressor map on **phase space** using an artificial “**window frame**” distribution:

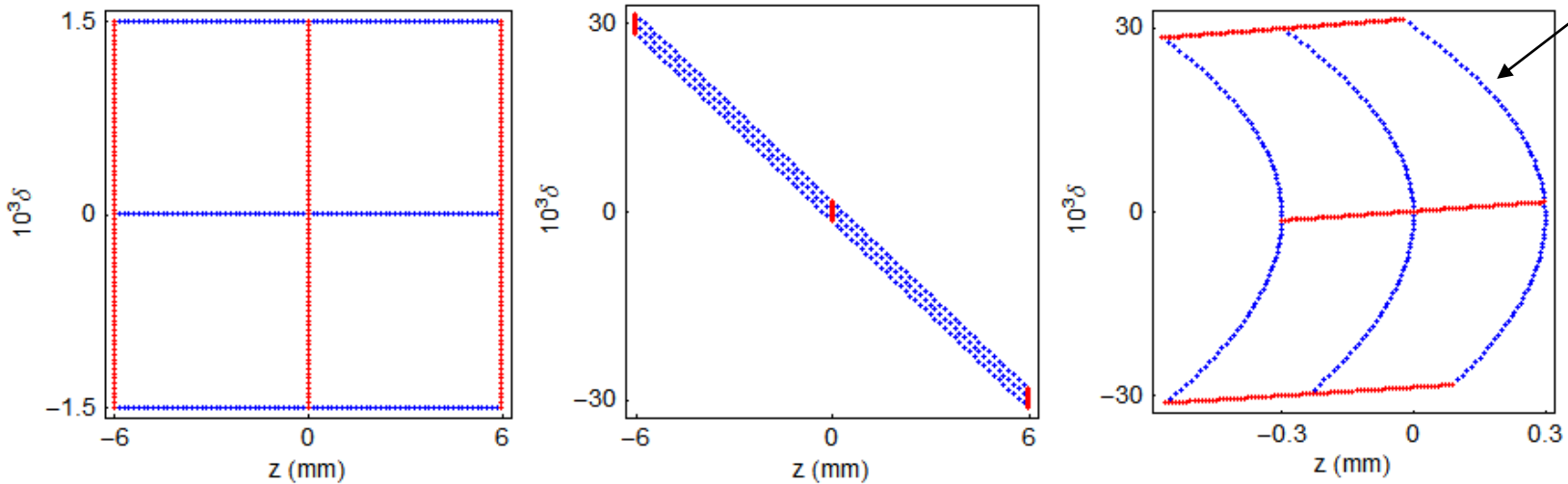


- The rms **bunch length** is **reduced** by a factor of 20 as required, but the **rms energy spread** is **increased** by the same factor, because the transfer map is **symplectic**, so phase space areas are conserved under the transformation

- Let's apply now the **full nonlinear map** for the bunch compressor.
- We need first to make some **assumptions** for the **RF voltage** and **frequency**, and the dipole **bending angle** and chicane **length** in order for the coefficient R_{65} and R_{56} to have the appropriate values

Beam (reference) energy	E_0	5 GeV
RF frequency	f_{rf}	1.3 GHz
RF voltage	V_{rf}	916 MV
Dipole bending angle	θ_0	3°
Dipole spacing	L_1	36.3 m

- As before, we illustrate the effect of the bunch compressor map on phase space using a “window frame” distribution:



- Although the **bunch length** has been **reduced**, there is significant **distortion** of the distribution: the **rms bunch length** will be **significantly longer** than what we are aiming for
- To reduce the distortion, we need to understand where it comes from
- In the phase space shown above, we see a **quadratic dependence** of the final particle position z_2 on the initial particle position z_0 .

- Consider a particle entering the bunch compressor with initial phase space co-ordinates z_0 and δ_0 . We can write the co-ordinates z_1 and δ_1 of the particle after the **RF cavity** to **2nd order** in z_0 and δ_0 :

$$\begin{aligned} z_1 &= z_0 , \\ \delta_1 &= \delta_0 + R_{65}z_0 + T_{655}z_0^2 \end{aligned}$$

- The co-ordinates of the particle after the **chicane** are (to **2nd order**):

$$\begin{aligned} z_2 &= z_1 + R_{56}\delta_1 + T_{566}\delta_1^2 , \\ \delta_2 &= \delta_1 \end{aligned}$$

- If we **combine** the **maps** for the RF and the chicane, we get:

$$\begin{aligned} z_2 &= (1 + R_{56}R_{65})z_0 + R_{56}\delta_0 \\ &\quad + (R_{56}T_{655} + R_{65}^2T_{566})z_0^2 \\ &\quad + 2R_{65}T_{566}z_0\delta_0 + T_{566}\delta_0^2, \\ \delta_2 &= \delta_0 + R_{65}z_0 + T_{655}z_0^2 \end{aligned}$$

- In order to eliminate the strong **non-linear distortion**, we have to **eliminate the second term**, i.e.

$$R_{56}T_{655} + R_{65}^2 T_{566} = 0$$

- By expanding the original map,

$$z_2 = z_1 + 2L_1 \left(\frac{1}{\cos \theta_0} - \frac{1}{\cos(\theta(\delta_1))} \right)$$

as a Taylor series in δ , we find that for small angles:

$$T_{566} \approx -3L_1\theta_0^2$$

- Now, it remains to determine T_{655} , i.e. the **coefficient** for the **second-order** dependence of the **energy deviation** on **longitudinal position**

- The **map** of the **energy deviation**

$$\delta_1 = \delta_0 - \frac{eV}{E_0} \sin\left(\frac{\omega z_0}{c}\right)$$

contains only **odd order terms** unless the RF cavity is operated **out of phase**, i.e.

$$\delta_1 = \delta_0 - \frac{eV}{E_0} \sin\left(\frac{\omega z_0}{c} + \phi_0\right)$$

- The **first** and **second order** coefficients in the transfer map for the **energy deviation** are:

$$R_{65} = -\frac{eV}{E_0} \frac{\omega}{c} \cos \phi_0 \quad \text{and} \quad T_{655} = -\frac{1}{2} \frac{eV}{E_0} \left(\frac{\omega}{c}\right)^2 \sin \phi_0$$

- Recall that $R_{65} = -4.9937 \text{ m}^{-1}$ and $R_{56} = 0.19975 \text{ m}$
- We also obtain

$$T_{566} \approx -3L_1\theta_0^2 = -0.29963 \text{ m}$$

- By imposing $R_{56}T_{655} + R_{65}^2T_{566} = 0$, we have that

$$T_{655} = 37.406 \text{ m}^{-2}$$

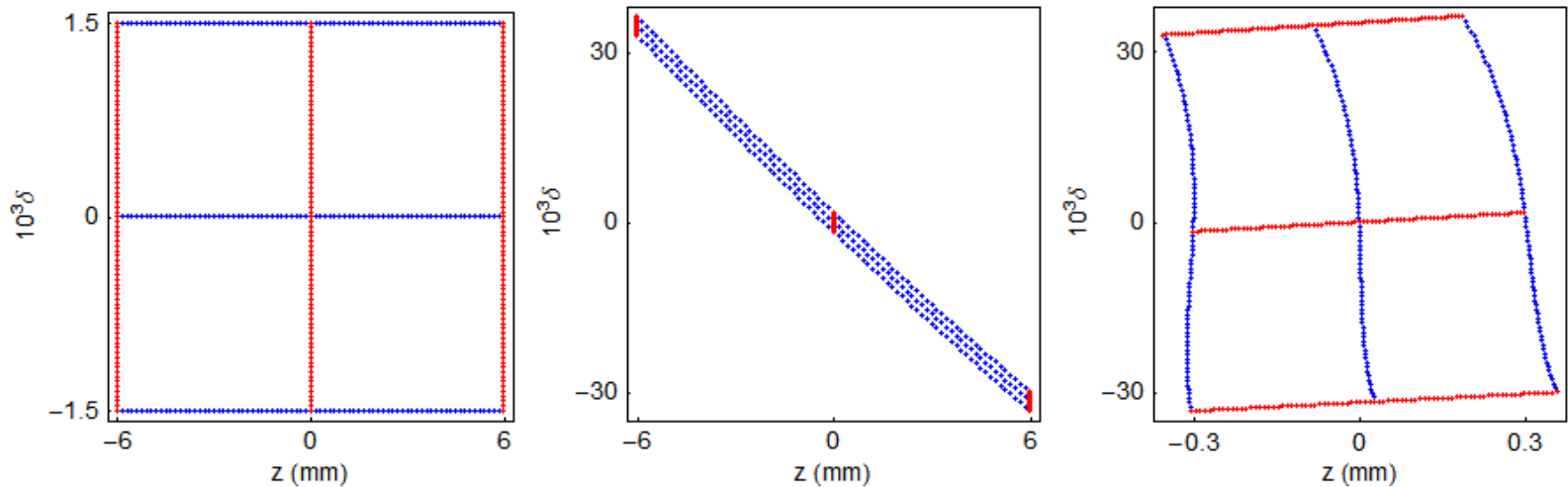
- Using the expressions

$$R_{65} = -\frac{eV}{E_0} \frac{\omega}{c} \cos \phi_0 \quad \text{and} \quad T_{655} = -\frac{1}{2} \frac{eV}{E_0} \left(\frac{\omega}{c}\right)^2 \sin \phi_0$$

the **voltage** and **phase** are determined as

$$V = 1046 \text{ MV} \quad \text{and} \quad \phi_0 = 28.8^\circ$$

- As before, we illustrate the effect of the bunch compressor on phase space using a “window frame” distribution, using the parameters determined above, to try to compress by a factor 20, while minimising the second-order distortion:



- The **dominant distortion** now appears to be **3rd order**, and looks **small enough** that it should not significantly affect the performance