LONGITUDINAL beam DYNAMICS in circular accelerators

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The goal of an accelerator is to provide a **stable particle beam**.

The particles nevertheless perform transverse betatron oscillations. We will see that they also perform (so-called **synchrotron** oscillations in the longitudinal plane and in energy.

We will look at the stability of these oscillations, their dynamics and derive some basic equations.

**More related lectures:**
- Linacs
- RF Systems
- Electron Beam Dynamics
- Non-Linear longitudinal Beam Dynamics

**Hands-on calculations** - longitudinal in the second week !!!
Motivation for circular accelerators

- Linear accelerators scale in size and cost(!) linearly with the energy.
- Circular accelerators can each turn reuse
  - the accelerating system
  - the vacuum chamber
  - the bending/focusing magnets
  - beam instrumentation, ...

-> economic solution to reach higher particle energies
But each accelerator has a limited energy range.
The accelerating system will depend upon the evolution of the particle velocity:

- **electrons** reach a constant velocity (~speed of light) at relatively low energy
- **heavy particles** reach a constant velocity only at very high energy
  
  - need different types of resonators, optimized for different velocities
  
  - the revolution frequency will vary, so the RF frequency will be changing
  
  - magnetic field needs to follow the momentum increase

**Particle rest mass \( m_0 \):**

- electron \( 0.511 \) MeV
- proton \( 938 \) MeV
- \( ^{239}\text{U} \) \( \sim 220000 \) MeV

**Total Energy:** \( E = \gamma m_0 c^2 \)

**Relativistic gamma factor:**

\[
\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - \beta^2}}
\]

**Momentum:**

\[
p = m v = \frac{E}{c^2} \beta c = \beta \frac{E}{c} = \beta \gamma m_0 c
\]
The revolution and RF frequency will be changing during acceleration. Much more important for lower energies (values are kinetic energy - protons).

**PS Booster:**
- 50 MeV ($\beta = 0.314$) -> 1.4 GeV ($\beta = 0.915$)  
  602 kHz -> 1746 kHz => **190% frequency increase**
- (post LS2): 160 MeV ($\beta = 0.520$) -> 2 GeV ($\beta = 0.948$) => **95% increase**

**PS:**
- 1.4 GeV ($\beta = 0.915$) -> 25.4 GeV ($\beta = 0.9994$)  
  437 KHz -> 477 kHz => **9% increase**
- (post LS2): 2 GeV ($\beta = 0.948$) -> 25.4 GeV ($\beta = 0.9994$) => **5% increase**

**SPS:**
- 25.4 GeV -> 450 GeV ($\beta = 0.999998$)  
  => **0.06% frequency increase**

**LHC:**
- 450 GeV -> 7 TeV ($\beta = 0.999999991$)  
  => only **2 $10^{-6}$ increase**

RF system needs more flexibility in lower energy accelerators.

**Question:** What about electrons and positrons?
To accelerate, we need a force in the direction of motion!

**Newton-Lorentz Force** on a charged particle: \( \vec{F} = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}) \)  
2\textsuperscript{nd} term always perpendicular to motion => no acceleration

Hence, it is necessary to have an electric field \( E \) (preferably) along the direction of the initial momentum (\( z \)), which changes the momentum \( p \) of the particle.

In relativistic dynamics, total energy \( E \) and momentum \( p \) are linked by
\[
E^2 = E_0^2 + p^2 c^2 \quad \Rightarrow \quad dE = v dp \quad \text{(} 2E dE = 2c^2 p dp \Leftrightarrow dE = c^2 mv / E \ dp = v dp \text{)}
\]

The rate of energy gain per unit length of acceleration (along \( z \)) is then:
\[
\frac{dE}{dz} = v \frac{dp}{dz} = \frac{dp}{dt} = q E_z
\]

and the kinetic energy gained from the field along the \( z \) path is:
\[
dW = dE = q E_z \ dz \quad \rightarrow \quad W = q \int E_z \ dz = q V \quad - \ V \text{ is a potential} \quad - \ q \text{ the charge}
\]

May the force be with you!
Unit of Energy

Today’s accelerators and future projects work/aim at the TeV energy range.

- LHC: 7 TeV -> 14 TeV
- CLIC: 380 GeV -> 3 TeV
- FCC: 100 TeV

In fact, this energy unit comes from acceleration:

1 eV (electron Volt) is the energy that 1 elementary charge $e$ (like one electron or proton) gains when it is accelerated in a potential (voltage) difference of 1 Volt.

Basic Unit: eV (electron Volt)
- keV = 1000 eV = $10^3$ eV
- MeV = $10^6$ eV
- GeV = $10^9$ eV
- TeV = $10^{12}$ eV

LHC = ~450 Million km of batteries!!!
3x distance Earth-Sun
**Methods of Acceleration in circular accelerators**

Electrostatic field limited by insulation, magnetic field doesn't accelerate at all.  

**Circular machine: DC acceleration impossible** since $\oint \mathbf{E} \cdot d\mathbf{s} = 0$

The electric field is derived from a scalar potential $\phi$ and a vector potential $A$

The time variation of the magnetic field $\mathbf{H}$ generates an electric field $\mathbf{E}$

The solution: $\Rightarrow$ time varying electric fields
- Induction
- RF frequency fields

\[ \oint \mathbf{E} \cdot d\mathbf{s} = -\int \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} \]
**Acceleration by Induction: The Betatron**

It is based on the principle of a transformer:
- **primary side**: large electromagnet
- **secondary side**: electron beam.

The ramping magnetic field is used to guide particles on a circular trajectory as well as for acceleration.

**Limited by saturation in iron (~300 MeV e-)**

**Used in industry and medicine, as they are compact accelerators for electrons**

---

Donald Kerst with the first betatron, invented at the University of Illinois in 1940
Circular accelerators

Cyclotron
Synchrotron
Circular accelerators: Cyclotron

Courtesy: EdukiteLearning, https://youtu.be/cNnNM2ZqIsc
Circular accelerators: Cyclotron

Used for protons, ions

\[ B = \text{constant} \]
\[ \omega_{RF} = \text{constant} \]

Synchronism condition

\[ \omega_s = \omega_{RF} \]
\[ 2\pi \rho = v_s T_{RF} \]

Cyclotron frequency

\[ \omega = \frac{qB}{m_0 \gamma} \]

1. \( \gamma \) increases with the energy
   \( \Rightarrow \) no exact synchronism
2. if \( v << c \) \( \Rightarrow \) \( \gamma \approx 1 \)

Animation: https://phyanim.sciences.univ-nantes.fr/Meca/Charges/cyclotron.php
Circular accelerators: Cyclotron

Courtesy Berkeley Lab,
https://www.youtube.com/watch?v=cutKuFxeXmQ
Synchrocyclotron: Same as cyclotron, except a modulation of $\omega_{RF}$
- $B = \text{constant}$
- $\gamma \omega_{RF} = \text{constant}$
- $\omega_{RF}$ decreases with time

The condition:

$$\omega_s(t) = \omega_{RF}(t) = \frac{q B}{m_0 \gamma(t)}$$

Allows to go beyond the non-relativistic energies

More in lectures by Mike Seidel
1. **Constant orbit** during acceleration

2. To keep particles on the closed orbit, \( B \) should increase with time

3. \( \omega \) and \( \omega_{RF} \) increase with energy

RF frequency can be multiple of revolution frequency

\[
\omega_{RF} = h\omega
\]

\[
T_s = h T_{RF}
\]

\[
\frac{2\pi R}{v_s} = h T_{RF}
\]

\( h \) integer, harmonic number: number of RF cycles per revolution

\( h \) is the maximum number of bunches in the synchrotron. Normally less bunches due to gaps for kickers, collision constraints,…
Circular accelerators: The Synchrotron

Examples of different proton and electron synchrotrons at CERN

+ LHC (of course!)
The magnetic field (dipole current) is increased during the acceleration.

Time from start of injection (s)

Dipole current (A)

Start of the ramp

Injection phase

Preparation and access

Beam dump

Energy ramp

Coast

Coast

450 GeV

7 TeV

L. Bottura
The synchrotron is a synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:

\[ eV \sin \phi \quad \Rightarrow \quad \text{Energy gain per turn} \]

\[ \Phi = \Phi_s \quad \Rightarrow \quad \text{Synchronous particle} \]

\[ \omega_{RF} = h\omega \quad \Rightarrow \quad \text{RF synchronism (} h \text{ - harmonic number)} \]

\[ \rho = \text{cte} \quad R = \text{cte} \quad \Rightarrow \quad \text{Constant orbit} \]

\[ B\rho = P/e \Rightarrow B \quad \Rightarrow \quad \text{Variable magnetic field} \]

If \( v \approx c \), \( \omega \) hence \( \omega_{RF} \) remain constant (ultra-relativistic \( e^- \))
1. For circular accelerators, the origin of time is taken at the zero crossing of the RF voltage with positive slope.

2. For linear accelerators, the origin of time is taken at the positive crest of the RF voltage.

Time t = 0 chosen such that:

\[ E_1(t) = E_0 \sin(\omega_{RF} t) \]
\[ E_2(t) = E_0 \cos(\omega_{RF} t) \]

3. I will stick to convention 1 in the following to avoid confusion.
The Synchrotron – Energy ramping

Energy ramping by increasing the B field (frequency has to follow v):

\[ p = eB \rho \quad \Rightarrow \quad \frac{dp}{\rho \text{ const.}} = e \rho \dot{B} \quad \Rightarrow \quad (\Delta p)_{\text{turn}} = e \rho \dot{B} T_r = \frac{2\pi e \rho R \dot{B}}{v} \]

With

\[ E^2 = E_0^2 + p^2 c^2 \quad \Rightarrow \quad \Delta E = v \Delta p \]

\[ (\Delta E)_{\text{turn}} = (\Delta W)_s = 2\pi e \rho R \dot{B} = e \hat{V} \sin \phi_s \]

Synchronous phase \( \phi_s \) changes during energy ramping

\[ \sin \phi_s = 2\pi \rho R \frac{\dot{B}}{\hat{V}_{RF}} \quad \Rightarrow \quad \phi_s = \arcsin\left(2\pi \rho R \frac{\dot{B}}{\hat{V}_{RF}}\right) \]

• The synchronous phase depends on
• the change of the magnetic field
• and the RF voltage
The Synchrotron - Frequency change

During the energy ramping, the RF frequency increases to follow the increase of the revolution frequency:

\[ \frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi} \frac{e c^2}{E_s(t) R_s} \rho B(t) \]

Hence:

\[ \omega = \frac{\omega_{RF}}{h} = \omega(B, R_s) \]

( using \( p(t) = eB(t)\rho, \ E = mc^2 \) )

Since \( E^2 = (m_0 c^2)^2 + p^2 c^2 \) the RF frequency must follow the variation of the B field with the law

\[ \frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s} \left\{ \frac{B(t)^2}{(m_0 c^2 / ec \rho)^2 + B(t)^2} \right\}^{1/2} \]

RF frequency program during acceleration determined by B-field!
During the energy ramping, the B-field and the revolution frequency increase.

Example: PS - Field / Frequency change

- **B-field**
- **kinetic energy**
- **'B-dot'**
- **B-field change**
- **revolution frequency**

![Graphs showing B-field, B-dot, kinetic energy, and revolution frequency changes over time.](image)
Overtaking in a Formula 1 Race
Overtaking in a Formula 1 Race
Overtaking in a Formula 1 Race

A F1 car wants to overtake another car! It will have a
- a different track length due to a 'dispersion orbit'
- and a different velocity.

\[
T = \frac{L}{v} = \frac{2\pi R}{v} \quad \text{and} \quad f_r = \frac{1}{T} = \frac{v}{2\pi R}
\]

\[
\Rightarrow \frac{\Delta T}{T} = \frac{\Delta R}{R} - \frac{\Delta v}{v}
\]

The winner depends on the relative change in speed compared to the relative change in track length!

If the relative change in speed is larger than the relative change in track length \( \Rightarrow \) the red car will win!
Overtaking in a Synchrotron

A particle slightly shifted in momentum will have a
- dispersion orbit and a different orbit length
- a different velocity.

As a result of both effects the revolution period $T$ changes with a "slip factor" $\eta$:

$$\eta = \frac{dT/T}{dp/p}$$

Note: you also find $\eta$ defined with a minus sign!

The "momentum compaction factor" is defined as relative orbit length change with momentum:

$$\alpha_c = \frac{dL/L}{dp/p}$$

$p$=particle momentum  
$R$=synchrotron physical radius  
$T$=revolution period
Momentum Compaction Factor

\[ \alpha_c = \frac{p \, dL}{L \, dp} \quad ds_0 = \rho d\theta \]

\[ ds = (\rho + x) \, d\theta \]

The elementary path difference from the two orbits is:

\[ \frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{\rho} = \frac{D_x}{\rho} \frac{dp}{p} \]

leading to the total change in the circumference:

\[ dL = \int_C dl = \int_C \frac{x}{\rho} ds_0 = \int_C \frac{D_x}{\rho} \frac{dp}{p} ds_0 \]

\[ \alpha_c = \frac{1}{L} \int_C \frac{D_x(s)}{\rho(s)} \, ds_0 \]

\[ \alpha_c = \frac{\langle D_x \rangle_m}{R} \]

With \( \rho = \infty \) in straight sections we get:

\(< >_m \) means that the average is considered over the bending magnet only.

Property of the transverse beam optics!
Dispersion Effects - Revolution Period

The two effects of the orbit length and the particle velocity change the revolution period as:

\[ T = \frac{L}{\beta c} \quad \Rightarrow \quad \frac{dT}{T} = \frac{dL}{L} - \frac{d\beta}{\beta} = \alpha_c \frac{dp}{p} - \frac{d\beta}{\beta} \]

Definition of momentum compaction factor:

\[ \frac{dT}{T} = \left( \alpha_c - \frac{1}{\gamma^2} \right) \frac{dp}{p} \]

\[ p = mv = \beta \gamma \frac{E_0}{c} \quad \Rightarrow \quad \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1 - \beta^2\right)^{-\frac{1}{2}}}{\gamma^2} = \left(1 - \beta^2\right)^{-1} \frac{d\beta}{\beta} \]

Slip factor:

\[ \eta = \alpha_c - \frac{1}{\gamma^2} \quad \text{or} \quad \eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} \quad \text{with} \quad \gamma_t = \frac{1}{\sqrt{\alpha_c}} \]

At transition energy, \( \eta = 0 \), the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.
Let's consider a succession of accelerating gaps, operating in the $2\pi$ mode, for which the synchronism condition is fulfilled for a phase $\Phi_s$.

$$eV_s = e\hat{\nu} \sin \Phi_s$$

is the energy gain in one gap for the particle to reach the next gap with the same RF phase: $P_1, P_2, \ldots$ are fixed points.

**RECAP: Principle of Phase Stability (Linac)**

If an energy increase is transferred into a velocity increase $\Rightarrow$

- $M_1$ & $N_1$ will move towards $P_1 \Rightarrow$ stable
- $M_2$ & $N_2$ will go away from $P_2 \Rightarrow$ unstable

(Highly relativistic particles have no significant velocity change)

For a $2\pi$ mode, the electric field is the same in all gaps at any given time.
Phase Stability in a Synchrotron

From the definition of $\eta$ it is clear that an increase in momentum gives
- below transition ($\eta < 0$) a higher revolution frequency
  (increase in velocity dominates) while
- above transition ($\eta > 0$) a lower revolution frequency ($v \approx c$ and longer path) where the momentum compaction (generally $> 0$) dominates.

![Diagram showing phase stability in a synchrotron with equation $\eta = \alpha_c - \frac{1}{\gamma^2}$]
At transition, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Crossing transition during acceleration makes the previous stable synchronous phase unstable. The RF system needs to make a rapid change of the RF phase, a ‘phase jump’.

\[ \alpha_c \sim \frac{1}{Q_x^2} \quad \gamma_t = \frac{1}{\sqrt{\alpha_c}} \sim Q_x \]

In the PS: \( \gamma_t \) is at \( \sim 6 \) GeV
In the SPS: \( \gamma_t = 22.8 \), injection at \( \gamma = 27.7 \)
=> no transition crossing!
In the LHC: \( \gamma_t \) is at \( \sim 55 \) GeV, also far below injection energy

Transition crossing not needed in leptons machines, why?
Dynamics: Synchrotron oscillations

Simple case (no accel.): $B = \text{const.}$, below transition $\gamma < \gamma_t$

The phase of the synchronous particle must therefore be $\phi_0 = 0$.

- $\Phi_1$ - The particle $B$ is accelerated
  - Below transition, an energy increase means an increase in revolution frequency
  - The particle arrives earlier - tends toward $\phi_0$

- $\phi_2$ - The particle is decelerated
  - decrease in energy - decrease in revolution frequency
  - The particle arrives later - tends toward $\phi_0$
Particle B is performing **Synchrotron Oscillations** around synchronous particle A.

The amplitude depends on the initial phase and energy.

The **oscillation frequency** is much **slower than** in the transverse plane. It takes a large number of revolutions for one complete oscillation. Restoring electric force smaller than magnetic force.
The Potential Well

Cavity voltage

Potential well

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The energy - phase oscillations can be drawn in phase space:

The particle trajectory in the phase space ($\Delta p/p, \phi$) describes its longitudinal motion.

Emittance: phase space area including all the particles

NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)
Particle B oscillates around particle A

This is a synchrotron oscillation

Plotting this motion in longitudinal phase space gives:

Longitudinal Phase Space Motion
*Synchrotron oscillations - No acceleration*

Phase space picture

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Synchrotron motion in phase space

The restoring force is non-linear.
⇒ speed of motion depends on position in phase-space

Remark:
Synchrotron frequency much smaller than betatron frequency.
**Synchrotron motion in phase space**

\[ \Delta E - \phi \text{ phase space of a stationary bucket} \]

(when there is no acceleration)

Bucket area: area enclosed by the separatrix

The area covered by particles is the longitudinal emittance

Dynamics of a particle
Non-linear, conservative oscillator $\to$ e.g. pendulum

Particle inside the separatrix:

Particle at the unstable fix-point

Particle outside the separatrix:
The bunches of the beam fill usually a part of the bucket area.

**Bucket area** = **longitudinal Acceptance** [eVs]

**Bunch area** = **longitudinal beam emittance** (rms) = \( \pi \sigma_E \sigma_t \) [eVs]

**Attention:** Different definitions are used!

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Synchrotron oscillations (with acceleration)

Case with acceleration $B$ increasing

$\gamma < \gamma_t$

$V_{RF}$

$\phi = \omega_{RF} t$

$\phi_s < \phi < \pi - \phi_s$

Phase space picture

The symmetry of the case $B = \text{const.}$ is lost

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The areas of stable motion (closed trajectories) are called “BUCKET”. The number of circulating buckets is equal to “h”.

The phase extension of the bucket is maximum for \( \phi_s = 180^\circ \) (or \( 0^\circ \)) which means no acceleration.

During acceleration, the buckets get smaller, both in length and energy acceptance.

=> Injection preferably without acceleration.
Longitudinal Motion with Synchrotron Radiation

Synchrotron radiation energy-loss energy dependant:

\[ U_0 = \frac{4}{3} \pi \frac{r_{ep}}{(m_0 c^2)^3} \frac{E^4}{\rho} \]

During one period of synchrotron oscillation:

- when the particle is in the upper half-plane, it loses more energy per turn, its energy gradually reduces

- when the particle is in the lower half-plane, it loses less energy per turn, but receives \( U_0 \) on the average, so its energy deviation gradually reduces

The phase space trajectory spirals towards the origin (limited by quantum excitations)

\[ \Delta E = \alpha \frac{\sigma_{\epsilon}}{\Omega_s (E)} \]

\( \approx \) The synchrotron motion is damped toward an equilibrium bunch length and energy spread.

More details in the lectures on *Electron Beam Dynamics*
Longitudinal Dynamics in Synchrotrons

Now we will look more quantitatively at the “synchrotron motion”.

The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle.

Since there is a well defined synchronous particle which has always the same phase $\phi_s$, and the nominal energy $E_0$, it is sufficient to follow other particles with respect to that particle.

So let’s introduce the following reduced variables:

- particle RF phase : $\Delta \phi = \phi - \phi_s$
- particle momentum : $\Delta p = p - p_0$
- particle energy : $\Delta E = E - E_0$
- angular frequency : $\Delta \omega = \omega - \omega_0$
- azimuth orbital angle: $\Delta \theta = \theta - \theta_s$

Look at difference from synchronous particle
First Energy-Phase Equation

\[ f_{RF} = h f_r \quad \Rightarrow \quad \Delta \phi = -h \Delta \theta \quad \text{with} \quad \theta = \int \omega \ dt \]

Particle ahead arrives earlier
\[ \Rightarrow \text{smaller RF phase} \]

For a given particle with respect to the reference one:

\[ \Delta \omega = \frac{d}{dt} (\Delta \theta) = -\frac{1}{h} \frac{d}{dt} (\Delta \phi) = -\frac{1}{h} \frac{d\phi}{dt} \]

Since:

\[ \eta = -\frac{p_0}{\omega_0} \left( \frac{d\omega}{dp} \right)_s \]

And

\[ E^2 = E_0^2 + p^2 c^2 \]

\[ \Delta E = v_s \Delta p = \omega_0 R \Delta p \]

One gets:

\[ \frac{\Delta E}{\omega_0} = \frac{p_0 R}{h \eta \omega_0} \frac{d(\Delta \phi)}{dt} = \frac{p_0 R}{h \eta \omega_0} \dot{\phi} \]
The rate of energy gained by a particle is:
\[
\frac{dE}{dt} = e \hat{V} \sin \phi \frac{\omega_r}{2\pi}
\]

The rate of relative energy gain with respect to the reference particle is then:
\[
2\pi \Delta \left( \frac{\dot{E}}{\omega_0} \right) = e \hat{V} (\sin \phi - \sin \phi_s)
\]

Expanding the left-hand side to first order:
\[
\Delta \left( \dot{E} T_r \right) \approx \dot{E} \Delta T_r + T_{rs} \Delta \dot{E} = \Delta E \dot{T}_r + T_{rs} \Delta \dot{E} = \frac{d}{dt} \left( T_{rs} \Delta E \right)
\]

leads to the second energy-phase equation:
\[
2\pi \frac{d}{dt} \left( \frac{\Delta E}{\omega_0} \right) = e \hat{V} (\sin \phi - \sin \phi_s)
\]
Equations of Longitudinal Motion

\[ \Delta E \frac{\dot{\phi}}{\omega_0} = \frac{p_0 R}{h\eta \omega_0} \dot{\phi} \]

\[ 2\pi \frac{d}{dt} \left( \frac{\Delta E}{\omega_0} \right) = e \hat{V} (\sin \phi - \sin \phi_s) \]

deriving and combining

\[ \frac{d}{dt} \left[ -\frac{p_0 R}{h\eta \omega_0} d\phi \right] + \frac{e \hat{V}}{2\pi} (\sin \phi - \sin \phi_s) = 0 \]

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time.

We will study some cases in the following...
Small Amplitude Oscillations

Let's assume constant parameters R, p₀, ω₀ and η:

\[
\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s} (\sin\phi - \sin\phi_s) = 0
\]

with

\[
\Omega_s^2 = \frac{-q\dot{V}_{RF}\eta h\omega_0}{2\pi Rp_0} \cos\phi_s
\]

Consider now small phase deviations from the reference particle:

\[
\sin\phi - \sin\phi_s = \sin(\phi_s + \Delta\phi) - \sin\phi_s \approx \cos\phi_s \Delta\phi
\]

(for small \(\Delta\phi\))

and the corresponding linearized motion reduces to a harmonic oscillation:

\[
\ddot{\phi} + \Omega_s^2 \Delta\phi = 0
\]

where \(\Omega_s\) is the synchrotron angular frequency.

The synchrotron tune \(\nu_s\) is the number of synchrotron oscillations per revolution:

\[
\nu_s = \Omega_s / \omega_0
\]

Typical values are \(\ll 1\), as it takes several 10 - 1000 turns per oscillation.

- proton synchrotrons of the order \(10^{-3}\)
- electron storage rings of the order \(10^{-1}\)
Stability is obtained when \( \Omega_s \) is real and so \( \Omega_s^2 \) positive:

\[
\Omega_s^2 = \frac{-q \hat{V}_{RF} \eta h \omega_0}{2 \pi R p_0} \cos \phi_s
\]

\( \Omega_s^2 = \omega_0^2 \frac{-q \hat{V}_{RF} \eta h}{2 \pi \beta^2 E} \cos \phi_s \)

with

\[
R p = \frac{\beta^2 E}{\omega}
\]

**Stability** is obtained when \( \Omega_s \) is real and so \( \Omega_s^2 \) positive:

\[
\Omega_s^2 > 0
\]

\[\Downarrow\]

\[\eta \cos \phi_s < 0\]

Stable in the region if:

\[\gamma < \gamma_t\]

\[\eta < 0\]

acceleration

\[\gamma > \gamma_t\]

\[\eta > 0\]

\[\gamma > \gamma_t\]

\[\eta > 0\]

\[\gamma < \gamma_t\]

\[\eta < 0\]

deceleration
Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

\[
\ddot{\phi} + \frac{\Omega_s^2}{\cos \phi_s} (\sin \phi - \sin \phi_s) = 0 \quad (\Omega_s \text{ as previously defined})
\]

Multiplying by \( \dot{\phi} \) and integrating gives an invariant of the motion:

\[
\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = I
\]

which for small amplitudes reduces to:

\[
\frac{\dot{\phi}^2}{2} + \Omega_s^2 \frac{(\Delta \phi)^2}{2} = I' \quad \text{(the variable is } \Delta \phi, \text{ and } \phi_s \text{ is constant)}
\]

Similar equations exist for the second variable: \( \Delta E \propto \dot{\phi}/\dot{t} \)
When $\phi$ reaches $\pi-\phi_s$ the force goes to zero and beyond it becomes non restoring. Hence $\pi-\phi_s$ is an extreme amplitude for a stable motion which in the phase space $(\frac{\dot{\phi}}{\Omega_s}, \Delta \phi)$ is shown as closed trajectories.

Equation of the separatrix:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = - \frac{\Omega_s^2}{\cos \phi_s} (\cos(\pi - \phi_s) + (\pi - \phi_s) \sin \phi_s)$$

Second value $\phi_m$ where the separatrix crosses the horizontal axis:

$$\cos \phi_m + \phi_m \sin \phi_s = \cos(\pi - \phi_s) + (\pi - \phi_s) \sin \phi_s$$
Energy Acceptance

From the equation of motion it is seen that $\dot{\phi}$ reaches an extreme at $\phi = \phi_s$. Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}_{\text{max}}^2 = 2\Omega_s^2 \left\{ 2 + (2\phi_s - \pi)\tan\phi_s \right\}$$

That translates into an energy acceptance:

$$\left( \frac{\Delta E}{E_0} \right)_{\text{max}} = \pm \beta \sqrt{-q\hat{V}} \frac{G(\phi_s)}{\pi h\eta E_0}$$

$$G(\phi_s) = \left[ 2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s \right]$$

This “RF acceptance” depends strongly on $\phi_s$ and plays an important role for the capture at injection, and the stored beam lifetime. It’s largest for $\phi_s=0$ and $\phi_s=\pi$ (no acceleration, depending on $\eta$). It becomes smaller during acceleration, when $\phi_s$ is changing. Need a higher RF voltage for higher acceptance. For the same RF voltage it is smaller for higher harmonics $h$. 
RF Acceptance versus Synchronous Phase

The areas of stable motion (closed trajectories) are called “BUCKET”. The number of circulating buckets is equal to “h”.

The phase extension of the bucket is maximum for \( \phi_s = 180^\circ \) (or 0°) which means no acceleration.

During acceleration, the buckets get smaller, both in length and energy acceptance.

=> Injection preferably without acceleration.
Stationary Bucket - Separatrix

This is the case \( \sin \phi_s = 0 \) (no acceleration) which means \( \phi_s = 0 \) or \( \pi \). The equation of the separatrix for \( \phi_s = \pi \) (above transition) becomes:

\[
\frac{\dot{\phi}^2}{2} + \Omega_s^2 \cos \phi = \Omega_s^2
\]

Replacing the phase derivative by the (canonical) variable \( W \):

\[
W = \frac{\Delta E}{\omega_0} = \frac{p_0 R}{h \eta \omega_0} \dot{\phi}
\]

and introducing the expression for \( \Omega_s \) leads to the following equation for the separatrix:

\[
W = \pm \frac{R}{c} \sqrt{\frac{2q\hat{V}E_0}{\pi h \eta}} \sin \frac{\phi}{2} = \pm W_{bk} \sin \frac{\phi}{2}
\]

with \( C = 2\pi R \)
Bucket height - bucket area

Setting $\phi=\pi$ in the previous equation gives the height of the stationary bucket:

$$W_{bk} = \frac{R}{c} \sqrt{\frac{2q\hat{V}E_0}{\pi h|\eta|}}$$

The bucket area is:

$$A_{bk} = 2 \int_0^{2\pi} W d\phi$$

Since:

$$\int_0^{2\pi} \sin\frac{\phi}{2} d\phi = 4$$

one gets:

$$A_{bk} = 8 W_{bk} = \frac{8R}{c} \sqrt{\frac{2q\hat{V}E_0}{\pi h|\eta|}}$$

For an accelerating bucket, this area gets reduced by a factor depending on $\Phi_s$:

$$\alpha(\Phi_s) \approx \frac{1 - \sin \Phi_s}{1 + \sin \Phi_s}$$
Potential Energy Function

The longitudinal motion is produced by a force that can be derived from a scalar potential:

\[
\frac{d^2\phi}{dt^2} = F(\phi) \quad F(\phi) = -\frac{\partial U}{\partial \phi}
\]

\[
U = -\int_0^\phi F(\phi)d\phi = -\frac{\Omega_s^2}{\cos\phi_s}(\cos\phi + \phi\sin\phi_s) - F_0
\]

The sum of the potential energy and kinetic energy is constant and by analogy represents the total energy of a non-dissipative system.
Introducing a new convenient variable, \( W \), leads to the 1\textsuperscript{st} order equations:

\[
W = \frac{\Delta E}{\omega_0}
\]

\[
\begin{align*}
\frac{d\phi}{dt} &= \frac{\hbar \eta \omega_0}{p_0 R} W \\
\frac{dW}{dt} &= \frac{e\hat{V}}{2\pi} \left( \sin \phi - \sin \phi_s \right)
\end{align*}
\]

The two variables \( \phi, W \) are canonical since these equations of motion can be derived from a Hamiltonian \( H(\phi, W, t) \):

\[
H(\phi, W) = \frac{1}{2} \frac{\hbar \eta \omega_0}{p_0 R} W^2 + \frac{e\hat{V}}{2\pi} \left[ \cos \phi - \cos \phi_s + (\phi - \phi_s) \sin \phi_s \right]
\]
Hamiltonian of Longitudinal Motion

What does it represent? The total energy of the system!

Surface of $H(\phi,W)$

Contours of constant $H$ are particle trajectories in phase space! ($H$ is conserved)

Hamiltonian Mechanics can help us understand some fairly complicated dynamics (multiple harmonics, bunch splitting, ...)

Introductory CAS, Santa Susanna, Sept/Oct 2023
Injection: Bunch-to-bucket transfer

- Bunch from sending accelerator into the bucket of receiving

Advantages:

→ Particles always subject to longitudinal focusing
→ No need for RF capture of de-bunched beam in receiving accelerator
→ No particles at unstable fixed point
→ Time structure of beam preserved during transfer
Effect of a Mismatch

Injected bunch: short length and large energy spread after 1/4 synchrotron period: longer bunch with a smaller energy spread.

For larger amplitudes, the angular phase space motion is slower (1/8 period shown below) => can lead to filamentation and emittance growth.
Effect of a Mismatch (2)

- Long. emittance is only preserved for correct RF voltage

Matched case

\[ \Delta \phi \text{ [rad]} \]

\[ \Delta E \text{ [MeV]} \]

- Bunch is fine, longitudinal emittance remains constant

Longitudinal mismatch

\[ \Delta \phi \text{ [rad]} \]

\[ \Delta E \text{ [MeV]} \]

- Dilution of bunch results in increase of long. emittance
Effect of a Mismatch (3)

Evolution of an injected beam for the first 100 turns.
For a mismatched transfer, the emittance increases (right).
Longitudinal matching - Beam profile

**Matched case**

\[ \Delta \phi = 0, \frac{V_{\text{inj}}}{V_{\text{RF}}} = 1 \]

- Bunch is fine, longitudinal emittance remains constant

**Longitudinal mismatch**

\[ \Delta \phi = 0, \frac{V_{\text{inj}}}{V_{\text{RF}}} = 2 \]

- Dilution of bunch results in increase of long. emittance
Matching quiz!

- Find the difference!

→ -45° phase error at injection
→ Can be easily corrected by bucket phase

→ Equivalent energy error
→ Phase does not help: requires beam energy change
We can reconstruct the phase space distribution of the beam.

- Longitudinal bunch profiles over a number of turns
- Parameters determining $\Omega_s$
Phase space motion can be used to make short bunches.
Start with a long bunch and extract or recapture when it’s short.
Capture of a Debunched Beam with Fast Turn-On
Capture of a Debunched Beam with Adiabatic Turn-On
Generating a 25ns LHC Bunch Train in the PS

- **Longitudinal bunch splitting (basic principle)**
  - Reduce voltage on principal RF harmonic and simultaneously rise voltage on multiple harmonics (adiabatically with correct phase, etc.)

- PS ejection: 72 bunches in 1 turn
  - 320 ns beam gap
  - Quadruple splitting at 25 GeV
    - 72 bunches on h=84
    - 40 MHz fixed RF = 40.0 MHz
      - 1.1 x 10^{11} ppb
      - 20 MHz fixed RF = 20.0 MHz
      - 2.2 x 10^{11} ppb

- Acceleration to 25 GeV
  - 18 bunches on h=21
  - 10 MHz system RF = 9.18 MHz
    - 4.4 x 10^{11} ppb

- PS injection: 4+2 bunches in 2 batches
  - 6 bunches on h=7
  - 10 MHz system RF = 3.06 MHz
    - 13.2 x 10^{11} ppb

R. Garoby
1. Inject four bunches \( \sim 180 \) ns, 1.3 eVs

Wait 1.2 s for second injection

2. Inject two bunches

3. Triple split after second injection \( \sim 0.7 \) eVs

4. Accelerate from 1.4 GeV (\( E_{\text{kin}} \)) to 26 GeV
5. During acceleration: longitudinal emittance blow-up: 0.7 – 1.3 eVs

6. Double split ($h_{21} \rightarrow h_{42}$)

7. Double split ($h_{42} \rightarrow h_{84}$)  ~ 0.35 eVs, 4 ns

10. Fine synchronization, bunch rotation → Extraction!
The LHC25 (ns) cycle in the PS

Triple splitting after 2\textsuperscript{nd} injection

Inject 4+2 bunches

$h = 7$

Split in four at flat-top energy

$Eject 72$ bunches

$\gamma_{tr}$

Each bunch from the Booster divided by $12 \rightarrow 6 \times 3 \times 2 \times 2 = 72$
Triple splitting in the PS
Two times double splitting in the PS

Two times double splitting and bunch rotation:

- Bunch is divided twice using RF systems at $h = 21/42$ (10/20 MHz) and $h = 42/84$ (20/40 MHz)

- Bunch rotation: first part $h=84$ only + $h=168$ (80 MHz) for final part
Summary

- **Cyclotrons/Synchrocyclotrons** for low energy
- **Synchrotrons** for high energies, constant orbit, synchronously rising field and frequency
- Particles with higher energy have a longer orbit (normally) but a higher velocity
  - at low energies (below transition) velocity increase dominates
  - at high energies (above transition) velocity almost constant
- Particles perform **oscillations around synchronous phase**
  - synchronous phase depending on acceleration
  - below or above transition
- **Hamiltonian** approach can deal with fairly complicated dynamics
- **Bucket** is the stable region in phase space inside the **separatrix**
- **Matching** the shape of the bunch to the bucket is essential
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End of our crash course in longitudinal dynamics

www.formula1.com
Appendix: Relativity + Energy Gain

Newton-Lorentz Force
\[ \vec{F} = \frac{d\vec{p}}{dt} = e \left( \vec{E} + \vec{v} \times \vec{B} \right) \]

2nd term always perpendicular to motion => no acceleration

Relativistics Dynamics
\[ \beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \]
\[ \gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - \beta^2}} \]
\[ p = mv = \frac{E}{c^2} \beta c = \beta \frac{E}{c} = \beta \gamma m_0 c \]
\[ E^2 = E_0^2 + p^2 c^2 \quad \rightarrow \quad dE = v dp \]
\[ \frac{dE}{dz} = v \frac{dp}{dz} = \frac{dp}{dt} = eE_z \]
\[ dE = dW = eE_z dz \quad \rightarrow \quad W = e \int E_z dz \]

RF Acceleration
\[ E_z = \hat{E}_z \sin \omega_{RF} t = \hat{E}_z \sin \phi(t) \]
\[ \int \hat{E}_z \, dz = \hat{V} \]
\[ W = e\hat{V} \sin \phi \]
(neglecting transit time factor)

The field will change during the passage of the particle through the cavity
=> effective energy gain is lower
Phase Space Trajectories inside Stationary Bucket

A particle trajectory inside the separatrix is described by the equation:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = I$$

The points where the trajectory crosses the axis are symmetric with respect to $\phi_s = \pi$

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2 \cos \phi = I$$

$$\dot{\phi} = \pm \Omega_s \sqrt{2 (\cos \phi_m - \cos \phi)}$$

$$W = \pm W_{bk} \sqrt{\cos^2 \frac{\phi_m}{2} - \cos^2 \frac{\phi}{2}}$$

$$\cos(\phi) = 2 \cos^2 \frac{\phi}{2} - 1$$
The Synchrotron - Energy ramping

Energy ramping by increasing the B field (frequency has to follow v).
Stable phase $\varphi_s$ changes during energy ramping

$$\sin \phi_s = 2\pi \rho R \frac{\dot{B}}{V_{RF}}$$

$$\phi_s = \arcsin \left( 2\pi \rho R \frac{\dot{B}}{V_{RF}} \right)$$

Energy Gain

$eV_s$

$\eta = \alpha_c - \frac{1}{\gamma^2}$

$\eta > 0$

stable synchr. particle for above transition

$\eta < 0$

early $\rightarrow$ late

$\phi = \omega_{RF} t$
Case with acceleration $B$ increasing

$\gamma < \gamma_t$

$V_{RF}$

$\phi = \omega_{RF} t$

$\phi_s < \phi < \pi - \phi_s$

Phase space picture

$\frac{\Delta p}{p}$

Particles will accumulate energy difference each turn

stable region

unstable region

separatrix