Longitudinal calculations cheat sheet

Relativistic relationships

- Mass energy

$$E_0 = m_0 c^2$$

- Total energy

$$E = E_{\text{kin}} + E_0$$

- Momentum

$$E = \sqrt{(pc)^2 + E_0^2}$$

$$p = \gamma m_0 \beta c$$

- Relativistic velocity

$$\beta = \frac{v}{c} = \frac{pc}{E} = \sqrt{1 - \frac{1}{\gamma^2}}$$

- Lorentz factor

$$\gamma = \frac{E}{E_0} = \frac{1}{\sqrt{1-\beta^2}}$$

- Differential relationships

$$\frac{dp}{p} = \frac{1}{\beta^2}\frac{dE}{E} = \gamma^2 \frac{d\beta}{\beta}$$

Machine parameters

- Magnetic rigidity

$$B\rho = \frac{p}{q}$$

- Revolution period/frequency

$$T = \frac{2\pi R}{v} = \frac{1}{f} = \frac{2\pi}{\omega}$$

- RF period/frequency

$$f_{\text{rf}} = hf = \frac{\omega_{\text{rf}}}{2\pi} = \frac{1}{T_{\text{rf}}}$$

- Linear momentum compaction factor (and transition factor)

$$\alpha_c = \frac{\Delta R/R}{\Delta p/p} = \frac{1}{\gamma_t^2}$$

- Linear phase slippage factor

$$\eta = \frac{\Delta T/T}{\Delta p/p} = -\frac{\Delta f/f}{\Delta p/p} = \alpha_c - \frac{1}{\gamma^2}$$

- Other useful relationship

$$pR = \frac{\beta^2 E}{\omega}$$

Longitudinal equations of motion

- Equation of motion 1 - Phase slippage (drift) along the ring (continuous)

$$\frac{d\phi}{dt} = \frac{h\eta\omega}{pR}\left(\frac{\Delta E}{\omega}\right) = \frac{h\eta\omega^2}{\beta^2 E}\left(\frac{\Delta E}{\omega}\right)$$

- Equation of motion 2 - Energy kick with single RF cavity (continuous)

$$\frac{d}{dt}\left(\frac{E}{\omega}\right) = \frac{qV}{2\pi}(\sin\phi - \sin\phi_s)$$

- Equation of motion 1 - Phase slippage (drift) along the ring (discretized over T)

$$\phi_{n+1} = \phi_n + 2\pi h\eta\frac{\Delta E_n}{\beta^2 E}$$

- Equation of motion 2 - Energy kick in cavity (discretized over T, $\dot{\omega}$ neglected)

$$\Delta E_{n+1} = \Delta E_n + qV\sin(\phi_{n+1}) - U_0$$

- Beam energy gain per turn

$$U_0 = qV\sin\phi_s$$

Bucket parameters and synchrotron motion

- Bucket height

$$\Delta E_{\max} = \beta\sqrt{\frac{2qVE}{\pi h|\eta|}}\,Y(\phi_s)$$

- Bucket height reduction factor

$$Y(\phi_s) = \left|\cos\phi_s - \frac{\pi - 2\phi_s}{2}\sin\phi_s\right|^{1/2}$$

- Bucket area

$$A_{\mathrm{bk}} = 16\frac{\beta}{\omega_{\mathrm{rf}}}\sqrt{\frac{qVE}{2\pi h|\eta|}}\,\alpha(\phi_s)$$

- Bucket area reduction factor

$$\alpha(\phi_s) \approx \frac{1 - \sin\phi_s}{1 + \sin\phi_s}$$

- Angular synchrotron frequency (small amplitudes)

$$\Omega_s^2 = 2\pi f_s = -\omega^2\frac{h\eta qV\cos\phi_s}{2\pi\beta^2 E}$$

- Non-linear synchrotron frequency for a maximum amplitude in phase ϕ_u

$$\frac{\Omega(\phi_u)}{\Omega_s} \approx 1 - \frac{\phi_u^2}{16}$$

- Synchrotron tune

$$Q_s = \frac{\Omega_s}{\omega}$$

Differential relationships

C. Bovet et al., *A selection of formulae and data useful for the design of A.G. synchrotrons*, CERN-MPS-SI-Int-DL-70-4 (https://cds.cern.ch/record/104153/files/MPS-SI-Int-DL-70-4.pdf)

Variables	Equations
B, p, R	$\frac{dp}{p} = \gamma_t^2 \frac{dR}{R} + \frac{dB}{B}$
f, p, R	$\frac{dp}{p} = \gamma^2 \frac{df}{f} + \gamma^2 \frac{dR}{R}$
B, f, p	$\frac{dB}{B} = \gamma_t^2 \frac{df}{f} + \frac{\gamma^2 - \gamma_t^2}{\gamma^2} \frac{dp}{p}$
B, f, R	$\frac{dB}{B} = \gamma^2 \frac{df}{f} + (\gamma^2 - \gamma_t^2) \frac{dR}{R}$