Synchrotron Light:

Electron Beam Dynamics

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Curved orbit of electrons in magnet field

Accelerated charge → Electromagnetic radiation
Electromagnetic waves or photons
Synchrotron radiation: some dates

- 1873  Maxwell’s equations
- 1887  Hertz: electromagnetic waves
- 1898  Liénard: retarded potentials
- 1900  Wiechert: retarded potentials
- 1908  Schott: Adams Prize Essay

... waiting for accelerators ...
1940: 2.3 MeV betatron, Kerst, Serber
War es ein Gott, der diese Zeichen schrieb
Die mit geheimnisvoll verborg’nem Trieb
Die Kräfte der Natur um mich enthüllen
Und mir das Herz mit stiller Freude füllen.
Ludwig Boltzmann

Was it a God whose inspiration
Led him to write these fine equations
Nature’s fields to me he shows
And so my heart with pleasure glows.
translated by John P. Blewett
Synchrotron radiation: some dates

- 1873  Maxwell’s equations
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THEORETICAL UNDERSTANDING →

1873  Maxwell’s equations

→ made evident that changing charge densities would result in electric fields that would radiate outward

1887  Heinrich Hertz demonstrated such waves:

It’s of no use whatsoever[...] this is just an experiment that proves Maestro Maxwell was right—we just have these mysterious electromagnetic waves that we cannot see with the naked eye. But they are there.
Synchrotron radiation: some dates

- 1873: Maxwell’s equations
- 1887: Hertz: electromagnetic waves
- 1898: Liénard: retarded potentials
- 1900: Wiechert: retarded potentials
- 1908: Schott: Adams Prize Essay (330 pages)

... waiting for accelerators ...

1940: 2.3 MeV betatron, Kerst, Serber
Donald Kerst: first betatron (1940)

"Ausserordentlich hochgeschwindigkeitelektronen entwickelnden schwerarbeitsbeigollitron"
Synchrotron radiation: some dates

- 1946  Blewett observes energy loss due to synchrotron radiation, 100 MeV betatron
- 1947  First visual observation of SR, 70 MeV synchrotron, GE Lab
- 1949  Schwinger PhysRev paper
- 1976  Madey: first demonstration of Free Electron laser
Why do they radiate?
Synchrotron Radiation is not as simple as it seems

... I will try to show that it is much simpler
Charge at rest
Coulomb field, no radiation
Uniformly moving charge does not radiate

\[ v = \text{constant} \]

But! Cerenkov!
Free isolated electron cannot emit a photon

Easy proof using 4-vectors and relativity

- momentum conservation if a photon is emitted

\[ P_i = P_f + P_\gamma \]

- square both sides

\[ m^2 = m^2 + 2P_f \cdot P_\gamma + 0 \Rightarrow P_f \cdot P_\gamma = 0 \]

- in the rest frame of the electron

\[ P_f = (m, 0) \quad P_\gamma = (E_\gamma, p_\gamma) \]

this means that the photon energy must be zero.
We need to separate the field from charge
Bremsstrahlung
or
“braking” radiation
Transition Radiation

\[ c_1 = \frac{1}{\sqrt{\varepsilon_1 \mu_1}} \quad c_2 = \frac{1}{\sqrt{\varepsilon_2 \mu_2}} \]
Liénard–Wiechert potentials

\[ \varphi(t) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r(1 - \mathbf{n} \cdot \mathbf{\beta})_{ret}} \]

\[ \vec{A}(t) = \frac{q}{4\pi\varepsilon_0 c^2} \left[ \frac{\vec{v}}{r(1 - \mathbf{n} \cdot \mathbf{\beta})_{ret}} \right] \]

and the electromagnetic fields:

\[ \nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} = 0 \quad \text{(Lorentz gauge)} \]

\[ \vec{B} = \nabla \times \vec{A} \]

\[ \vec{E} = -\nabla \varphi - \frac{\partial \vec{A}}{\partial t} \]
Fields of a moving charge

\[
\vec{E}(t) = \frac{q}{4\pi\varepsilon_0} \left[ \frac{\hat{n} - \beta}{(1 - \hat{n} \cdot \beta)^3 \gamma^2} \cdot \frac{1}{r^2} \right]_{ret} + \text{“near field”}
\]

\[
\frac{q}{4\pi\varepsilon_0 c} \left[ \hat{n} \times \left( \hat{n} - \beta \right) \times \beta \right] \cdot \frac{1}{r} \right]_{ret} \text{ “far field”}
\]

\[
\vec{B}(t) = \frac{1}{c} \left[ \hat{n} \times \vec{E} \right]
\]
Energy flow integrated over a sphere

*Power* ~ $E^2 \cdot \text{Area}$

$$A = 4\pi r^2$$

Near field:
$$P \propto \frac{1}{r^4} r^2 \propto \frac{1}{r^2}$$

Far field:
$$P \propto \frac{1}{r^2} r^2 \propto \text{const}$$

*Radiation* = constant flow of energy to infinity
Transverse acceleration

Radiation field quickly separates itself from the Coulomb field
Radiation field cannot separate itself from the Coulomb field
Synchrotron Radiation
Basic Properties
Beams of ultra-relativistic particles: e.g. a race to the Moon

An electron with energy of a few GeV emits a photon... a race to the Moon!

\[ \Delta t = \frac{L}{\beta c} - \frac{L}{c} = \frac{L}{\beta c} (1 - \beta) \approx \frac{L}{\beta c} \cdot \frac{1}{2\gamma^2} \]

Electron will lose
- by only 8 meters
- the race will last only 1.3 seconds

\[ \Delta L = L (1 - \beta) \approx \frac{L}{2\gamma^2} \]

\[ \beta \equiv \frac{v}{c} \]

\[ \gamma \equiv \frac{E}{mc^2} = \frac{1}{\sqrt{1 - \beta^2}} \]
Moving Source of Waves: Doppler effect

“redshift”

“blueshift”
Time compression

Electron with velocity $\beta$ emits a wave with period $T_{\text{emit}}$ while the observer sees a different period $T_{\text{obs}}$ because the electron was moving towards the observer.

\[ T_{\text{obs}} = (1 - \mathbf{n} \cdot \beta) T_{\text{emit}} \]

The wavelength is shortened by the same factor in ultra-relativistic case, looking along a tangent to the trajectory since

\[ \lambda_{\text{obs}} = (1 - \beta \cos \theta) \lambda_{\text{emit}} \]

\[ \lambda_{\text{obs}} = \frac{1}{2\gamma^2} \lambda_{\text{emit}} \]

\[ 1 - \beta = \frac{1 - \beta^2}{1 + \beta} \approx \frac{1}{2\gamma^2} \]
Radiation is emitted into a narrow cone

\[ \theta = \frac{1}{\gamma} \cdot \theta_e \]

- \( v \approx c \)
- \( V \ll C \)
- \( V \approx C \)
Sound waves (non-relativistic)

Angular collimation

\[ \theta = \frac{v_{s\perp}}{v_{s||} + v} \approx \theta_e \cdot \frac{1}{1 + \frac{v}{v_s}} \]

Doppler effect (moving source of sound)

\[ \lambda_{\text{heard}} = \lambda_{\text{emitted}} \left( 1 - \frac{v}{v_s} \right) \]
Synchrotron radiation power

Power emitted is proportional to:

\[ P \propto E^2 B^2 \]

\[ P_\gamma = \frac{cC_\gamma \cdot E^4}{2\pi \rho^2} \]

\[ C_\gamma = \frac{4\pi}{3} \frac{r_e}{(m_e c^2)^3} = 8.858 \cdot 10^{-5} \left[ \frac{m}{\text{GeV}^3} \right] \]
The power is all too real!
Synchrotron radiation power

Power emitted is proportional to:

\[ P_\gamma = \frac{c C_\gamma}{2\pi} \cdot \frac{E^4}{\rho^2} \]

\[ C_\gamma = \frac{4\pi}{3} \frac{r_e}{(m_e c^2)^{\frac{3}{2}}} = 8.858 \cdot 10^{-5}\left[\frac{m}{\text{GeV}^3}\right] \]

Energy loss per turn:

\[ U_0 = C_\gamma \cdot \frac{E^4}{\rho} \]

Energy emitted is proportional to:

\[ P \propto E^2 B^2 \]

\[ P_\gamma = \frac{2}{3} \alpha \hbar c^2 \cdot \frac{\gamma^4}{\rho} \]

\[ \alpha = \frac{1}{137} \]

\[ \hbar c = 197 \text{ Mev} \cdot \text{fm} \]

\[ U_0 = \frac{4\pi}{3} \alpha \hbar c \gamma^4 \frac{\rho}{\rho} \]
Typical frequency of synchrotron light

Due to extreme collimation of light observer sees only a small portion of electron trajectory (a few mm)

\[ l \sim \frac{2\rho}{\gamma} \]

Pulse length: difference in times it takes an electron and a photon to cover this distance

\[ \Delta t \sim \frac{l}{\beta c} - \frac{l}{c} = \frac{l}{\beta c}(1 - \beta) \]

\[ \omega \sim \frac{1}{\Delta t} \sim \gamma^3 \omega_0 \]

\[ \Delta t \sim \frac{2\rho}{\gamma c} \cdot \frac{1}{2\gamma^2} \]
Short magnet: higher energy photons

When Lorentz factor is not very high (e.g. protons)...

\[ l \ll \frac{2\rho}{\gamma} \]

Pulse length: difference in times it takes an electron and a photon to cover this distance

\[ \Delta t \sim \frac{l}{c} \frac{1}{2\gamma^2} \]

\[ \Delta t \sim \frac{l}{\beta c} - \frac{l}{c} = \frac{l}{\beta c} (1 - \beta) \]
Spectrum of synchrotron radiation

- Synchrotron light comes in a series of flashes every $T_0$ (revolution period)

- The spectrum consists of harmonics of

- Flashes are extremely short: harmonics reach up to very high frequencies

- At high frequencies the individual harmonics overlap

$$\omega_0 = \frac{1}{T_0}$$

$$\omega_{\text{typ}} \approx \gamma^3 \omega_0$$

$$\omega_0 \sim 1 \text{ MHz}$$

$$\gamma \sim 4000$$

$$\omega_{\text{typ}} \sim 10^{16} \text{ Hz}!$$

Continuous spectrum!
**Wavelength continuously tunable!**

![Electromagnetic Spectrum Diagram](image-url)

<table>
<thead>
<tr>
<th>Wavelength Size</th>
<th>Soccer Field</th>
<th>House</th>
<th>Baseball</th>
<th>This Period</th>
<th>Cell</th>
<th>Bacteria</th>
<th>Virus</th>
<th>Protein</th>
<th>Water Molecule</th>
</tr>
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<tbody>
<tr>
<td>Meter</td>
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<td></td>
<td>$10^3$</td>
<td>$10^2$</td>
<td>$10^1$</td>
<td>$10^{-1}$</td>
<td>$10^{-2}$</td>
<td>$10^{-3}$</td>
<td>$10^{-4}$</td>
<td>$10^{-5}$</td>
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</table>

<table>
<thead>
<tr>
<th>Common Name</th>
<th>Radio Waves</th>
<th>Infrared</th>
<th>Ultraviolet</th>
<th>&quot;Hard&quot; X Rays</th>
<th>Microwaves</th>
<th>&quot;Soft&quot; X Rays</th>
<th>Gamma Rays</th>
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<tr>
<th>Accelerator-Based Light Sources</th>
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<tbody>
<tr>
<td>AM Radio</td>
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<thead>
<tr>
<th>Common Sources</th>
<th>PEOPLE</th>
<th>LIGHT BULB</th>
<th>UV LAMP</th>
<th>X-RAY MACHINE</th>
</tr>
</thead>
</table>

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<tr>
<th>Frequency (waves per second)</th>
<th>$10^6$</th>
<th>$10^7$</th>
<th>$10^8$</th>
<th>$10^9$</th>
</tr>
</thead>
</table>

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<tr>
<th>Energy of One Photon (eV)</th>
<th>$10^{-9}$</th>
<th>$10^{-8}$</th>
<th>$10^{-7}$</th>
<th>$10^{-6}$</th>
</tr>
</thead>
</table>

Wavelength continuously tunable!
\[ \frac{dP}{d\omega} = \frac{P_{\text{tot}}}{\omega_c} S \left( \frac{\omega}{\omega_c} \right) \]
\[ S(x) = \frac{9\sqrt{3}}{8\pi} x \int_x^\infty K_{\gamma}^3(x') dx' \]
\[ \int_0^\infty S(x') dx' = 1 \]

\[ P_{\text{tot}} = \frac{2}{3} \hbar c^2 \alpha \gamma^4 \rho^2 \]

\[ \omega_c = \frac{3}{2} c \gamma^3 \rho \]

\[ \varepsilon_{\text{eV}} = 665 E^2 \text{[GeV]} B \text{[T]} \]
Beamstrahlung

Synchrotron radiation in the collective field of the bunch

The onset of the quantum regime: the critical photon energy, calculated with classical formulae can exceed the electron energy! Need to take into account the recoil.

\[ \varepsilon_c [GeV] = 0.664 \cdot E^2 [TeV] \cdot B [T] \]

- Center of mass collision energy is not well defined
- Backgrounds: direct synchrotron radiation
- Backgrounds: pair production from high energy photons
Fields of a long bunch (linear charge density $\lambda$)

Transverse electric field: from Gauss law

$$E_r = \frac{\lambda}{2\pi \varepsilon_0 r}$$

$$2\pi r \cdot E_r = \frac{\lambda}{\varepsilon_0}$$

Transverse magnetic field: from Ampere law

$$B_\theta = \frac{\mu_0 \lambda}{2\pi r} \cdot \frac{v}{c^2} = \frac{\lambda}{2\pi \varepsilon_0 r} \cdot \frac{v}{c^2}$$

$$2\pi r \cdot B_\theta = \mu_0 I$$

$$\mu_0 \varepsilon_0 = \frac{1}{c^2}$$

$$\mu_0 = 4\pi \cdot 10^{-7} \frac{V \cdot s}{A \cdot m}$$

$$\varepsilon_0 = 8.85 \cdot 10^{-12} \frac{C}{V \cdot m}$$
Fields in the bunch

- Round uniform distribution

\[ E_r = \frac{eN}{2\pi\varepsilon_0 l} \cdot \frac{1}{r} \quad r > a \]
\[ E_r = \frac{eN}{2\pi\varepsilon_0 l} \cdot \frac{r}{a^2} \quad r < a \]

- Round Gaussian distribution

\[ E_r = \frac{eN}{2\pi\varepsilon_0 l\sigma} \left[ 1 - e^{-\frac{1}{2}(\frac{r}{\sigma})^2} \right] \]
Useful books and references

H. Wiedemann, *Synchrotron Radiation*
Springer-Verlag Berlin Heidelberg 2003

H. Wiedemann, *Particle Accelerator Physics*
Springer, 2015 [Open Access]

A. Hofmann, *The Physics of Synchrotron Radiation*
Cambridge University Press 2004

Radiation is emitted into a narrow cone

\[ \theta = \frac{1}{\gamma} \cdot \theta_e \]
Radiation effects in electron storage rings

Average radiated power restored by RF

- Electron loses energy each turn to synchrotron radiation
- RF cavities accelerate electrons back to the nominal energy

Radiation damping

- Average rate of energy loss produces **DAMPING** of electron oscillations in all three degrees of freedom (if properly arranged!)

Quantum fluctuations

- Statistical fluctuations in energy loss (from quantized emission of radiation) produce **RANDOM EXCITATION** of these oscillations

Equilibrium distributions

- The balance between the damping and the excitation of the electron oscillations determines the equilibrium distribution of particles in the beam

\[ U_0 \approx 10^{-3} \text{ of } E_0 \]

\[ V_{RF} > U_0 \]
Radiation damping

Transverse oscillations
**Average energy loss and gain per turn**

- Every turn electron radiates small amount of energy
  
  \[ E_1 = E_0 - U_0 = E_0 \left(1 - \frac{U_0}{E_0}\right) \]

- Only the amplitude of the momentum changes
  
  \[ P_1 = P_0 - \frac{U_0}{c} = P_0 \left(1 - \frac{U_0}{E_0}\right) \]

- Only the longitudinal component of the momentum is increased in the RF cavity

- Energy of betatron oscillation
  
  \[ E_\beta \propto A^2 \]

\[ A_1^2 = A_0^2 \left(1 - \frac{U_0}{E_0}\right) \quad \text{or} \quad A_1 \approx A_0 \left(1 - \frac{U_0}{2E_0}\right) \]
Damping of vertical oscillations

- But this is just the exponential decay law!
  \[
  \frac{\Delta A}{A} = -\frac{U_0}{2E}
  \]
  \[
  A = A_0 \cdot e^{-t/\tau}
  \]

- The oscillations are exponentially damped with the damping time (milliseconds!)
  \[
  \tau = \frac{2ET_0}{U_0}
  \]
  the time it would take particle to ‘lose all of its energy’

- In terms of radiation power
  \[
  \tau = \frac{2E}{P_\gamma}
  \]
  and since
  \[
  P_\gamma \propto E^4
  \]
  \[
  \tau \propto \frac{1}{E^3}
  \]
Adiabatic damping in linear accelerators

In a **linear accelerator**: 

\[ x' = \frac{p_{\perp}}{p} \text{ decreases } \propto \frac{1}{E} \]

In a **storage ring** beam passes many times through same RF cavity

- Clean loss of energy every turn (no change in \( x' \))
- Every turn is re-accelerated by RF (\( x' \) is reduced)
- Particle energy on average remains constant
Emittance damping in linacs:

\[ \varepsilon \propto \frac{1}{\gamma} \]

or

\[ \gamma \varepsilon = \text{const.} \]
Radiation damping

Longitudinal oscillations
Longitudinal motion: compensating radiation loss $U_0$

- RF cavity provides accelerating field with frequency
  - $h$ - harmonic number

- The energy gain:
  
  $$U_{RF} = eV_{RF}(\tau)$$

- Synchronous particle:
  - has design energy
  - gains from the RF on the average as it loses per turn $U_0$
Longitudinal motion: phase stability

- **Particle ahead of synchronous one**
  - gets too much energy from the RF
  - goes on a longer orbit (not enough B)
    - takes longer to go around
  - comes back to the RF cavity closer to synchronous part.

- **Particle behind the synchronous one**
  - gets too little energy from the RF
  - goes on a shorter orbit (too much B)
  - catches up with the synchronous particle
Longitudinal motion: energy-time oscillations

energy deviation from the design energy, or the energy of the synchronous particle

longitudinal coordinate measured from the position of the synchronous electron
Orbit Length

Length element depends on $x$

$$dl = (1 + \frac{x}{\rho})ds$$

Horizontal displacement has two parts:

- To first order $x_\beta$ does not change $L$
- $x_\epsilon$ – has the same sign around the ring

Length of the off-energy orbit

$$L_\epsilon = \int dl = \int (1 + \frac{x_\epsilon}{\rho})ds = L_0 + \Delta L$$

$$\Delta L = \delta \cdot \int \frac{D(s)}{\rho(s)} ds$$ where \( \delta = \frac{\Delta p}{p} = \frac{\Delta E}{E} \)

$$\frac{\Delta L}{L} = \alpha \cdot \delta$$
Something funny happens on the way around the ring...

Revolution time changes with energy

\[ \Delta T = \frac{\Delta L}{T} = \frac{\Delta \beta}{\beta} \]

- Particle goes faster (not much!)

- while the orbit length increases (more!)

- The “slip factor” \( \eta \approx \alpha \) since \( \alpha \gg \frac{1}{\gamma^2} \)

\[ \frac{\Delta T}{T} = (\alpha - \frac{1}{\gamma^2}) \cdot \frac{dp}{p} = \eta \cdot \frac{dp}{p} \]

- Ring is above “transition energy”

isochronous ring: \( \eta = 0 \) or \( \gamma = \gamma_{tr} \)

\[ T_0 = \frac{L_0}{c\beta} \]

\[ \frac{d\beta}{\beta} = \frac{1}{\gamma^2} \cdot \frac{dp}{p} \] (relativity)

\[ \frac{\Delta L}{L} = \alpha \cdot \frac{dp}{p} \]

\[ \alpha \equiv \frac{1}{\gamma_{tr}^2} \]

\[ \eta = 0 \] or \( \gamma = \gamma_{tr} \)
Not only accelerators work above transition

Dante Alighieri
Divine Comedy
RF Voltage

\[ V(\tau) = \hat{V} \sin(h\omega_0 \tau + \psi_s) \]

here the synchronous phase

\[ \psi_s = \arcsin\left(\frac{U_0}{e\hat{V}}\right) \]
Momentum compaction factor

Like the tunes $Q_x$, $Q_y$ - $\alpha$ depends on the whole optics

- A quick estimate for separated function guide field:

$$\alpha = \frac{1}{L_0 \rho_0} \int_{mag} D(s) ds = \frac{1}{L_0 \rho_0} \langle D \rangle \cdot L_{mag}$$

- But

$$L_{mag} = 2\pi \rho_0$$

- Since dispersion is approximately

$$D \approx \frac{R}{Q^2} \Rightarrow \alpha \approx \frac{1}{Q^2} \text{ typically < 1%}$$

and the orbit change for $\sim 1\%$ energy deviation

$$\frac{\Delta L}{L} = \frac{1}{Q^2} \cdot \delta \approx 10^{-4}$$
Energy balance

Energy gain from the RF system: \[ U_{RF} = eV_{RF}(\tau) = U_0 + eV_{RF} \cdot \tau \]

- synchronous particle \((\tau = 0)\) will get exactly the energy loss per turn
- we consider only linear oscillations
- Each turn electron gets energy from RF and loses energy to radiation within one revolution time \(T_0\)

\[ \Delta \varepsilon = (U_0 + eV_{RF} \cdot \tau) - (U_0 + U' \cdot \varepsilon) \]

\[ \frac{d\varepsilon}{dt} = \frac{1}{T_0} (eV_{RF} \cdot \tau - U' \cdot \varepsilon) \]

- An electron with an energy deviation will arrive after one turn at a different time with respect to the synchronous particle

\[ \frac{d\tau}{dt} = -\alpha \frac{\varepsilon}{E_0} \]
Synchrotron oscillations: damped harmonic oscillator

Combining the two equations

\[ \frac{d^2 \varepsilon}{dt^2} + 2 \alpha \frac{d \varepsilon}{dt} + \Omega^2 \varepsilon = 0 \]

- where the oscillation frequency
  \[ \Omega^2 \equiv \frac{\alpha e V_{RF}}{T_0 E_0} \]

- the damping is slow:
  \[ \alpha \varepsilon \equiv \frac{U''}{2T_0} \]
  typically \( \alpha \varepsilon << \Omega \)

- the solution is then:
  \[ \varepsilon(t) = \hat{\varepsilon}_0 e^{-\alpha \varepsilon t} \cos (\Omega t + \theta) \]

- similarly, we can get for the time delay:
  \[ \tau(t) = \hat{\tau}_0 e^{-\alpha \varepsilon t} \cos (\Omega t + \theta) \]
Synchrotron (time - energy) oscillations

The ratio of amplitudes at any instant

\[ \hat{\tau} = \frac{\alpha}{\Omega E_0} \hat{\epsilon} \]

Oscillations are 90 degrees out of phase

\[ \theta_{\epsilon} = \theta_{\tau} + \frac{\pi}{2} \]

The motion can be viewed in the phase space of conjugate variables

\[ \{ \epsilon, \tau \} \quad \{ \frac{\alpha \epsilon}{E_0}, \Omega \tau \} \]

\[ \hat{\epsilon}, \hat{\tau} \quad \hat{\epsilon}, \hat{\tau} \]

\[ \epsilon \quad \tau \quad \frac{\alpha \epsilon}{E_0} \quad \Omega \tau \]
Stable regime

$V_o \sin \phi_s$

Separatrix

$\frac{d(\Delta \phi)}{dt}$

Longitudinal Phase Space
During one period of synchrotron oscillation:

- when the particle is in the upper half-plane, it loses more energy per turn, its energy gradually reduces
- when the particle is in the lower half-plane, it loses less energy per turn, but receives $U_0$ on the average, so its energy deviation gradually reduces

The synchrotron motion is damped

- the phase space trajectory is spiraling towards the origin
Robinson theorem: Damping partition numbers

- Transverse betatron oscillations are damped with
- Synchrotron oscillations are damped twice as fast
- The total amount of damping (Robinson theorem) depends only on energy and loss per turn

\[ \frac{1}{\tau_x} + \frac{1}{\tau_y} + \frac{1}{\tau_\epsilon} = \frac{2U_0}{ET_0} = \frac{U_0}{2ET_0}(J_x + J_y + J_\epsilon) \]

the sum of the partition numbers

\[ J_x + J_z + J_\epsilon = 4 \]
Radiation loss

Displaced off the design orbit particle sees fields that are different from design values

- **energy deviation** $\varepsilon$
  - different energy:
    $$P_\gamma \propto E^2$$
  - different magnetic field $B$
    particle moves on a different orbit, defined by the off-energy or dispersion function $D_x$

both contribute to linear term in

- **betatron oscillations**: zero on average
Radiation loss

To first order in $\varepsilon$

$$U_{\text{rad}} = U_0 + U' \cdot \varepsilon$$

electron energy changes slowly, at any instant it is moving on an orbit defined by $D_x$

after some algebra one can write

$$U' = \frac{U_0}{E_0}(2 + D)$$

$D \neq 0$ only when $\frac{k}{\rho} \neq 0$
Damping partition numbers

- Typically we build rings with no vertical dispersion
  \[ J_z = 1 \quad J_x + J_\varepsilon = 3 \]

- Horizontal and energy partition numbers can be modified via \( D \):
  \[ J_x = 1 - D \quad J_\varepsilon = 2 + D \]

- Use of combined function magnets

- Shift the equilibrium orbit in quads with RF frequency
Equilibrium beam sizes
**Radiation effects in electron storage rings**

**Average radiated power restored by RF**
- Electron loses energy each turn to synchrotron radiation
- RF cavities accelerate electrons back to the nominal energy

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\[ V_{RF} > U_0 \]

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**Quantum fluctuations**
- Statistical fluctuations in energy loss (from quantized emission of radiation) produce **RANDOM EXCITATION** of these oscillations

**Equilibrium distributions**
- The balance between the damping and the excitation of the electron oscillations determines the equilibrium distribution of particles in the beam
Quantum nature of synchrotron radiation

Damping only

• If damping was the whole story, the beam emittance (size) would shrink to microscopic dimensions!*

• Lots of problems! (e.g. coherent radiation)

* How small? On the order of electron wavelength

\[ E = \gamma mc^2 = h\nu = \frac{hc}{\lambda_e} \Rightarrow \lambda_e = \frac{1}{\gamma} \frac{h}{mc} = \frac{\lambda_C}{\gamma} \]

\[ \lambda_C = 2.4 \times 10^{-12} m \] — Compton wavelength

Diffraction limited electron emittance

\[ \varepsilon \geq \frac{\lambda_C}{4\pi\gamma} (\times N^{1/3} - \text{fermions}) \]
Quantum nature of synchrotron radiation

Quantum fluctuations

- Because the radiation is emitted in quanta, radiation itself takes care of the problem!

- It is sufficient to use quasi-classical picture:
  - Emission time is very short
  - Emission times are statistically independent (each emission - only a small change in electron energy)

Purely stochastic (Poisson) process
Visible quantum effects

I have always been somewhat amazed that a purely quantum effect can have gross macroscopic effects in large machines;

and, even more,

that Planck’s constant has just the right magnitude needed to make practical the construction of large electron storage rings.

A significantly larger or smaller value of $\hbar$

would have posed serious -- perhaps insurmountable -- problems for the realization of large rings.

Mathew Sands
Quantum excitation of energy oscillations

Photons are emitted with typical energy at the rate (photons/second)

\[ N = Pg \]

Fluctuations in this rate excite oscillations

During a small interval \( \Delta t \) electron emits photons losing energy of

Actually, because of fluctuations, the number is resulting in \textit{spread in energy loss}

For large time intervals RF compensates the energy loss, providing damping towards the design energy \( E_0 \)

Steady state: typical deviations from \( E_0 \)

\[ \approx \text{typical fluctuations in energy during a damping time } \tau_e \]
Equilibrium energy spread: rough estimate

We then expect the rms energy spread to be

$$\sigma_\varepsilon \approx \sqrt{N \cdot \tau_\varepsilon \cdot u_{ph}}$$

and since

$$\tau_\varepsilon \approx \frac{E_0}{P_\gamma}$$

and

$$P_\gamma = N \cdot u_{ph}$$

Relative energy spread can be written then as:

$$\frac{\sigma_\varepsilon}{E_0} \approx \gamma \sqrt{\frac{\lambda_e}{\rho}}$$

$$\lambda_e = \frac{\hbar}{m_e c} \approx 4 \cdot 10^{-13} m$$

it is roughly constant for all rings

• typically

$$\rho \propto E^2$$

$$\frac{\sigma_\varepsilon}{E_0} \sim const \sim 10^{-3}$$
Equilibrium energy spread

More detailed calculations give

- for the case of an ‘isomagnetic’ lattice

\[ \rho(s) = \rho_0 \quad \text{in dipoles} \]
\[ \rho(s) = \infty \quad \text{elsewhere} \]

\[
\left( \frac{\sigma_E}{E} \right)^2 = \frac{C_q E^2}{J_\epsilon \rho_0}
\]

with

\[ C_q = \frac{55}{32 \sqrt{3}} \frac{\hbar c}{(m_e c^2)^3} = 1.468 \cdot 10^{-6} \frac{m}{\text{GeV}^2} \]

It is difficult to obtain energy spread < 0.1%

- limit on undulator brightness!
Equilibrium bunch length

Bunch length is related to the energy spread

■ Energy deviation and time of arrival (or position along the bunch) are conjugate variables (synchrotron oscillations)

■ recall that

\[
\Omega_s \propto \sqrt{V_{RF}}
\]

Two ways to obtain short bunches:

■ RF voltage (power!)

■ Momentum compaction factor in the limit of \( \alpha = 0 \)

isochronous ring: particle position along the bunch is frozen

\[
\sigma_\tau \propto \frac{1}{\sqrt{V_{RF}}}
\]

\[
\hat{\tau} = \frac{\alpha}{\Omega_s \left( \frac{\dot{E}}{E} \right)}
\]
Excitation of betatron oscillations

\[ x = x_\beta + x_\varepsilon \]

\[ x_\varepsilon = D \cdot \frac{\varepsilon}{E} \]

[Diagram showing betatron oscillations with labels for equil. orbit, photon emission, foc. quad, defoc. quad, and bending magnet]

\[ x' = x'_\beta + x'_\varepsilon \]

\[ \Delta x = \Delta x_\beta + \Delta x_\varepsilon = 0 \]

\[ \Delta x_\beta = -D \cdot \frac{\varepsilon_\gamma}{E} \]

[Box labeled Courant Snyder invariant]

\[ \Delta x'_\beta = -D' \cdot \frac{\varepsilon_\gamma}{E} \]

\[ \Delta \varepsilon = \gamma \Delta x_\beta^2 + 2\alpha \Delta x_\beta \Delta x'_\beta + \beta \Delta x'_\beta^2 = \left[ \gamma D^2 + 2\alpha DD' + \beta D'^2 \right] \cdot \left( \frac{\varepsilon_\gamma}{E} \right)^2 \]
Electron emitting a photon
- at a place with non-zero dispersion
- starts a betatron oscillation around a new reference orbit
Horizontal oscillations: equilibrium

Emission of photons is a random process
- Again we have random walk, now in $x$. How far particle will wander away is limited by the radiation damping
- The balance is achieved on the time scale of the damping time $\tau_x = 2 \tau_\varepsilon$

$$\sigma_{x\beta} \approx \sqrt{N \cdot \tau_x \cdot D \cdot \frac{\varepsilon}{E}} = \sqrt{2 \cdot D \cdot \frac{\sigma_\varepsilon}{E}}$$

- Typical horizontal beam size $\sim 1 \text{ mm}$

Quantum effect visible to the naked eye!

- Vertical size - determined by coupling
**Beam emittance**

**Betatron oscillations**
- Particles in the beam execute betatron oscillations with different amplitudes.

**Transverse beam distribution**
- Gaussian (electrons)
- “Typical” particle: 1 - σ ellipse
  (in a place where \( \alpha = \beta' = 0 \))

\[
\text{Emittance } \equiv \frac{\sigma_x^2}{\beta}
\]

Units of \( \varepsilon \) [m \cdot rad]

\[
\varepsilon = \sigma_x \cdot \sigma_{x'}
\]

\[
\sigma_x = \sqrt{\varepsilon \beta}
\]

\[
\sigma_{x'} = \sqrt{\varepsilon / \beta}
\]

\[
\beta = \frac{\sigma_x}{\sigma_{x'}}
\]

Area = \( \pi \cdot \varepsilon \)
Equilibrium horizontal emittance

Detailed calculations for isomagnetic lattice

\[ \mathcal{E}_{x0} \equiv \frac{\sigma_x \beta}{\beta} = \frac{C_q E^2}{J_x} \cdot \frac{\langle \mathcal{H} \rangle_{mag}}{\rho} \]

where

\[ \mathcal{H} = \gamma D^2 + 2\alpha DD' + \beta D'^2 \]
\[ = \frac{1}{\beta} \left[ D^2 + (\beta D' + \alpha D)^2 \right] \]

and \[ \langle \mathcal{H} \rangle_{mag} \] is average value in the bending magnets
2-D Gaussian distribution

Electron rings emittance definition

- 1 - $\sigma$ ellipse

$$n(x)dx = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} dx$$

Area = $\pi \varepsilon_x$

- Probability to be inside 1-\(\sigma\) ellipse

$$P_1 = 1 - e^{-1/2} = 0.39$$

- Probability to be inside n-\(\sigma\) ellipse

$$P_n = 1 - e^{-n^2/2}$$
FODO cell lattice
FODO lattice emittance

\[ \mathcal{H} \sim \frac{D^2}{\beta} \sim \frac{R}{Q^3} \]

\[ \epsilon_{x0} \approx \frac{C_q E^2}{J_x} \cdot \frac{R}{\beta} \cdot \frac{1}{Q^3} \]

\[ \epsilon \propto \frac{E^2}{J_x} \theta^3 F_{\text{FODO}}(\mu) \]
Ionization cooling

similar to radiation damping, but there is multiple scattering in the absorber that blows up the emittance

to minimize the blow up due to multiple scattering in the absorber we can focus the beam

\[ \sigma' = \sqrt{\sigma_0'^2 + \sigma_{MS}^{'2}} \]

\[ \sigma_0' >> \sigma_{MS}' \]
Minimum emittance lattices

\[ \sigma_0 \quad \sigma'_0 \]

\[ \mathcal{E}_{x0} = \frac{C_q E^2}{J_x} \cdot \theta^3 \cdot F_{\text{latt}} \]

\[ F_{\text{min}} = \frac{1}{12\sqrt{15}} \]
Quantum limit on emittance

- Electron in a storage ring’s dipole fields is accelerated, interacts with vacuum fluctuations: «accelerated thermometers show increased temperature»

- Synchrotron radiation opening angle is $\sim 1/\gamma$ -> a lower limit on equilibrium vertical emittance

- Independent of energy

\[
\epsilon_y = \frac{13}{55} C_q \frac{\oint \beta_y(s)|G^3(s)|ds}{\oint G^2(s)ds}
\]

$G(s) =$ curvature, $C_q = 0.384$ pm

- In case of SLS: 0.2 pm

\[
\epsilon_y = 0.09 \text{ pm} \cdot \frac{\langle \beta_y \rangle_{\text{Mag}}}{\rho}
\]

Isomagnetic lattice
Vertical emittance record

Beam size $3.6 \pm 0.6 \, \mu\text{m}$

Emittance $0.9 \pm 0.4 \, \text{pm}$

SLS beam cross section compared to a human hair:
Summary of radiation integrals

Momentum compaction factor

\[ \alpha = \frac{I_1}{2\pi R} \]

Energy loss per turn

\[ U_0 = \frac{1}{2\pi} C_\gamma E^4 \cdot I_2 \]

\[ C_\gamma = \frac{4\pi}{3} \frac{r_e}{(m_e c^2)^3} = 8.858 \cdot 10^{-5} \left[ \frac{m}{\text{GeV}^3} \right] \]
Summary of radiation integrals (2)

Damping parameter

\[ D = \frac{I_4}{I_2} \]

Damping times, partition numbers

\[ J_\varepsilon = 2 + D, \quad J_x = 1 - D, \quad J_y = 1 \]

\[ \tau_i = \frac{\tau_0}{J_i} \]

\[ \tau_0 = \frac{2ET_0}{U_0} \]

Equilibrium energy spread

\[ \left( \frac{\sigma_\varepsilon}{E} \right)^2 = \frac{C_q E^2}{J_\varepsilon} \cdot \frac{I_3}{I_2} \]

Equilibrium emittance

\[ \varepsilon_{x0} = \frac{\sigma^2_{x\beta}}{\beta} = \frac{C_q E^2}{J_x} \cdot \frac{I_5}{I_2} \]

\[ C_q = \frac{55}{32\sqrt{3}} \frac{\hbar c}{(m_e c^2)^3} = 1.468 \cdot 10^{-6} \left[ \frac{m}{\text{GeV}^2} \right] \]

\[ H = \gamma D^2 + 2\alpha DD' + \beta D'^2 \]
**Damping wigglers**

Increase the radiation loss per turn $U_0$ with WIGGLERS

- reduce damping time
  
  $\tau = \frac{E}{P_\gamma + P_{wig}}$

- emittance control

  wigglers at high dispersion: blow-up emittance
  e.g. storage ring colliders for high energy physics

  wigglers at zero dispersion: decrease emittance
  e.g. damping rings for linear colliders
  e.g. synchrotron light sources (PETRAIII, 1 nm.rad)
END OF LECTURES
FOR DREAMING AND DISCUSSION
Does charged particle radiate in gravitational field?

Yes, charged particles can radiate in a gravitational field, but the circumstances under which this happens are different from radiation in an electromagnetic field.

In electromagnetism, an accelerating charged particle emits radiation. This is well-established and is the basis for many phenomena and applications, such as radio wave transmission.

In general relativity, which describes gravitation, things are a bit more nuanced. A free-falling charged particle in a gravitational field, following a geodesic (the equivalent of a “straight line” in curved spacetime), does not radiate from the perspective of another free-falling observer. This is because, from the viewpoint of the falling particle, it isn’t accelerating—it’s in free fall.

However, if a charged particle is held stationary in a gravitational field (against the gravitational pull, perhaps by a tether or some other force), or if it is made to move in some non-geodesic way, then it can be seen as accelerating and can emit radiation.

This kind of radiation in a gravitational field is not as commonly encountered or as well-understood in practical terms as electromagnetic radiation from accelerating charges in empty space, but the underlying principles from general relativity suggest that radiation would indeed be emitted under the right circumstances.
Future Circular Collider Study: 100 km, 100 TeV
Some quick calculations for FCC-hh

• FCC-hh center-of-mass energy 100 TeV. Bending radius for 16 Tesla field?
• Proton Lorentz factor?
• Dipole field 16 Tesla. Critical photon energy?
• Loss per turn:
• Damping time:
• In LEP we shortened the damping time using wigglers. Could we do it here?