# Synchrotron Light:

# Electron Beam Dynamics


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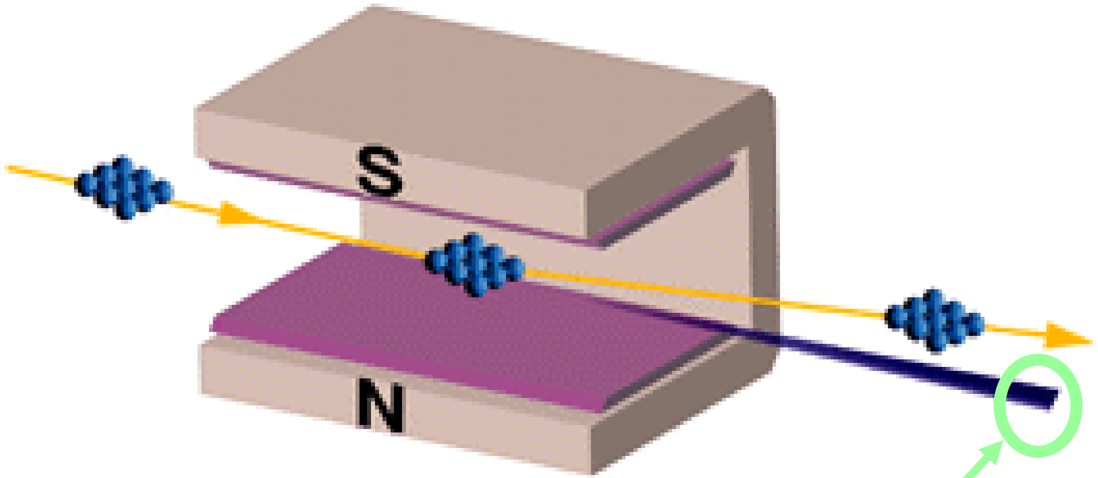
# Curved orbit of electrons in magnet field

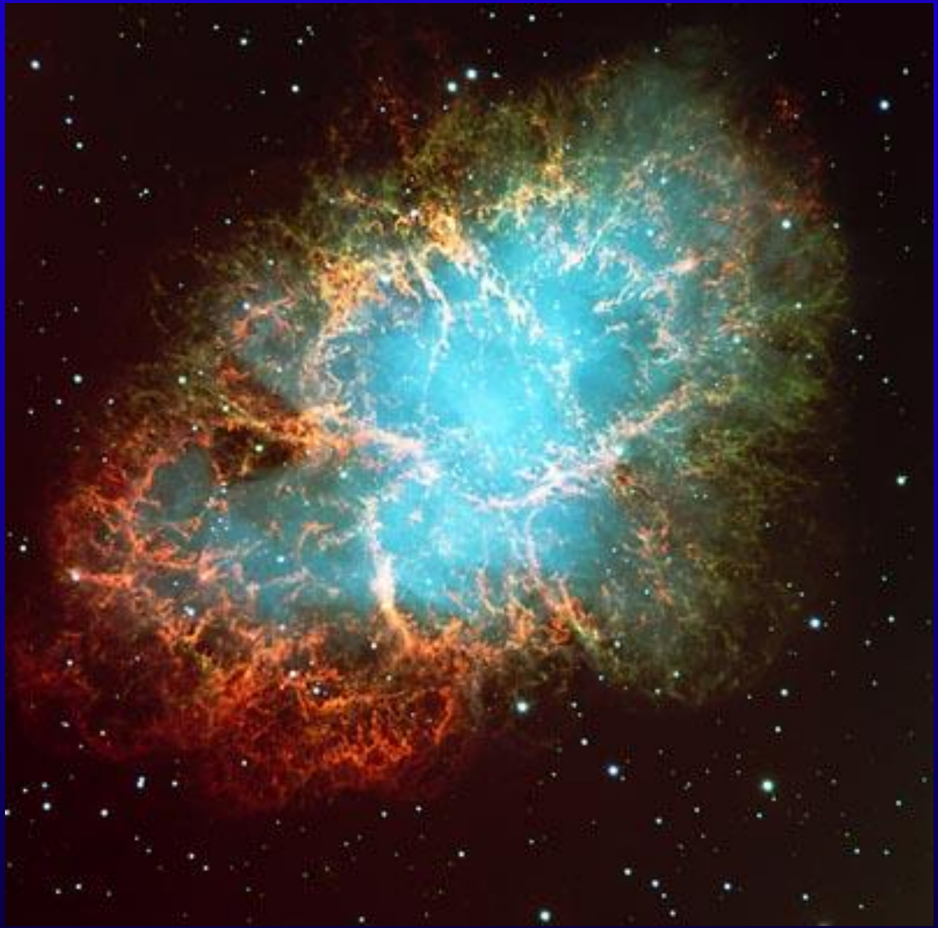



Accelerated charge → Electromagnetic radiation

# Electromagnetic waves or photons

## Crab Nebula
## 6000 light years away

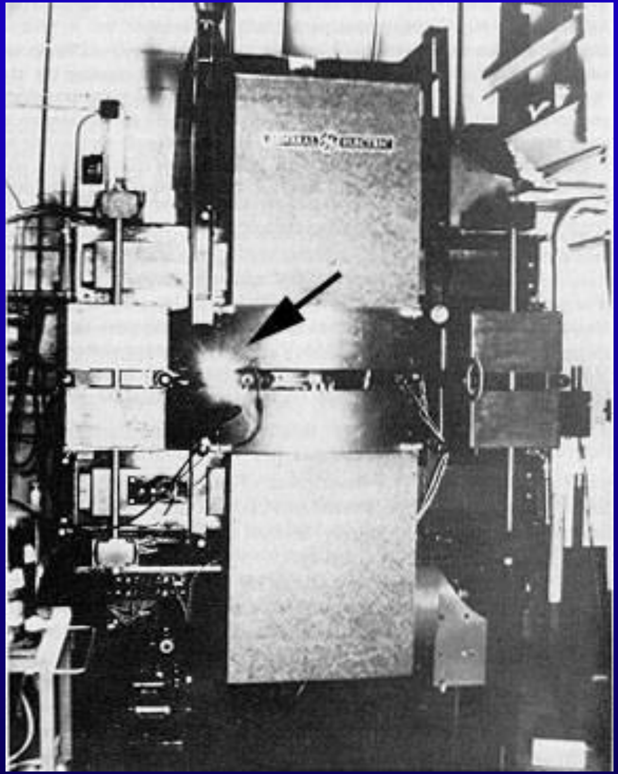
C R E A T I O N



**First light observed**
**1054 AD**

## GE Synchrotron
## New York State

G E N E R A T I O N



**First light observed**
**24 April, 1947**

# Synchrotron radiation: some dates

- 1873 Maxwell's equations
- 1887 Hertz: electromagnetic waves
- 1898 Liénard: retarded potentials
- 1900 Wiechert: retarded potentials
- 1908 Schott: Adams Prize Essay

... waiting for accelerators …
1940: 2.3 MeV betatron,Kerst, Serber

# Maxwell equations (poetry)

*War es ein Gott, der diese Zeichen schrieb*
*Die mit geheimnisvoll verborg'nem Trieb*
*Die Kräfte der Natur um mich enthüllen*
*Und mir das Herz mit stiller Freude füllen.*
Ludwig Boltzman

*Was it a God whose inspiration*
*Led him to write these fine equations*
*Nature's fields to me he shows*
*And so my heart with pleasure glows.*
translated by John P. Blewett

# Synchrotron radiation: some dates

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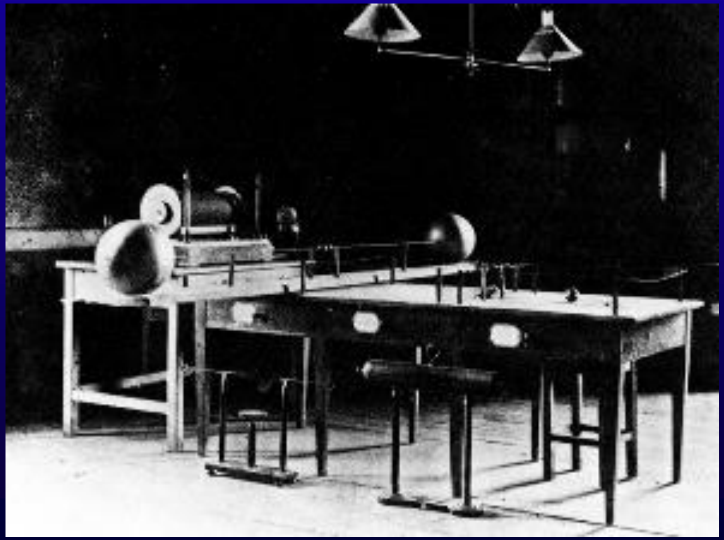
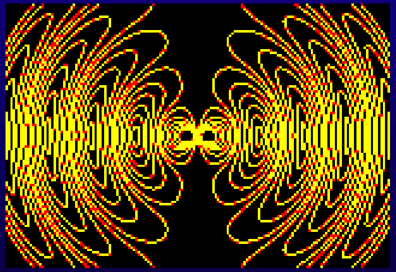
# THEORETICAL UNDERSTANDING →

**1873** **Maxwell’s equations**

**→ made evident that changing charge densities would result in electric fields that would radiate outward**

**1887** **Heinrich Hertz demonstrated such waves:**





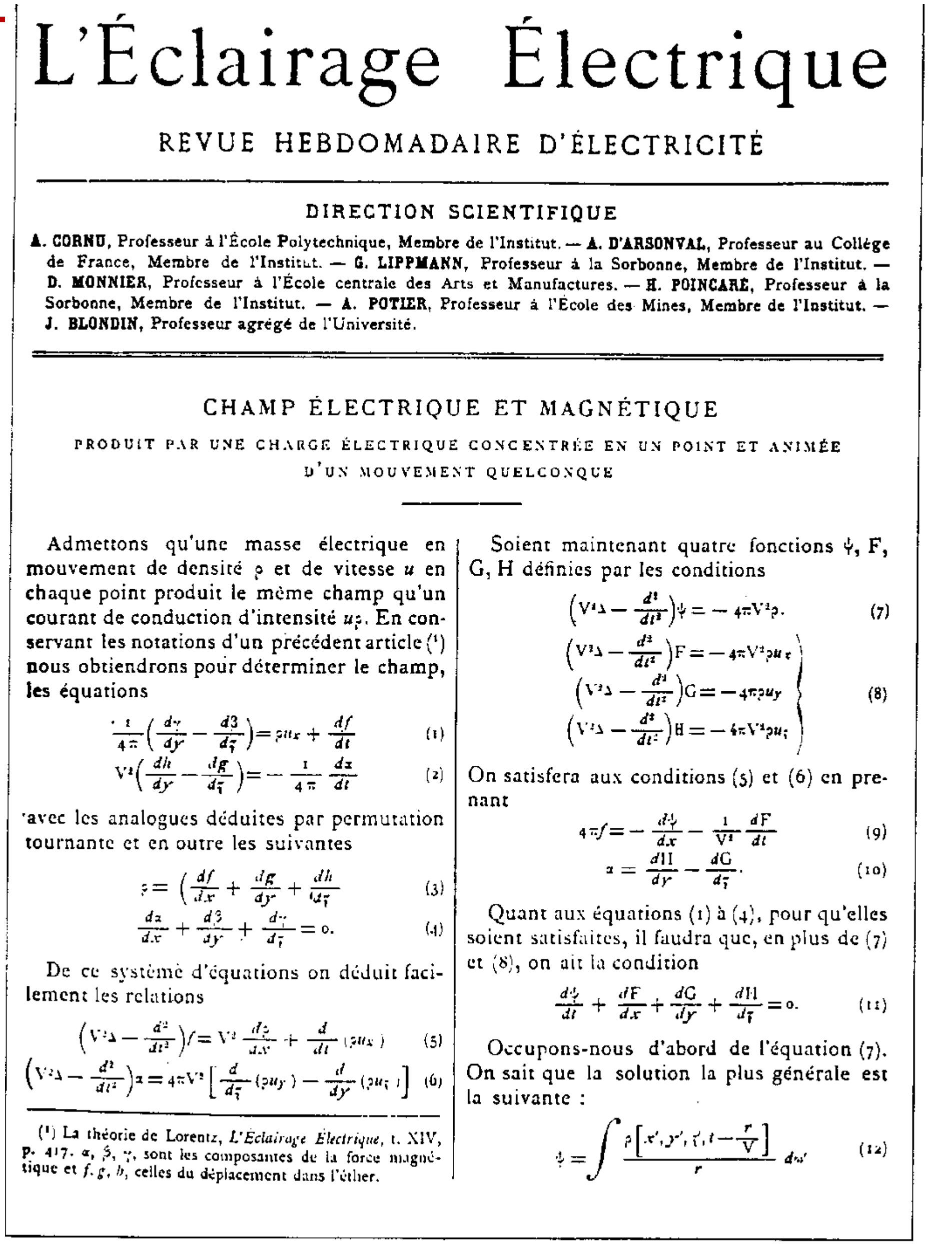
*It's of no use whatsoever[...] this is just an experiment that proves Maestro Maxwell was right—we just have these mysterious electromagnetic waves that we cannot see with the naked eye. But they are there.*

# Synchrotron radiation: some dates

- 1873 Maxwell's equations
- 1887 Hertz: electromagnetic waves
- **1898 Liénard: retarded potentials**
- **1900 Wiechert: retarded potentials**
- 1908 Schott: Adams Prize Essay (330 pages)

... waiting for accelerators …

1940: 2.3 MeV betatron,Kerst, Serber



# L'Éclairage Électrique

REVUE HEBDOMADAIRE D'ÉLECTRICITÉ

DIRECTION SCIENTIFIQUE

**A. CORNU**, Professeur à l'École Polytechnique, Membre de l'Institut. — **A. D'ARSONVAL**, Professeur au Collège de France, Membre de l'Institut. — **G. LIPPMANN**, Professeur à la Sorbonne, Membre de l'Institut. — **D. MONNIER**, Professeur à l'École centrale des Arts et Manufactures. — **H. POINCARÉ**, Professeur à la Sorbonne, Membre de l'Institut. — **A. POTIER**, Professeur à l'École des Mines, Membre de l'Institut. — **J. BLONDIN**, Professeur agrégé de l'Université.

## CHAMP ÉLECTRIQUE ET MAGNÉTIQUE

PRODUIT PAR UNE CHARGE ÉLECTRIQUE CONCENTRÉE EN UN POINT ET ANIMÉE D'UN MOUVEMENT QUELCONQUE

Admettons qu'une masse électrique en mouvement de densité $\rho$ et de vitesse $u$ en chaque point produit le même champ qu'un courant de conduction d'intensité $u\rho$. En conservant les notations d'un précédent article (1) nous obtiendrons pour déterminer le champ, les équations

$$\frac{1}{4\pi}\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right)=\rho u_x+\frac{df}{dt} \quad (1)$$

$$V^2\left(\frac{dh}{dy}-\frac{dg}{dz}\right)=-\frac{1}{4\pi}\frac{d\alpha}{dt} \quad (2)$$

avec les analogues déduites par permutation tournante et en outre les suivantes

$$\rho=\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right. \quad (3)$$

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0. \quad (4)$$

De ce système d'équations on déduit facilement les relations

$$\left(V^2\Delta-\frac{d^2}{dt^2}\right)f=V^2\frac{d\rho}{dx}+\frac{d}{dt}(\rho u_x) \quad (5)$$

$$\left(V^2\Delta-\frac{d^2}{dt^2}\right)\alpha=4\pi V^2\left[\frac{d}{dz}(\rho u_y)-\frac{d}{dy}(\rho u_z)\right] \quad (6)$$

Soient maintenant quatre fonctions $\psi$, F, G, H définies par les conditions

$$\left(V^2\Delta-\frac{d^2}{dt^2}\right)\psi=-4\pi V^2\rho. \quad (7)$$

$$\left.\begin{aligned}\left(V^2\Delta-\frac{d^2}{dt^2}\right)F&=-4\pi V^2\rho u_x\\ \left(V^2\Delta-\frac{d^2}{dt^2}\right)G&=-4\pi\rho u_y\\ \left(V^2\Delta-\frac{d^2}{dt^2}\right)H&=-4\pi V^2\rho u_z\end{aligned}\right\} \quad (8)$$

On satisfera aux conditions (5) et (6) en prenant

$$4\pi f=-\frac{d\psi}{dx}-\frac{1}{V^2}\frac{dF}{dt} \quad (9)$$

$$\alpha=\frac{dH}{dy}-\frac{dG}{dz}. \quad (10)$$

Quant aux équations (1) à (4), pour qu'elles soient satisfaites, il faudra que, en plus de (7) et (8), on ait la condition

$$\frac{d\psi}{dt}+\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=0. \quad (11)$$

Occupons-nous d'abord de l'équation (7). On sait que la solution la plus générale est la suivante :

$$\psi=\int\frac{\rho\left[x',y',z',t-\frac{r}{V}\right]}{r}\,d\omega' \quad (12)$$

(1) La théorie de Lorentz, *L'Eclairage Electrique*, t. XIV, p. 417. $\alpha$, $\beta$, $\gamma$, sont les composantes de la force magnétique et $f$, $g$, $h$, celles du déplacement dans l'éther.

Fig. 1. First page of Liénard's 1898 paper.

# Donald Kerst: first betatron (1940)

*"Ausserordentlichhochgeschwindigkeitelektronenentwickelnden schwerarbeitsbeigollitron"*

# Synchrotron radiation: some dates

- 1946 Blewett observes energy loss due to synchrotron radiation
  100 MeV betatron
- 1947 First visual observation of SR
  70 MeV **synchrotron**, GE Lab

  ***NAME!***

- 1949 Schwinger PhysRev paper

…

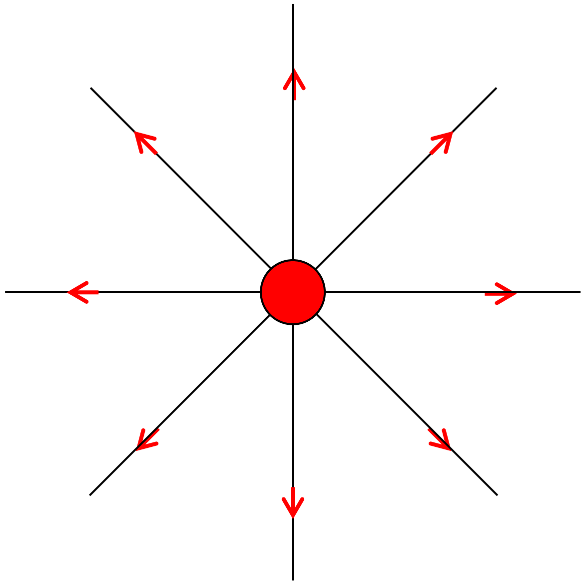
- 1976 Madey: first demonstration of
  Free Electron laser

# Why do they radiate?

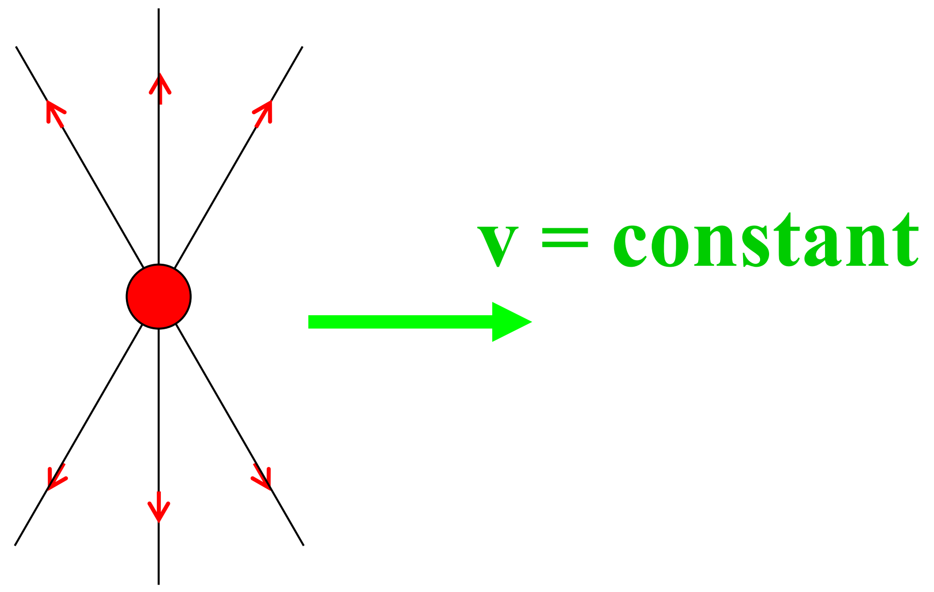
Synchrotron Radiation is
not as simple as it seems

… I will try to show
that it is much simpler

# Charge at rest
# Coulomb field, no radiation



# Uniformly moving charge does not radiate



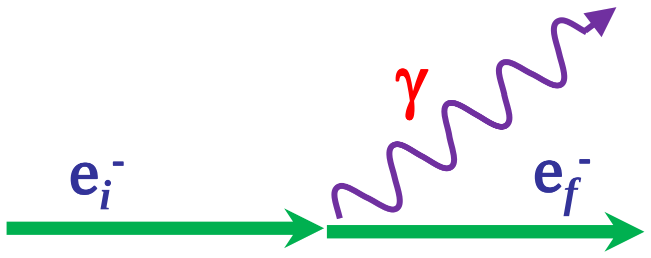

But! Cerenkov!

# Free isolated electron cannot emit a photon

Easy proof using 4-vectors and relativity

- momentum conservation if a photon is emitted

$$\boldsymbol{P}_i = \boldsymbol{P}_f + \boldsymbol{P}_\gamma$$




- square both sides

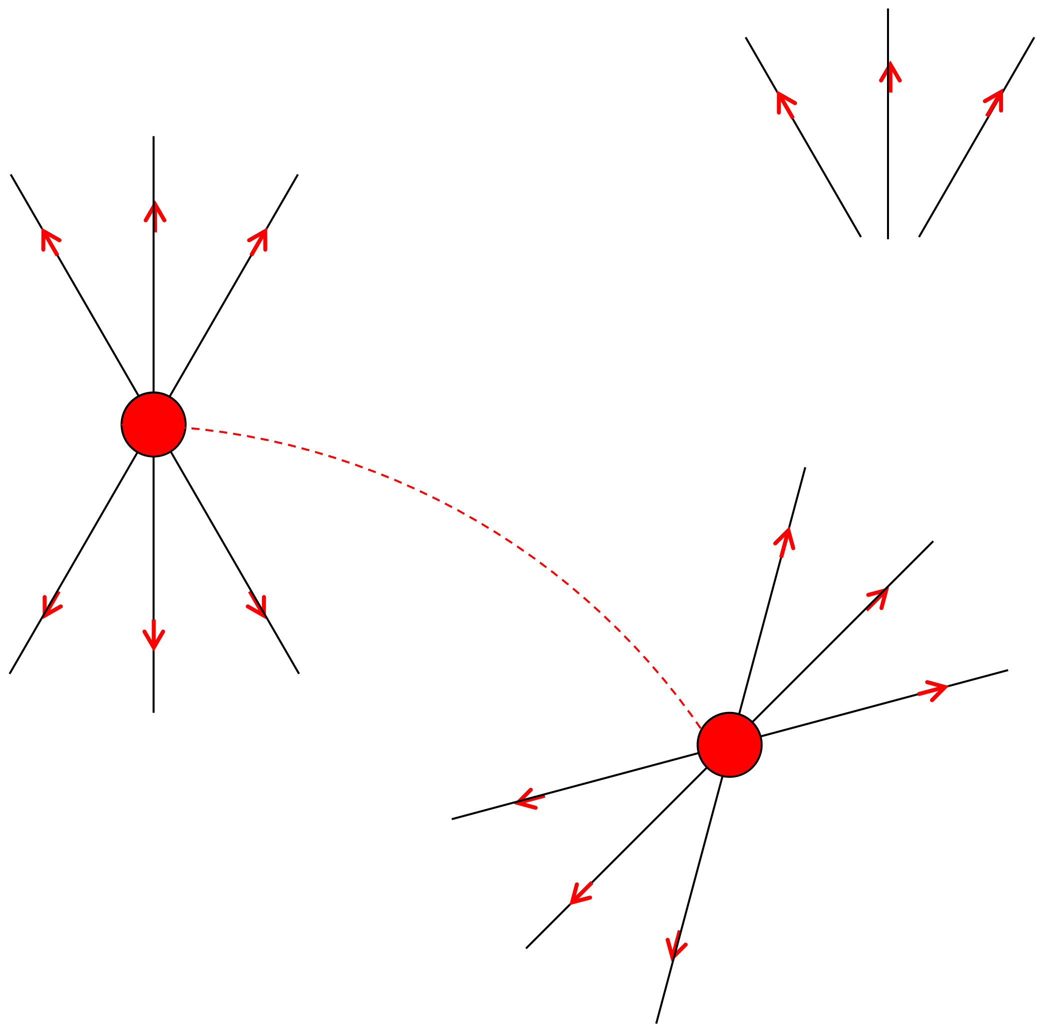
$$m^2 = m^2 + 2\boldsymbol{P}_f \cdot \boldsymbol{P}_\gamma + 0 \;\Rightarrow\; \boldsymbol{P}_f \cdot \boldsymbol{P}_\gamma = 0$$

- in the rest frame of the electron

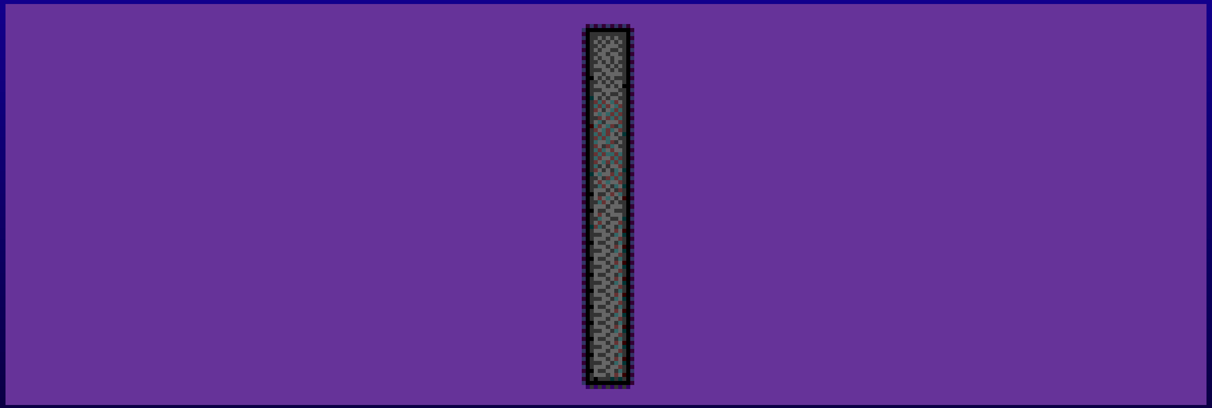
$$\boldsymbol{P}_f = (m, 0) \qquad \boldsymbol{P}_\gamma = (E_\gamma, p_\gamma)$$

this means that the photon energy must be zero.

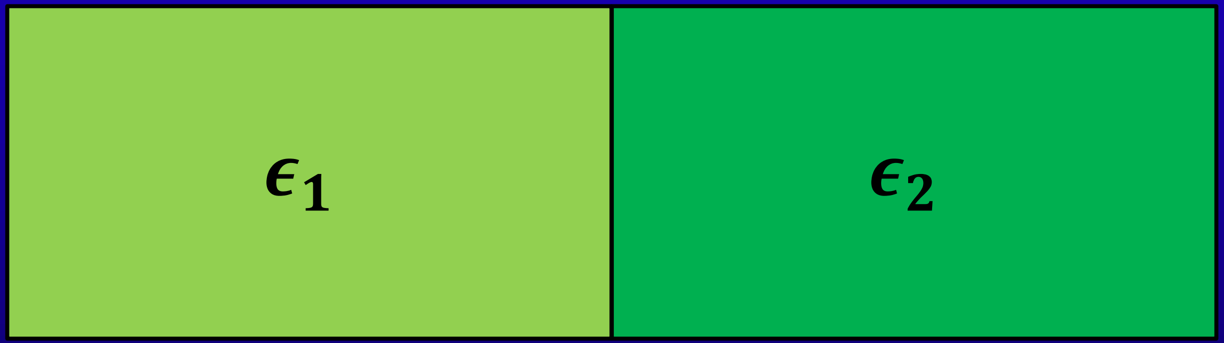
# We need to separate the field from charge



# Bremsstrahlung
# or
# “braking” radiation



# Transition Radiation




$$c_1 = \frac{1}{\sqrt{\epsilon_1 \mu_1}} \qquad c_2 = \frac{1}{\sqrt{\epsilon_2 \mu_2}}$$

# Liénard–Wiechert potentials

$$\varphi(t) = \frac{1}{4\pi\varepsilon_0} \frac{q}{\left[r\left(1 - \vec{\mathbf{n}} \cdot \vec{\boldsymbol{\beta}}\right)\right]_{ret}} \qquad \vec{\mathbf{A}}(t) = \frac{q}{4\pi\varepsilon_0 c^2}\left[\frac{\vec{\mathbf{v}}}{r\left(1 - \vec{\mathbf{n}} \cdot \vec{\boldsymbol{\beta}}\right)}\right]_{ret}$$

and the electromagnetic fields:

$$\nabla \cdot \vec{\mathbf{A}} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0$$ (Lorentz gauge)

$$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}}$$

$$\vec{\mathbf{E}} = -\nabla \varphi - \frac{\partial \vec{\mathbf{A}}}{\partial t}$$

# Fields of a moving charge

$$\vec{\mathbf{E}}(t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\vec{\mathbf{n}}-\vec{\boldsymbol{\beta}}}{\left(1-\vec{\mathbf{n}}\cdot\vec{\boldsymbol{\beta}}\right)^3\gamma^2}\cdot\boxed{\frac{\mathbf{1}}{\mathbf{r}^2}}\right]_{ret} +$$

"near field"

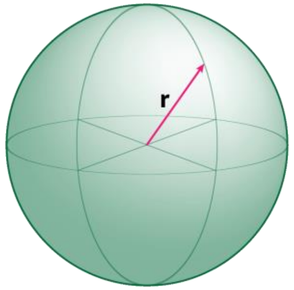
$$\frac{q}{4\pi\varepsilon_0 c}\left[\frac{\vec{\mathbf{n}}\times\left[\left(\vec{\mathbf{n}}-\vec{\boldsymbol{\beta}}\right)\times\boxed{\dot{\vec{\boldsymbol{\beta}}}}\right]}{\left(1-\vec{\mathbf{n}}\cdot\vec{\boldsymbol{\beta}}\right)^3\gamma^2}\cdot\boxed{\frac{\mathbf{1}}{\mathbf{r}}}\right]_{ret}$$

"far field"

$$\vec{\mathbf{B}}(\mathrm{t}) = \frac{1}{\mathrm{c}}\left[\vec{\mathbf{n}}\times\vec{\mathbf{E}}\right]$$

# Energy flow integrated over a sphere

$$\boxed{Power \sim E^2 \cdot \mathrm{Area}}$$



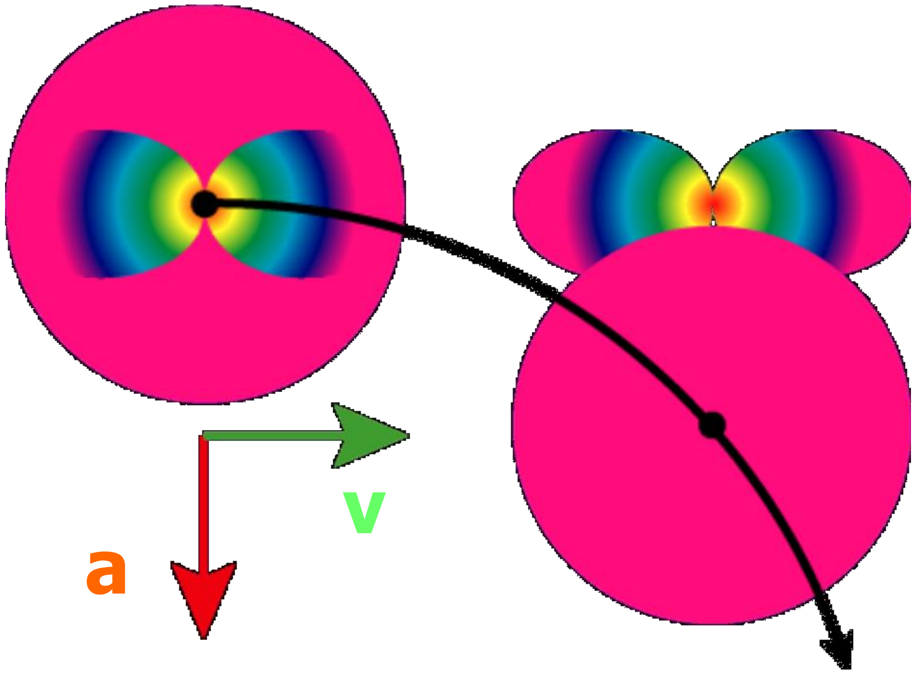

$$A = 4\pi r^2$$

Near field $$P \propto \frac{1}{r^4} r^2 \propto \frac{1}{r^2}$$

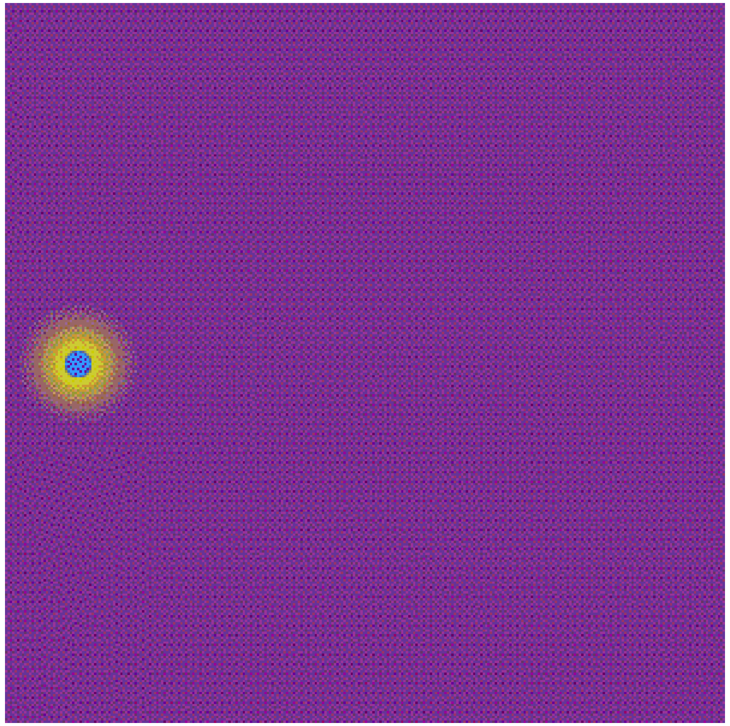
Far field $$\boxed{P \propto \frac{1}{r^2} r^2 \propto const}$$

*Radiation = constant flow of energy to infinity*

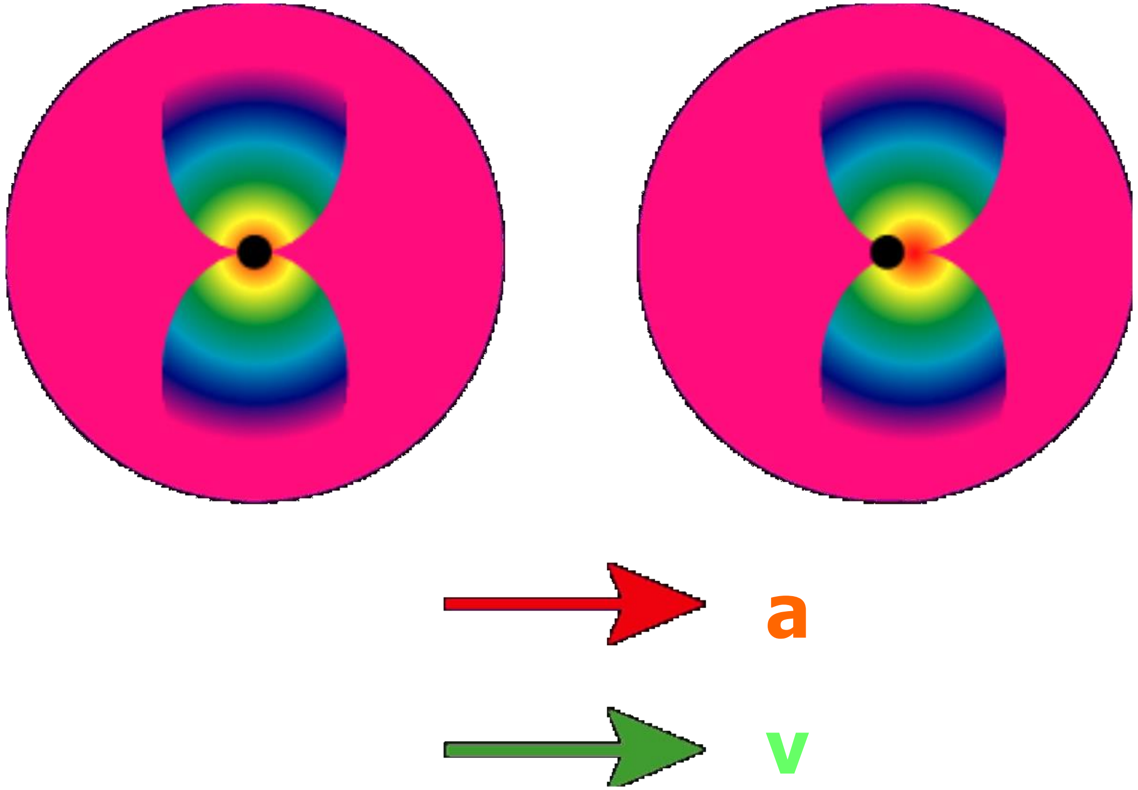
# Transverse acceleration




**Radiation field quickly separates itself from the Coulomb field**



# Longitudinal acceleration

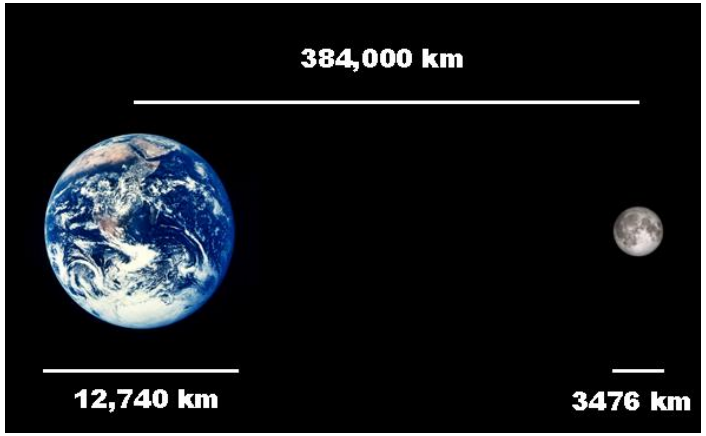



**Radiation field cannot separate itself from the Coulomb field**

# Synchrotron Radiation
# Basic Properties

# Beams of ultra-relativistic particles: e.g. a race to the Moon

An electron with energy of a few GeV emits a photon… a race to the Moon!




$$\Delta t = \frac{L}{\beta c} - \frac{L}{c} = \frac{L}{\beta c}(1-\beta) \sim \frac{L}{\beta c} \cdot \frac{1}{2\gamma^2}$$

$$\beta \equiv \frac{v}{c}$$

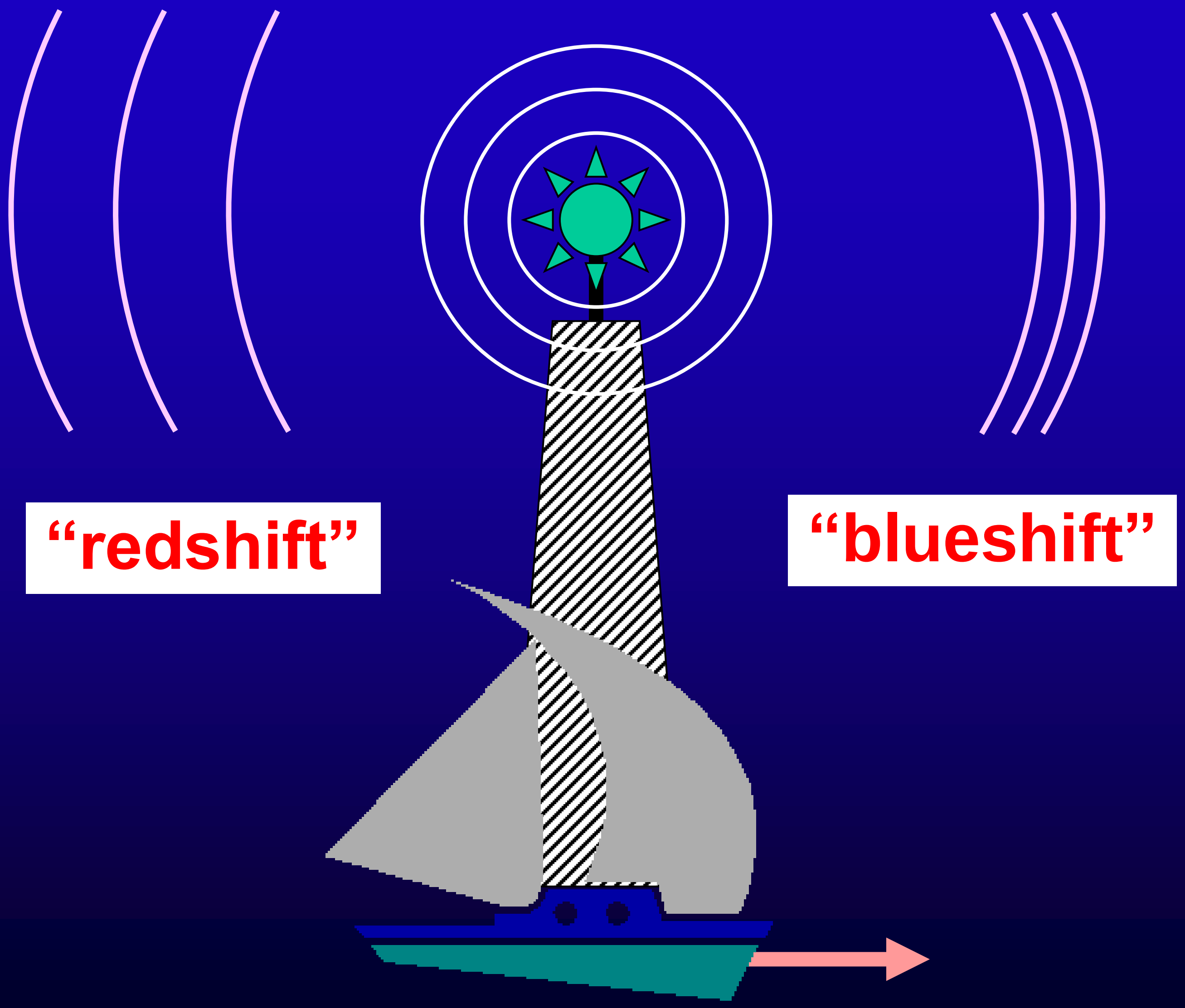
Electron will lose

- by only 8 meters
- the race will last only 1.3 seconds

$$\Delta L = L(1-\beta) \cong \frac{L}{2\gamma^2}$$

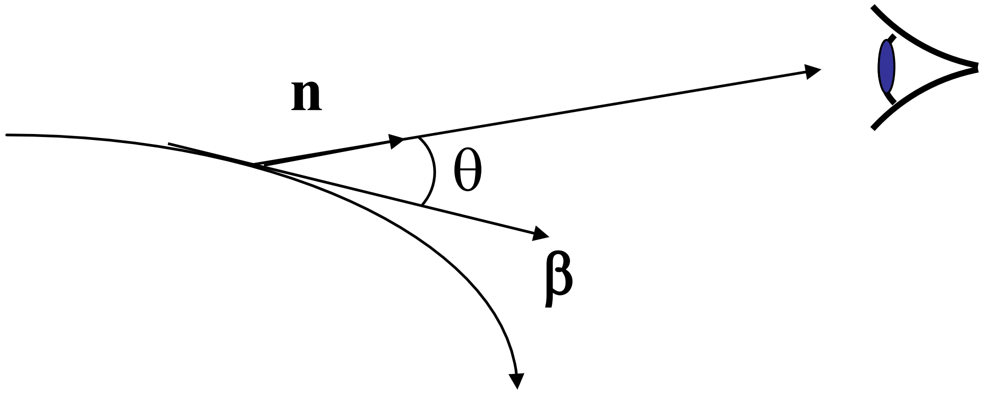
$$\gamma \equiv \frac{E}{mc^2} = \frac{1}{\sqrt{1-\beta^2}}$$

# Moving Source of Waves: Doppler effect



# Time compression

**Electron with velocity β emits a wave with period $T_{emit}$ while the observer sees a different period $T_{obs}$ because the electron was moving towards the observer**




$$T_{obs} = (1 - \mathbf{n} \cdot \boldsymbol{\beta})\, T_{emit}$$

**The wavelength is shortened by the same factor**

$$\lambda_{obs} = (1 - \beta \cos\theta)\, \lambda_{emit}$$

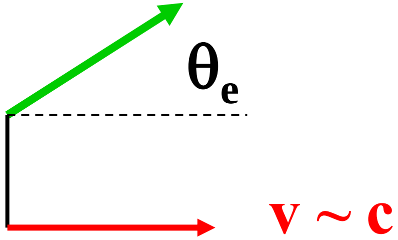
**in ultra-relativistic case, looking along a tangent to the trajectory**

$$\boxed{\lambda_{obs} = \frac{1}{2\gamma^2}\lambda_{emit}}$$

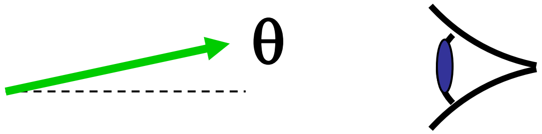
**since**

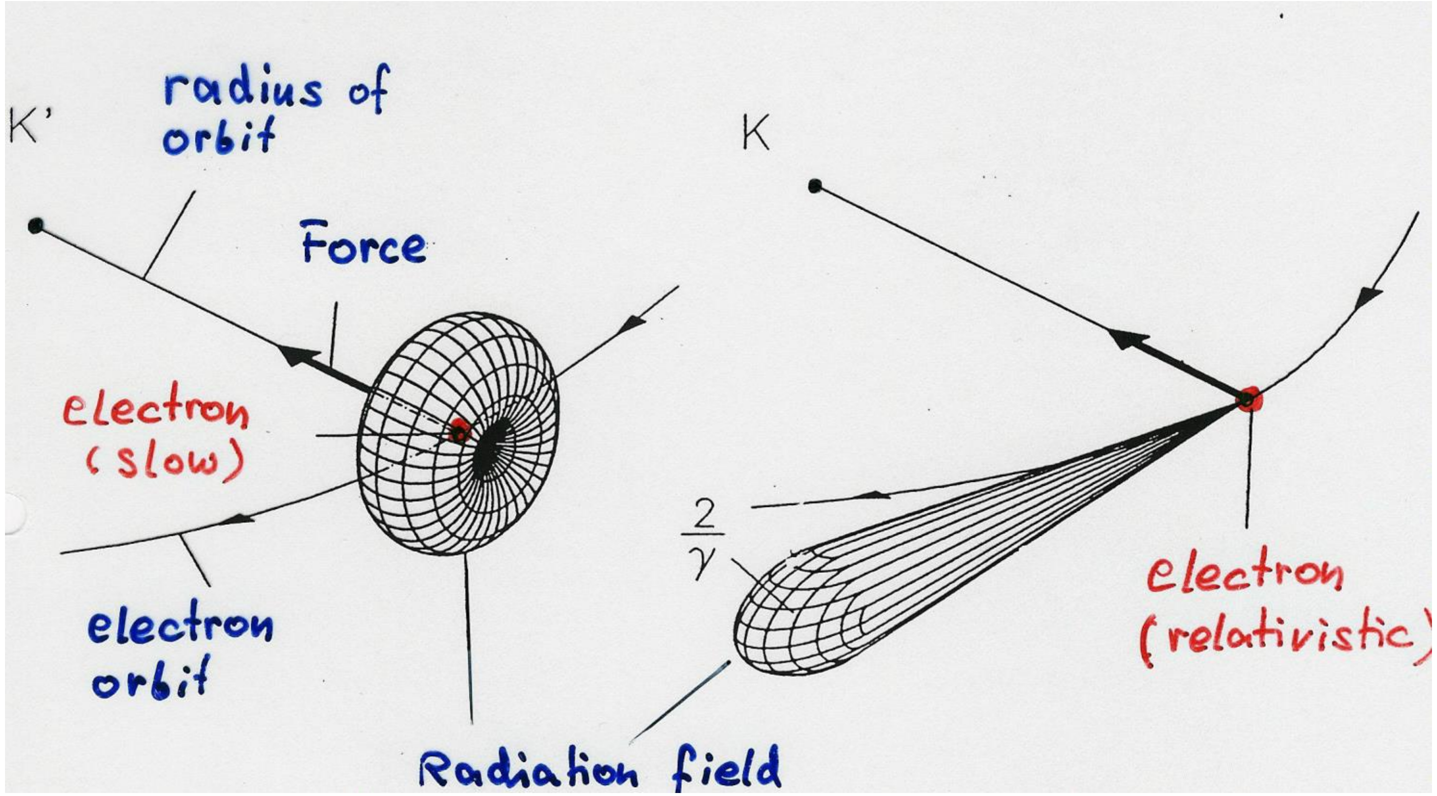
$$\boxed{1 - \beta = \frac{1 - \beta^2}{1 + \beta} \cong \frac{1}{2\gamma^2}}$$

# Radiation is emitted into a narrow cone




$$\theta = \frac{1}{\gamma} \cdot \theta_e$$



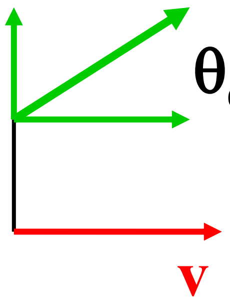



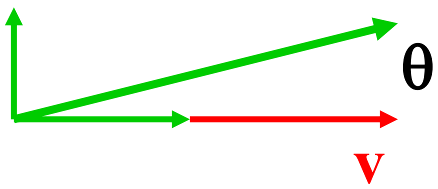

**v << c**

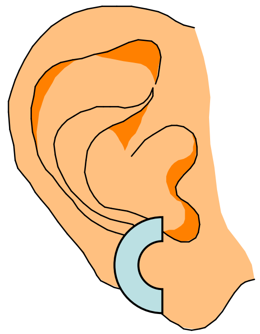
**v ≈ c**

# Sound waves (non-relativistic)

## Angular collimation



$$\theta = \frac{v_{s\perp}}{v_{s||} + \mathbf{v}} = \frac{v_{s\perp}}{v_{s||}} \cdot \frac{1}{1 + \frac{\mathbf{v}}{v_s}} \approx \theta_e \cdot \frac{1}{1 + \frac{\mathbf{v}}{v_s}}$$

## Doppler effect (moving source of sound)

$$\lambda_{heard} = \lambda_{emitted}\left(1 - \frac{\mathbf{v}}{v_s}\right)$$

# Synchrotron radiation power

Power emitted is proportional to:

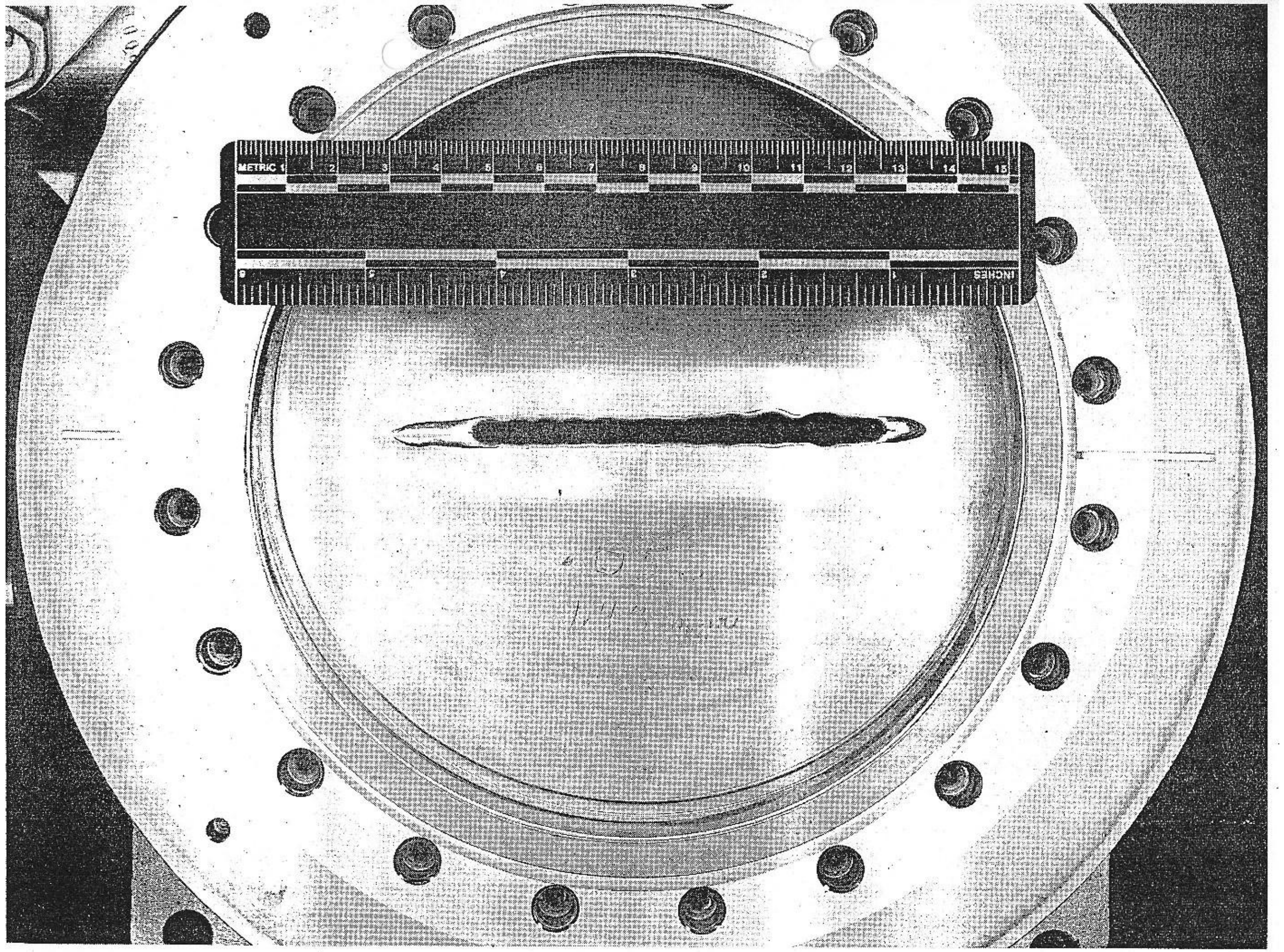
$$\boxed{P \propto E^2 B^2}$$

$$\boxed{P_\gamma = \frac{cC_\gamma}{2\pi} \cdot \frac{E^4}{\rho^2}}$$

$E$ = Energy!

$$C_\gamma = \frac{4\pi}{3} \frac{r_e}{(m_e c^2)^3} = 8.858 \cdot 10^{-5} \left[ \frac{\mathrm{m}}{\mathrm{GeV}^3} \right]$$

# The power is all too real!




ig. 12. Damaged X-ray ring front end gate valve. The power incident on the valve was approximately 1 kW for a duration estimated to 2–10 min and drilled a hole through the valve plate.

# Synchrotron radiation power

Power emitted is proportional to:

$$\boxed{P \propto E^2 B^2}$$

Energy

Magnetic field

$$P_\gamma = \frac{cC_\gamma}{2\pi} \cdot \frac{E^4}{\rho^2}$$

$$P_\gamma = \frac{2}{3}\alpha\hbar c^2 \cdot \frac{\gamma^4}{\rho^2}$$

$$C_\gamma = \frac{4\pi}{3}\frac{r_e}{(m_e c^2)^3} = 8.858 \cdot 10^{-5}\left[\frac{\mathrm{m}}{\mathrm{GeV}^3}\right]$$

$$\alpha = \frac{1}{137}$$

$$\hbar c = 197\ \mathrm{Mev} \cdot \mathrm{fm}$$

Energy loss per turn:

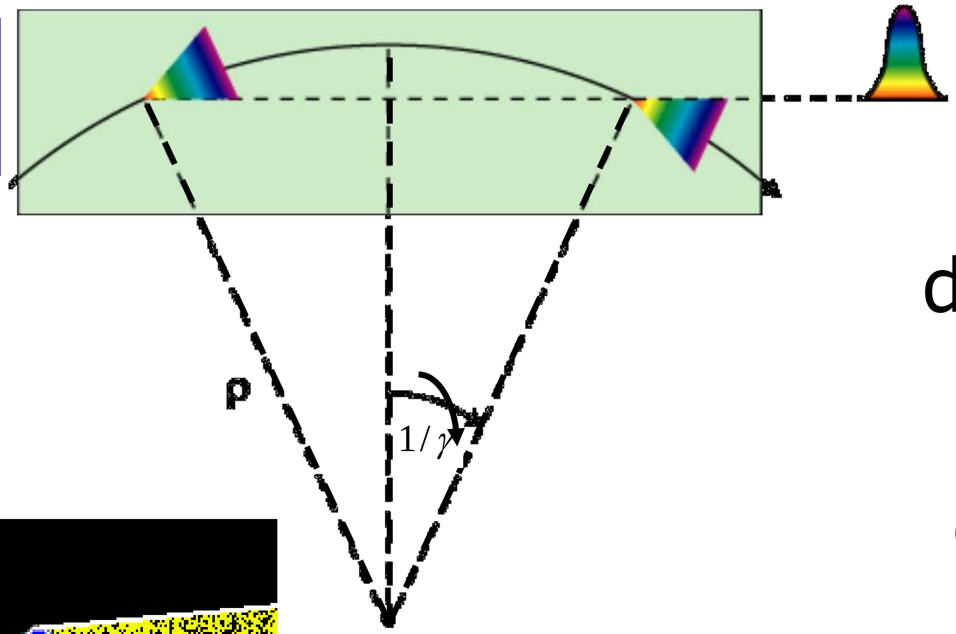
$$U_0 = C_\gamma \cdot \frac{E^4}{\rho}$$

$$U_0 = \frac{4\pi}{3}\alpha\hbar c\frac{\gamma^4}{\rho}$$

# Typical frequency of synchrotron light

Due to extreme collimation of light observer sees only a small portion of electron trajectory **(a few mm)**

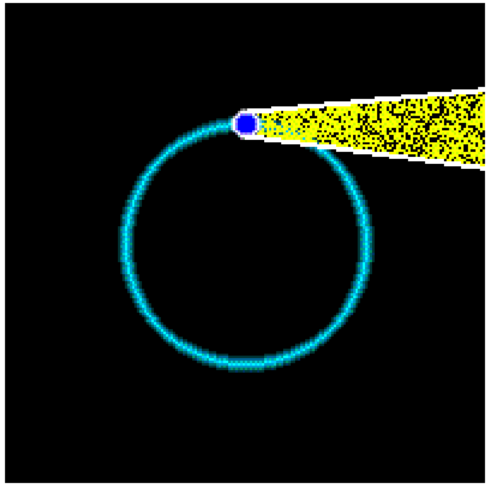
$$l \sim \frac{2\rho}{\gamma}$$




Pulse length: difference in times it takes an electron and a photon to cover this distance

$$\Delta t \sim \frac{l}{\beta c} - \frac{l}{c} = \frac{l}{\beta c}(1-\beta)$$

$$\Delta t \sim \frac{2\rho}{\gamma c} \cdot \frac{1}{2\gamma^2}$$

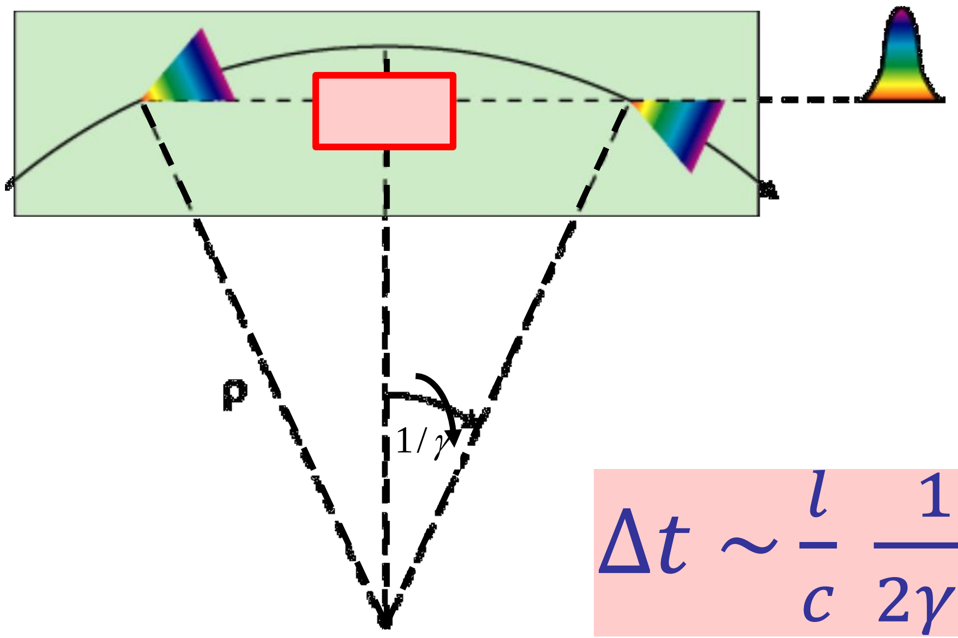


$$\omega \sim \frac{1}{\Delta t} \sim \gamma^3 \omega_0$$

# Short magnet: higher energy photons

When Lorentz factor is not very high (e.g. protons)…

$$l \ll \frac{2\rho}{\gamma}$$




$$\Delta t \sim \frac{l}{c}\frac{1}{2\gamma^2}$$

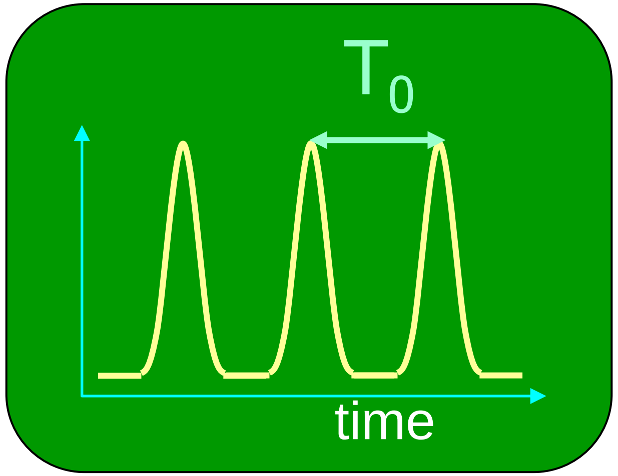
Other ideas?

Pulse length: difference in times it takes an electron and a photon to cover this distance

$$\Delta t \sim \frac{l}{\beta c} - \frac{l}{c} = \frac{l}{\beta c}(1-\beta)$$

# Spectrum of synchrotron radiation

- Synchrotron light comes in a series of flashes every $T_0$ (revolution period)




- the spectrum consists of harmonics of

$$\omega_0 = \frac{1}{T_0}$$

- flashes are extremely short: harmonics reach up to very high frequencies

$$\omega_{typ} \cong \gamma^3 \omega_0$$

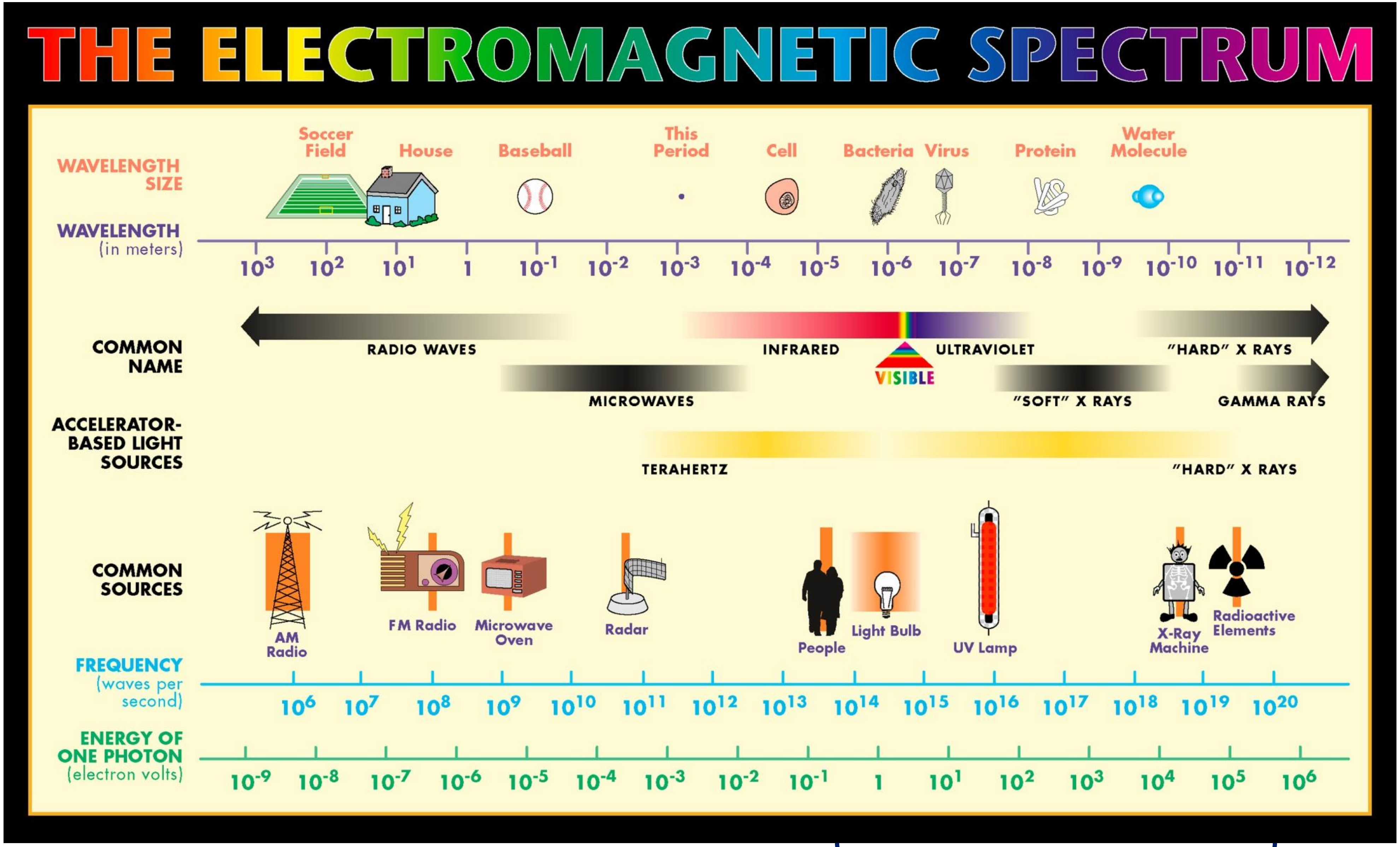
$$\omega_0 \sim 1\ \mathrm{MHz}$$

$$\gamma \sim 4000$$

$$\omega_{\mathrm{typ}} \sim 10^{16}\ \mathrm{Hz}\ !$$

- At high frequencies the individual harmonics overlap

continuous spectrum !



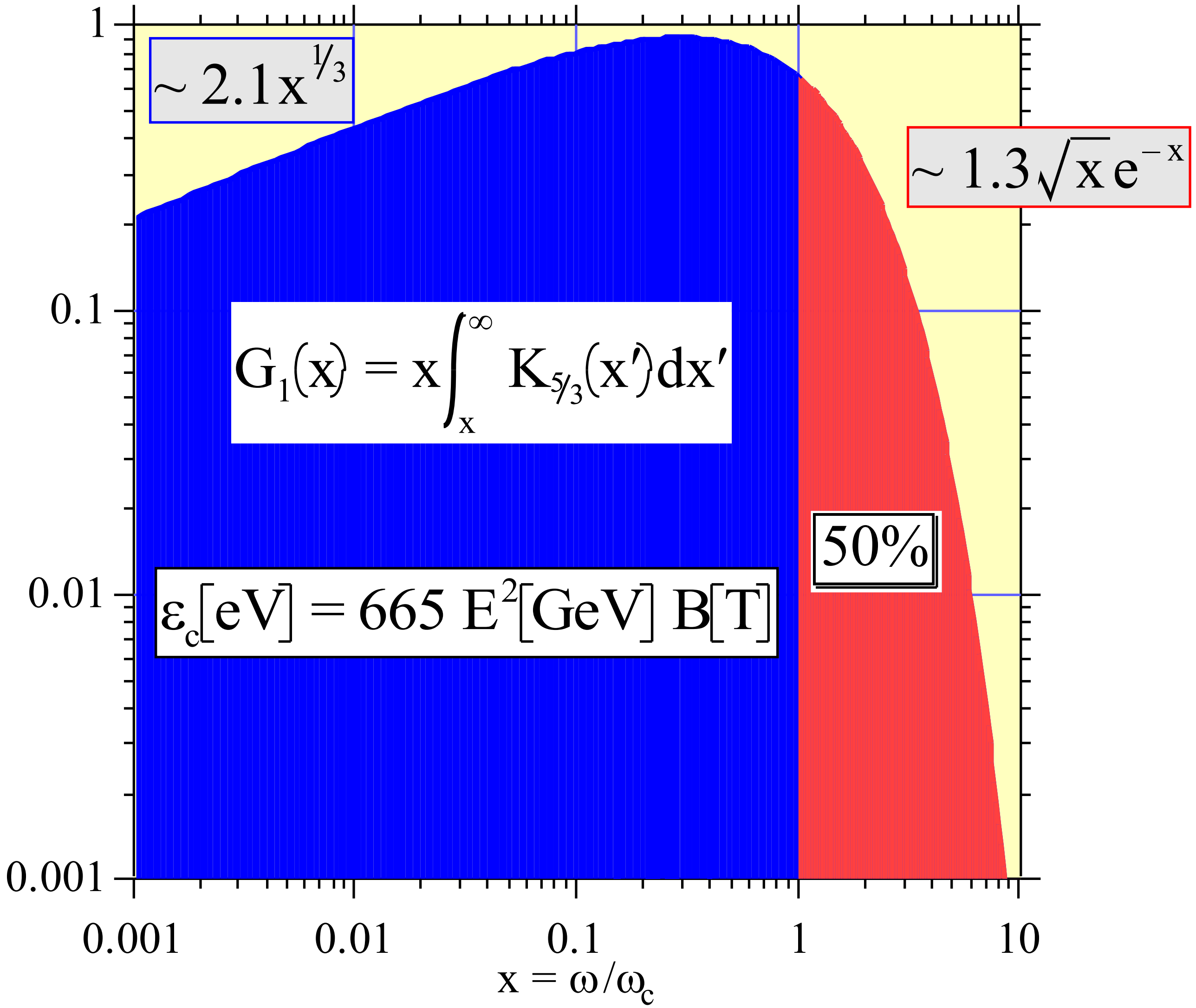

**Wavelength continuously tunable !**

$$\boxed{\frac{dP}{d\omega} = \frac{P_{tot}}{\omega_c} S\left(\frac{\omega}{\omega_c}\right)}$$

$$S(x) = \frac{9\sqrt{3}}{8\pi}\, x \int_x^{\infty} K_{5/3}(x')\,dx' \qquad \int_0^{\infty} S(x')\,dx' = 1$$

$$P_{tot} = \frac{2}{3}\hbar c^2 \alpha \frac{\gamma^4}{\rho^2}$$

$$\omega_c = \frac{3}{2}\frac{c\gamma^3}{\rho}$$



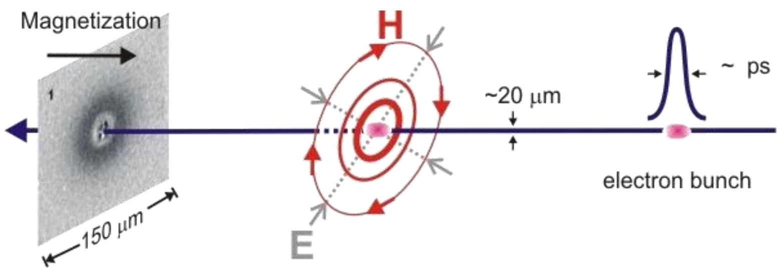
# Beamstrahlung

Synchrotron radiation in the collective field of the bunch

The onset of the quantum regime: the critical photon energy, calculated with classical formulae can exceed the electron energy! Need to take into account the recoil.

$$\varepsilon_c[GeV] = 0.664 \cdot E^2[TeV] \cdot B[T]$$

- Center of mass collision energy is not well defined
- Backgrounds: direct synchrotron radiation
- Backgrounds: pair production from high energy photons

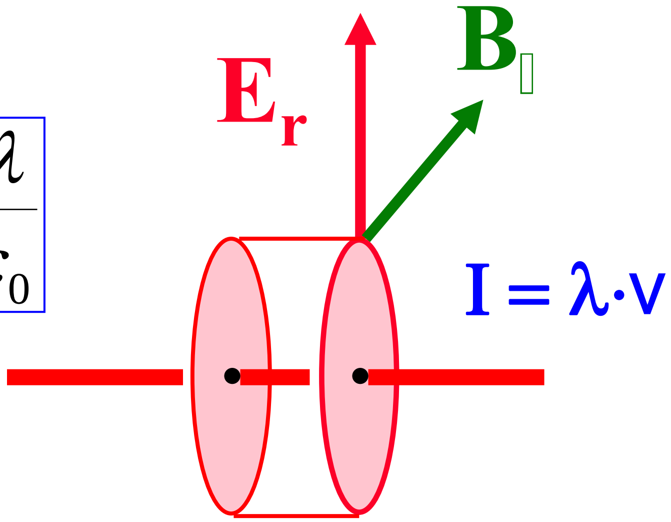


# Fields of a long bunch (linear charge density λ)

Transverse electric field: from Gauss law

$$\boxed{E_r = \frac{\lambda}{2\pi\varepsilon_0 r}}$$

$$\boxed{2\pi r \cdot E_r = \frac{\lambda}{\varepsilon_0}}$$




Transverse magnetic field: from Ampere law

$$\boxed{2\pi r \cdot B_\theta = \mu_0 I}$$

$$\boxed{B_\theta = \frac{\mu_0 \lambda}{2\pi r} \mathrm{v} = \frac{\lambda}{2\pi\varepsilon_0 r} \cdot \frac{\mathrm{v}}{c^2}}$$

$$\mu_0 \varepsilon_0 = \frac{1}{c^2}$$

$$\boxed{B_\theta[T] = \frac{1}{c} E_r \left[\frac{V}{m}\right]}$$

$$\mu_0 = 4\pi \cdot 10^{-7}\ {}^{V\cdot s}/_{A\cdot m}$$

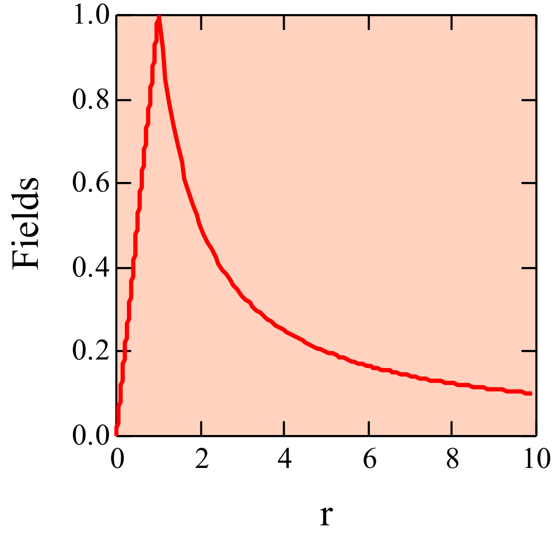
$$\varepsilon_0 = 8.85 \cdot 10^{-12}\ {}^{C}/_{V\cdot m}$$

# Fields in the bunch

- Round uniform distribution

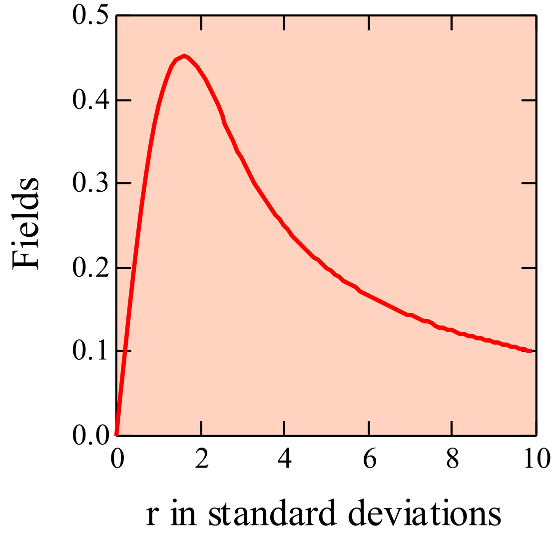
$$E_r = \frac{eN}{2\pi\varepsilon_0 l}\cdot\frac{1}{r} \qquad r > a$$

$$E_r = \frac{eN}{2\pi\varepsilon_0 l}\cdot\frac{r}{a^2} \qquad r < a$$




- Round Gaussian distribution

$$E_r = \frac{eN}{2\pi\varepsilon_0 l\sigma}\left[\frac{1-e^{-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2}}{\frac{r}{\sigma}}\right]$$



# Useful books and references

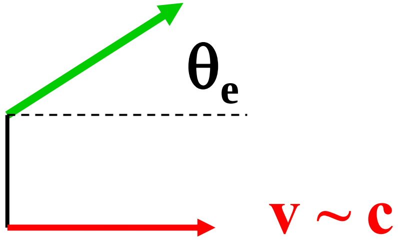
H. Wiedemann, *Synchrotron Radiation*
Springer-Verlag Berlin Heidelberg 2003

H. Wiedemann, *Particle Accelerator Physics*
Springer, 2015 Open Access

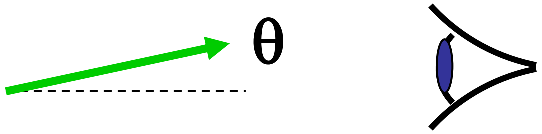
A.Hofmann, *The Physics of Synchrotron Radiation*
Cambridge University Press 2004

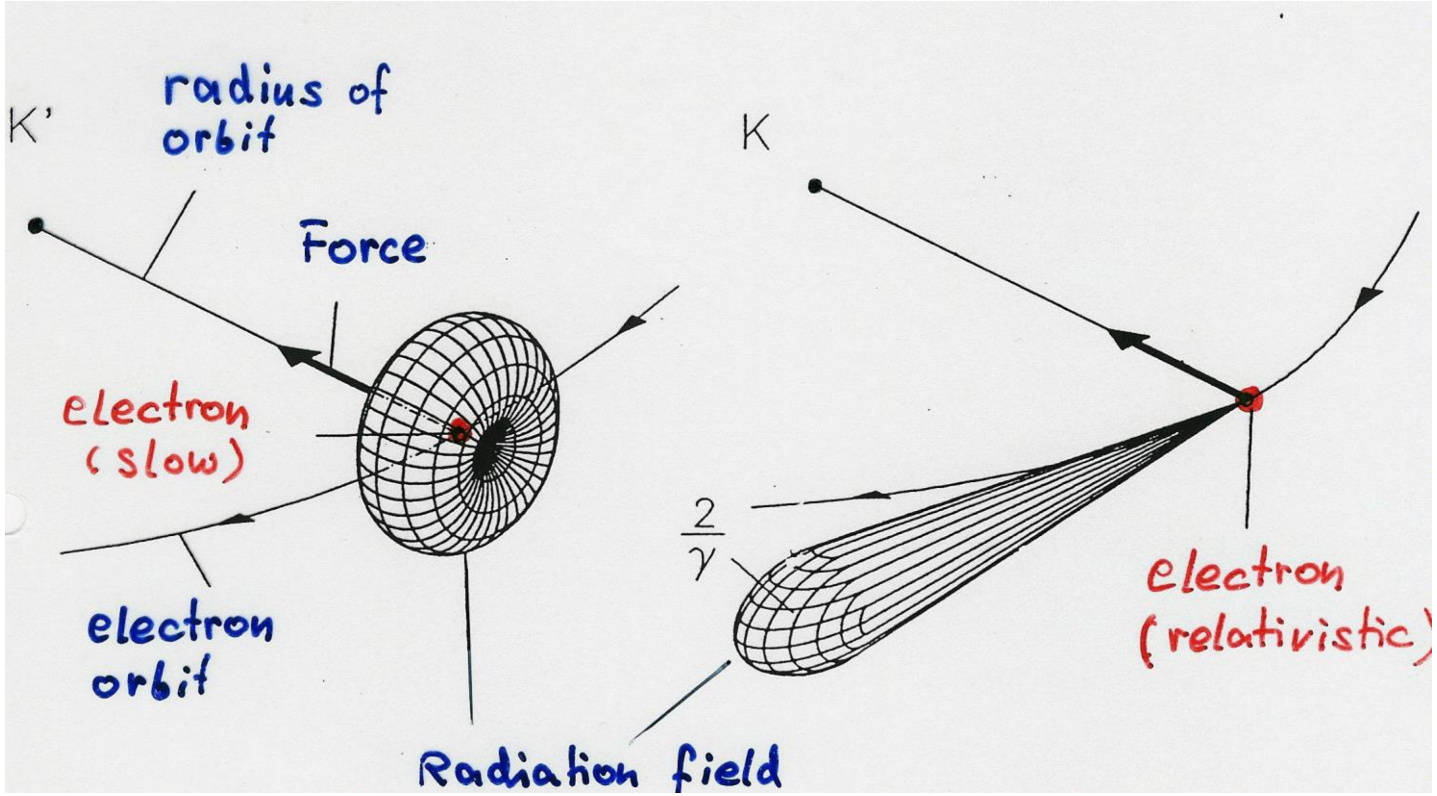
A. W. Chao, M. Tigner, *Handbook of Accelerator Physics and Engineering*, World Scientific 2013

# Radiation is emitted into a narrow cone




$$\boxed{\theta = \frac{1}{\gamma} \cdot \theta_e}$$




**v << c**

**v ≈ c**

# Radiation effects in electron storage rings

## Average radiated power restored by RF

- Electron loses energy each turn to synchrotron radiation $\boxed{U_0 \cong 10^{-3} \text{ of } E_0}$
- RF cavities accelerate electrons back to the nominal energy $\boxed{V_{RF} > U_0}$

## Radiation damping

- Average rate of energy loss produces **DAMPING** of electron oscillations in all three degrees of freedom (if properly arranged!)

## Quantum fluctuations

- Statistical fluctuations in energy loss (from quantized emission of radiation) produce **RANDOM EXCITATION** of these oscillations

## **Equilibrium** distributions

- The balance between the damping and the excitation of the electron oscillations determines the equilibrium distribution of particles in the beam

# Radiation damping

## Transverse oscillations

# Average energy loss and gain per turn

- Every turn electron radiates small amount of energy

$$E_1 = E_0 - U_0 = E_0\left(1 - \frac{U_0}{E_0}\right)$$

- only the amplitude of the momentum changes

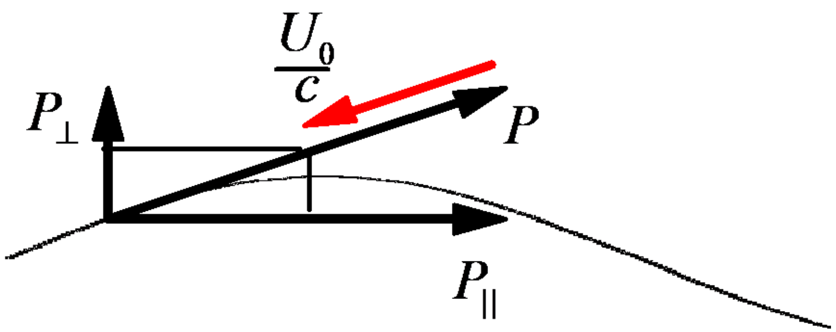
$$P_1 = P_0 - \frac{U_0}{c} = P_0\left(1 - \frac{U_0}{E_0}\right)$$

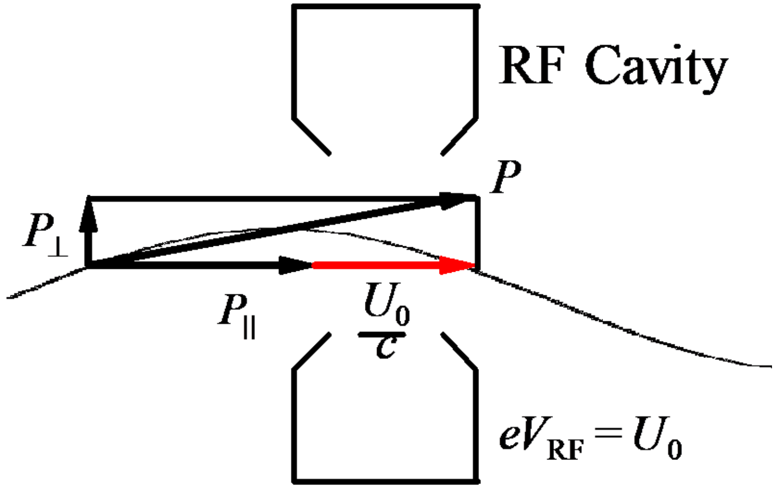
- Only the longitudinal component of the momentum is increased in the RF cavity

- Energy of betatron oscillation

$$\boxed{E_\beta \propto A^2}$$

$$\boxed{A_1^2 = A_0^2\left(1 - \frac{U_0}{E_0}\right) \quad \text{or} \quad A_1 \cong A_0\left(1 - \frac{U_0}{2E_0}\right)}$$



# Damping of vertical oscillations

- But this is just the exponential decay law!

$$\frac{\Delta A}{A} = -\frac{U_0}{2E}$$

$$A = A_0 \cdot e^{-t/\tau}$$

- The oscillations are exponentially **damped** with the **damping time** **(milliseconds!)**

$$\tau = \frac{2ET_0}{U_0}$$

the time it would take particle to 'lose all of its energy'

- In terms of radiation power

$$\tau = \frac{2E}{P_\gamma}$$

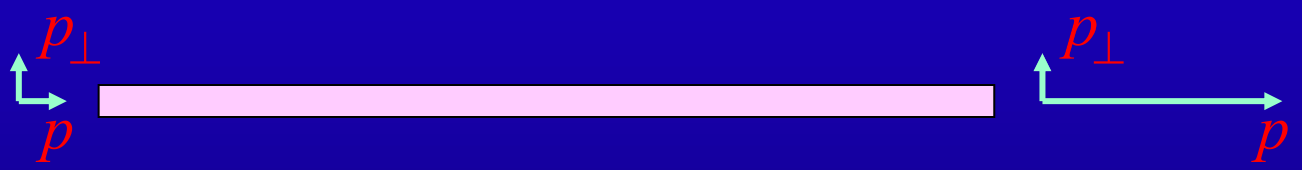
and since $P_\gamma \propto E^4$

$$\tau \propto \frac{1}{E^3}$$

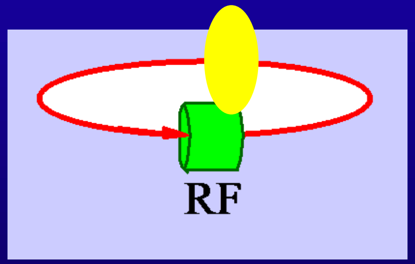
# Adiabatic damping in linear accelerators

In a **linear accelerator**:

$$x' = \frac{p_\perp}{p} \quad \text{decreases} \quad \propto \frac{1}{E}$$




In a **storage ring** beam passes many times through same RF cavity

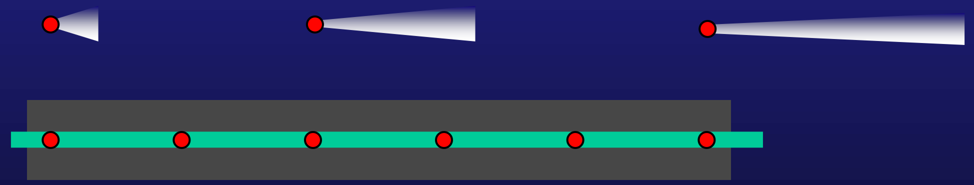



- Clean loss of energy every turn (no change in x')
- Every turn is re-accelerated by RF (x' is reduced)
- Particle energy on average remains constant

# Emittance damping in linacs:

$\varepsilon$ $\qquad \frac{\varepsilon}{2}$ $\qquad \frac{\varepsilon}{4}$

$\Omega$ $\qquad \frac{\Omega}{2}$ $\qquad \frac{\Omega}{4}$



$\gamma$ $\qquad 2\gamma$ $\qquad 4\gamma$

$$\varepsilon \propto \frac{1}{\gamma}$$

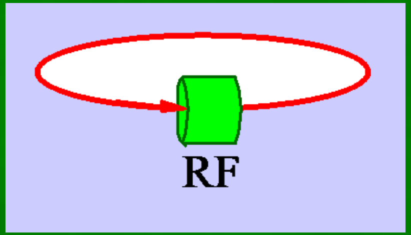
or

$$\gamma\varepsilon = \text{const.}$$

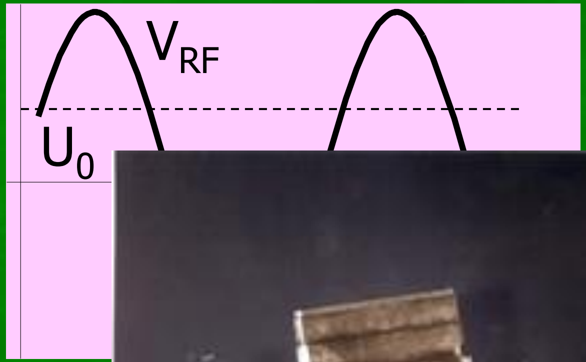
# Radiation damping

## Longitudinal oscillations

# Longitudinal motion: compensating radiation loss $U_0$




- RF cavity provides accelerating field with frequency $f_{RF} = h \cdot f_0$
  - h – harmonic number

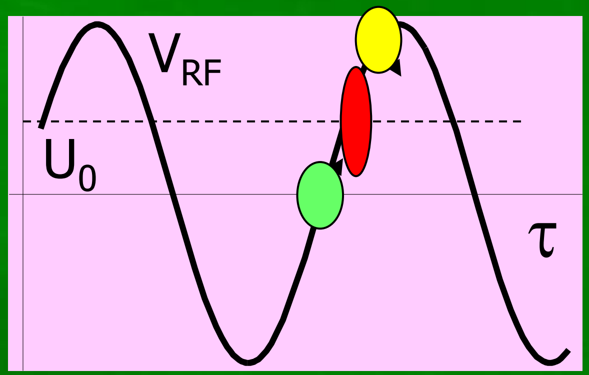



- The energy gain:

$$\boxed{U_{RF} = eV_{RF}(\tau)}$$

- Synchronous particle:
  - has design energy
  - gains from the RF on the average as much as it loses per turn $U_0$

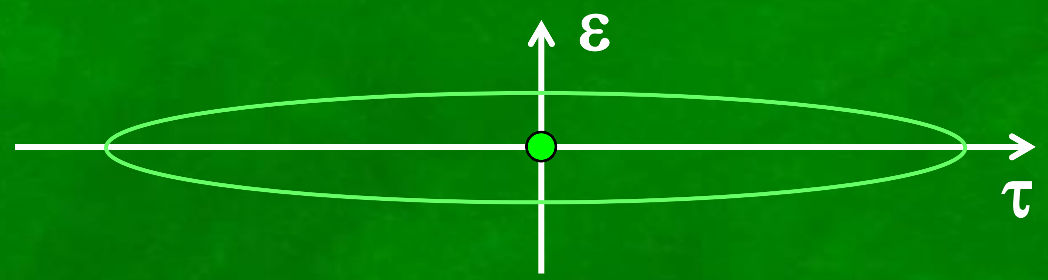
# *Longitudinal motion: phase stability*




- Particle ahead of synchronous one
  - gets too much energy from the RF
  - goes on a longer orbit (not enough B)
    >> takes longer to go around
  - comes back to the RF cavity closer to synchronous part.
- Particle behind the synchronous one
  - gets too little energy from the RF
  - goes on a shorter orbit (too much B)
  - catches-up with the synchronous particle

# Longitudinal motion: energy-time oscillations

energy deviation from the design energy,
or the energy of the synchronous particle

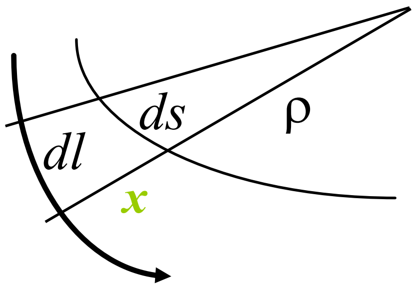



longitudinal coordinate measured from the
position of the synchronous electron

# Orbit Length

Length element depends on x

$$dl = \left(1 + \frac{x}{\rho}\right) ds$$




Horizontal displacement has two parts:

$$x = x_\beta + x_\varepsilon$$

- To first order $x_\beta$ does not change L
- $x_\varepsilon$ – has the same sign around the ring

Length of the off-energy orbi

$$L_\varepsilon = \oint dl = \oint \left(1 + \frac{x_\varepsilon}{\rho}\right) ds = L_0 + \Delta L$$

$$\Delta L = \delta \cdot \oint \frac{D(s)}{\rho(s)} ds \quad \text{where} \quad \delta = \frac{\Delta p}{p} = \frac{\Delta E}{E}$$

$$\frac{\Delta L}{L} = \alpha \cdot \delta$$

# Something funny happens on the way around the ring...

Revolution time changes with energy

$$T_0 = \frac{L_0}{c\beta}$$

$$\frac{\Delta T}{T} = \frac{\Delta L}{L} - \frac{\Delta\beta}{\beta}$$

- Particle goes faster (not much!)

$$\frac{d\beta}{\beta} = \frac{1}{\gamma^2} \cdot \frac{dp}{p} \qquad \text{(relativity)}$$

- while the orbit length increases (more!)

$$\frac{\Delta L}{L} = \alpha \cdot \frac{dp}{p}$$

- The "slip factor" $\quad \eta \cong \alpha \quad$ since $\quad \alpha >> \frac{1}{\gamma^2}$

$$\frac{\Delta T}{T} = \left(\alpha - \frac{1}{\gamma^2}\right) \cdot \frac{dp}{p} = \eta \cdot \frac{dp}{p}$$

- Ring is above "transition energy"

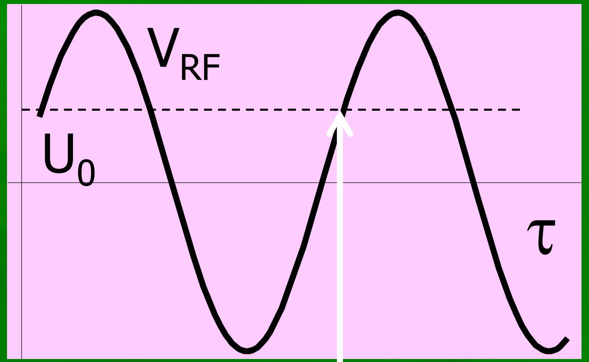
$$\alpha \equiv \frac{1}{\gamma_{tr}^2}$$

isochronous ring: $\quad \eta = 0 \ \text{ or } \ \gamma = \gamma_{tr}$

# Not only accelerators work above transition


Dante Alighieri
Divine Comedy

# RF Voltage




$$V(\tau) = \hat{V}\sin\left(h\omega_0\tau + \psi_s\right)$$

here the synchronous phase

$$\psi_s = \arcsin\left(\frac{U_0}{e\hat{V}}\right)$$

# Momentum compaction factor

$$\alpha \equiv \frac{1}{L}\oint \frac{D(s)}{\rho(s)} ds$$

Like the tunes $Q_x$, $Q_y$ - $\alpha$ depends on the whole optics

- A quick estimate for separated function guide field:

$$\alpha = \frac{1}{L_0\rho_0}\int_{mag} D(s)\,ds = \frac{1}{L_0\rho_0}\langle D\rangle \cdot L_{mag}$$

$\rho = \rho_0$ in dipoles
$\rho = \infty$ elsewhere

- But

$$L_{mag} = 2\pi\rho_0$$

$$\alpha = \frac{\langle D\rangle}{R}$$

- Since dispersion is approximately

$$D \approx \frac{R}{Q^2} \Rightarrow \alpha \approx \frac{1}{Q^2} \quad \text{typically} \; < 1\%$$

and the orbit change for ~ 1% energy deviation

$$\frac{\Delta L}{L} = \frac{1}{Q^2}\cdot\delta \approx 10^{-4}$$

# Energy balance

Energy gain from the RF system: $U_{RF} = eV_{RF}(\tau) = U_0 + e\dot{V}_{RF} \cdot \tau$

- synchronous particle ($\tau = 0$) will get exactly the energy loss per turn
- we consider only linear oscillations $\dot{V}_{RF} = \left.\frac{dV_{RF}}{d\tau}\right|_{\tau=0}$

- Each turn electron gets energy from RF and loses energy to radiation within one revolution time $T_0$

$$\Delta\varepsilon = \left(U_0 + e\dot{V}_{RF} \cdot \tau\right) - \left(U_0 + U' \cdot \varepsilon\right)$$

$$\frac{d\varepsilon}{dt} = \frac{1}{T_0}\left(e\dot{V}_{RF} \cdot \tau - U' \cdot \varepsilon\right)$$

- An electron with an energy deviation will arrive after one turn at a different time with respect to the synchronous particle

$$\frac{d\tau}{dt} = -\alpha \frac{\varepsilon}{E_0}$$

# Synchrotron oscillations: damped harmonic oscillator

Combining the two equations

$$\boxed{\frac{d^2\varepsilon}{dt^2} + 2\alpha_\varepsilon \frac{d\varepsilon}{dt} + \Omega^2 \varepsilon = 0}$$

- where the oscillation frequency

$$\Omega^2 \equiv \frac{\alpha e \dot{V}_{RF}}{T_0 E_0}$$

- the damping is slow:

$$\alpha_\varepsilon \equiv \frac{U'}{2T_0} \qquad \text{typically} \qquad \alpha_\varepsilon << \Omega$$

- the solution is then:

$$\varepsilon(t) = \hat{\varepsilon}_0 e^{-\alpha_\varepsilon t} \cos\left(\Omega t + \theta_\varepsilon\right)$$

- similarly, we can get for the time delay:

$$\tau(t) = \hat{\tau}_0 e^{-\alpha_\varepsilon t} \cos\left(\Omega t + \theta_\tau\right)$$

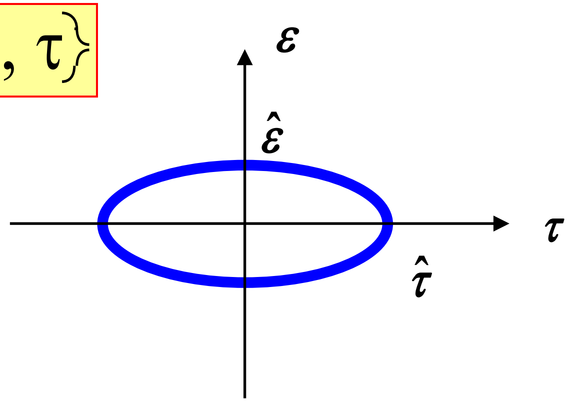
# Synchrotron (time - energy) oscillations

The ratio of amplitudes at any instant $\boxed{\hat{\tau} = \dfrac{\alpha}{\Omega E_0}\hat{\varepsilon}}$

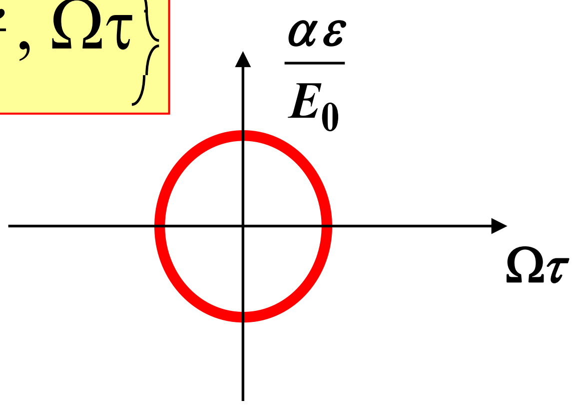
Oscillations are 90 degrees out of phase $\boxed{\theta_\varepsilon = \theta_\tau + \dfrac{\pi}{2}}$

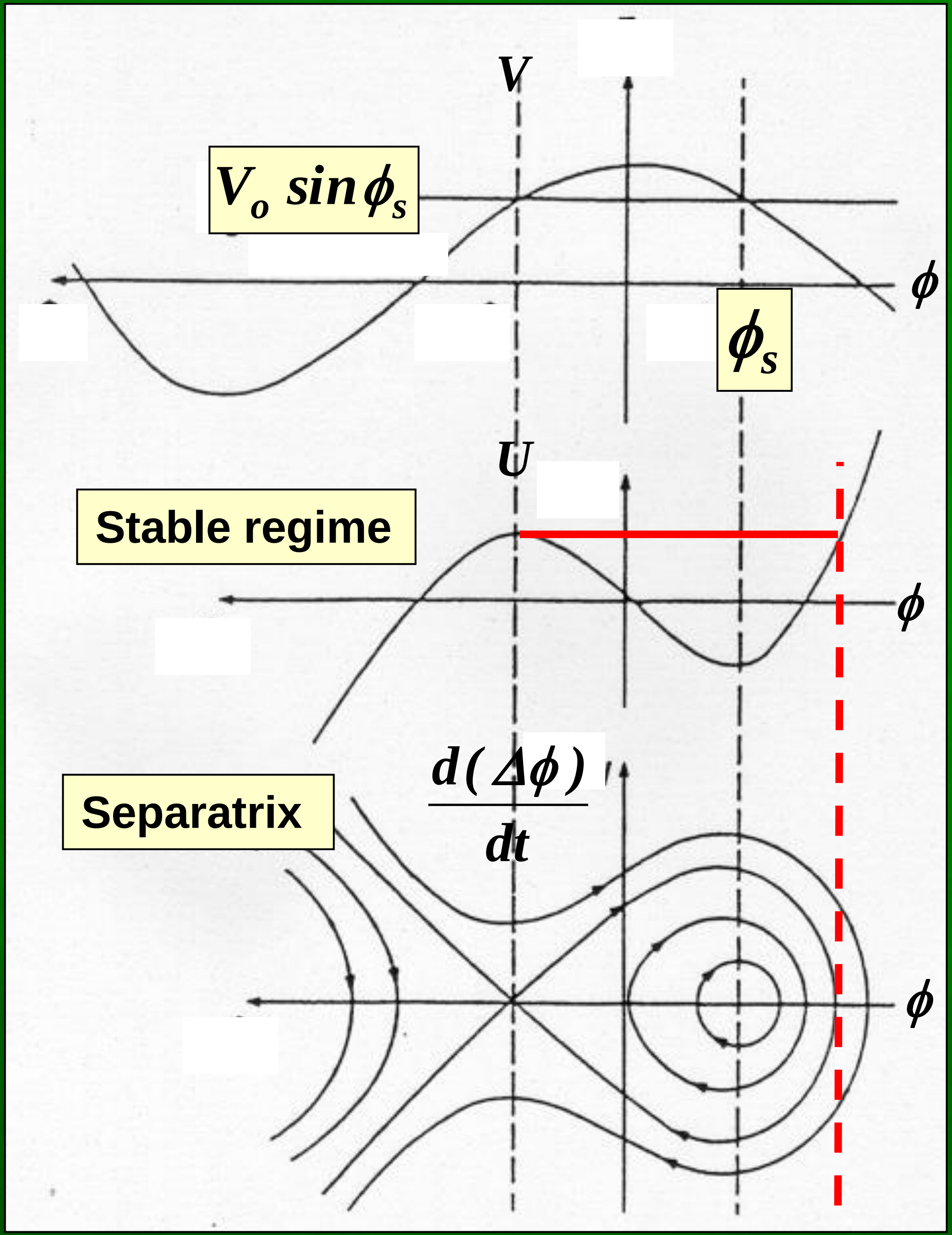
The motion can be viewed in the phase space of conjugate variables

$\boxed{\{\varepsilon, \tau\}}$




$\boxed{\left\{\dfrac{\alpha\varepsilon}{E_0}, \Omega\tau\right\}}$





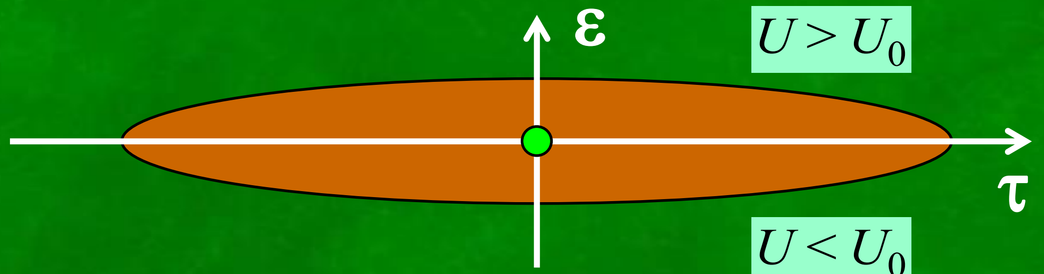

Longitudinal
Phase
Space

# Longitudinal motion: damping of synchrotron oscillations

$P_\gamma \propto \boldsymbol{E}^2 B^2$

During one period of synchrotron oscillation:

- when the particle is in the upper half-plane, it loses more energy per turn, its energy gradually reduces




- when the particle is in the lower half-plane, it loses less energy per turn, but receives $U_0$ on the average, so its energy deviation gradually reduces

The synchrotron motion is damped

- the phase space trajectory is spiraling towards the origin

# Robinson theorem: Damping partition numbers

- Transverse betatron oscillations are damped with

$$\tau_x = \tau_z = \frac{2ET_0}{U_0}$$

- Synchrotron oscillations are damped twice as fast

$$\tau_\varepsilon = \frac{ET_0}{U_0}$$

- The total amount of damping (Robinson theorem) depends only on energy and loss per turn

$$\boxed{\frac{1}{\tau_x} + \frac{1}{\tau_y} + \frac{1}{\tau_\varepsilon} = \frac{2U_0}{ET_0} = \frac{U_0}{2ET_0}\left(J_x + J_y + J_\varepsilon\right)}$$

the sum of the partition numbers $\boxed{J_x + J_z + J_\varepsilon = 4}$

# Radiation loss

$$P_\gamma \propto E^2 B^2$$

Displaced off the design orbit particle sees fields that are different from design values

- **energy deviation ε**
  - different energy: $P_\gamma \propto E^2$
  - different magnetic field **B**
    particle moves on a different orbit, defined by the **off-energy** or **dispersion** function $D_x$

  **both contribute to linear term in** $P_\gamma(\varepsilon)$

- **betatron oscillations**: zero on *average*

# Radiation loss

$$P_\gamma \propto E^2 B^2$$

To first order in $\varepsilon$

$$U_{rad} = U_0 + U' \cdot \varepsilon$$

*electron energy changes slowly, at any instant it is moving on an orbit defined by* $D_x$

$$U' \equiv \left.\frac{dU_{rad}}{dE}\right|_{E_0}$$

*after some algebra one can write*

$$U' = \frac{U_0}{E_0}\left(2 + \mathcal{D}\right)$$

$$\mathcal{D} \neq 0 \quad \textit{only when} \quad \frac{k}{\rho} \neq 0$$

# Damping partition numbers

$$\boxed{J_x + J_z + J_\varepsilon = 4}$$

- Typically we build rings with no vertical dispersion

$$\boxed{J_z = 1} \qquad \boxed{J_x + J_\varepsilon = 3}$$

- Horizontal and energy partition numbers can be modified via $\mathcal{D}$:

$$\boxed{J_x = 1 - \mathcal{D}} \qquad \boxed{J_\varepsilon = 2 + \mathcal{D}}$$

  - Use of combined function magnets
  - Shift the equilibrium orbit in quads with RF frequency

# Equilibrium beam sizes

# Radiation effects in electron storage rings

## Average radiated power restored by RF

- Electron loses energy each turn to synchrotron radiation $\boxed{U_0 \cong 10^{-3} \text{ of } E_0}$
- RF cavities accelerate electrons back to the nominal energy $\boxed{V_{RF} > U_0}$

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## Quantum fluctuations

- Statistical fluctuations in energy loss (from quantized emission of radiation) produce **RANDOM EXCITATION** of these oscillations

## **Equilibrium** distributions

- The balance between the damping and the excitation of the electron oscillations determines the equilibrium distribution of particles in the beam

# Quantum nature of synchrotron radiation

## Damping only

- If damping was the whole story, the beam emittance (size) would shrink to microscopic dimensions!*
- Lots of problems! (e.g. **coherent radiation**)

---

* How small? On the order of electron wavelength

$$E = \gamma mc^2 = h\nu = \frac{hc}{\lambda_e} \quad \Rightarrow \quad \lambda_e = \frac{1}{\gamma}\frac{h}{mc} = \frac{\lambda_C}{\gamma}$$

$\lambda_C = 2.4 \cdot 10^{-12} m$ – Compton wavelength

Diffraction limited electron emittance $\quad \varepsilon \geq \frac{\lambda_C}{4\pi\gamma}\left(\times N^{1/3} - \text{fermions}\right)$

# Quantum nature of synchrotron radiation

## Quantum fluctuations

- Because the radiation is emitted in quanta, radiation itself takes care of the problem!
- It is sufficient to use quasi-classical picture:
  - *Emission time is very short*
  - *Emission times are statistically independent (each emission - only a small change in electron energy)*

**Purely stochastic (Poisson) process**

# Visible quantum effects

*I have always been somewhat amazed that a purely quantum effect can have gross macroscopic effects in large machines;*

*and, even more,*

*that Planck's constant has just the right magnitude needed to make practical the construction of large electron storage rings.*

*A significantly larger or smaller value of* $\hbar$

*would have posed serious -- perhaps insurmountable -- problems for the realization of large rings.*

*Mathew Sands*

# Quantum excitation of energy oscillations

Photons are emitted with typical energy $u_{ph} \approx \hbar\omega_{typ} = \hbar c \frac{\gamma^3}{\rho}$

at the rate (photons/second) $\mathcal{N} = \frac{P_\gamma}{u_{ph}}$

**Fluctuations in this rate excite oscillations**

During a small interval $\Delta t$ electron emits photons $N = \mathcal{N} \cdot \Delta t$

losing energy of $N \cdot u_{ph}$

Actually, because of fluctuations, the number is $N \pm \sqrt{N}$

resulting in **spread in energy loss** $\pm\sqrt{N} \cdot u_{ph}$

For large time intervals RF compensates the energy loss, providing damping towards the design energy $\boldsymbol{E_0}$

**Steady state**: typical deviations from $\boldsymbol{E_0}$
$\approx$ typical fluctuations in energy during a damping time $\boldsymbol{\tau_\varepsilon}$

# Equilibrium energy spread: rough estimate

We then expect the rms energy spread to be $\sigma_\varepsilon \approx \sqrt{N \cdot \tau_\varepsilon \cdot u_{ph}}$

and since $\tau_\varepsilon \approx \dfrac{E_0}{P_\gamma}$ and $P_\gamma = N \cdot u_{ph}$

$\sigma_\varepsilon \approx \sqrt{E_0 \cdot u_{ph}}$ geometric mean of the electron and photon energies!

Relative energy spread can be written then as:

$$\frac{\sigma_\varepsilon}{E_0} \approx \gamma \sqrt{\frac{\lambda\!\!\!^{-}_e}{\rho}} \qquad \lambda\!\!\!^{-}_e = \frac{\hbar}{m_e c} \simeq 4 \cdot 10^{-13} m$$

it is roughly constant for all rings

- typically $\rho \propto E^2$

$$\frac{\sigma_\varepsilon}{E_0} \sim const \sim 10^{-3}$$

# Equilibrium energy spread

More detailed calculations give

- for the case of an 'isomagnetic' lattice $\rho(s) = \begin{matrix} \rho_0 & \text{in dipoles} \\ \infty & \text{elsewhere} \end{matrix}$

$$\left(\frac{\sigma_\varepsilon}{E}\right)^2 = \frac{C_q E^2}{J_\varepsilon \rho_0}$$

with $$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar c}{(m_e c^2)^3} = 1.468 \cdot 10^{-6} \left[\frac{\mathrm{m}}{\mathrm{GeV}^2}\right]$$

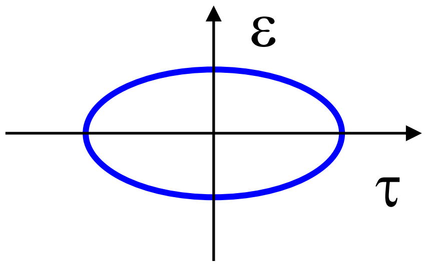
It is difficult to obtain energy spread < 0.1%

- limit on undulator brightness!

# Equilibrium bunch length

Bunch length is related to the energy spread

- Energy deviation and time of arrival
  (or position along the bunch)
  are conjugate variables (synchrotron oscillations)




$$\hat{\tau} = \frac{\alpha}{\Omega_s}\left(\frac{\hat{\varepsilon}}{E}\right)$$

$$\boxed{\sigma_\tau = \frac{\alpha}{\Omega_s}\left(\frac{\sigma_\varepsilon}{E}\right)}$$

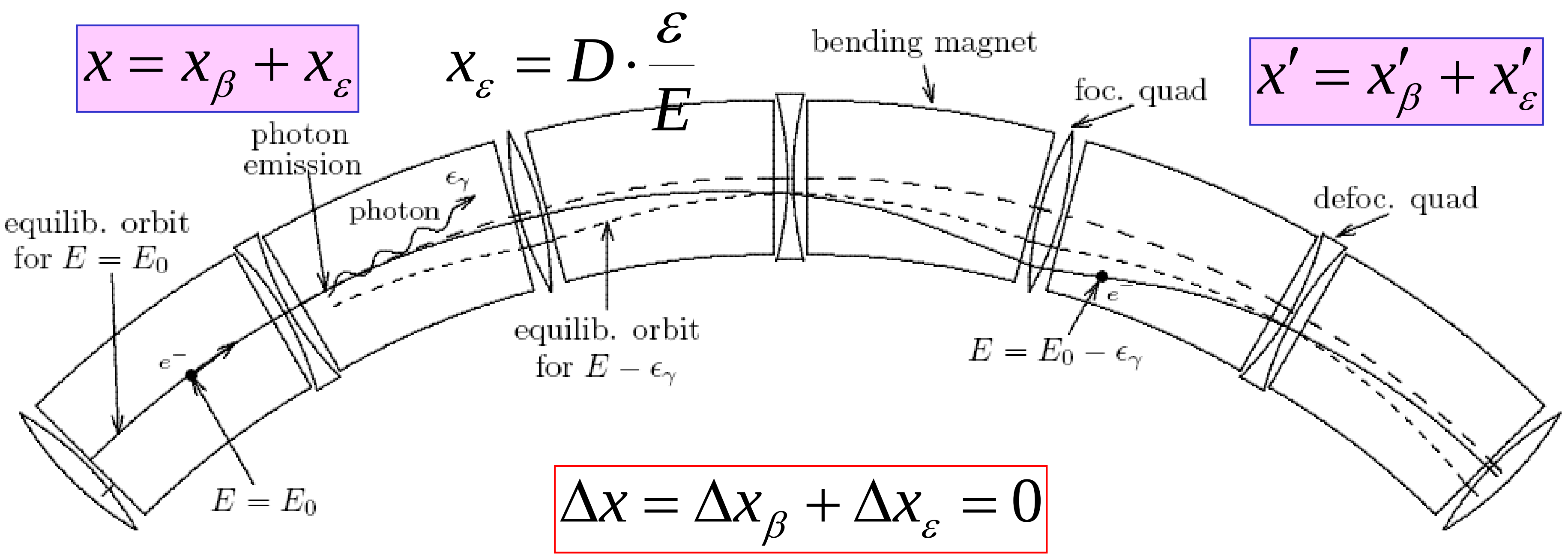
- recall that $\boxed{\Omega_s \propto \sqrt{V_{RF}}}$

Two ways to obtain short bunches:

- RF voltage (power!) $\boxed{\sigma_\tau \propto 1/\sqrt{V_{RF}}}$
- Momentum compaction factor in the limit of $\alpha = 0$
  isochronous ring: particle position along the bunch is frozen

$$\boxed{\sigma_\tau \propto \alpha}$$

# Excitation of betatron oscillations




$$\boxed{x = x_\beta + x_\varepsilon} \qquad x_\varepsilon = D \cdot \frac{\varepsilon}{E} \qquad \boxed{x' = x'_\beta + x'_\varepsilon}$$

$$\boxed{\Delta x = \Delta x_\beta + \Delta x_\varepsilon = 0}$$

$$\boxed{\Delta x_\beta = -D \cdot \frac{\varepsilon_\gamma}{E}} \qquad \boxed{\Delta x'_\beta = -D' \cdot \frac{\varepsilon_\gamma}{E}}$$

Courant Snyder invariant

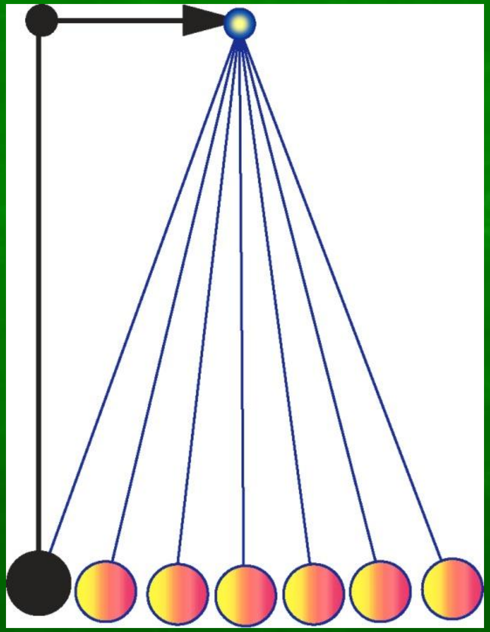
$$\boxed{\Delta\varepsilon = \gamma \Delta x_\beta^2 + 2\alpha \Delta x_\beta \Delta x'_\beta + \beta \Delta x'^2_\beta = \left[\gamma D^2 + 2\alpha D D' + \beta D'^2\right] \cdot \left(\frac{\varepsilon_\gamma}{E}\right)^2}$$

# Excitation of betatron oscillations

Electron emitting a photon

- at a place with non-zero dispersion
- starts a betatron oscillation around a new reference orbit

$$\boxed{x_\beta \approx D \cdot \frac{\varepsilon_\gamma}{E}}$$



# Horizontal oscillations: equilibrium

Emission of photons is a random process

- Again we have random walk, now in **x**. How far particle will wander away is limited by the radiation damping
- The balance is achieved on the time scale of the damping time $\tau_x = 2\ \tau_\varepsilon$

$$\sigma_{x\beta} \approx \sqrt{\mathcal{N} \cdot \tau_x} \cdot D \cdot \frac{\varepsilon_\gamma}{E} = \sqrt{2} \cdot D \cdot \frac{\sigma_\varepsilon}{E}$$

- Typical horizontal beam size ~ 1 mm

**Quantum effect visible to the naked eye!**

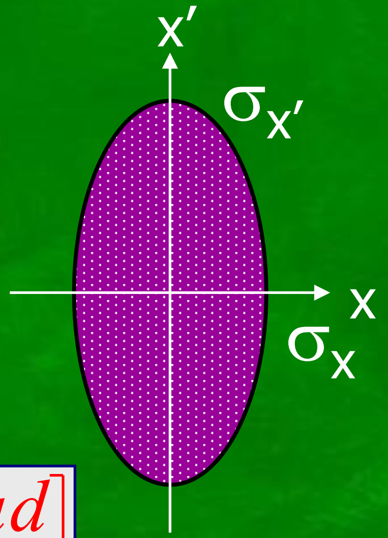
- Vertical size - determined by coupling

# *Beam emittance*

$\text{Area} = \pi \cdot \varepsilon$

## Betatron oscillations

- Particles in the beam execute betatron oscillations with different amplitudes.




## Transverse beam distribution

- Gaussian (electrons)
- "Typical" particle: 1 - $\sigma$ ellipse
  (in a place where $\alpha = \beta' = 0$)

Units of $\varepsilon$ $[m \cdot rad]$

$$\text{Emittance} \equiv \frac{\sigma_x^2}{\beta}$$

$$\varepsilon = \sigma_x \cdot \sigma_{x'}$$

$$\sigma_x = \sqrt{\varepsilon\,\beta}$$

$$\sigma_{x'} = \sqrt{\varepsilon / \beta}$$

$$\beta = \frac{\sigma_x}{\sigma_{x'}}$$

# Equilibrium horizontal emittance

Detailed calculations for isomagnetic lattice

$$\varepsilon_{x0} \equiv \frac{\sigma_{x\beta}^2}{\beta} = \frac{C_q E^2}{J_x} \cdot \frac{\langle \mathcal{H} \rangle_{mag}}{\rho}$$

where

$$\begin{aligned} \mathcal{H} &= \gamma D^2 + 2\alpha D D' + \beta D'^2 \\ &= \frac{1}{\beta}\left[D^2 + (\beta D' + \alpha D)^2\right] \end{aligned}$$

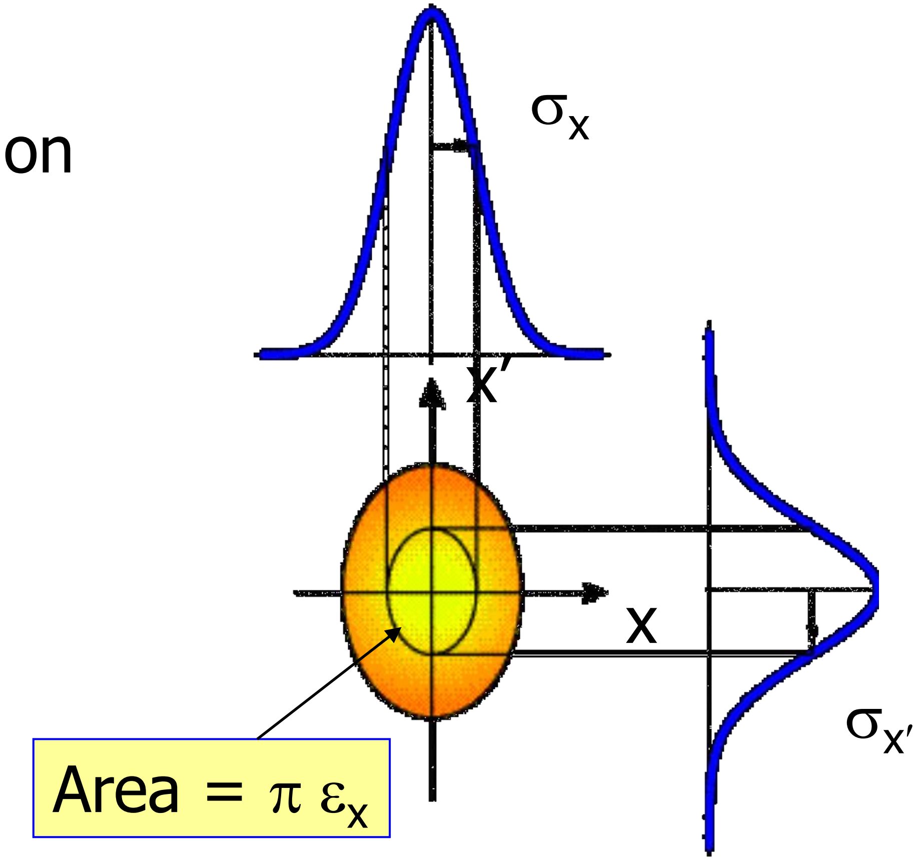
and $\langle \mathcal{H} \rangle_{mag}$ is average value in the bending magnets

# 2-D Gaussian distribution

Electron rings emittance definition

- 1 - $\sigma$ ellipse

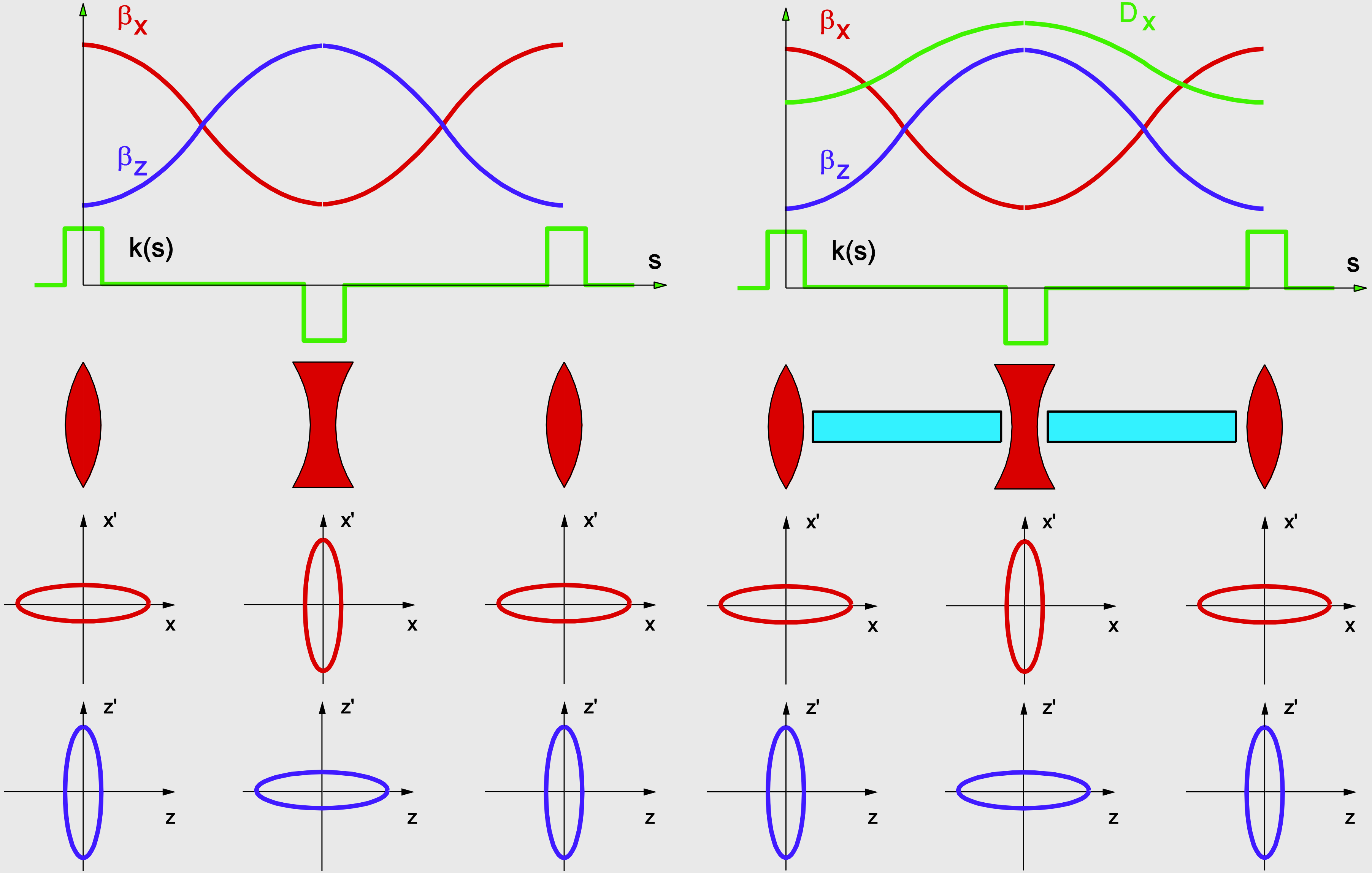
$$n(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2} dx$$



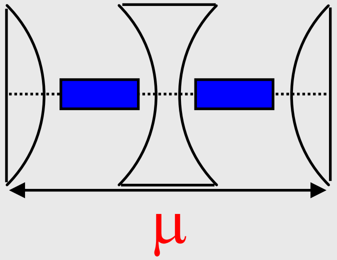

- Probability to be inside 1-$\sigma$ ellipse $P_1 = 1 - e^{-1/2} = 0.39$

- Probability to be inside n-$\sigma$ ellipse $P_n = 1 - e^{-n^2/2}$

# FODO cell lattice



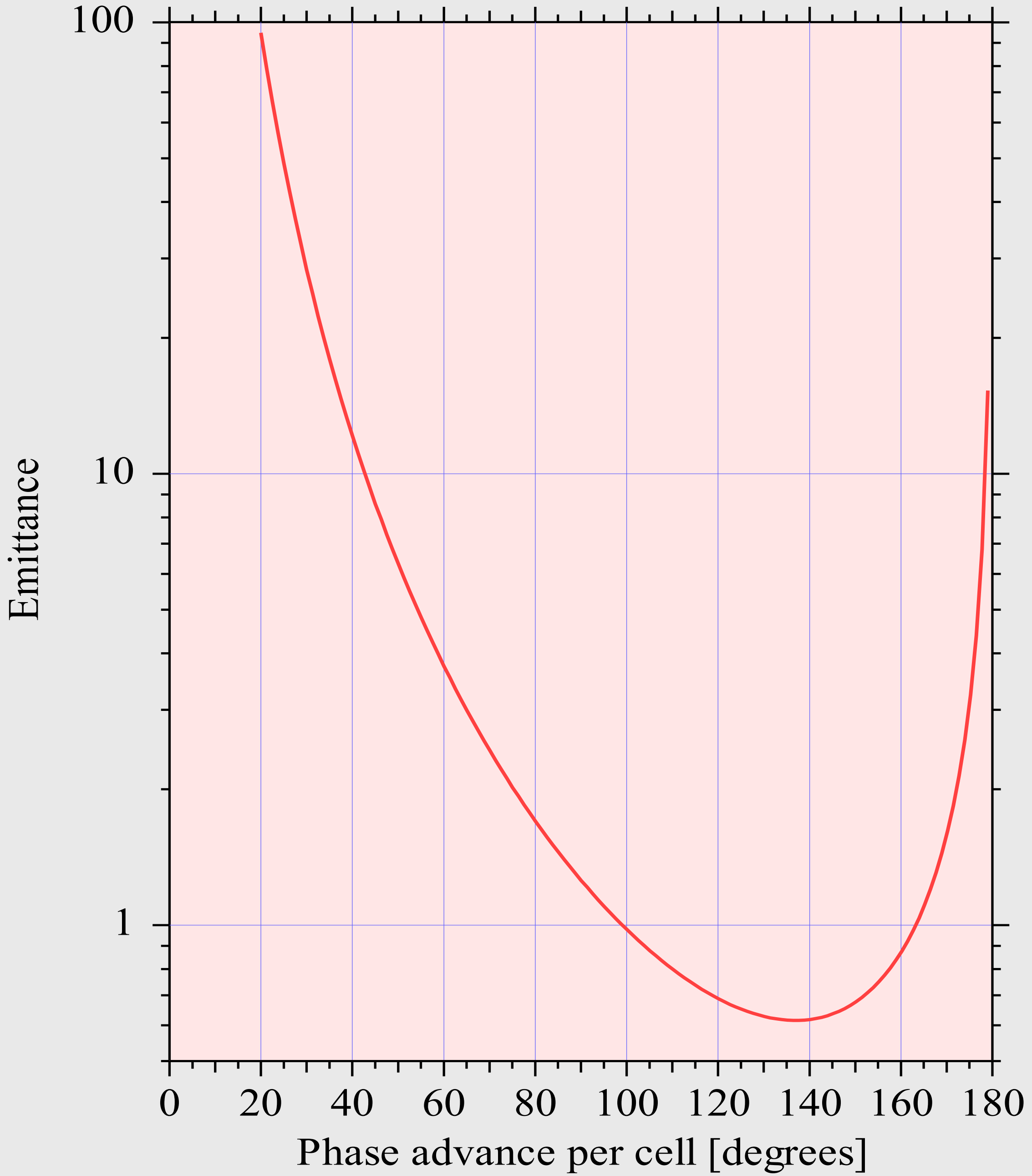
# FODO lattice emittance



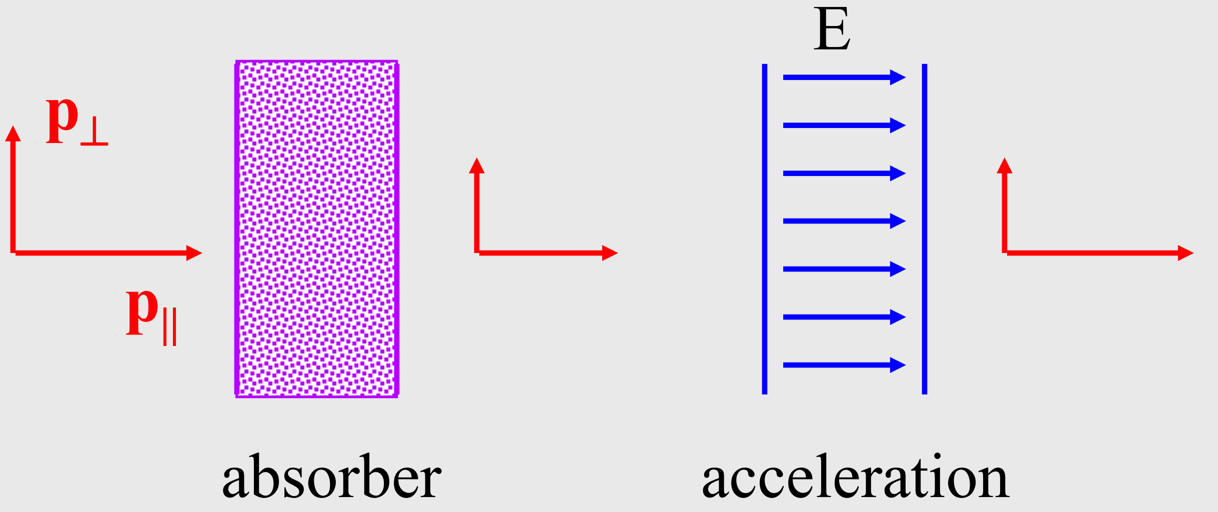

$$\boxed{\mathcal{H} \sim \frac{D^2}{\beta} \sim \frac{R}{Q^3}}$$

$$\boxed{\varepsilon_{x0} \approx \frac{C_q E^2}{J_x} \cdot \frac{R}{\rho} \cdot \frac{1}{Q^3}}$$

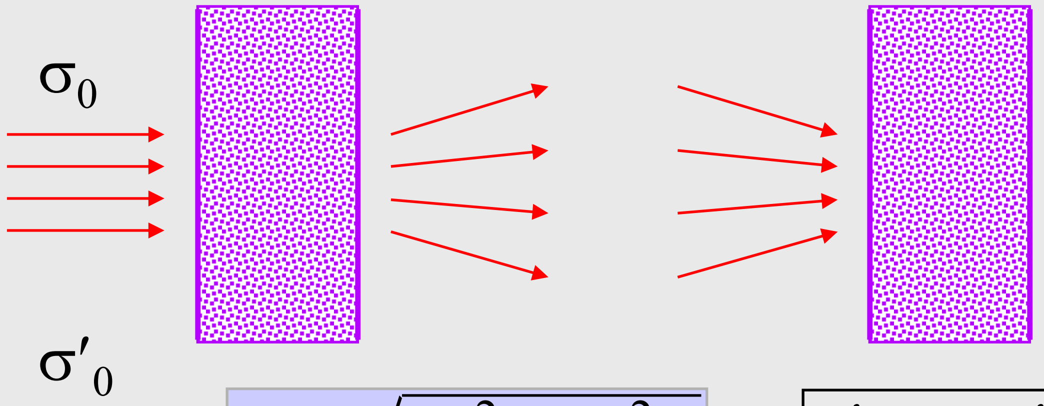
$$\boxed{\varepsilon \propto \frac{\mathbf{E}^2}{\mathbf{J_x}} \theta^3 F_{FODO}(\mu)}$$



# Ionization cooling




similar to radiation damping, but there is multiple scattering in the absorber that blows up the emittance

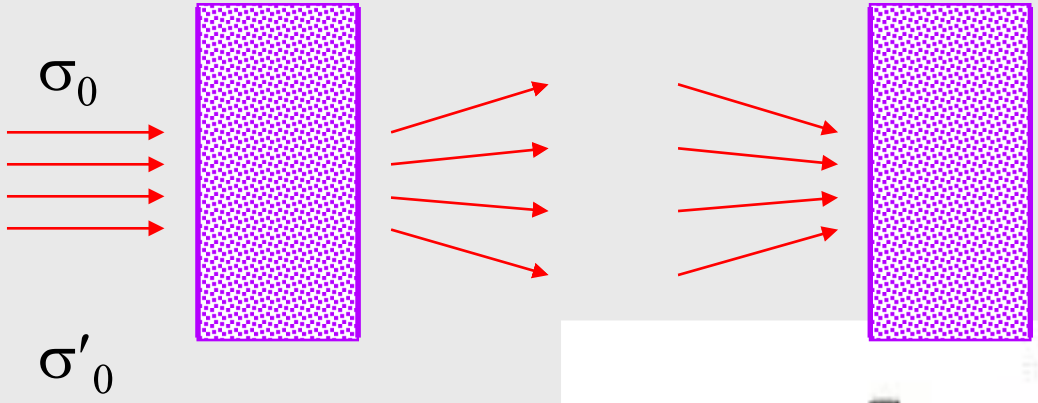



$$\sigma' = \sqrt{\sigma'^2_0 + \sigma'^2_{MS}}$$

$$\sigma'_0 >> \sigma'_{MS}$$

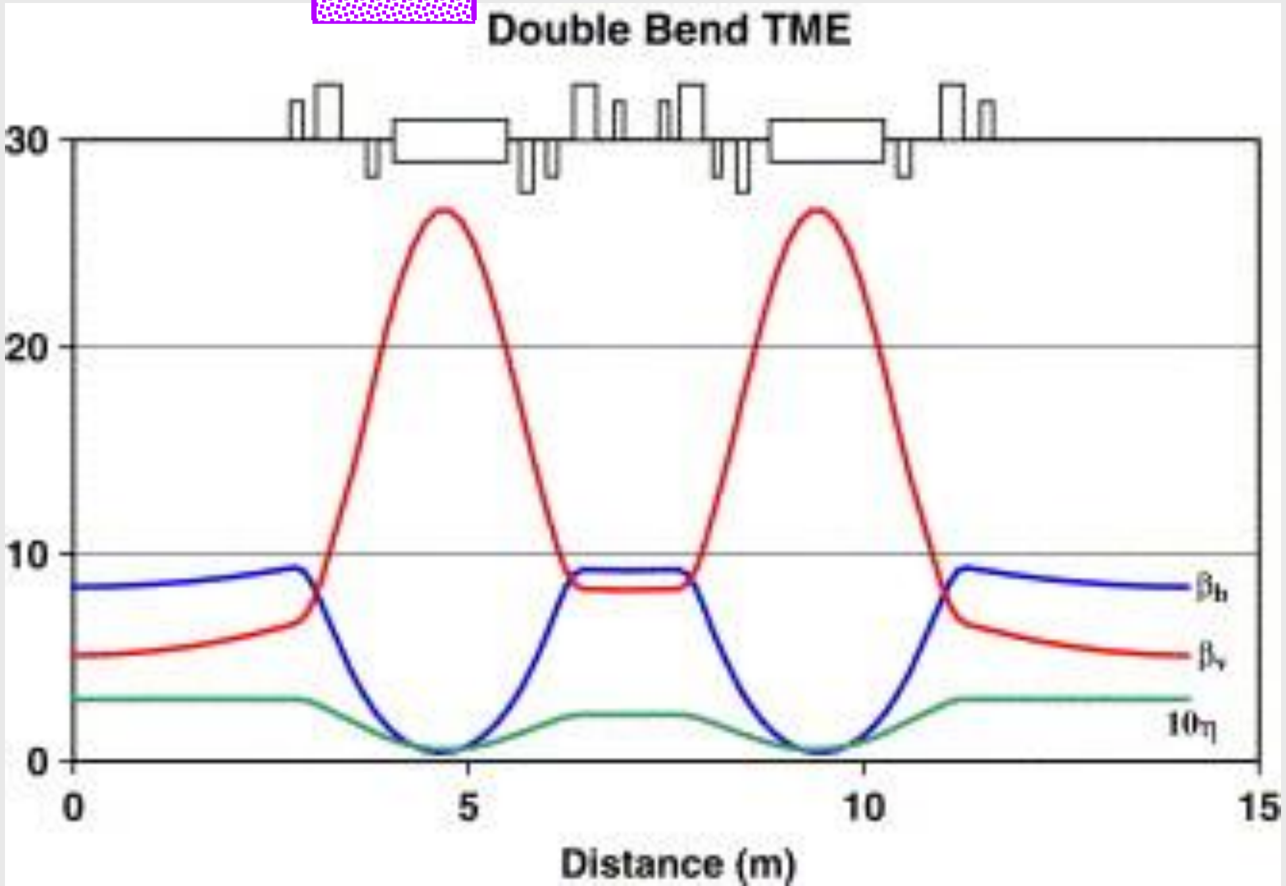
to minimize the blow up due to multiple scattering in the absorber we can **focus** the beam

# Minimum emittance lattices




$$\varepsilon_{x0} = \frac{C_q E^2}{J_x} \cdot \theta^3 \cdot F_{\mathrm{latt}}$$

$$F_{\min} = \frac{1}{12\sqrt{15}}$$



# Quantum limit on emittance

- Electron in a storage ring’s dipole fields is accelerated, interacts with vacuum fluctuations: «accelerated thermometers show increased temperature»

- synchrotron radiation opening angle is ~ $1/\gamma$ -> a lower limit on equilibrium vertical emittance

- independent of energy

$$\epsilon_y = \frac{13}{55} C_q \frac{\oint \beta_y(s)|G^3(s)|ds}{\oint G^2(s)ds}$$

$G(s)$ =curvature, $C_q$ = 0.384 pm

- in case of SLS: 0.2 pm

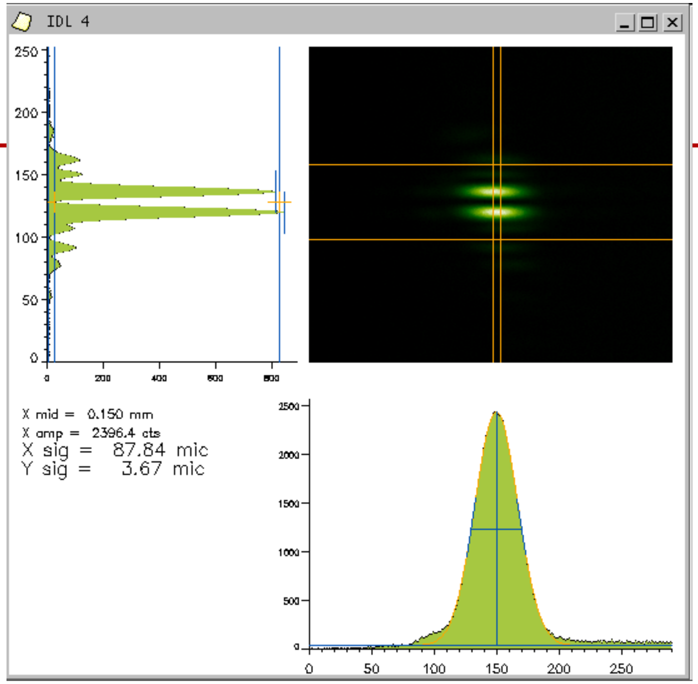
isomagnetic lattice

$$\varepsilon_y = 0.09\,\mathrm{pm} \cdot \frac{\langle \beta_y \rangle_{\mathrm{Mag}}}{\rho}$$

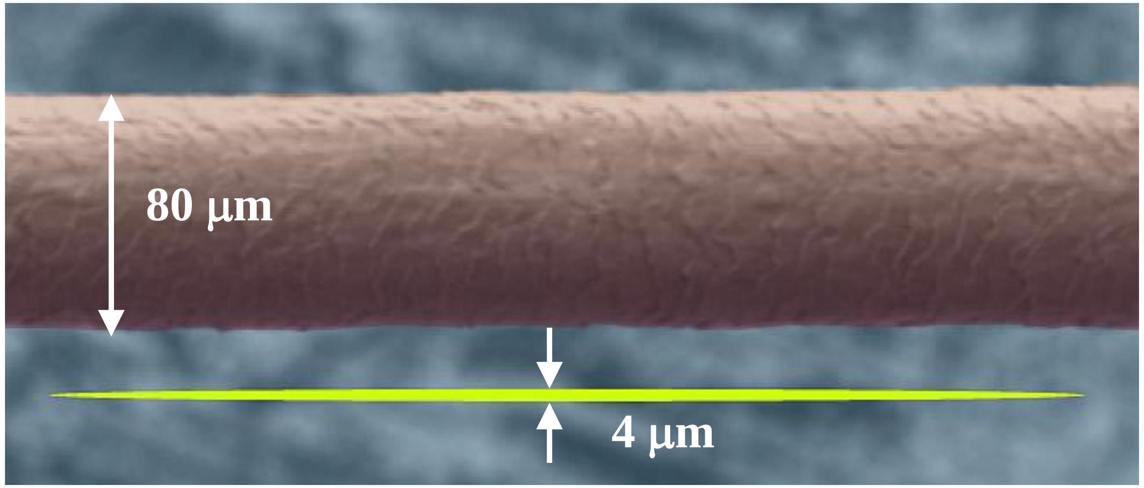
# Vertical emittance record

Beam size **3.6 ± 0.6 μm**

Emittance **0.9 ± 0.4 pm**




SLS beam cross section compared to a human hair:



# Summary of radiation integrals

Momentum compaction factor

$$\boxed{\alpha = \frac{I_1}{2\pi R}}$$

Energy loss per turn

$$\boxed{U_0 = \frac{1}{2\pi} C_\gamma E^4 \cdot I_2}$$

$$C_\gamma = \frac{4\pi}{3} \frac{r_e}{(m_e c^2)^3} = 8.858 \cdot 10^{-5} \left[\frac{\mathrm{m}}{\mathrm{GeV}^3}\right]$$

$$I_1 = \oint \frac{D}{\rho}\, ds$$

$$I_2 = \oint \frac{ds}{\rho^2}$$

$$I_3 = \oint \frac{ds}{|\rho^3|}$$

$$I_4 = \oint \frac{D}{\rho}\left(2k + \frac{1}{\rho^2}\right) ds$$

$$I_5 = \oint \frac{\mathcal{H}}{|\rho^3|}\, ds$$

# Summary of radiation integrals (2)

Damping parameter

$$\boxed{\mathcal{D} = \frac{I_4}{I_2}}$$

Damping times, partition numbers

$$\boxed{J_\varepsilon = 2 + \mathcal{D}\,, \quad J_x = 1 - \mathcal{D}\,, \quad J_y = 1}$$

$$\boxed{\tau_i = \frac{\tau_0}{J_i}} \qquad \boxed{\tau_0 = \frac{2ET_0}{U_0}}$$

Equilibrium energy spread

$$\boxed{\left(\frac{\sigma_\varepsilon}{E}\right)^2 = \frac{C_q E^2}{J_\varepsilon} \cdot \frac{I_3}{I_2}}$$

Equilibrium emittance

$$\boxed{\varepsilon_{x0} = \frac{\sigma_{x\beta}^2}{\beta} = \frac{C_q E^2}{J_x} \cdot \frac{I_5}{I_2}}$$

$$I_1 = \oint \frac{D}{\rho}\, ds$$

$$I_2 = \oint \frac{ds}{\rho^2}$$

$$I_3 = \oint \frac{ds}{|\rho^3|}$$

$$I_4 = \oint \frac{D}{\rho}\left(2k + \frac{1}{\rho^2}\right) ds$$

$$I_5 = \oint \frac{\mathcal{H}}{|\rho^3|}\, ds$$

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar c}{(m_e c^2)^3} = 1.468 \cdot 10^{-6} \left[\frac{\mathrm{m}}{\mathrm{GeV}^2}\right]$$

$$\mathcal{H} = \gamma D^2 + 2\alpha D D' + \beta D'^2$$

# Damping wigglers

Increase the radiation loss per turn $U_0$ with WIGGLERS

- reduce damping time

$$\boxed{\tau = \frac{E}{P_\gamma + P_{wig}}}$$

- emittance control

wigglers at high dispersion: blow-up emittance

e.g. storage ring colliders for high energy physics

wigglers at zero dispersion: decrease emittance

e.g. damping rings for linear colliders
e.g. synchrotron light sources (PETRAIII, 1 nm.rad)

# END OF LECTURES

# FOR DREAMING AND DISCUSSION

# Does charged particle radiate in gravitational field?

ChatGPT

Yes, charged particles can radiate in a gravitational field, but the circumstances under which this happens are different from radiation in an electromagnetic field.

In electromagnetism, an accelerating charged particle emits radiation. This is well-established and is the basis for many phenomena and applications, such as radio wave transmission.

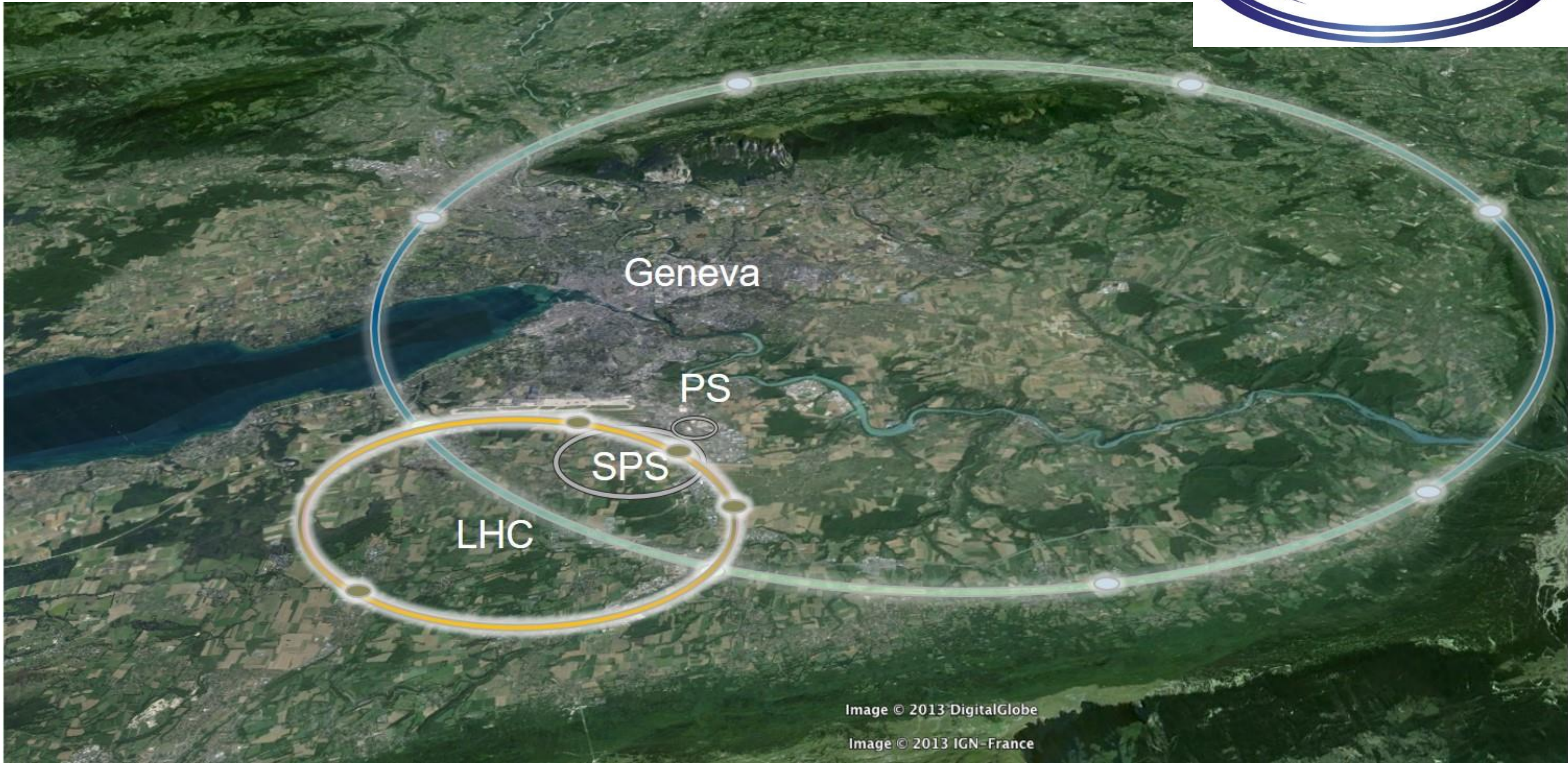
In general relativity, which describes gravitation, things are a bit more nuanced. A free-falling charged particle in a gravitational field, following a geodesic (the equivalent of a “straight line” in curved spacetime), does not radiate from the perspective of another free-falling observer. This is because, from the viewpoint of the falling particle, it isn’t accelerating—it’s in free fall.

However, if a charged particle is held stationary in a gravitational field (against the gravitational pull, perhaps by a tether or some other force), or if it is made to move in some non-geodesic way, then it can be seen as accelerating and can emit radiation.

This kind of radiation in a gravitational field is not as commonly encountered or as well-understood in practical terms as electromagnetic radiation from accelerating charges in empty space, but the underlying principles from general relativity suggest that radiation would indeed be emitted under the right circumstances.

CAN YOU DO BETTER?

# Future Circular Collider Study: 100 km, 100 TeV



# Some quick calculations for FCC-hh

- FCC-hh center-of-mass energy 100 TeV. Bending radius for 16 Tesla field?
- Proton Lorentz factor?
- Dipole field 16 Tesla. Critical photon energy?
- Loss per turn:
- Damping time:
- In LEP we shortened the damping time using wigglers. Could we do it here?