Beam Instrumentation & Diagnostics Part 2

CAS Introduction to Accelerator Physics
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2nd part of this lecture covers:

➢ Transverse profile techniques
➢ Emittance determination at transfer lines
➢ Diagnostics for bunch shape determination
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Measurement of Beam Profile

The beam width can be changed by focusing via quadruples. 
Transverse matching between ascending accelerators is done by focusing.
→ Profiles have to be controlled at many locations.

**Synchrotrons**: Lattice functions $\beta(s)$ and $D(s)$ are fixed $\Rightarrow$ width $\sigma$ and emittance $\varepsilon$ are:

$$
\sigma_x^2(s) = \varepsilon_x \beta_x(s) + \left( D(s) \frac{\Delta p}{p} \right)^2 
$$

and

$$
\sigma_y^2(s) = \varepsilon_y \beta_y(s) 
$$

(no vertical bend)

**Transfer lines**: Lattice functions are ‘smoothly’ defined due to variable input emittance.

**Typical beam sizes**:
- $e^-$-beam: typically $\varnothing$ 0.01 to 3 mm,
- protons: typically $\varnothing$ 1 to 30 mm

**A great variety of devices are used**:

- **Optical techniques**: Scintillating screens (all beams),
  synchrotron light monitors ($e^-$), optical transition radiation ($e^-$, high-energetic $p$),
  ionization profile monitors (protons)

- **Electronics techniques**: Secondary electron emission SEM grids, wire scanners (all)
Measurement of Beam Profile

Outline:

- Scintillation screens:
  - emission of light, universal usage, limited dynamic range
- Optical Transition Radiation
- SEM-Grid
- Wire scanner
- Ionization Profile Monitor
- Synchrotron Light Monitors
- Summary
**Scintillation Screen**

**Scintillation:** Particle’s energy loss in matter causes emission of light → the most direct way of profile observation as used from the early days on!

Pneumatic drive with Ø70 mm screen:

In beam dynamics and by Volker Ziemann: Profile measurement is called ‘Screens’
Example of Screen based Beam Profile Measurement

**Advantage of screens:**
- Direct 2-dim measurement
- High spatial resolution
- Cheap realization
  ⇒ widely used at transfer lines

**Disadvantage of screens:**
- Intercepting device
- Some material might be brittle
- Possible low dynamic range
- Might be destroyed by the beam (radiation damage)

Observation with CMOS camera

**Scintillation Screen (beam stopped)**

**Example:** GSI LINAC, 4 MeV/u, low current, YAG:Ce screen
Light output from various Scintillating Screens

Example: Color CCD camera: Images at different particle intensities determined for U at 300 MeV/u

- Alumina: $\text{Al}_2\text{O}_3$
- CsI:Tl
- Chromox: $\text{Al}_2\text{O}_3$:Cr
- P43
- YAG:Ce
- Quartz
- Quartz:Ce
- $\text{ZrO}_2$:Mg

- Very different light yield i.e. photons per ion’s energy loss
- Different wavelength of emitted light
Material Properties for Scintillating Screens

Some materials and their basic properties:

<table>
<thead>
<tr>
<th>Type</th>
<th>Name</th>
<th>Material</th>
<th>Activ.</th>
<th>Max. λ</th>
<th>Decay</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ceramics</td>
<td>Chromox</td>
<td>Al₂O₃</td>
<td>Cr</td>
<td>700nm</td>
<td>≈ 10ms</td>
</tr>
<tr>
<td></td>
<td>Alumina</td>
<td>Al₂O₃</td>
<td>Non</td>
<td>380nm</td>
<td>≈ 10ns</td>
</tr>
<tr>
<td>Crystal</td>
<td>YAG:Ce</td>
<td>Y₃Al₅O₁₂</td>
<td>Ce</td>
<td>550nm</td>
<td>200ns</td>
</tr>
<tr>
<td></td>
<td>YAG:Ce</td>
<td>Y₃Al₅O₁₂</td>
<td>Ce</td>
<td>530nm</td>
<td>300ns</td>
</tr>
<tr>
<td></td>
<td>LYSO</td>
<td>Lu₁.₈Y₂SiO₅</td>
<td>Ce</td>
<td>420nm</td>
<td>40ns</td>
</tr>
<tr>
<td>Powder</td>
<td>P43</td>
<td>Gd₂O₃S</td>
<td>Tb</td>
<td>545nm</td>
<td>1ms</td>
</tr>
<tr>
<td></td>
<td>P46</td>
<td>Y₃Al₅O₁₂</td>
<td>Ce</td>
<td>530nm</td>
<td>300ns</td>
</tr>
<tr>
<td></td>
<td>P47</td>
<td>Y₂SiO₅</td>
<td>Ce&amp;Tb</td>
<td>400nm</td>
<td>100ns</td>
</tr>
</tbody>
</table>

Properties of a good scintillator:

- Large light output at optical wavelength → standard camera can be used
- Large dynamic range → usable for different currents
- Short decay time → observation of variations
- Radiation hardness → long lifetime
- Good mechanical properties → typ. size up to Ø 10 cm (Phosphor Pxx grains of Ø ≈ 10 μm on glass or metal).
Outline:

➢ Scintillation screens:
   emission of light, universal usage, limited dynamic range

➢ Optical Transition Radiation:
   light emission due to crossing material boundary, mainly for relativistic beams

➢ SEM-Grid

➢ Wire scanner

➢ Ionization Profile Monitor

➢ Synchrotron Light Monitors

➢ Summary
Optical Transition Radiation: Depictive Description

Optical Transition Radiation OTR for a single charge $e$:

Assuming a charge $e$ approaches an ideal conducting boundary e.g. metal foil:

- Image charge is created by electric field
- Dipole type field pattern
- Field distribution depends on velocity $\beta$ and Lorentz factor $\gamma$ due to relativistic trans. field increase
- Penetration of charge through surface within $t < 10$ fs: sudden change of source distribution
- Emission of radiation with dipole characteristic

Other physical interpretation: Impedance mismatch at boundary leads to radiation
Optical Transition Radiation: Depictive Description

Optical Transition Radiation OTR can be described in classical physics:

Approximated formula for normal incidence & in-plane polarization:

\[
\frac{d^2 W}{d\theta \, d\omega} \approx \frac{2e^2 \beta^2}{\pi c} \cdot \frac{\sin^2 \theta \cdot \cos^2 \theta}{(1 - \beta^2 \cos^2 \theta)^2}
\]

W: radiated energy \( \omega \): frequency of wave

Angular distribution of radiation in optical spectrum:

- Lope emission pattern depends on velocity or Lorentz factor \( \gamma \)
- Peak at angle \( \theta \approx 1/\gamma \)
- Emitted energy i.e. amount of photons scales with \( W \propto \beta^2 \)
- Broad wave length spectrum (i.e. no dependence on \( \omega \))

→ Suited for high energy electrons
Technical Realization of Optical Transition Radiation OTR

OTR is emitted by charged particle passage through a material boundary.

Photon distribution:
\[
\frac{dN_{\text{photon}}}{d\Omega} = N_{\text{beam}} \cdot \frac{2e^2 \beta^2}{\pi c} \cdot \log\left(\frac{\lambda_{\text{begin}}}{\lambda_{\text{end}}}\right) \cdot \frac{\theta^2}{(\gamma^{-2} + \theta^2)^2}
\]

- Detection: Optical 400 nm < \(\lambda\) < 800 nm
- Larger signal for relativistic beam \(\gamma \gg 1\)
- Low divergence for \(\gamma \gg 1\) \(\Rightarrow\) large signal
- Well suited for \(e^-\) beams
- \(p\)-beam used for \(E_{\text{kin}} \gtrsim 10\) GeV \(\iff\) \(\gamma \gtrsim 10\)

- Insertion of thin Al-foil under 45°
- Observation of low light by CCD.
Optical Transition Radiation compared to Scintillation Screen

Installation of OTR and scintillation screens on same drive:

Example: ALBA LINAC 100 MeV

Results:

➢ Much more light from YAG:Ce for 100 MeV ($\gamma \approx 200$) electrons
  light output $I_{\text{YAG}} \approx 10^5 I_{\text{OTR}}$

➢ Broader image from YAG:Ce due to finite YAG:Ce thickness

Courtesy of U. Iriso et al., DIPAC’09
Comparison between Scintillation Screens and OTR

**OTR:** electrodynamic process → beam intensity linear to # photons, high radiation hardness

**Scint. Screen:** complex atomic process → saturation possible, for some low radiation hardness

**OTR:** thin foil Al or Al on Mylar, down to 0.25 μm thickness

→ minimization of beam scattering (Al is low Z-material e.g. plastics like Mylar)

**Scint. Screen:** thickness ≈ 1 mm inorganic, fragile material, not always radiation hard

**OTR:** low number of photons → sensitive & expensive camera required

**Scint. Screen:** large number of photons → simple camera is sufficient

**OTR:** large γ needed → e⁻-beam with $E_{kin} > 100$ MeV, proton-beam with $E_{kin} > 100$ GeV

**Scint. Screen:** for all beams

**OTR:** complex angular photon distribution → resolution limited

**Scint. Screen:** isotropic photon distribution → simple interpretation

**Remark:**

1. **OTR:** beam angular distribution measurable → beam emittance

2. **OTR** **not** suited for LINAC-FEL due to **coherent** light emission (not covered here) but scintillation screens can be used.
Measurement of Beam Profile

Outline:

➢ Scintillation screens:
  emission of light, universal usage, limited dynamic range

➢ Optical Transition Radiation:
  light emission due to crossing material boundary, mainly for relativistic beams

➢ SEM-Grid:
  emission of electrons, workhorse, limited resolution

➢ Wire scanner

➢ Ionization Profile Monitor

➢ Synchrotron Light Monitors

➢ Summary
Secondary Electron Emission by Ion Impact

Energy loss of ions in metals close to a surface:

Closed collision with large energy transfer: $\rightarrow$ fast $e^-$ with $E_{kin} \gg 100$ eV

Distant collision with low energy transfer: $\rightarrow$ slow $e^-$ with $E_{kin} \leq 10$ eV

$\rightarrow$ ‘diffusion’ & scattering with other $e^-$: scattering length $L_s \approx 1 - 10$ nm

$\rightarrow$ at surface $\approx 90\%$ probability for escape

Secondary electron yield and energy distribution comparable for all metals!

$\Rightarrow Y = \text{const.} \times \frac{dE}{dx}$ (Sternglass formula)

Secondary Electron Emission Grids = SEM-Grid

**Beam surface interaction**: $e^-$ emission $\rightarrow$ measurement of current.

*Example: 15 wire spaced by 1.5 mm:*

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Typ. value</th>
</tr>
</thead>
<tbody>
<tr>
<td># wires per plane</td>
<td>10 ...100</td>
</tr>
<tr>
<td>Active area</td>
<td>(5...20 cm)$^2$</td>
</tr>
<tr>
<td>Wire $\varnothing$</td>
<td>25....100 $\mu$m</td>
</tr>
<tr>
<td>Spacing</td>
<td>0.3...2 mm</td>
</tr>
<tr>
<td>Material</td>
<td>e.g. W or Carbon</td>
</tr>
<tr>
<td>Max. beam power</td>
<td>1 W/mm</td>
</tr>
</tbody>
</table>

*SEM-Grid drive on $\varnothing$ 200 mm flange:*

- Pneumatic drive
- Beam $\approx$ 1 m

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[Image of SEM-Grid and beam interaction diagram]
Secondary Electron Emission Grids = SEM-Grid

**Beam surface interaction:** $e^{-}$ emission $\rightarrow$ measurement of current.

*Example:* 15 wire spaced by 1.5 mm:

Each wire is equipped with one I/U converter different ranges settings by $R_i$

$\rightarrow$ very large dynamic range up to $10^6$. 

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**Diagram:**

- **SEM-grid**
- **Range select**
- **Trans-impedance amplifier** one per wire
- **ADC** one channel per wire
- **Digital electronics**

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Peter Forck, CAS 2023, Santa Susanna
Example of Profile Measurement with SEM-Grids

Even for low energies, several SEM-Grid can be used due to the $\approx 80\%$ transmission $\Rightarrow$ frequently used instrument beam optimization: setting of quadrupoles, energy.

*Example:* $C^6^+$ beam of 11.4 MeV/u at different locations at GSI-LINAC
Measurement of Beam Profile

Outline:

➢ Scintillation screens:
  emission of light, universal usage, limited dynamic range
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  light emission due to crossing material boundary, mainly for relativistic beams
➢ SEM-Grid:
  emission of electrons, workhorse, limited resolution
➢ **Wire scanner:**
  emission of electrons, workhorse, scanning method
➢ Ionization Profile Monitor
➢ Synchrotron Light Monitors
➢ Summary
Slow, linear Wire Scanner

Idea: One wire is scanned through the beam!
Wire diameter 100 µm < $d_{wire}$ < 10 µm

Slow, linear scanner are used for:

➢ Low energy protons
➢ High resolution measurements for $e^-$ beam by de-convolution $\sigma^2_{beam} = \sigma^2_{meas} - d^2_{wire}$ ⇒ resolution down to 1 µm range can be reached
➢ Detection of beam halo

Example: Wires scanner at CERB LINAC4
**Fast, Flying Wire Scanner**

In a synchrotron **one** wire is scanned though the beam as fast as possible.

Fast pendulum scanner for synchrotrons; sometimes it is called ‘flying wire’:

*From* [https://twiki.cern.ch/twiki/bin/viewauth/BWSUpgrade/](https://twiki.cern.ch/twiki/bin/viewauth/BWSUpgrade/)
Usage of Flying Wire Scanners

**Material:** Carbon or SiC → low Z-material for low energy loss and high temperature.

**Thickness:** Down to 10 μm → high resolution.

**Detection:** High energy **secondary particles** with a detector like a beam loss monitor

**Secondary particles:**
- **Proton beam** → hadrons shower (π, n, p...)
- **Electron beam** → Bremsstrahlung photons.

**Proton impact on scanner at CERN-PS Booster:**

Rest mass:
- \( m_{\pi^\pm} = 140 \text{ MeV}/c^2 \)
- \( m_{\pi^0} = 135 \text{ MeV}/c^2 \)

**Kinematics of flying wire:**
- Velocity during passage typi. 10 m/s = 36 km/h & typical beam size \( \Theta \) 10 mm ⇒ time for traversing the beam \( t \approx 1 \text{ ms} \)

**Challenges:** Wire stability for fast movement with high acceleration
Comparison between SEM-Grid and **slow linear** Wire Scanners

**Grid:** Measurement at a single moment in time

**Scanner:** Scanning through beam ⇒ fast variations can not be monitored

⇒ for pulsed LINACs precise synchronization is needed

**Grid:** Resolution of a grid is fixed by the wire distance (typically 1 mm)

**Scanner:** For slow scanners the resolution is about the wire thickness (down to 10 μm)

⇒ used for e⁻--beams having small sizes (down to 10 μm)

**Grid:** Needs one electronics channel per wire

⇒ expensive electronics and data acquisition

**Scanner:** Needs a precise movable feed-through ⇒ expensive mechanics.

**Flying wire:**

**Grid:** **Not** adequate at synchrotrons for stored beam parameters

**Scanner:** **At high energy synchrotrons:** flying wire scanners are nearly non-destructive
Outline:

- **Scintillation screens:**
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- **Optical Transition Radiation:**
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- **SEM-Grid:**
  emission of electrons, workhorse, limited resolution

- **Wire scanner:**
  emission of electrons, workhorse, scanning method

- **Ionization Profile Monitor:**
  secondary particle detection from interaction beam-residual gas

- **Synchrotron Light Monitors**

- **Summary**
Ionization Profile Monitor at GSI Synchrotron

**Non-destructive** device for proton synchrotron:

- Beam ionizes the residual gas by electronic stopping
- Gas ions or $e^-$ accelerated by $E$-field $\approx 1 \text{ kV/cm}$
- Spatial resolved single particle detection

**Realization at GSI synchrotron:**

**One monitor per plane**

- IPM with $175 \times 175 \text{ mm clearance}$
- HV–electrode
- MCP: $100 \times 30 \text{ mm}^2$
- 63 wires, $2 \text{ mm} \text{ spacing}$
- 300 mm flange

**Typical vacuum pressure:**

Transfer line: $p = 10^{-8} \ldots 10^{-6} \text{ mbar} \Leftrightarrow n = 3 \cdot 10^8 \ldots 3 \cdot 10^{10} \text{ cm}^{-3}$

Synchrotron: $p = 10^{-11} \ldots 10^{-9} \text{ mbar} \Leftrightarrow n = 3 \cdot 10^5 \ldots 3 \cdot 10^7 \text{ cm}^{-3}$
Ionization Profile Monitor Realization

The realization for the heavy ion storage ring ESR at GSI:

Realization at GSI synchrotron:
One monitor per plane

**Horizontal IPM:**
- E-field box electrodes
- E-field separation disks
- View port Ø150 mm

**Vertical IPM:**
- MCP
- HV-electrode

**Typical vacuum pressure:**
- Transfer line: $p = 10^{-8} \ldots 10^{-6} \text{ mbar} \iff n = 3 \cdot 10^8 \ldots 3 \cdot 10^{10} \text{ cm}^{-3}$
- Synchrotron: $p = 10^{-11} \ldots 10^{-9} \text{ mbar} \iff n = 3 \cdot 10^5 \ldots 3 \cdot 10^7 \text{ cm}^{-3}$
Ionization Profile Monitor Realization

The realization for the heavy ion storage ring ESR at GSI:

**Horizontal IPM:**
- E-field box electrodes
- View port Ø150 mm
- MCP
- 175mm

**Vertical IPM**
- Insertion 650 mm
- Ø250 mm

Realization at COSY synchrotron for one plane:
Measurement of Beam Profile

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➢ Wire scanner:
  emission of electrons, workhorse, scanning method

➢ Ionization Profile Monitor:
  secondary particle detection from interaction beam-residual gas

➢ Synchrotron Light Monitors:
  photon detection of emitted synchrotron light in optical and X-ray range

➢ Summary
Synchrotron Radiation Monitor

An electron bent (i.e. accelerated) by a dipole magnet emit synchrotron light
see lecture ‘Electron Beam Dynamics’ by Lenny Rivkin

This light is emitted into a cone of opening $2/\gamma$ in lab-frame.

$\Rightarrow$ Well suited for rel. e$^-$
For protons:
Only for energies $E_{kin} > 100$ GeV

The light is focused to a intensified CCD.

**Advantage:**
Signal anyhow available!

Rest frame of electron:
Laboratory frame:

$$P \propto \gamma^4/\rho^2$$

$d\mathbf{p}/dt$
Realization of a Synchrotron Radiation Monitor

Extracting out of the beam’s plane by a (cooled) mirror
→ Focus to a slit + wavelength filter for optical wavelength
→ Image intensified CCD camera

**Example:** ESRF monitor from dipole with bending radius 22 m (blue or near UV)

![Diagram of synchrotron radiation monitor](image-url)
**Example:** Synchrotron radiation facility APS accumulator ring and blue wavelength:

**Advantage:** Direct measurement of 2-dim distribution, good optics for visible light

**Realization:** Optics outside of vacuum pipe

**Disadvantage:** Resolution limited by the diffraction due to finite apertures in the optics.

B.X. Yang (ANL) et al. PAC’97
‘Adiabatic Damping’ for an Electron Beam

**Example:** Booster at the light source ALBA acceleration from 0.1 → 3 GeV within 130 ms

Profiles from synchrotron radiation monitor:

- Adiabatic damping
- Longitudinal momentum contribution via dispersion $D(s) \Rightarrow \Delta x_D(s) = D(s) \cdot \frac{\Delta p}{p}$
- Quantum fluctuation due to light emission

The beam emittance is influenced by:

\[
\sigma_{tot}(s) = \sqrt{\varepsilon\beta(s) + \left(D(s) \cdot \frac{\Delta p}{p}\right)^2}
\]

- Adiabatic damping
- Longitudinal momentum contribution

**Profiles from synchrotron radiation monitor:**

Horizontal width $\sigma_x$

Vertical width $\sigma_y$

**Courtesy U. Iriso & M. Pont (ALBA) et al. IPAC 2011**
Diffraction Limit of Synchrotron Light Monitor

Limitations:
Diffraction limits the resolution due to Fraunhofer diffraction
Pattern width for 1:1 image:

\[ \sigma \approx \frac{\lambda}{2D/L} \approx 0.6 \cdot \left(\frac{\lambda^2}{q}\right)^{1/3} \]

\[ \Rightarrow \sigma \approx 100 \, \mu m \] for typical cases

Improvements:

➢ **Shorter wavelength:**
  Using X-rays and an aperture of Ø 1mm
  \[ \to \] ‘X-ray pin hole camera’,
  achievable resolution \[ \sigma \approx 10 \, \mu m \]

➢ **Interference technique:**
  At optical wavelength using a double slit
  \[ \to \] interference fringe blurring compared to point source
  achievable resolution \[ \sigma \approx 1 \, \mu m \].
Appendix: The Synchrotron Light Facility ALBA: Profile Measurement

Transverse profile:
- Many location in transport line
- Single location in ring
- Different devices used

Abbreviation:
- **FS**: Fluorescence Screen
- **OTR**: Optical Trans. Rad. Screen
- FS & OTR are destructive
- **SRM**: Synchr. Radiation Monitor
- **XSR**: X-ray pin hole camera
  both non-destructive

From U. Iriso, ALBA
Summary for Beam Profile Measurement

*Different techniques are suited for different beam parameters:*

- **e⁻-beam**: typically Ø 0.01 to 3 mm, **protons**: typically Ø 1 to 30 mm

*Intercepting ↔ non-intercepting methods*

*Direct observation of electrodynamics processes:*
- Optical synchrotron radiation monitor: non-destructive, for e⁻-beams, complex, limited res.
- X-ray synchrotron radiation monitor: non-destructive, for e⁻-beams, very complex
- OTR screen: nearly non-destructive, large relativistic γ needed, e⁻-beams mainly

*Detection of secondary photons, electrons or ions:*
- Scintillation screen: destructive, large signal, simple setup, all beams
- Ionization profile monitor: non-destructive, expensive, limited resolution, for protons

*Wire based electronic methods:*
- SEM-grid: partly destructive, large signal and dynamic range, limited resolution
- Wire scanner: partly destructive, large signal and dynamics, high resolution, slow scan.
Measurement of transverse Emittance

The emittance characterizes the whole beam quality, assuming linear behavior as described by second order differential equation.

It is defined within the phase space as: 

\[ \varepsilon_x = \frac{1}{\pi} \int_A dx dx' \]

The measurement is based on determination of:

*Either* profile width \( \sigma_x \) and angular width \( \sigma_x' \) at one location

*Or* profile width \( \sigma_x \) at different locations and linear transformations.

Different devices are used at transfer lines:

- Lower energies \( E_{\text{kin}} < 100 \text{ MeV/u} \): slit-grid device, pepper-pot (suited in case of non-linear forces).
- All beams: Quadrupole variation method using linear transformations (not well suited in the presence of non-linear forces)

**Synchrotron**: lattice functions results in stability criterion

\[ \Rightarrow \text{beam width delivers emittance: } \varepsilon_x = \frac{1}{\beta_x(s)} \left[ \sigma_x^2 - \left( D(s) \frac{\Delta p}{p} \right) \right] \text{ and } \varepsilon_y = \frac{\sigma_y^2}{\beta_y(s)} \]
Trajectory and Characterization of many Particles

➢ Single particle trajectories are forming a beam
➢ They have a distribution of start positions and angles
⇒ Characteristic quantity is the beam envelope
➢ Goal:
Transformation of envelope ↔ behavior of whole ensemble

see lecture ‘Transverse linear Beam Dynamics’ by Wolfgang Hillert

 Courtesy K.Wille
Definition of Coördinates and basic Equations

The basic vector is 6 dimensional:

\[ \vec{x} = \begin{pmatrix} x \\ x' \\ y \\ y' \\ l \\ \delta \end{pmatrix} = \begin{pmatrix} \text{hori. spatial deviation} \\ \text{horizontal divergence} \\ \text{vert. spatial deviation} \\ \text{vertical divergence} \\ \text{long. deviation} \\ \text{momentum deviation} \end{pmatrix} = \begin{pmatrix} [\text{mm}] \\ [\text{mrad}] \\ [\text{mm}] \\ [\text{mrad}] \\ [\text{mm}] \\ [10^{-3}] \end{pmatrix} \]

The transformation of a single particle from a location \( s_0 \) to \( s_1 \) is given by the Transfer Matrix \( R \):

\[ \vec{x}(s_1) = R(s) \cdot \vec{x}(s_0) \]

The transformation of the envelope from a location \( s_0 \) to \( s_1 \) is given by the Beam Matrix \( \sigma \):

\[ \sigma(s_1) = R(s) \cdot \sigma(s_0) \cdot R^T(s) \]

6-dim Beam Matrix with decoupled hor., vert. and long. plane:

\[ \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 & 0 & 0 & 0 \\ \sigma_{12} & \sigma_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} & \sigma_{34} & 0 & 0 \\ 0 & 0 & \sigma_{34} & \sigma_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{55} & \sigma_{56} \\ 0 & 0 & 0 & 0 & \sigma_{56} & \sigma_{66} \end{pmatrix} \]

Beam width for the three coordinates:

- Horizontal beam matrix:
  \[ \sigma_{11} = \langle x^2 \rangle \]
  \[ \sigma_{12} = \langle x x' \rangle \]
  \[ \sigma_{22} = \langle x'^2 \rangle \]
- Horizontal
  \[ x_{rms} = \sqrt{\sigma_{11}} \]
- Vertical
  \[ y_{rms} = \sqrt{\sigma_{33}} \]
- Longitudinal
  \[ l_{rms} = \sqrt{\sigma_{55}} \]
- Horizontal-lon. coupling
  \[ x'_{rms} = \sqrt{\sigma_{56}} \]
- Vertical-lon. coupling
  \[ y'_{rms} = \sqrt{\sigma_{66}} \]
Measurement of transverse Emittance

Outline:

➢ Definition and some properties of transverse emittance
➢ Quadrupole strength variation and position measurement
  emittance from several profile measurement and beam optical calculation
➢ Slit-Grid device: scanning method
Emittance Measurement by Quadrupole Variation

From a profile determination, the emittance can be calculated via linear transformation, if a well known and constant distribution (e.g. Gaussian) is assumed.

- Measurement of beam width
  \[ x^2_{\text{max}} = \sigma_{11}(s_1, k) \]

  matrix \( R(k) \) describes the focusing.

- With the drift matrix the transfer is
  \[ R(k_i) = R_{\text{drift}} \cdot R_{\text{focus}}(k_i) \]

- Transformation of the beam matrix
  \[ \sigma(s_1, k_i) = R(k_i) \cdot \sigma(s_0) \cdot R^T(k_i) \]

Task: Calculation of matrix \( \sigma(s_0) \) at entrance \( s_0 \), i.e. three elements

see lecture ‘Linear Imperfections’ by Volker Ziemann
Measurement of transverse Emittance

Using the ‘thin lens approximation’ i.e. the quadrupole has a focal length of \( f \):

\[
R_{\text{focus}}(K) = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ K & 1 \end{pmatrix} \Rightarrow R(L, K) = R_{\text{drift}}(L) \cdot R_{\text{focus}}(K) = \begin{pmatrix} 1 + LK & L \\ K & 1 \end{pmatrix}
\]

Measurement of matrix-element \( \sigma_{11}(s_1, K) \) from matrices \( \sigma(s_i, K_i) = R(K_i) \cdot \sigma(s_0) \cdot R^T(K_i) \)

**Example:** Square of the beam width at ELETTRA 100 MeV e\(^-\) Linac, YAG:Ce:

\[
\begin{align*}
\sigma_{11} & (1, K) = L^2 \sigma_{11}(0) \cdot K^2 \\
& + 2 \cdot (L \sigma_{11}(0) + L^2 \sigma_{12}(0)) \cdot K \\
& + L^2 \sigma_{22}(0) + \sigma_{11}(0) \\
& \equiv a \cdot K^2 - 2ab \cdot K + ab^2 + c
\end{align*}
\]

A fit delivers the beam matrix elements \( \sigma_{ij}(s_0) \)

**Assumptions:**
- ‘Regular’ phase space distribution
- Well aligned beam, no steering
- No emittance blow-up due to space charge

**Improved methods:**
Based on e.g. tomographic reconstruction
Outline:

➢ Definition and some properties of transverse emittance
➢ Quadrupole strength variation and position measurement
   emittance from several profile measurement and beam optical calculation
➢ Slit-Grid device: scanning method
   scanning slit → beam position & grid → angular distribution
The Emittance for Gaussian and non-Gaussian Beams

The beam distribution can be non-Gaussian, e.g. at:

- Beams behind ion source
- Space charged dominated beams at LINAC & synchrotron
- Cooled beams in storage rings

Generally: Emittance by 2nd statistical moments of 2-dim distribution:

Beam matrix: \( \sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{pmatrix} \)

Emittance: \( \varepsilon_{rms} = \sqrt{\det \sigma} \)

\[ = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2} \]

Variances Covariance i.e. correlation

It describes the value for 1 standard derivation.
The Slit-Grid Measurement Device

Slit-Grid: Direct determination of position and angle distribution.
Used for protons with $E_{\text{kin}} < 100$ MeV/u $\Rightarrow$ range $R < 1$ cm.

**Hardware**

- **Slit**: position $P(x)$ with typical width: 0.1 to 0.5 mm
- **Distance**: typ. 0.5 to 5 m (depending on beam energy 0.1 ... 100 MeV)
- **SEM-Grid**: angle distribution $P(x')$
Display of Measurement Results

The distribution is depicted as a function of position [mm] & angle [mrad]

The distribution can be visualized by
- Mountain plot
- Contour plot

**Calc. of 2\textsuperscript{nd} moments** $\langle x^2 \rangle, \langle x'^2 \rangle \& \langle xx' \rangle$

**Emittance value** $\epsilon_{\text{rms}}$ from

$$\epsilon_{\text{rms}} = \sqrt{\langle x^2 \rangle \cdot \langle x'^2 \rangle - \langle xx' \rangle^2}$$

**Problems:**
- Finite binning results in limited resolution
- Background $\rightarrow$ large influence on $\langle x^2 \rangle, \langle x'^2 \rangle \& \langle xx' \rangle$
- Or fit of distribution with an ellipse
  $\Rightarrow$ Effective emittance only

**Remark:** Behind a ion source the beam might be non-Gaussian due to plasma density and aberration at quadrupoles

**Beam:** $\text{Ar}^{4+}$, 60 keV, 15 $\mu$A at Spiral2 Phoenix ECR source.

P. Ausset, DIPAC 2009

See lecture ‘Sources’ by Klaus Knie
Summary for transverse Emittance Measurement

Emittance is the important quantity for comparison to theory.

It includes absolute value (value of $\mathbf{\varepsilon}$) & orientation in phase space ($\sigma_{ij}$ or $\alpha$, $\beta$ and $\gamma$)

three independent values $\varepsilon_{rms} = \sqrt{\sigma_{11} \cdot \sigma_{22} - \sigma_{12}} \equiv \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$

assuming no coupling between horizontal, vertical and longitudinal planes

Transfer line, all beams $\rightarrow$ profile measurement + linear transformation:

➢ Quadrupole variation: one location, different setting of a quadrupole

  Assumptions: ➢ well aligned beam, no steering
  ➢ no emittance blow-up due to space charge

Transfer line, low energy beams $\rightarrow$ direct measurement of $x$- and $x'$-distribution:

➢ Slit-grid: movable slit $\rightarrow$ $x$-profile, grid $\rightarrow$ $x'$-profile

➢ Requirement: Beam is stopped in $\approx 1\text{cm} \iff$ protons $E_{kin} \leq 100\text{ MeV}$

Remark: Non-linear transformation possible via tomographic reconstruction

Important remark: For a synchrotron with a stable beam storage,

width measurement is sufficient using $x_{rms} = \sqrt{\varepsilon_{rms} \cdot \beta}$
Measurement of longitudinal Parameters

Measurement of longitudinal parameter:
Bunch length measurement at

- Synchrotron light sources: Streak camera
- Linear light sources: Electro-optical modulator
- Summary

**Longitudinal ↔ transverse correspondences:**
- position relative to rf ↔ transverse center-of-mass
- bunch structure in time ↔ transverse profile
- momentum or energy spread ↔ transverse divergence
- longitudinal emittance ↔ transverse emittance.
The bunch position is given relative to the accelerating rf.

- e.g. $\varphi_{\text{ref}} = -30^\circ$ inside a rf cavity
- must be well aligned for optimal acceleration

Transverse correspondence: Beam position

**Example:** Pick-up signal for $f_{\text{rf}} = 36$ MHz rf at GSI-LINAC:

![Image of a graph showing the bunch position measured by a pick-up signal.](image)
Electron bunches are too short ($\sigma_t < 100$ ps) to be covered by the bandwidth of pick-ups ($f < 3$ GHz $\iff t_{rise} > 100$ ps) for structure determination.

→ Time resolved observation of synchr. light with a streak camera: Resolution $\approx 1$ ps.

**Scheme of a streak camera**
Technical Realization of a Streak Camera

acceleration  focusing  deflection

≈ 30 cm

Hardware of a streak camera

Time resolution down to 0.5 ps

≈ 60 cm
Results of Bunch Length Measurement by a Streak Camera

The streak camera delivers a fast scan in vertical direction (here 360 ps full scale) and a slower scan in horizontal direction (24 μs).

**Example:** Bunch length at the synchrotron light source SOLEIL for $U_{rf} = 2$ MV for slow direction 24 μs and scaling for fast scan 360 ps: measure $\sigma_t = 35$ ps.

Short bunches are desired by the users

**Example:** Bunch length $\sigma_t$ as a function of stored current (i.e. space charge de-focusing) at SOLEIL

Courtesy of M. Labat et al., DIPAC’07

See lecture ‘Electron Beam Dynamics’ by Lenny Rivkin
Measurement of longitudinal Parameters

Measurement of longitudinal parameter:

Bunch length measurement at

- Synchrotron light sources: Streak camera
- Linear light sources: Electro-optical modulators
- Summary
Bunch Length Measurement by electro-optical Method

For Free Electron Lasers → bunch length below 1 ps is achieved

➢ Below the resolution of streak camera
➢ Short laser pulses with $t \approx 10$ fs and electro-optical modulator

**Electro optical modulator:** Birefringent, rotation angle depends on external electric field

Relativistic electron bunch: transverse ele. field $E_{\perp,\text{lab}} = \gamma E_{\perp,\text{rest}}$ carries the time information

Scanning of delay between bunch and laser → time profile after several pulses.

See lecture ‘Synchrotron light circular machines & FELs’ by Eduard Prat
Bunch Length Measurement by electro-optical Method

For Free Electron Lasers $\rightarrow$ bunch length below 1 ps is achieved

Short laser pulse $\iff$ broad frequency spectrum (property of Fourier Transformation)

**Optical stretcher:** Separation of colors by different path length

$\implies$ different colors at different time $\implies$ **single-shot observation**

---

**Spectral Decoding**

---

Courtesy S.P.Jamison et al., EPAC 2006
Hardware of a spectral-decoded EOSD Scanning Setup

Laser: Commercial Ti:Sa or Yb-fibre, 10 fs duration, near IR,

Crystal: GaP or ZnTe, 100 μm thickness

Example: Bunch length at FLASH
100 fs bunch duration = 30 μm length!

S. Funkner et al., arXiv1912.01323 (2019)

B. Steffen et al, DIPAC 2009
Summary of longitudinal Measurements

Devices for bunch length at light sources:

*Streak cameras*:

- Time resolved monitoring of synchrotron radiation
  - for relativistic $e^-$-beams, $10\ \text{ps} < t_{\text{bunch}} < 1\ \text{ns}$
  - Time resolution limit of streak camera $\approx 1\ \text{ps}$

*Laser-based electro-optical modulation*:

- Electro-optical modulation of short laser pulse
  - very high time resolution down to some fs time resolution
  - Technical complex installation
Conclusion for Beam Diagnostics Course

Diagnostics is the ‘sensory organ’ for the beam.
It required for operation and development of accelerators

**Several categories of demands leads to different installations:**
- Quick, non-destructive measurements leading to a single number or simple plots
- Complex instrumentation used for hard malfunction and accelerator development
- Automated measurement and control of beam parameters i.e. feedback

The goal and a clear interpretation of the results is an important design criterion.

**General comments:**
- Quite different technologies are used, based on various physics processes
- Accelerator development goes parallel to diagnostics development
  (⇒ it makes fun as many skills are required).

Thank you for your attention!
General Reading on Beam Instrumentation

- Proceedings of several CERN Acc. Schools (introduction & advanced level, special topics).
- Contributions to conferences, in particular to International Beam Instrumentation Conference IBIC.
Backup slides
Broadening due to the Beam’s Space Charge: Ion Detection

Influence of the residual gas ion trajectory by:
- External electric field $E_{ex}$
- Electric field of the beam’s space charge $E_{space}$

E.g. Gaussian density distribution for round beam:

$$E_{space}(r) = \frac{1}{2\pi\varepsilon_0 l} \cdot \frac{q e N}{r} \cdot \frac{1}{r} \left[1 - \exp \left(-\frac{r^2}{2\sigma^2} \right) \right]$$

Estimation of correction:

$$\sigma_{corr}^2 \approx \frac{e^2 \ln 2}{4\pi\varepsilon_0 \sqrt{m_p c^2}} \cdot \frac{q N}{l} \cdot d_{gap} \cdot \sqrt{\frac{1}{e U_{ex}}} \propto N \cdot d_{gap} \cdot \sqrt{\frac{1}{U_{ex}}}$$

With the measured beam width is given by convolution:

$$\sigma_{meas}^2 = \sigma_{true}^2 + \sigma_{corr}^2$$

Example: $U^{73+}$, $10^9$ particles per 3 m bunch length, cooled beam with $\sigma_{true} = 1$ mm FWHM.
Electron Detection and Guidance by Magnetic Field

**Ion detection mode:**

- **Electron detection mode:**

  - For $E_{kin,\perp} = 10$ eV & $B = 0.1$ T $\Rightarrow r_c \approx 100$ µm
  
  - $E_{kin}$ from atomic physics, $\approx 100$ µm resolution of MCP

  - **Time-of-flight:** $\approx 1 - 2$ ns $\Rightarrow 2 - 3$ cycles.

  - **B-field:** Dipole with large aperture

  - $\Rightarrow$ IPM is expensive & large device!

- e$^-$ detection in an external magnetic field $\Rightarrow$ cyclotron radius $r_c = \frac{mv_{\perp}}{eB}$

- 7 $\times$ $10^{10}$ charges per 10 m bunch

- $\Rightarrow$ broadening by beam’s electric field
IPM: Magnet Design

**Magnetic field for electron guidance:**

Maximum image distortion:
5% of beam width ⇒ \( \Delta B/B < 1\% \)

**Challenges:**
- High \( B \)-field homogeneity of 1%
- Clearance up to 500 mm
- Correctors required to compensate beam steering
- Insertion length 2.5 m incl. correctors

**Equation:**
\[
B_{\text{cor}} \cdot I_{\text{corr}} = - \frac{1}{2} B_{\text{IPM}} \cdot I_{\text{IPM}}
\]

Remark: For MCP wire-array readout lower clearance required
**IPM: Magnet Design**

**Magnetic field for electron guidance:**

Maximum image distortion:

5% of beam width \( \Rightarrow \Delta B/B < 1 \%

**Challenges:**
- High \( B \)-field homogeneity of 1%
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- Correctors required to compensate beam steering
- Insertion length 2.5 m incl. correctors

**Remark:** For MCP wire-array readout lower clearance required

\[
B_{\text{cor}} \cdot I_{\text{cor}} = -\frac{1}{2} B_{\text{IPM}} \cdot I_{\text{IPM}}
\]

**Magnet:** \( B = 250 \text{ mT}, \) Gap 220 mm

**IPM:** Profile 32 strips, 2.5 mm width

**Remark for electron beams:**

Resolution of 50 µm is insufficient, but sometimes used for photon beams
**Example** of realization at TERATRON:

- Insertion of foil
  - e.g. 5 μm Kapton coated with 0.1 μm Al
  - **Advantage:** thin foil \( \Rightarrow \) low heating & straggling
  - 2-dim image visible

Results at FNAL-TEVATRON synchrotron with 150 GeV proton

Using fast camera: Turn-by-turn measurement

![Beam instrumentation and diagnostics](image)

- rad-hard camera
- Lens
- Filter wheel
- Window
- Beam pipe

Courtesy V.E. Scarpine (FNAL) et al., BIW’06
Coherent Optical Transition Radiation

Observation of coherent OTR for **compressed** bunches at LINAC based light sources

**Reason:** Coherent emission if bunch length $\approx$ wavelength ($t_{\text{bunch}} = 2$ fs $\iff l_{\text{bunch}} = 600$ nm)

or bunch fluctuations $\approx$ wavelength

Parameter reach for most LINAC-based FELs!

Beam parameter: FLASH, 700 MeV, 0.5 nC, with bunch compression

OTR screen  scint. screen

**in-coherent emission photons**

beam

OTR screen

coherent emission photons

beam

OTR screen

**prompt** emission for OTR and scint. screen

$\rightarrow$ coherent and in-coherent OTR

**100 ns delayed** emission

$\rightarrow$ no OTR as expected (classical process)

$\rightarrow$ emission by scint. screen due to lifetime

$\iff$ correct profile image!

Contrary of M. Yan et al., DIPAC’11 & S. Wesch, DIPAC’11
X-ray Pin-Hole Camera

The diffraction limit is \( \sigma \approx 0.6 \cdot \left( \frac{\lambda^2}{\rho} \right)^{1/3} \) \( \Rightarrow \) shorter wavelength by X-rays.

Example: PETRA III

Courtesy K. Wittenburg, DESY
Double Slit Interference for Radiation Monitors

The blurring of interference pattern due to finite size of the sources
⇒ spatial coherence parameter $\gamma$ delivers $\text{rms}$ beam size

i.e. ‘de-convolution’ of blurred image!
→ highest resolution, but complex method

Typical resolution for three methods:

➢ Direct optical observation: $\sigma \approx 100 \ \mu\text{m}$
➢ Direct x-ray observation: $\sigma \approx 10 \ \mu\text{m}$
➢ Interference optical observer: $\sigma \approx 1 \ \mu\text{m}$

SR source of finite width

![Diagram of double slit interference pattern](image)

Courtesy of V. Schlott PSI
Optical stretcher for electro-optical Method

For Free Electron Lasers → bunch length below 1 ps is achieved

Short laser pulse ↔ broad frequency spectrum (property of Fourier Transformation)

Optical stretcher: Separation of colors by different path length

⇒ different colors at different time ⇒ single-shot observation

Spectral Decoding

Courtesy S.P. Jamison et al., EPAC 2006
Bunch Structure at low $E_{kin}$: Not possible with Pick-Ups

**Pick-ups are used for:**
- precise for bunch-center relative to rf
- course image of bunch shape

**But:**
For $\beta << 1 \rightarrow$ long. $E$-field significantly modified:

\[ \sigma \approx \frac{R}{\gamma} \]

**Example:** Comparison pick-up – particle counter:
Ar beam of 1.4 MeV/u ($\beta = 5.5\%$), $f_{rf} = 108$ MHz

\[ \frac{1}{f_{rf}} = 9.2 \text{ ns} \]

⇒ the pick-up signal is insensitive to bunch ’fine-structure’
Bunch Structure using secondary Electrons for low $E_{\text{kin}}$ Protons

Secondary $e^-$ liberated from a wire carrying the time information.

→ Bunch Shape Monitor (BSM)

**Working principle:**

- insertion of a 0.1 mm wire at $\approx 10$ kV
- emission of secondary $e^-$ within less than 10 ps
- secondary $e^-$ are accelerated
- toward an rf-deflector
- rf-deflector as ’time-to-space’ converter
- detector with a thin slit
- slow shift of the phase
- resolution $\approx 10$ ps $\approx 1^\circ @ 280$ MHz
- Measurements are comparable to that obtained with particle detectors.

**Diagram:**
- Detector: SEM or FC
- Bunch shape
- rf-deflector (+ phase shifter)
- Aperture: about 1 mm
- Secondary electron from wire
- Beam

SEM: secondary electron multiplier
Realization of Bunch Shape Monitor at CERN LINAC2

Example: The bunch shape behind RFQ with 120 keV/u:

- **rf-deflector**
- **movable HV wire**
- **electron detector**
- **electrons**
- **Flange Ø 150 mm**
- **ion beam**

Graph showing counts per channel over time relative to rf [ns] with two peaks:
- High current: 5 mA
- Low current: 0.1 mA
‘Adiabatic’ Damping during Acceleration

The emittance $\varepsilon = \int dx dx'$ is defined via the position deviation and angle in lab-frame.

After acceleration the longitudinal velocity is increased $\Rightarrow$ angle $\varphi$ is smaller.

The angle is expressed in momenta: $x' = p_\perp / p_\parallel$, the emittance is $<xx'> = 0$:

$$\varepsilon = x \cdot x' = x \cdot p_\perp / p_\parallel$$

$\Rightarrow$ under ideal conditions the emittance can be normalized to the momentum $p_\parallel = \gamma \cdot m \cdot \beta c$.

$\Rightarrow$ normalized emittance $\varepsilon_{\text{norm}} = \beta \gamma \cdot \varepsilon$ is preserved with the Lorentz factor $\gamma$ and velocity $\beta = v/c$.

**Example:** Acceleration in GSI-synchrotron for C$^{6+}$ from 6.7 $\rightarrow$ 600 MeV/u ($\beta = 12 \rightarrow 79\%$) observed by IPM.

Theoretical width: $\langle x \rangle_f = \sqrt{\frac{\beta_i \cdot \gamma_i}{\beta_f \cdot \gamma_f}} \cdot \langle x \rangle_i$

$$= 0.33 \cdot \langle x \rangle_i$$

Measured width: $\langle x \rangle_f \approx 0.37 \cdot \langle x \rangle_i$

IPM is well suited for long time observations without beam disturbance $\Rightarrow$ mainly used at proton synchrotrons.
Measurement of transverse Emittance

Using the ‘thin lens approximation’ i.e. the quadrupole has a focal length of $f$:

$$
R_{focus}(K) = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ K & 1 \end{pmatrix} \quad \Rightarrow \quad R(L, K) = R_{drift}(L) \cdot R_{focus}(K) = \begin{pmatrix} 1 +LK & L \\ K & 1 \end{pmatrix}
$$

Measurement of matrix-element $\sigma_{11}(s, K)$ from matrices $\sigma(s_{1}, K_{j}) = R(K_{j}) \cdot \sigma(s_{0}) \cdot R^{T}(K_{j})$

**Example:** Square of the beam width at ELETTRA 100 MeV e$^-$ Linac, YAG:Ce:

For completeness: The relevant formulas

$$
\sigma_{11}(1, K) = L^{2}\sigma_{11}(0) \cdot K^{2} + 2 \cdot ( L\sigma_{11}(0) + L^{2}\sigma_{12}(0) ) \cdot K + L^{2}\sigma_{22}(0) + \sigma_{11}(0) \\
\equiv a \cdot K^{2} - 2ab \cdot K + ab^{2} + c \\
= a \cdot (K - b)^{2} + c
$$

The three matrix elements at the quadrupole:

$$
\sigma_{11}(0) = \frac{a}{L^{2}} \\
\sigma_{12}(0) = -\frac{a}{L^{2}} \left( \frac{1}{L} + b \right) \\
\sigma_{22}(0) = \frac{1}{L^{2}} \left( ab^{2} + c + \frac{2ab}{L} + \frac{a}{L^{2}} \right)
$$

$$
\varepsilon_{rms} = \sqrt{\det(\sigma(0))} = \sqrt{\sigma_{11}(0) \cdot \sigma_{22}(0) - \sigma_{12}^{2}(0)} = \sqrt{ac} / L^{2}
$$

G. Penco (ELETTRA) et al., EPAC’08
The Emittance for Gaussian and non-Gaussian Beams

The beam distribution can be non-Gaussian, e.g. at:

- Beams behind ion source
- Space charged dominated beams at LINAC & synchrotron
- Cooled beams in storage rings

General description of emittance by statistical moments of 2-dim distribution:

\[ \varepsilon_{\text{rms}} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2} \]

It describes the value for 1 standard derivation

For **Gaussian beams only**: \( \varepsilon_{\text{rms}} \) interpreted as area containing a fraction \( f \) of ions:

\[ \varepsilon(f) = -2\pi \varepsilon_{\text{rms}} \cdot \ln(1 - f) \]

**Care:**

No common definition of emittance concerning the fraction \( f \)

<table>
<thead>
<tr>
<th>Emittance ( \varepsilon(f) )</th>
<th>Fraction ( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 \cdot \varepsilon_{\text{rms}} )</td>
<td>15 %</td>
</tr>
<tr>
<td>( \pi \cdot \varepsilon_{\text{rms}} )</td>
<td>39 %</td>
</tr>
<tr>
<td>( 2\pi \cdot \varepsilon_{\text{rms}} )</td>
<td>63 %</td>
</tr>
<tr>
<td>( 4\pi \cdot \varepsilon_{\text{rms}} )</td>
<td>86 %</td>
</tr>
<tr>
<td>( 8\pi \cdot \varepsilon_{\text{rms}} )</td>
<td>98 %</td>
</tr>
</tbody>
</table>