CAS SCHOOL, SEP 2023

INTRODUCTION TO ELECTROMAGNETISM

Dr. Irina Shreyber, PhD

National Research Tomsk State University

COPIRIGHTS

MATERIALS

CAS Website

These slides and the video will be available the CAS school website

Books

- J. David Jackson, "Classical Electrodynamics”
- David J. Griffiths, “Introduction to Electrodynamics”
- Chabay, Sherwood "Matter & Interactions"

Variables and Units

$\mathbf{E}$		electric field [V/m]
$\mathbf{B}$		magnetic field [T]

q		electric charge [C]
ρ		electric charge density [C/m^3]
$\mathbf{j}$	$= \rho \mathbf{v}$	current density [A/m^2]

ϵ_0		permittivity of vacuum, $8.854 \cdot 10^{-12}$ [F/m]
μ_0	$= \frac{1}{\epsilon_0 c^2}$	permeability of vacuum, $4\pi \cdot 10^{-7}$ [H/m or N/A^2]
c		speed of light in vacuum, $2.99792458 \cdot 10^8$ [m/s]

Differentiation with vectors

We define operator “nabla” which we treat as a special vector

$$\nabla \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z} \right)$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Divergence

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \quad \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \quad \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

Curl

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \quad \frac{\partial \phi}{\partial y}, \quad \frac{\partial \phi}{\partial z} \right)$$

Gradient

WHY EM?

EM is our first example of a field theory

To work in the accelerator physics field you really should understand field theory and understand that well

EM teaches us about special relativity

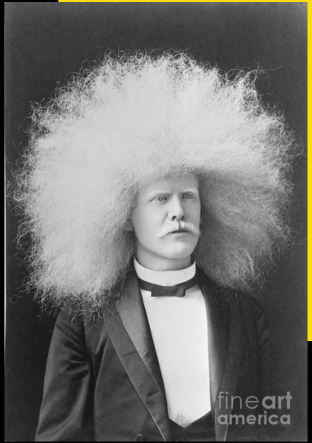
See Special Relativity lecture

Modern physics

Electromagnetism is the first example of using theories unification

Examples

Electric Force



Magnetic Force

WHY EM FOR ACCELERATORS?

Acceleration of Particles: this is achieved by creating a potential difference using an **electric field** which imparts energy to the particles

Steering and Focusing of Particles: once the particles are accelerated, they need to be guided along the desired path. This is done using **magnetic fields**.

Synchrotron Radiation: In circular accelerators, charged particles emit electromagnetic radiation known as synchrotron radiation when they are deflected by a magnetic field. This is a key consideration **in the design and operation of accelerators**, as it leads to energy loss that must be compensated for.

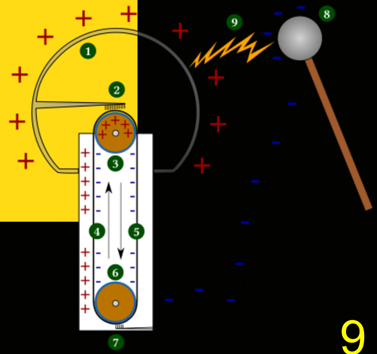
Particle Detection: after a collision, the resulting particles are detected based on their **electromagnetic interactions** and the **electromagnetic responses** of the detector materials.

Examples in accelerator physics

Charged particles in an electric field

Van de Graaff Generator

utilize the electric force to accumulate high voltages by transferring charge via a moving belt to a large dome, where the interplay of forces builds up a significant potential.



Examples in accelerator physics

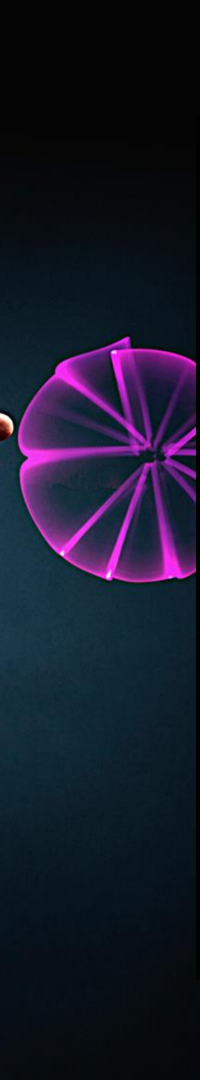
Charged particles in a magnetic field

Data Storage

Magnetic forces enable data storage in computer hard drives by magnetizing small regions of a disk to represent binary digits or bits

CONTENT OF THE COURSE

- INTRODUCTION
- ELECTROSTATICS
- MAGNETOSTATICS
- ELECTROMAGNETISM



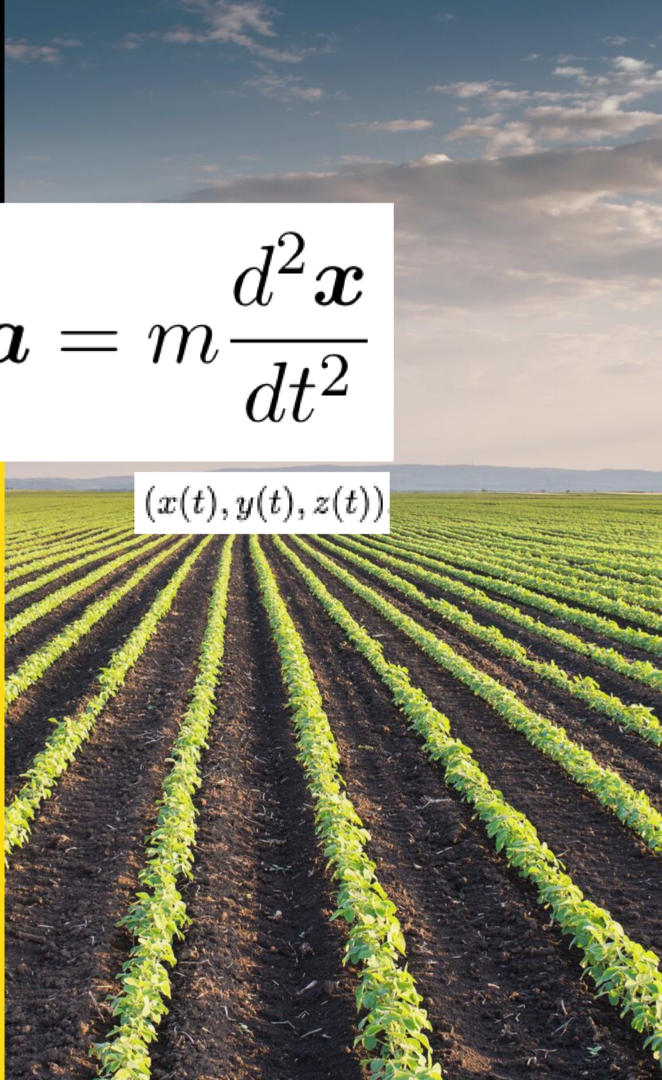
INTRODUCTION

- **Introduction to Fields**
- Charge and Current
- Conservation Law
- Lorentz Force
- Maxwell Equations

INTRODUCTION TO FIELDS

$$\boldsymbol{F} = m\boldsymbol{a} = m\frac{d^2\boldsymbol{x}}{dt^2}$$

$$(x(t), y(t), z(t))$$



GRAVITATIONAL FORCE

The force exerted by the earth on a particle.

GRAVITATIONAL FIELD

Instead of saying that the earth exerts a force on a falling object, it is more useful to say that the earth sets up **a gravitational force field**.

Any object near the earth is acted upon by the gravitational force field at that location.

INTRODUCTION TO FIELDS

F is the force acting on a particle of mass m and g – the acceleration due to gravity.

- *F and g are fields;*
- *the mass of the particle* m *is not a field*

$$F = mg$$

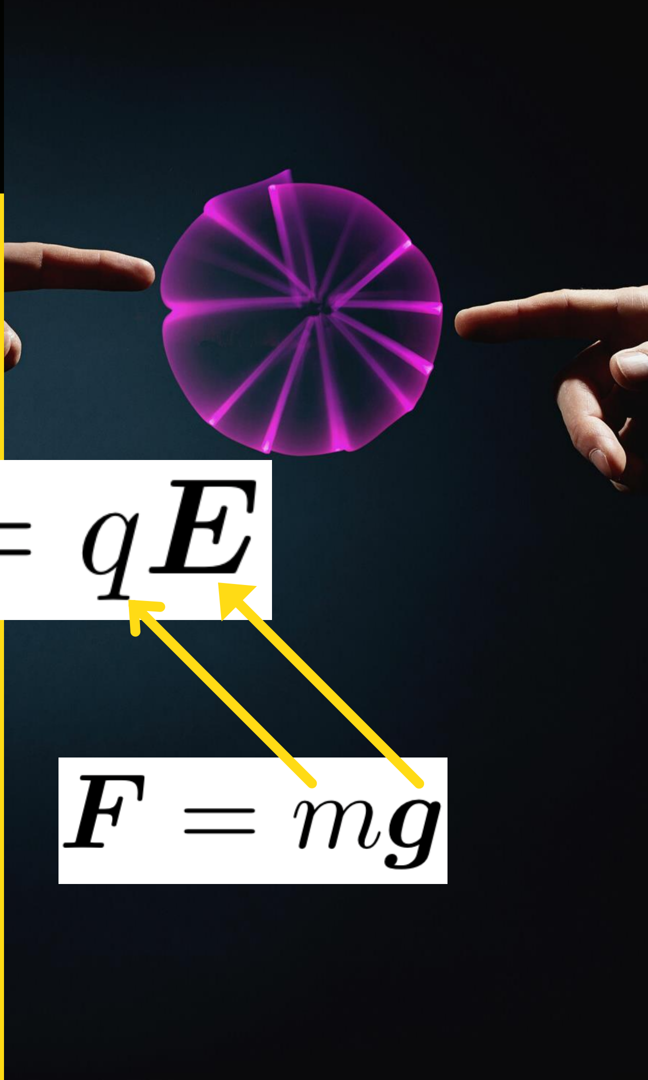
GRAVITATIONAL FORCE

- *We can split the system into a source which produces the field and an object which reacts to the field*
- *We treat both pieces separately*

INTRODUCTION TO FIELDS

ELECTRIC FORCE

The force between charged particles. Charged particles exert forces on each other



$$F = qE$$

$$F = mg$$

ELECTRIC FIELD

- The charge q of our particle replaces the mass m of our particle. q is a single number associated with the object that experiences the field.
- The electric field **E** replaces the gravitational field **g**

We are splitting things up into a source that produces a field and an object that experiences the field

INTRODUCTION TO FIELDS

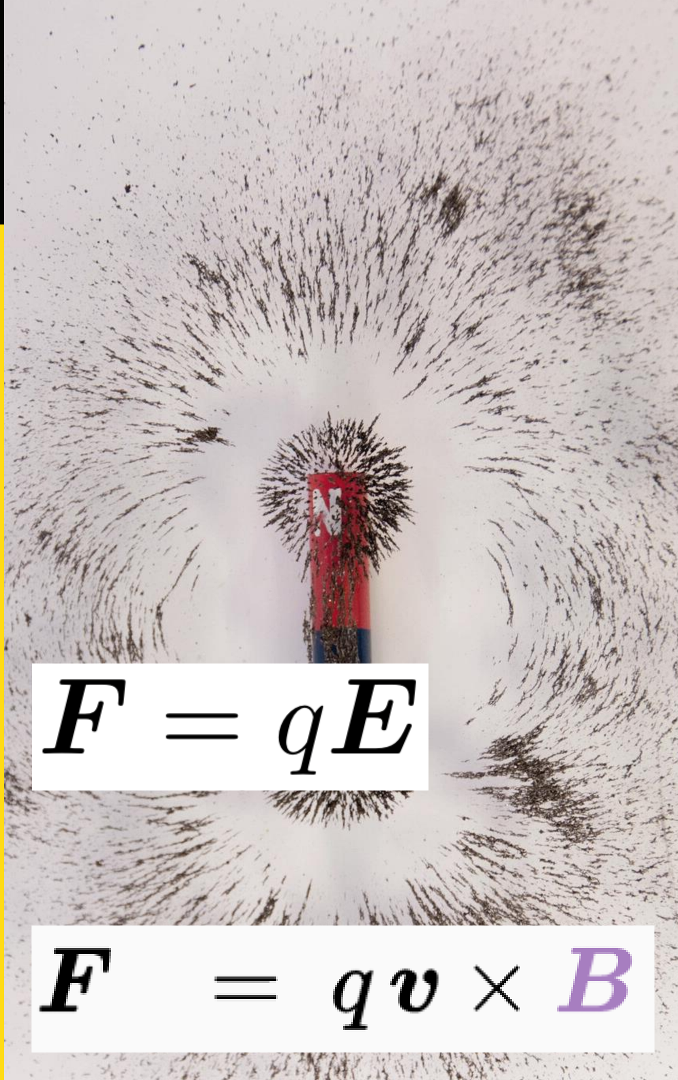
ELECTROMAGNETIC FORCE

To describe the **force of electromagnetism**, we need to introduce **two fields**:

ELECTRIC FIELD , **E** $\mathbf{E}(\mathbf{x}, t)$

AND

MAGNETIC FIELD , **B** $\mathbf{B}(\mathbf{x}, t)$



$$\boldsymbol{F} = q\boldsymbol{E}$$

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$$

INTRODUCTION

- Introduction to Fields
- **Charge and Current**
- Conservation Law
- Lorentz Force
- Maxwell Equations

CHARGE AND CURRENT

$$e = 1.602176634 \times 10^{-19}\ C$$

$$q = ne \qquad n \in \mathbf{Z}$$

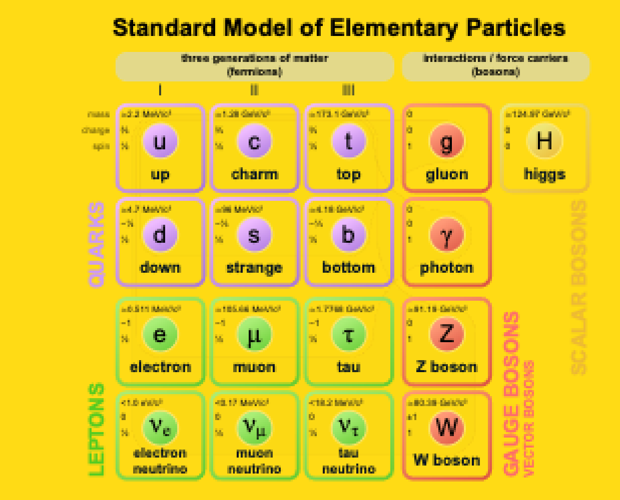
The SI unit of charge is the Coulomb, denoted by C

A much more natural unit . Then, proton/electron: n = ±1

$$q = -e/3$$

$$q = 2e/3$$

the charge of quarks



CHARGE AND CURRENT

$$\rho(\mathbf{x}, t)$$

$$Q = \int_V d^3x \; \rho(\mathbf{x}, t)$$

the charge density – charge per unit volume

the total charge Q in a given region V

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

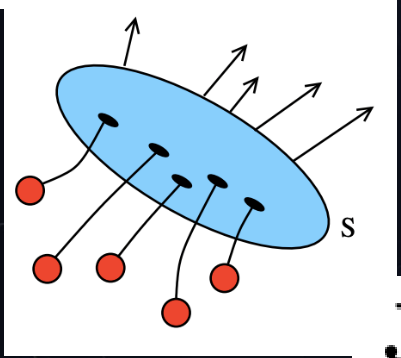
the movement of charge from one place to another is captured by the current density J.

I is called the current.

The current density is the current-per-unit-area

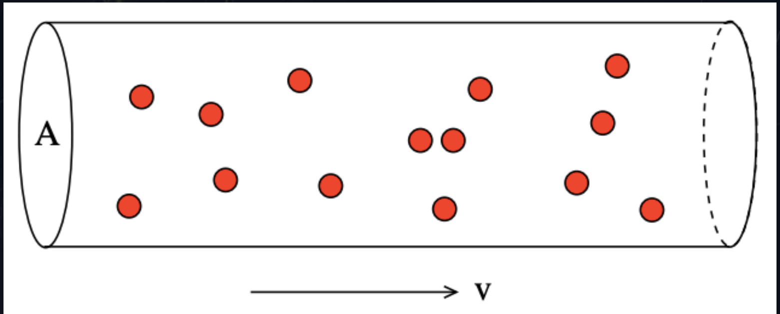
CHARGE AND CURRENT

Current flux



$$\mathbf{J} = \rho \mathbf{v}$$

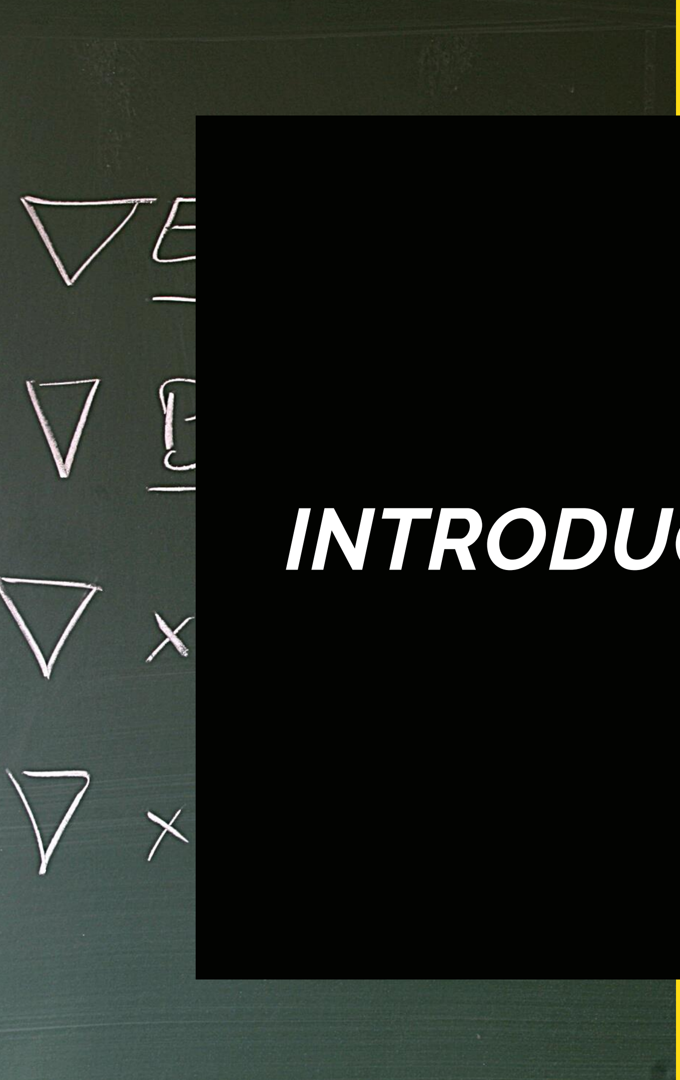
Move intuitive way:
A continuous charge distribution in which the velocity of a small volume, at point x, is given by v(x, t)



Electrons moving along a wire

$$\mathbf{J} = nq\mathbf{v}$$

$$I = |\mathbf{J}|A$$



INTRODUCTION

- Introduction to Fields
- Charge and Current
- **Conservation Laws**
- Lorentz Force
- Maxwell Equations

CONSERVATION LAWS

- **Conservation of Energy**: the total energy of an isolated system remains constant.
- **Conservation of Momentum**: the total momentum of an isolated system remains constant
- **Conservation of Angular Momentum**: the total angular momentum of a system remains constant unless an external torque is applied
- **Conservation of Charge**: the total electric charge in an isolated system remains constant
  - Continuity equation

CONTINUITY EQUATION

Continuity equation:
charge density can change in time only if there is a compensating current flowing into or out of that region

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

$$\frac{dQ}{dt} = \int_V d^3x\, \frac{\partial \rho}{\partial t} = -\int_V d^3x\, \nabla \cdot \mathbf{J} = -\int_S \mathbf{J} \cdot d\mathbf{S}$$

the change in the total charge **Q** contained in some region **V.** The minus sign is to ensure that if the net flow of current is outwards, then the total charge decreases.

If there is no current flowing out of the region, then $dQ/dt = 0$

INTRODUCTION

- Introduction to Fields
- Charge and Current
- Conservation Law
- **LORENTZ FORCE**
- Maxwell Equations

Lorentz Force

$$F = mg \rightarrow F = qE \rightarrow \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Lorentz Force

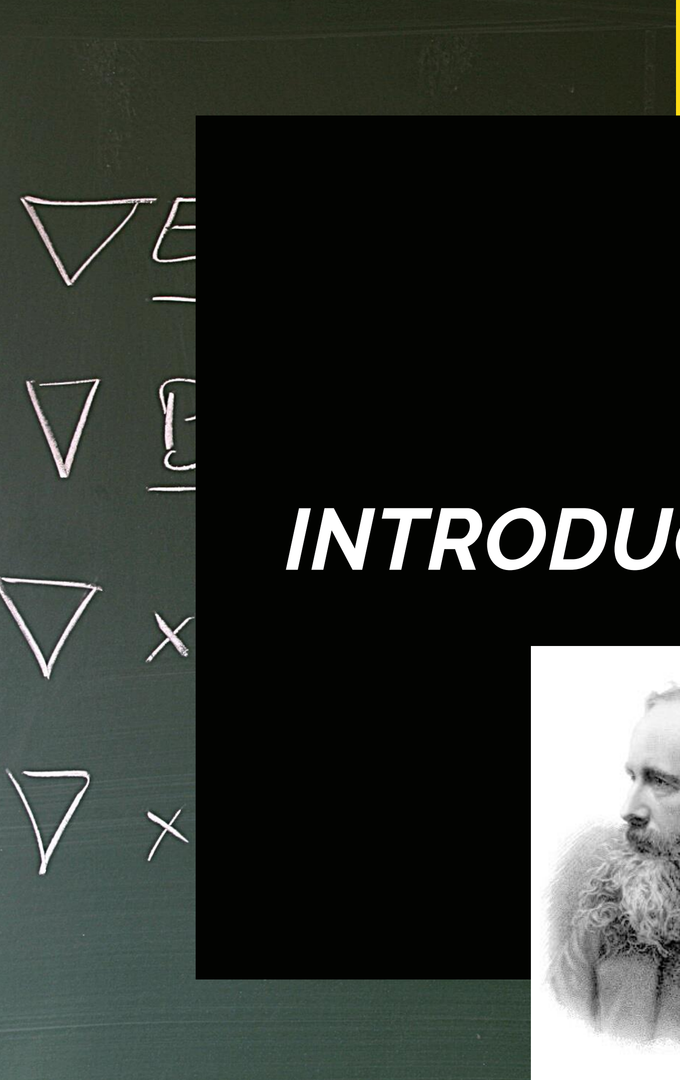
an electric field accelerates a particle in the direction **E**, while a magnetic field causes a particle to move in circles in the plane perpendicular to **B.**

Lorentz Force Law

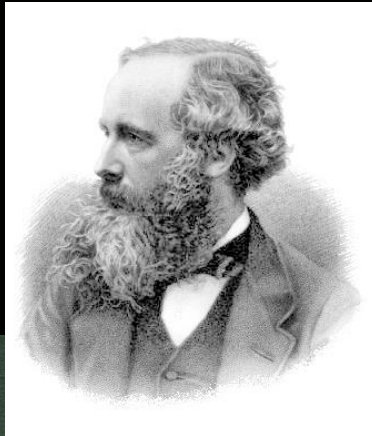
in terms of the charge distribution

Now we talk in terms of the force density **f**(**x**, t), which is the force acting on a small volume at point **x**

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$$



INTRODUCTION



- Introduction to Fields
- Charge and Current
- Conservation Law
- Lorentz Force
- **MAXWELL EQUATIONS**

MAXWELL EQUATIONS

DIFFERENTIAL FORM

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

GAUSS'S LAW FOR **E**

$$\nabla \cdot \mathbf{B} = 0$$

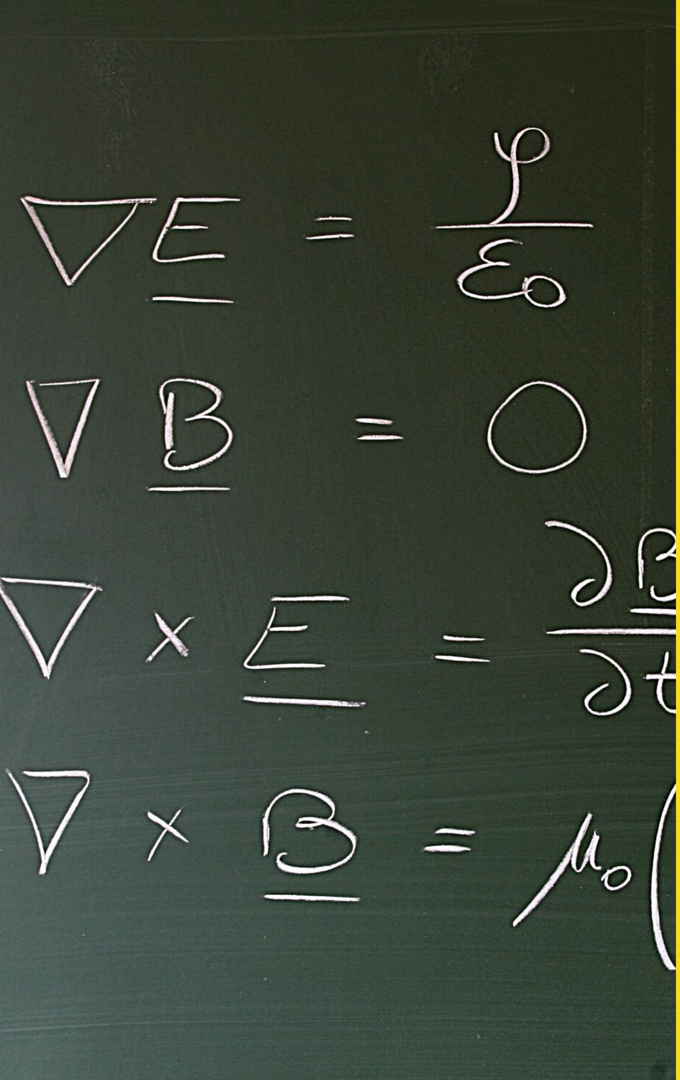
GAUSS'S LAW FOR **B**

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

FARADAY'S LAW
for time-varying magnetic fields

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

AMPERE(-MAXWELL) LAW
for time-varying electric fields



RECAP OF THE INTRODUCTION

- Introduction to Fields $\boldsymbol{F} = q\boldsymbol{E}$ $\boldsymbol{F}_{\mathrm{m}} = q\boldsymbol{v} \times \boldsymbol{B}$
- Charge and Current $\rho(\mathbf{x}, t)$ $Q = \int_V d^3x \, \rho(\mathbf{x}, t)$ $I = \int_S \mathbf{J} \cdot d\mathbf{S}$
- Continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$
- Lorentz Force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ $\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$

RECAP:
MAXWELL EQUATIONS

INTEGRAL FORM

$$\int_{S=\partial V} \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{Q}{\epsilon_0}$$

GAUSS'S LAW FOR E

$$\int_{\partial V} \boldsymbol{B} \cdot d\boldsymbol{S} = 0$$

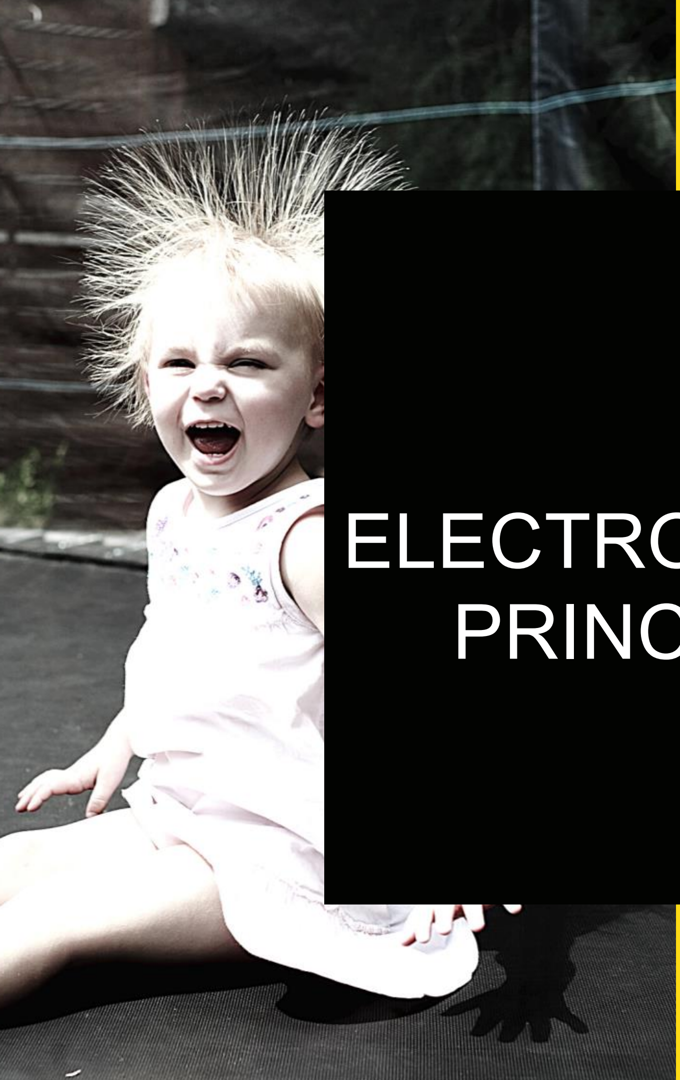
GAUSS'S LAW FOR B

$$\int_C \boldsymbol{E} \cdot d\boldsymbol{r} = \int_S \nabla \times \mathbf{E} \cdot d\boldsymbol{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\boldsymbol{S} = -\frac{d}{dt}\int_S \boldsymbol{B} \cdot d\boldsymbol{S}$$

FARADAY'S LAW

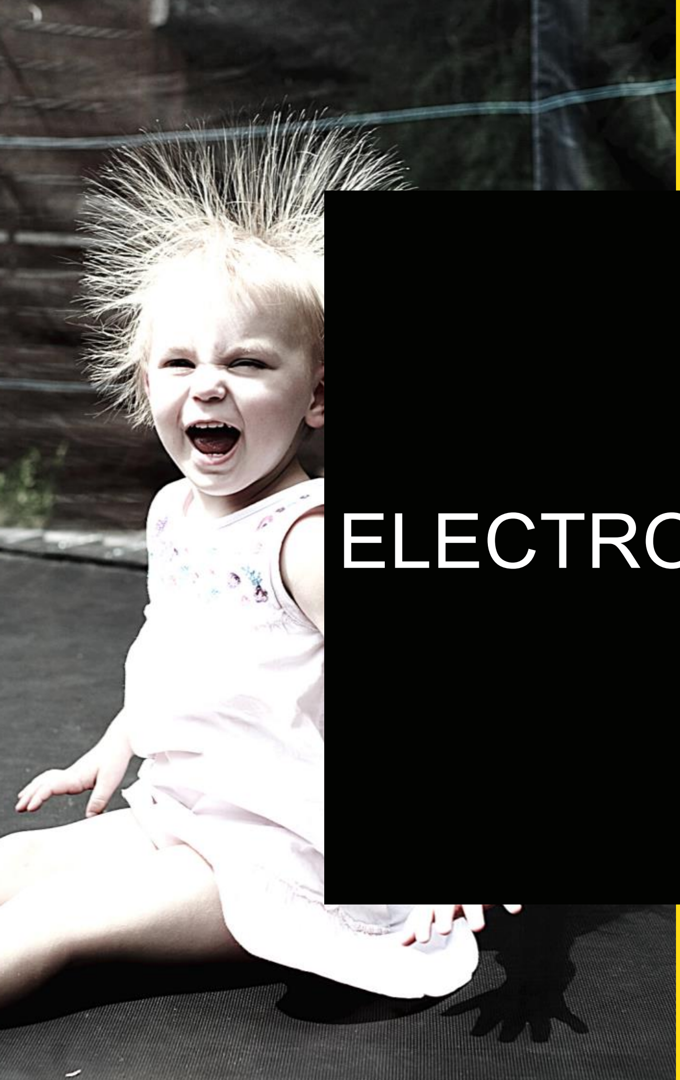
$$\int_S \nabla \times \mathbf{B} \cdot d\boldsymbol{S} = \mu_0 \int_S \boldsymbol{J} \cdot d\boldsymbol{S} + \mu_0 \epsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\boldsymbol{S}$$

AMPERE (-MAXWELL) LAW



ELECTROSTATICS PRINCIPLES

- **MAXWELL EQUATIONS**
- Coulomb force
- Electrostatic Potential
- Principle of Superposition
- Continuous distribution of charges



ELECTROSTATICS

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

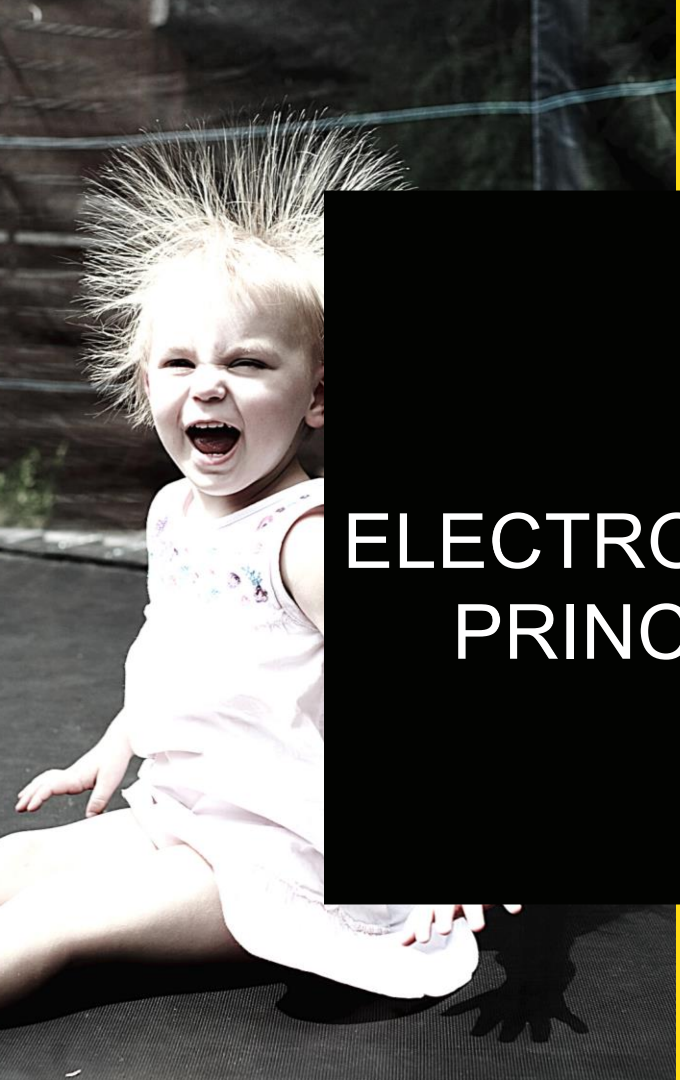
GAUSS'S LAW FOR E

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad 0$$

FARADAY'S LAW

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$



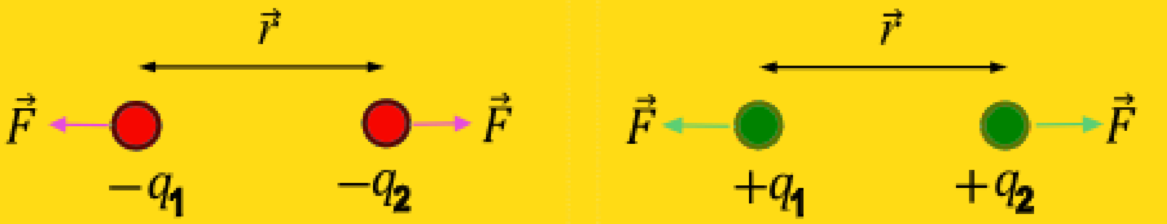
ELECTROSTATICS PRINCIPLES

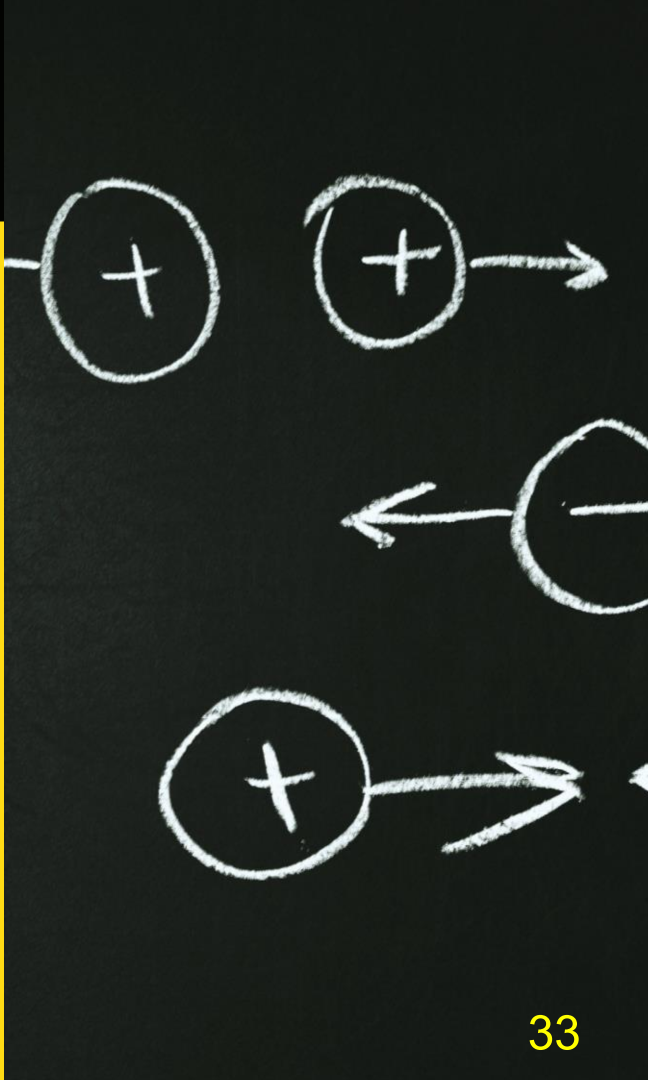
- Maxwell equations
- COULOMB FORCE
- Electrostatic Potential
- Principle of Superposition
- Continuous distribution of charges

COULOMB FORCE

$$F_E = K \cdot \frac{q_1 \cdot q_2}{r^2}$$

- Like charges repel and unlike charges attract;
- The force acts along the line joining the two-point charges





COULOMB FORCE

$$F_E = K \cdot \frac{q_1 \cdot q_2}{r^2}$$

ELECTROSTATIC FORCE

- Proportional to electric charge of each of the two interacting objects
- Inversely proportional to square of the distance
- Proportional to Coulomb constant **K,** which depends on medium type (vacuum, air, water, etc)

$$K = \frac{1}{4\pi\varepsilon_0} = 9 \cdot 10^9 \frac{C}{N \cdot m}$$

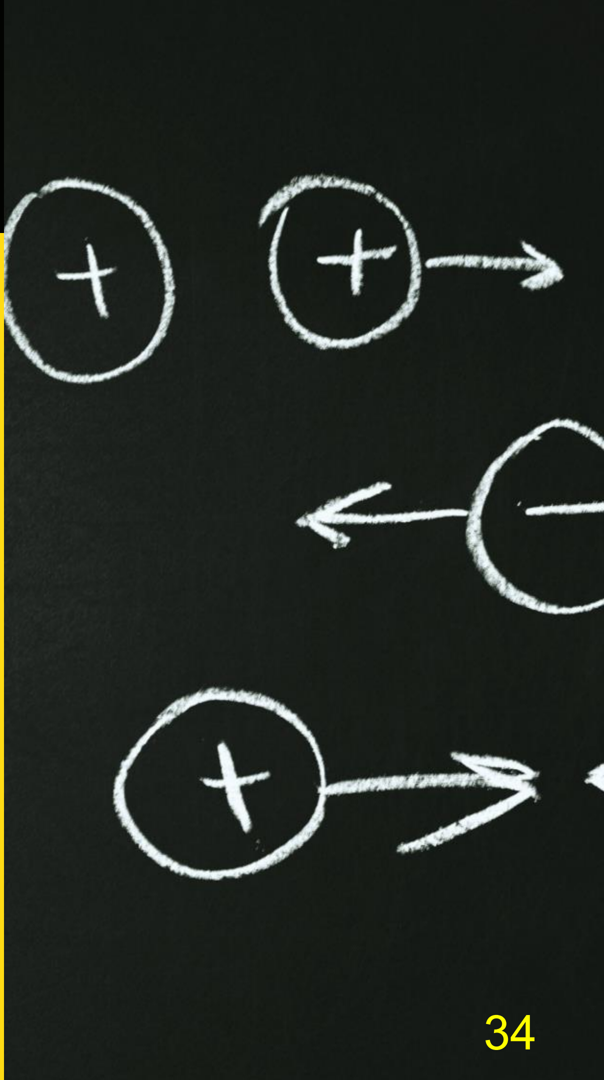
$$K = \frac{1}{4\pi\varepsilon}$$

material permittivity of dielectric

$$\varepsilon = \varepsilon_r \varepsilon_0 = (1 + \chi)\varepsilon_0$$

ε_r - relative permittivity

χ - susceptibility of the material



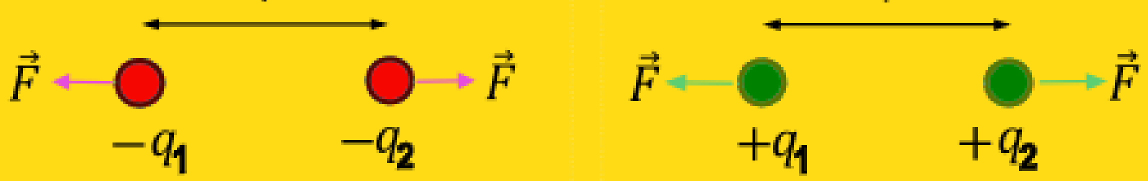
COULOMB FORCE

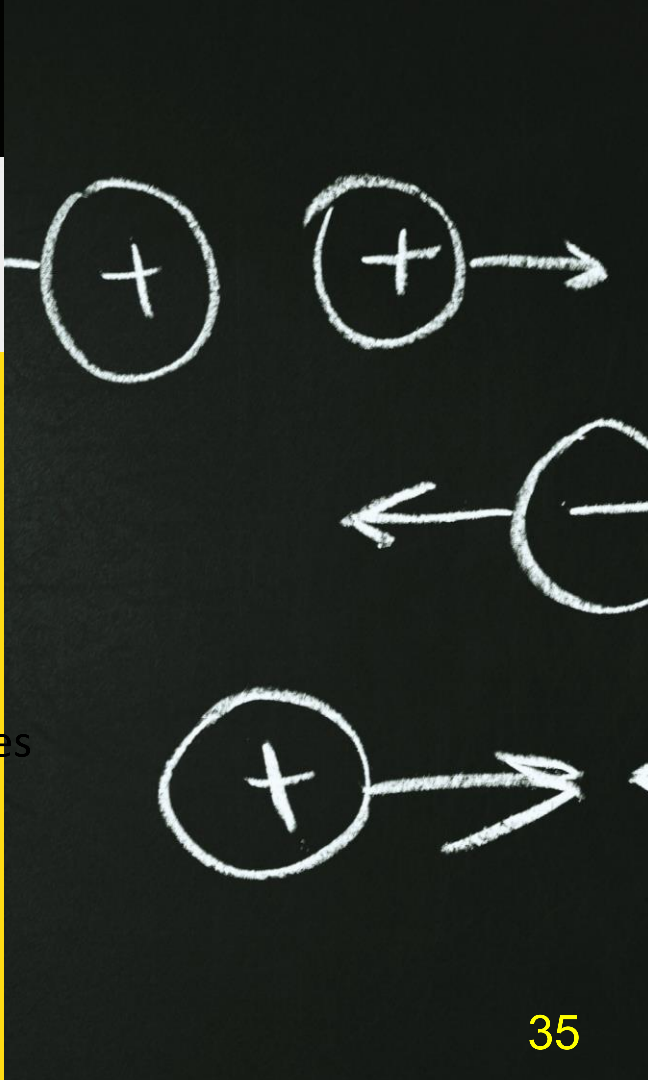
$$F_E = K \cdot \frac{q_1 \cdot q_2}{r^2}$$

- **Coulomb's Law,** the basis of the electric force

The force between two charges is directly proportional to the product of their charges and inversely proportional to the square of the distance between them.

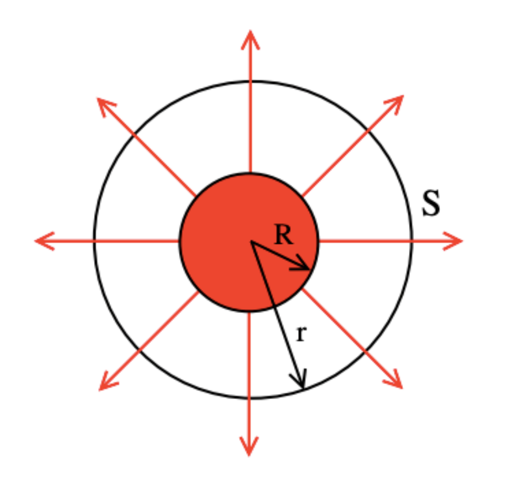
- It can be used to calculate the force between two particles in a cyclotron or other particle accelerator.





COULOMB FORCE VS GAUSS LAW

- Take a particle of charge **Q** and radius **R** and Gaussian surface **S** to be a sphere of radius **r**
- We want to know the electric field at some radius **r > R**



$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0}$$

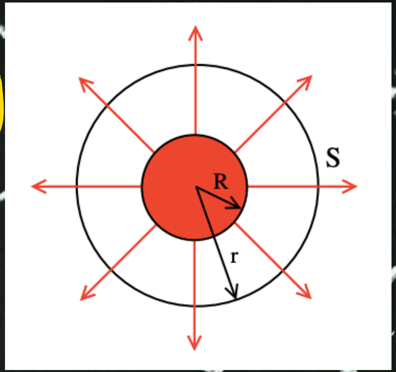
$$\mathbf{E}(\mathbf{x}) = E(r)\hat{\mathbf{r}}$$

$$\int_S \mathbf{E} \cdot d\mathbf{S} = E(r) \int_S \hat{\mathbf{r}} \cdot d\mathbf{S} = E(r)\, 4\pi r^2 = \frac{Q}{\epsilon_0}$$

COULOMB FORCE VS GAUSS LAW

$$\int_S \mathbf{E} \cdot d\mathbf{S} = E(r) \int_S \hat{\mathbf{r}} \cdot d\mathbf{S} = E(r)\, 4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$\mathbf{E}(\mathbf{x}) = E(r)\hat{\mathbf{r}}$$



$$\mathbf{E}(\mathbf{x}) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

electric field outside a sherically symmetric distribution of charge **Q**

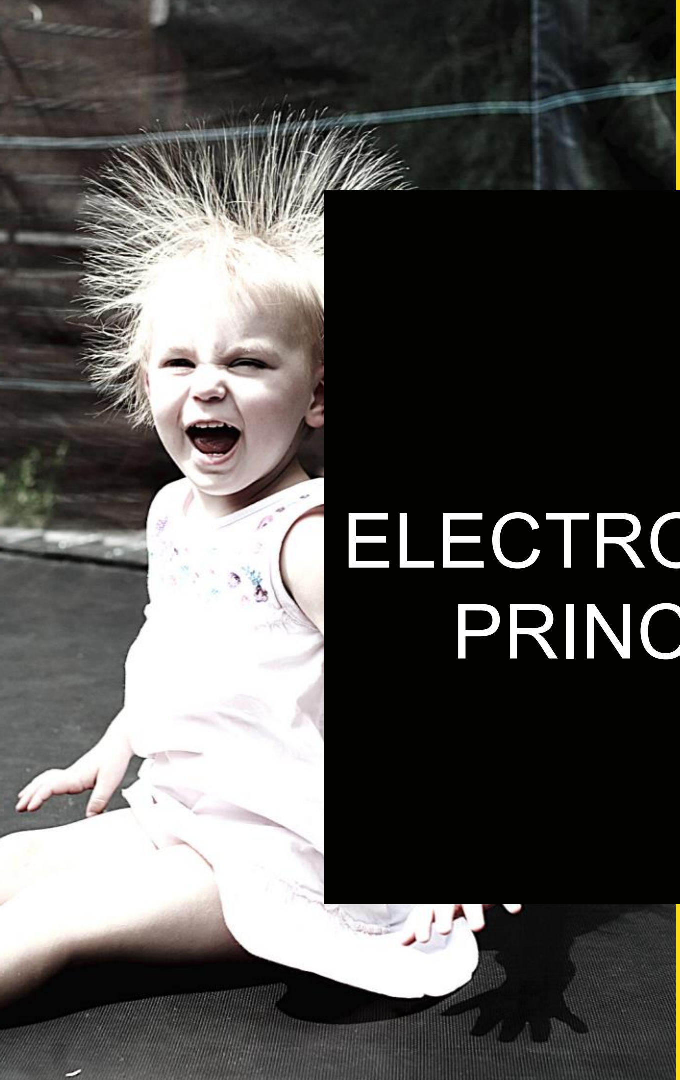
$$\mathbf{F} = \frac{Qq}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

$$F_E = K \cdot \frac{q_1 \cdot q_2}{r^2}$$

By the **Lorentz force law:** $\boldsymbol{F} = q\boldsymbol{E}$

force experienced by a test charge **q**

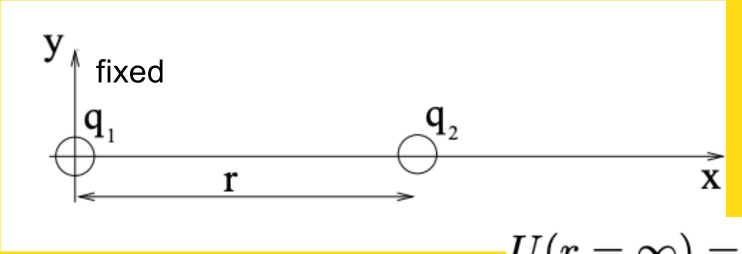
moving in the region **r>R**



ELECTROSTATICS PRINCIPLES

- Maxwell equations
- Coulomb force
- ELECTROSTATIC POTENTIAL
- Principle of Superposition
- Continuous distribution of charges

Electrostatic Potential energy



$$U(r=\infty)=0$$

Energy

If we let the charge **q2** move upon electrostatic force, then it starts accelerating and gain kinetic energy. Consequently, it will lose potential energy.

$$\mathbf{E}(\mathbf{x}) = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}}$$

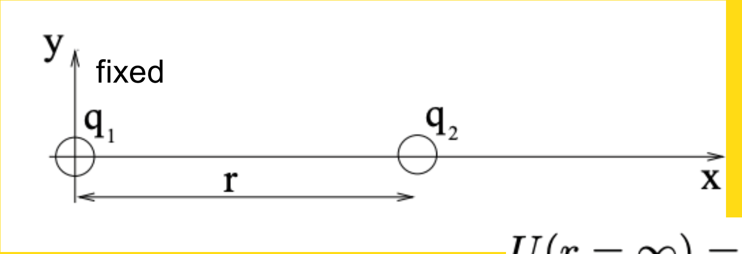
Potential Energy

Work needed to bring 2 point-like charges together (or to a distance **r**) .

$$W=\int_{\infty}^{r} \boldsymbol{F}\cdot d\boldsymbol{r} = q_1\int_{\infty}^{r} \boldsymbol{E}\cdot d\boldsymbol{r} = Kq_1q_2\int_{\infty}^{r}\frac{dr}{r^2} = Kq_1\,q_2\frac{1}{r}$$

$$U_E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1\cdot q_2}{r}$$

This work **W** is stored as potential energy **U**



$$U(r=\infty)=0$$

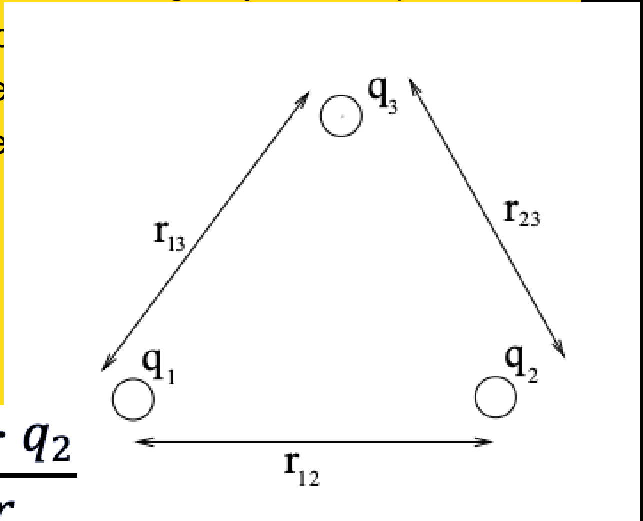
Energy

If we let the charge **q2** move upon

electro

accele

Conse



Potential Energy

The potential energy for a collection of point charges is the **sum of contributions** for each pair of particles.

$$U_E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 \cdot q_2}{r}$$

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{23}}$$

Electrostatic Potential energy

Electrostatic Potential

$$U = q\phi$$

volts

Electric Potential

the **electrical potential energy per charge** is the electric potential.
The scalar is called the **electrostatic potential** or **scalar potential** (or, sometimes, just the **potential)**.

Maxwell Equations: Electrostatics.

The two can be combined into the **Poisson equation**

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\mathbf{E} = -\nabla\phi$$

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

The Poisson's equation allows to compute the electric field generated by arbitrary charge distributions.

$$\nabla^2\phi = \cancel{-\frac{\rho}{\epsilon_0}}$$

Laplace equation

$$\nabla^2\phi = 0$$

Solutions to the Laplace equation are said to be **harmonic functions.**

$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Laplacian

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

Application of the Poisson's or Laplace's equation:

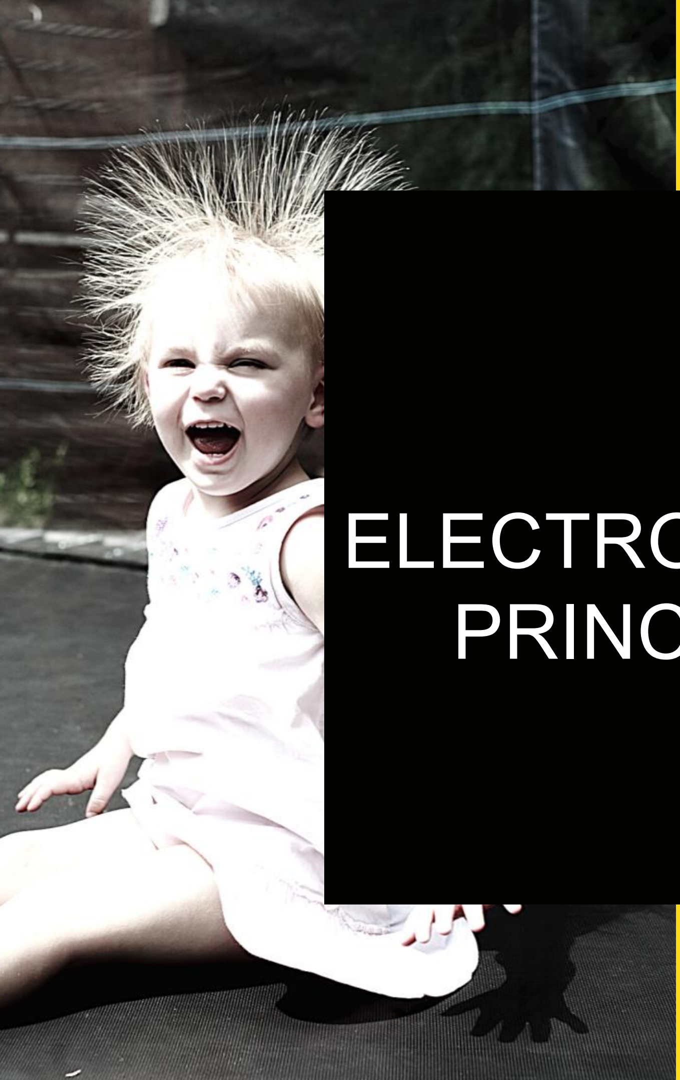
Boundary Value Problems: allow to solve boundary value problems to determine the electric potential everywhere in a region based on some boundary conditions.

Beam Dynamics: predict how space charge effects will impact the beam (poisson eq). For high-intensity beams, these effects can cause beam spreading, limiting the performance of the accelerator.

RF Cavity Design: The shape and structure of RF cavities are often optimized based on solving Laplace's and Poisson's equations to produce desired electric field configurations for efficient acceleration.

Beam Steering and Focusing

Diagnostics: understanding electric potential in BPM



ELECTROSTATICS PRINCIPLES

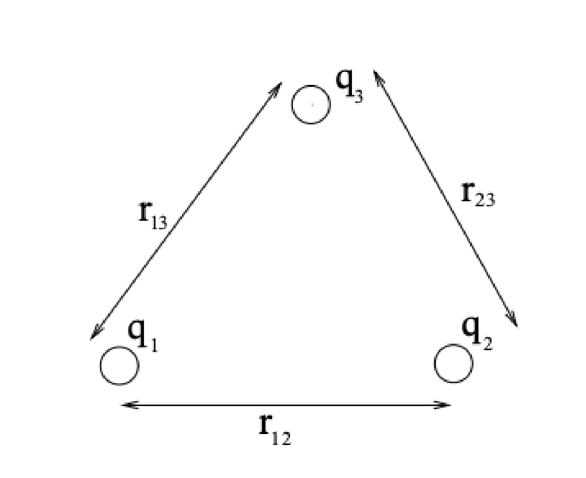
- Maxwell equations
- Coulomb force
- Electrostatic potential
- **PRINCIPLE OF SUPERPOSITION**
- Continuous distribution of charges

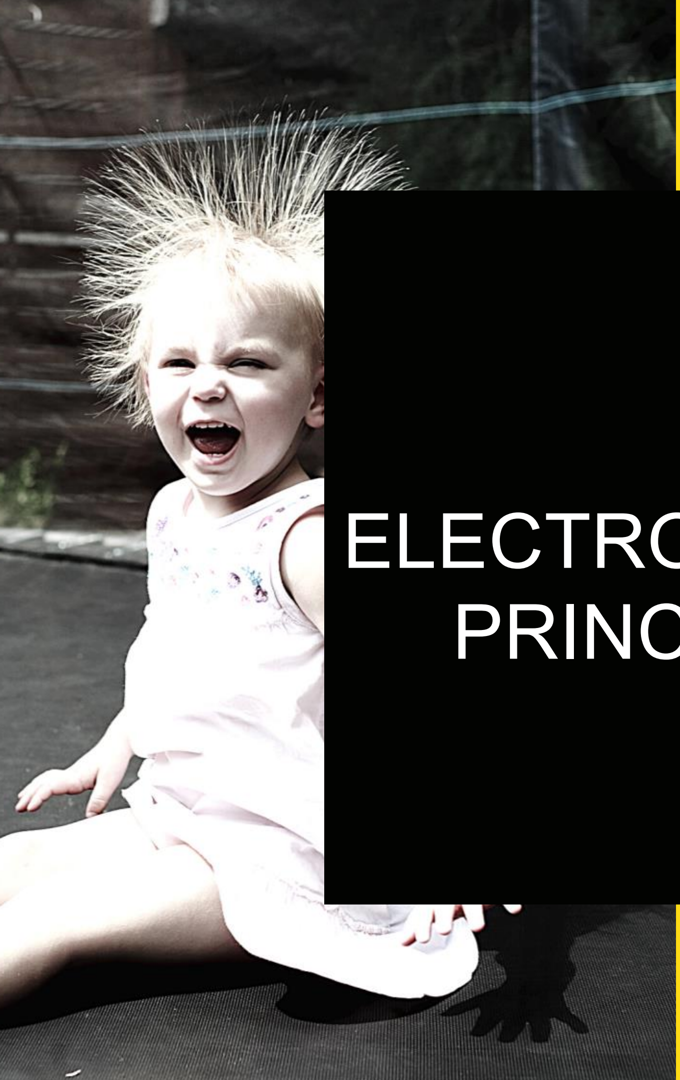
PRINCIPLE OF SUPERPOSITION

The net electric field at a location in space is equal to the vector sum of individual electric fields contributed by all charged particles located elsewhere.

Thus, the electric field contributed by a charged particle is unaffected by the presence of other charged particles.

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{23}}$$





ELECTROSTATICS PRINCIPLES

- Maxwell equations
- Coulomb force
- Electrostatic potential
- Principle of superposition
- **CONTINUOUS DISTRIBUTION OF CHARGES**

CONTINUOUS DISTRIBUTION OF CHARGE

The region in which charges are closely spaced is said to have

CONTINUOUS DISTRIBUTION OF CHARGE.

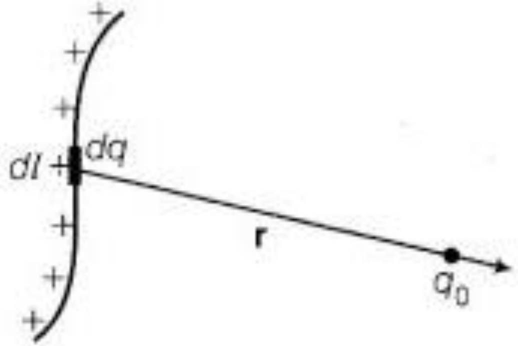
(i) Linear Charge Distribution

$$dq = \lambda \, dl$$

where, λ = linear charge density

$$d\mathbf{F} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_0 (dq)}{|\mathbf{r}|^2} \hat{\mathbf{r}} \Rightarrow d\mathbf{F} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_0 (\lambda \, dl)}{|\mathbf{r}|^2} \hat{\mathbf{r}}$$

Net force on charge q_0, $\mathbf{F} = \frac{q_0}{4\pi\varepsilon_0} \int_l \frac{\lambda \, dl}{|\mathbf{r}|^2} \hat{\mathbf{r}}$



CONTINUOUS DISTRIBUTION OF CHARGE

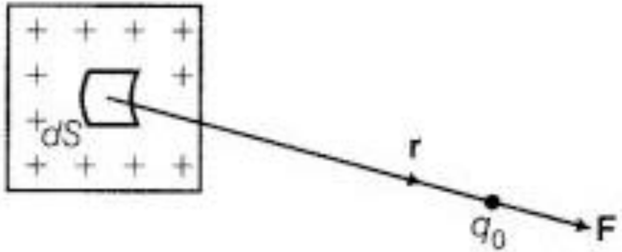
The region in which charges are closely spaced is said to have **CONTINUOUS DISTRIBUTION OF CHARGE.**

(*ii*) **Surface Charge Distribution**

$$dq = \sigma \, dS$$

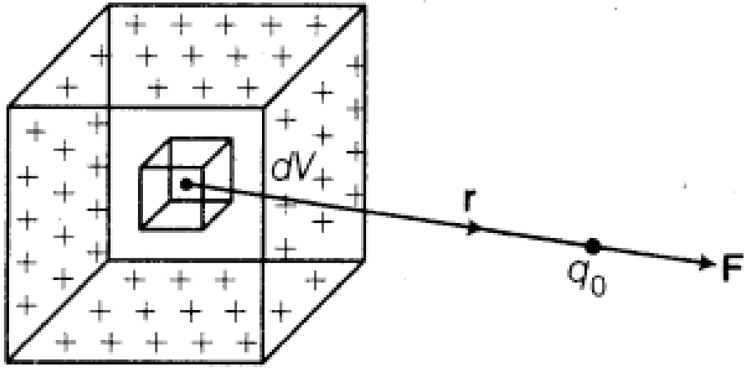
where, σ = surface charge density

Net force on charge q_0, $\mathbf{F} = \frac{q_0}{4\pi\varepsilon_0} \int_S \frac{\sigma \, dS}{|\mathbf{r}|^2} \hat{\mathbf{r}}$



CONTINUOUS DISTRIBUTION OF CHARGE

The region in which charges are closely spaced is said to have **CONTINUOUS DISTRIBUTION OF CHARGE.**



(iii) Volume Charge Distribution

$$dq = \rho\, dV$$

where, ρ = volume charge density

Net force on charge q_0, $\mathbf{F} = \dfrac{q_0}{4\pi\varepsilon_0}\displaystyle\int_V \dfrac{\rho\, dV}{|\mathbf{r}|^2}\,\hat{\mathbf{r}}$

$$\frac{1}{4\pi\epsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')\,\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

RECAP: ELECTROSTATICS

- Maxwell equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = 0$$

- Coulomb force vs Gauss law

$$F_E = K \cdot \frac{q_1 \cdot q_2}{r^2} \qquad \mathbf{F} = \frac{Qq}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

- Electrostatic potential, Poisson eq

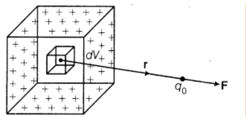
$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

- Principle of superposition

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{23}}$$

- Continuous distribution of charges

$$\frac{1}{4\pi\epsilon_0}\int_V d^3r' \, \rho(\mathbf{r}') \, \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$



MAGNETOSTATICS

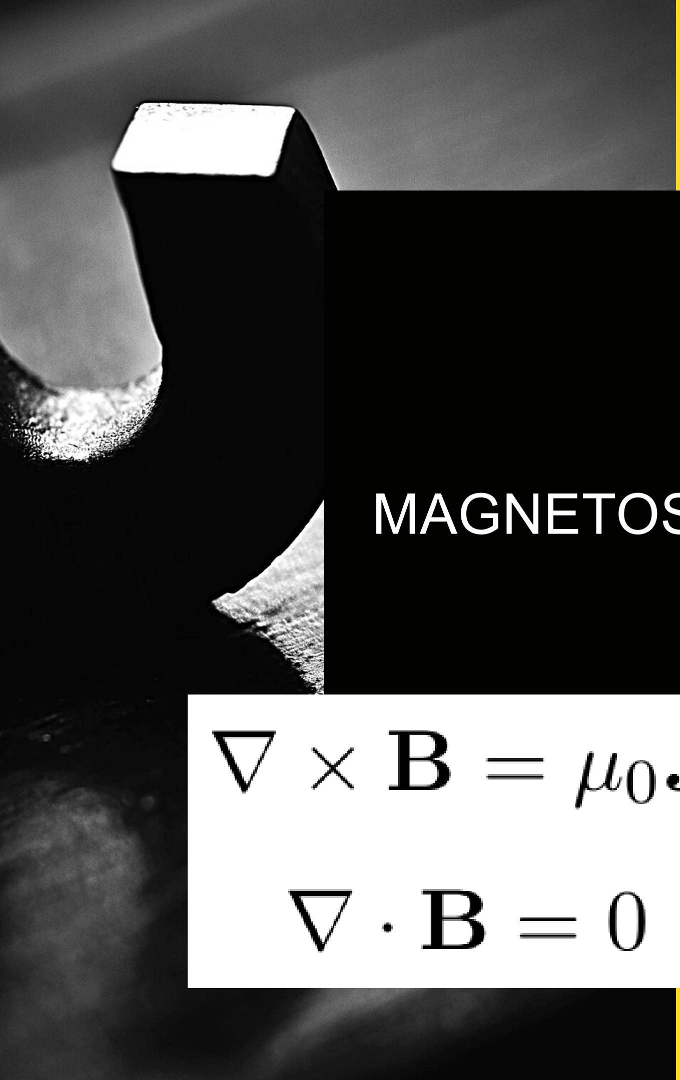
- Charges give rise to electric fields.
- Current give rise to magnetic fields.
- Moving charge particles make a magnetic field which is different from the electric field
- The magnetic field is induced by steady currents - continuous flow of charge

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$



MAGNETOSTATICS

$$\nabla \times \mathbf{B} = \mu_0 \boldsymbol{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

- **STEADY CURRENT**
- Ampère's Law
- Vector Potential
- Biot-Savart Law

Steady Current

Continuity equation, which captures the conservation of electric charge:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

charge density can change in time only if there is a compensating current flowing into or out of that region

Since the charge density is unchanging (and, indeed, vanishing)...

$$\cancel{\frac{\partial \rho}{\partial t}} + \nabla \cdot \mathbf{J} = 0$$

Steady Current

MATHEMATICALLY:

IF A CURRENT FLOWS INTO SOME REGION OF SPACE, AN EQUAL CURRENT MUST FLOW OUT TO AVOID THE BUILD UP OF CHARGE.

$$\cancel{\frac{\partial \rho}{\partial t}} + \nabla \cdot \mathbf{J} = 0$$

This is consistent with Maxwell Equations for magnetostatics

$$\nabla \times \mathbf{B} = \mu_0 \boldsymbol{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0$$

Steady Current

Bending Magnets: steady current-generated magnetic fields are used to bend the paths of charged particles.

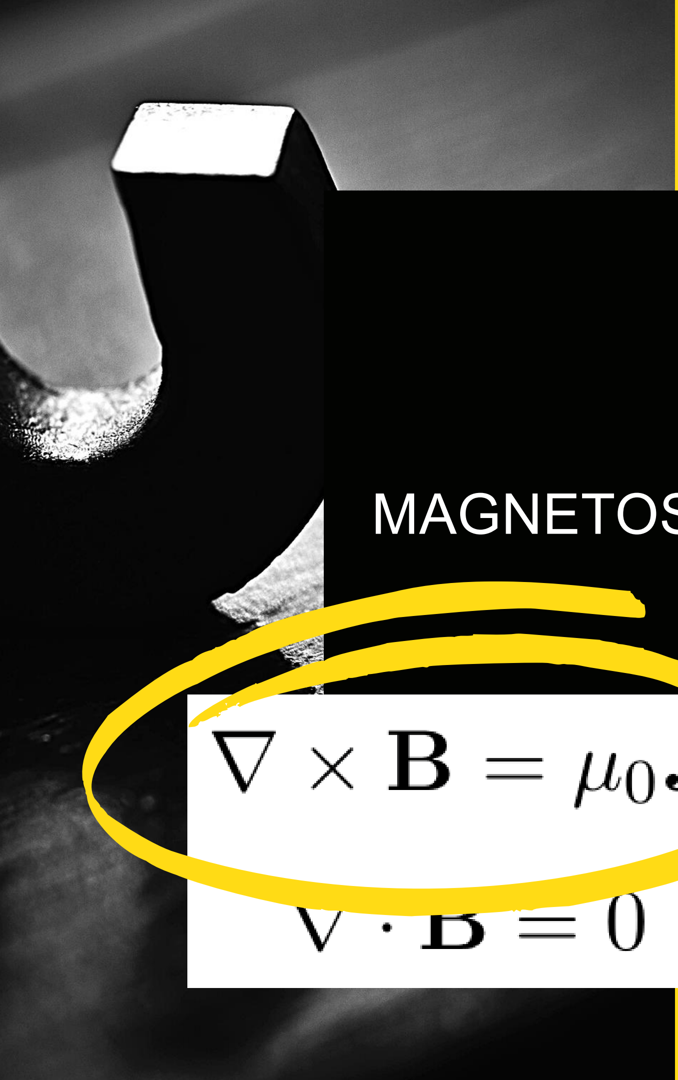
Quadrupole Magnets: steady currents are used in quadrupole magnets to focus the beam.

Sextupole and Higher Order Magnets: used to correct chromatic aberrations and other higher-order distortions in the beam's trajectory.

Beam Steering: dipole magnets with steady currents can be employed to make small adjustments to the trajectory of the beam

Stability: Time-independent magnetic fields, as opposed to oscillating ones, do not induce eddy currents in surrounding structures.

Magnetic Shielding and Corrections: magnetostatics principles are crucial when designing the shielding to prevent unwanted magnetic fields from affecting the accelerator's operation.



MAGNETOSTATICS

$$\nabla \times \mathbf{B} = \mu_0 \boldsymbol{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

- Steady Current
- **AMPÈRE LAW**
- Vector Potential
- Biot-Savart Law

AMPÈRE LAW

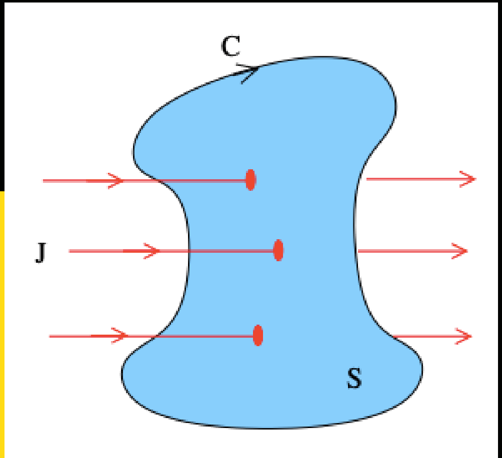
$$\nabla \times \mathbf{B} = \mu_0 \boldsymbol{J}$$

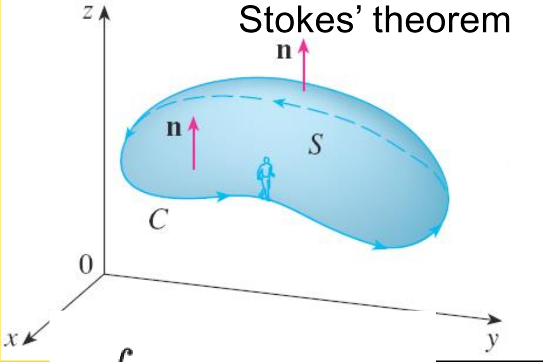
RELATIONSHIP BETWEEN A CURRENT AND THE MAGNETIC FIELD IT GENERATES

Ampere’s law states that the integral of the magnetic field around the contour **C** equals

$$\int_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}$$

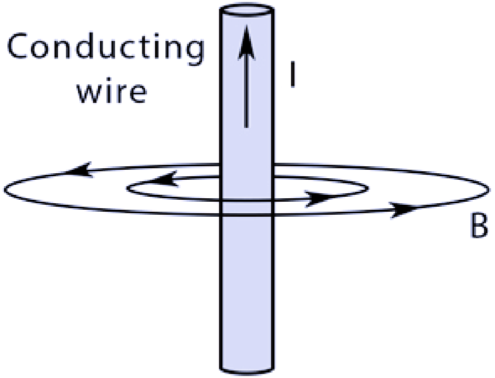
=



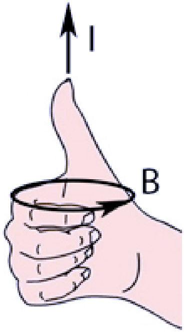


$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

Ampere's Law



Right hand thumb rule



Integral form: $\oint \vec{B}.\vec{dl} = \mu_o I$

Differential form: $\vec{\nabla} X \vec{B} = \mu_o \vec{J}$

I : Electric curent

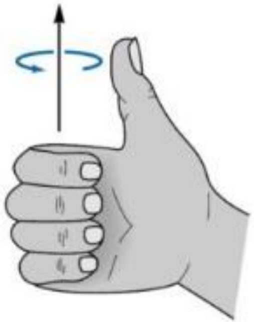
B : Magnetic field

μ_o: Permeability of free space

J : Current density

Thumb points in the direction of the electric current and fingers curl around the current indicating the direction of the magnetic field

Stokes' theorem



When the thumb points in the direction of $\hat{\mathbf{n}}$, the fingers curl in the forward direction around C

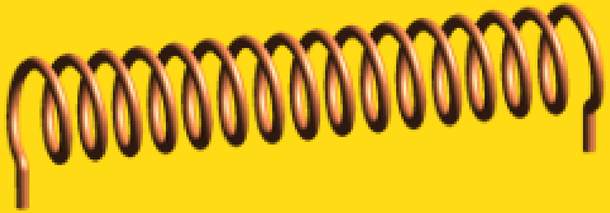
For positive current direction of magnetic field is determined with rule of right hand

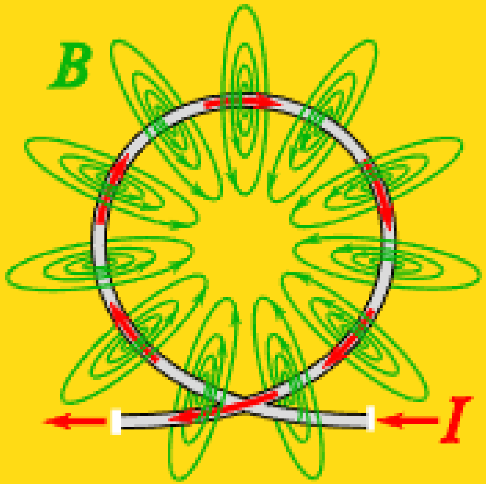
AMPÈRE LAW

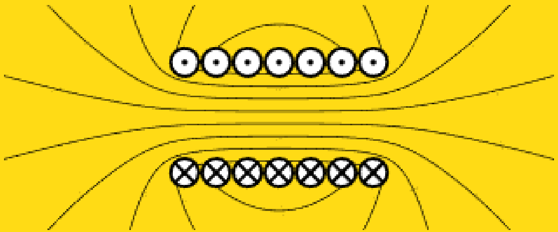
THE PRIMARY USAGE OF THE AMPERE LAW IS

CALCULATING THE MAGNETIC FIELD GENERATED BY AN ELECTRIC CURRENT

Ex: a long straight conducting wire, coaxial cable, cylindrical conductor, solenoid, and toroid



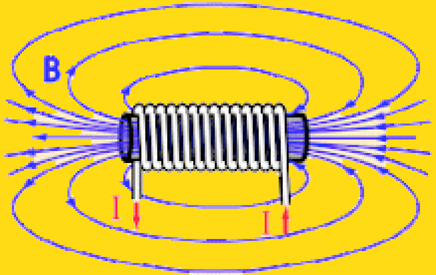




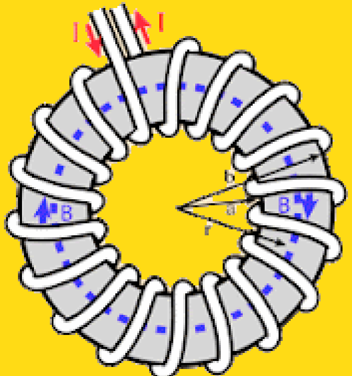
AMPÈRE LAW

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$BL = \mu NI$$
$$B = \mu \frac{N}{L} I$$
$$B = \mu nI$$



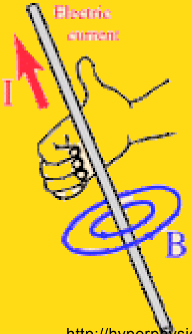
Magnetic field inside a long solenoic.



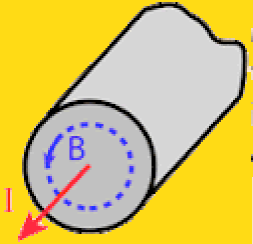
Magnetic field inside a toroidal coil.

$$B = \frac{\mu NI}{2\pi r}$$

$$B = \frac{\mu_0 I}{2\pi r}$$



Magnetic field from a long straight wire.



Magnetic field inside a conductor.

$$B = \frac{\mu Jr}{2} = \frac{\mu rI}{2\pi R^2}$$

which at the surface approaches:

$$B_{surface} = \frac{\mu I}{2\pi R}$$

Outside the surface,

$$B2\pi r' = \mu_0 I \quad \text{or} \quad B = \frac{\mu_0 I}{2\pi r'}$$

http://hyperphysics.phy-astr.gsu.edu

MAGNETOSTATICS

$$\nabla \cdot \mathbf{B} = 0$$

- Steady Current
- Ampère Law
- **VECTOR POTENTIAL**
- Biot-Savart Law

VECTOR POTENTIAL

To guaranteed a solution to $\nabla \cdot \mathbf{B} = 0$

we write the magnetic field as the curl of some vector field

$$\mathbf{B} = \nabla \times \mathbf{A}$$

A – is called the vector potential

While magnetic fields that can be written in this form certainly satisfy the given condition, the converse is also true

Ampère law becomes

$$\nabla \times \mathbf{B} = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = \mu_0 \mathbf{J}$$

This is the equation that we have to solve to determine A and, through that, **B**

MAGNETIC MONOPOLE

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \longrightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

It says that there are no magnetic charges.

A point-like magnetic charge g would source the magnetic field, giving rise a $1/r^2$ fall-off

$$\mathbf{B} = \frac{g\hat{\mathbf{r}}}{4\pi r^2}$$

object with this behaviour **– magnetic monopole**

Maxwell's equations says that they don't exist

MAGNETOSTATICS

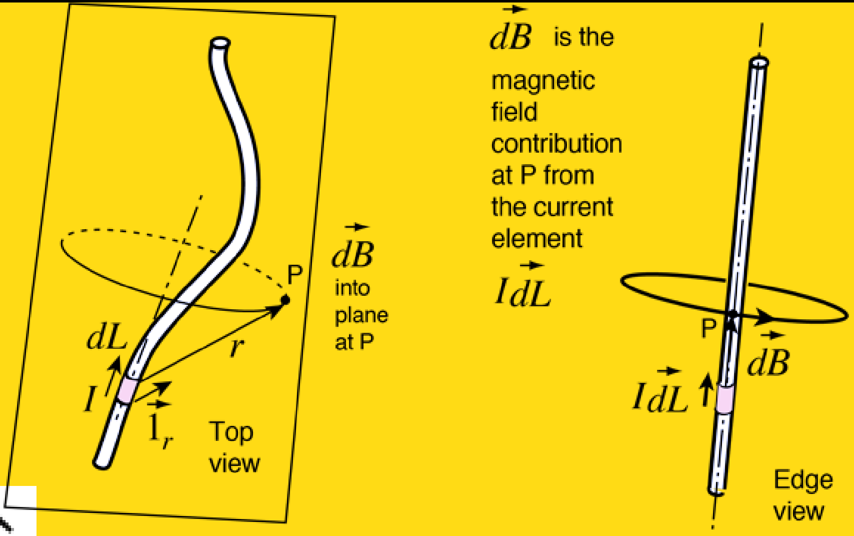
- Steady Current
- Ampère Law
- Vector Potential
- **BIOT-SAVART LAW**

BIOT-SAVART LAW

THE ANALOGOUS OF COULOMB LAW

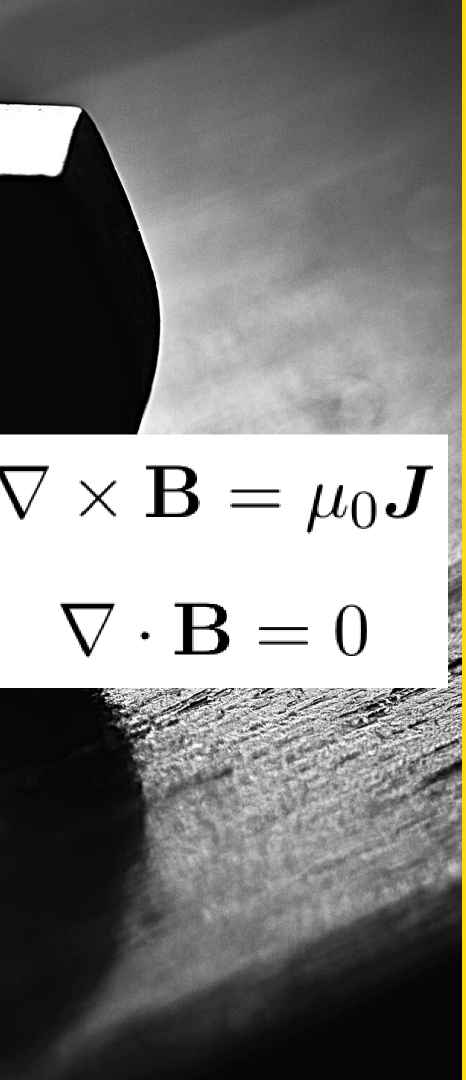
A segment of wire of length dl, carrying a current **I** sets up a magnetic field

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \hat{r}}{r^2}$$



Biot-Savart law for currents

http://hyperphysics.phy-astr.gsu.edu



RECAP: MAGNETOSTATICS

- **Steady Current** $\nabla \cdot \mathbf{J} = 0$ $\nabla \cdot (\nabla \times \mathbf{B}) = 0$

  if a current flows into some region of space, an equal current must flow out to avoid the build up of charge.

- **Ampère Law** $\nabla \times \mathbf{B} = \mu_0 \boldsymbol{J}$

  relationship between a current and the magnetic field it generates

- **Vector Potential** $\nabla \cdot \mathbf{B} = 0$ $\mathbf{B} = \nabla \times \mathbf{A}$

  magnetic fields can be represented using a vector potential.
  This way Ampere law becomes $\nabla \times \mathbf{B} = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = \mu_0 \mathbf{J}$

  This is the equation that we have to solve to determine A and, through that, B.

- **Biot-Savart law** $d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \hat{r}}{r^2}$

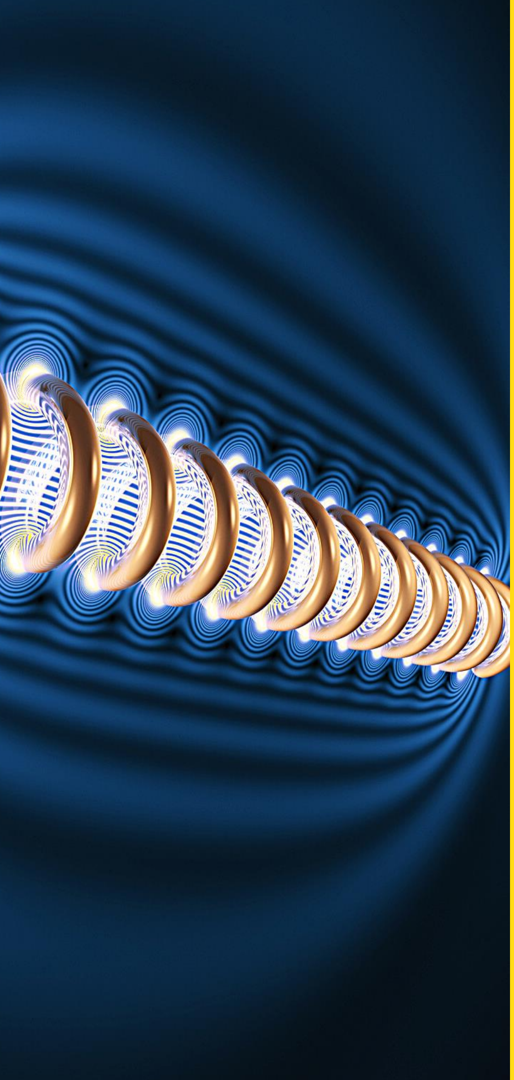
  It relates the magnetic field to the magnitude, direction, length, and proximity of the electric current

Summary of electro- and magneto-statics

One can compute the electric and the magnetic fields from the scalar and the vector potentials:

$$\vec{E} = -\nabla\phi$$
$$\vec{B} = \nabla \times \vec{A}$$

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}'$$
$$\vec{A}(r) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}'$$



ELECTRIC FORCE VS MAGNETIC FORCE

Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- To control a charged particle beam we use electromagnetic fields.
- In particle accelerators, transverse deflection is usually given by magnetic fields, whereas acceleration can only be given by electric fields.

$$\left|\vec{E}\right| = 1 \quad \text{MV/m}$$

$$\left|\vec{B}\right| = 1 \quad \text{T}$$

$$\frac{F_{\text{magnetic}}}{F_{\text{electric}}} = \frac{evB}{eE} = \frac{\beta cB}{E} \simeq \beta \frac{3 \cdot 10^8}{10^6} = 300\,\beta$$

- The magnetic force is much stronger then the electric one: in an accelerator, use magnetic fields whenever possible.

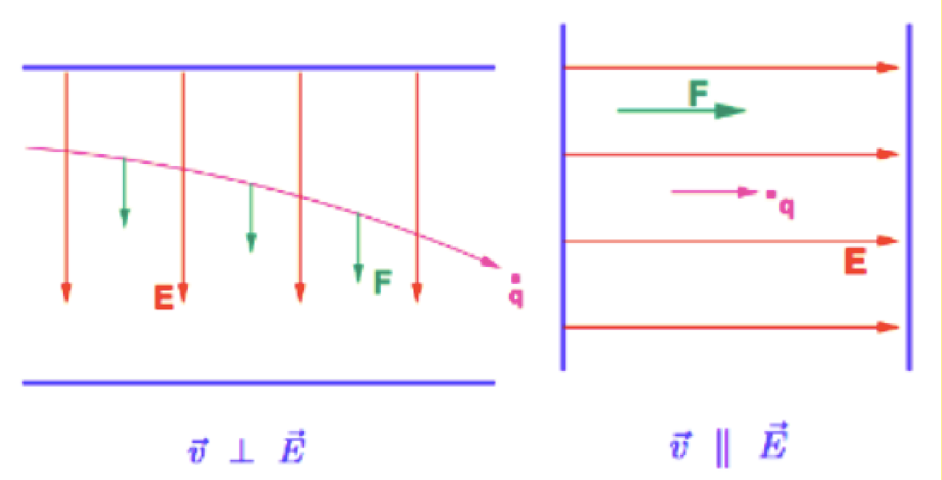
MOTION OF A CHARGED PARTICLE

Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

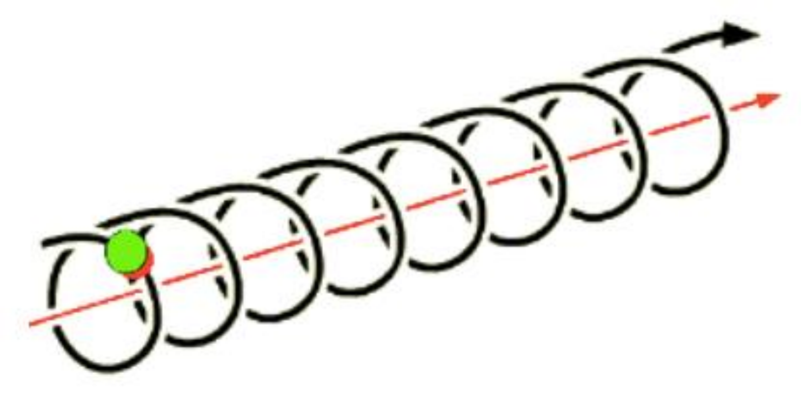
In case of an electric field, the force is always in the direction of the field, also for particles in rest.



MOTION OF A CHARGED PARTICLE

Lorentz force

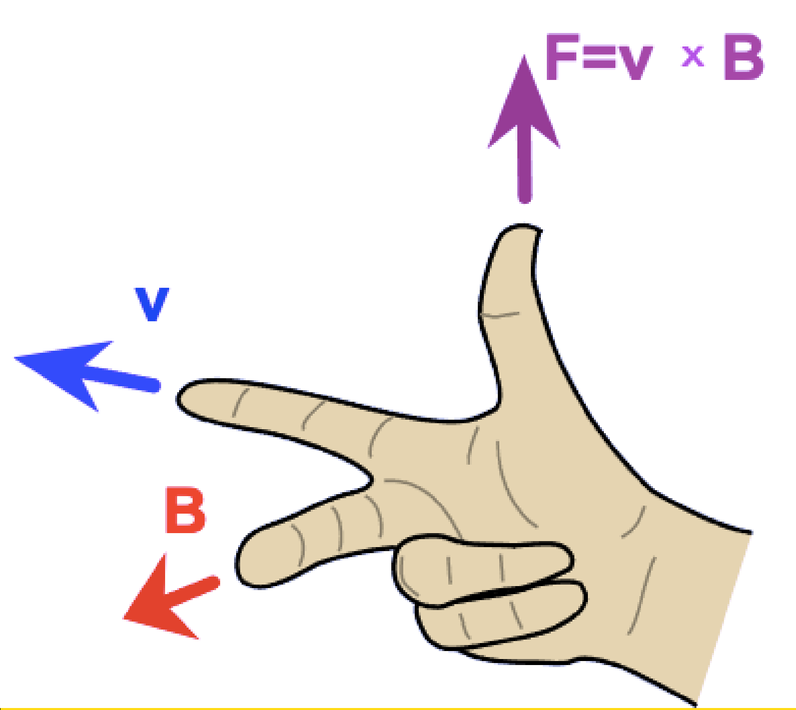
$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

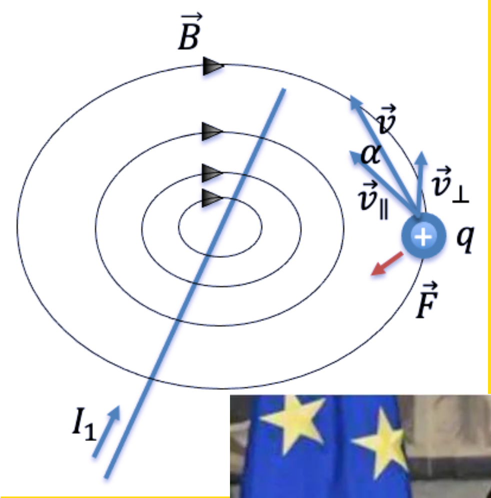


$$\mathbf{F} = q(\cancel{\mathbf{E}} + \mathbf{v} \times \mathbf{B})$$

In this case the force is perpendicular to both, **v** and **B**

MOTION OF A CHARGED PARTICLE





ELECTROMAGNETISM: NON-STATIC CASE

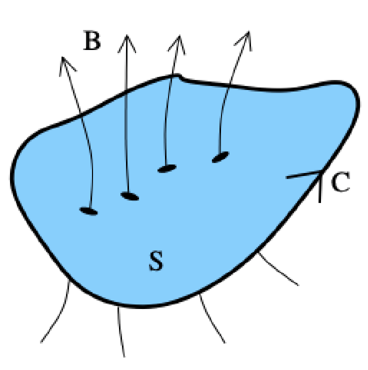
- **FARADAY'S LAW OF INDUCTION**
- Wave Function
- Propagation of electromagnetic waves

WORDS OF WISDOM

"I was at first almost frightened when I saw such mathematical force made to bear upon the subject, and then wondered to see that the subject stood it so well."

Faraday to Maxwell, 1857

Faraday's Law of Induction



The process of creating a current through changing magnetic fields is called INDUCTION.

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

Stokes theorem

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \rightarrow \int_C \mathbf{E} \cdot d\mathbf{r} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

$$\mathcal{E} = \int_C \mathbf{E} \cdot d\mathbf{r}$$

electromotive force

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

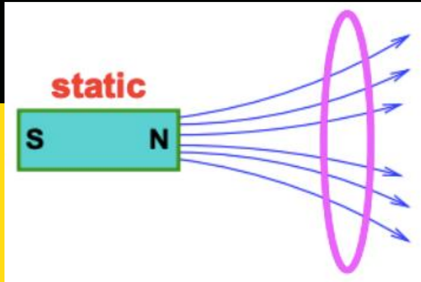
magnetic flux

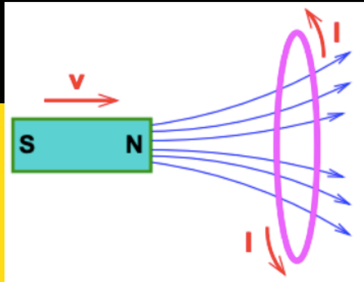
$$\mathcal{E} = -\frac{d\Phi}{dt}$$

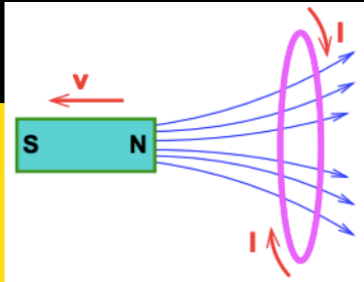
Faraday's Law

Faraday's Law of Induction

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

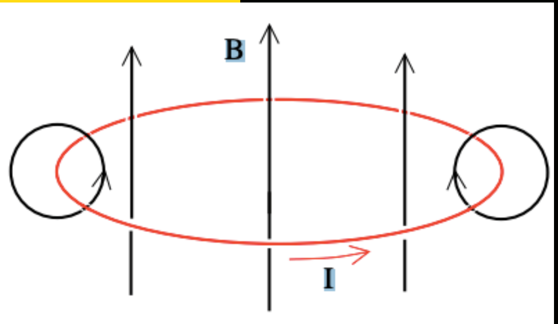






The electromotive force around a closed path is equal to the negative of the time rate of change of the magnetic flux enclosed by the path.

Secondary effect: When a current flows in C, it will create its own magnetic field. This induced magnetic field will always be in the direction that opposes the change. This is called **Lenz's law.**



ELECTROMAGNETISM: NON-STATIC CASE

$$\nabla \cdot \mathbf{E} = 0 \quad \text{and} \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- Faraday's Law of Induction
- **WAVE FUNCTION**
- Propagation of electromagnetic waves

WAVE FUNCTION

$$\nabla \cdot \mathbf{E} = 0 \quad \text{and} \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

ELECTRIC FIELD

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0$$

The wave equation

MAGNETIC FIELD

$$\frac{1}{c^2}\frac{\partial^2 \mathrm{B}}{\partial t^2} - \nabla^2 \mathrm{B} = 0$$

$$c = \sqrt{\frac{1}{\mu_0 \epsilon_0}}$$

SPEED OF LIGHT

- Probability amplitude:

$$\psi(t, \vec{x})$$

- Probability to find a particle at $(t, \vec{x})$:

$$0 \le \rho = |\psi|^2 \le 1$$

- Relativistic equation of motion:

$$E^2 - \vec{p}^2 = m^2 \quad \Rightarrow \quad -\frac{\partial^2}{\partial t^2}\psi + \vec{\nabla}^2\psi = m^2\psi$$

- Conserved electromagnetic current:

$$j_1^\mu = -2ep^\mu$$

NATURAL UNITS

Planck's constant $\hbar$ and speed of light c:

$$\hbar \equiv \frac{h}{2\pi} \simeq 1.055 \times 10^{-34}\,\text{J s}$$
$$c \simeq 2.998 \times 10^{8}\,\text{m/s}$$

Units in the international system:

$$[\hbar] = \frac{ML^2}{T} = \frac{\text{kg m}^2}{\text{s}}$$
$$[c] = \frac{L}{T} = \frac{\text{m}}{\text{s}}$$

"Natural" units:

$$\hbar = c \equiv 1 \quad ; \quad [\hbar] = [c] = 1$$
$$[e] = [\sqrt{\hbar c}] = [1] \quad ; \quad \alpha = \frac{\frac{1}{4\pi}\frac{e^2}{\hbar/mc}}{mc^2} =$$

National Research Tomsk State University

AND THERE WAS LIGHT

The velocity of transverse undulations in our hypothetical medium, calculated from the electro-magnetic experiments of MM. Kohlrausch and Weber, agrees so exactly with the velocity of light calculated from the optical experiments of M. Fizeau, that we can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena.

James Clerk Maxwell

WAVE FUNCTION

$\left\|\vec{k}\right\| = \frac{2\pi}{\lambda} = \frac{\omega}{c}$	wave-number vector
$\lambda = \frac{c}{f}$	wave length
f	frequency
$\omega = 2\pi f$	angular frequency

$$\mathbf{E} = \mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \quad \text{and} \quad \mathbf{B} = \mathbf{B}_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

k – the wave-number vector with **|k|** = **k**, which gives the direction of propagation of the wave.

ω is more properly called the angular frequency **(f –** frequency)

$$\omega^2 = c^2 k^2$$

dispersion relation

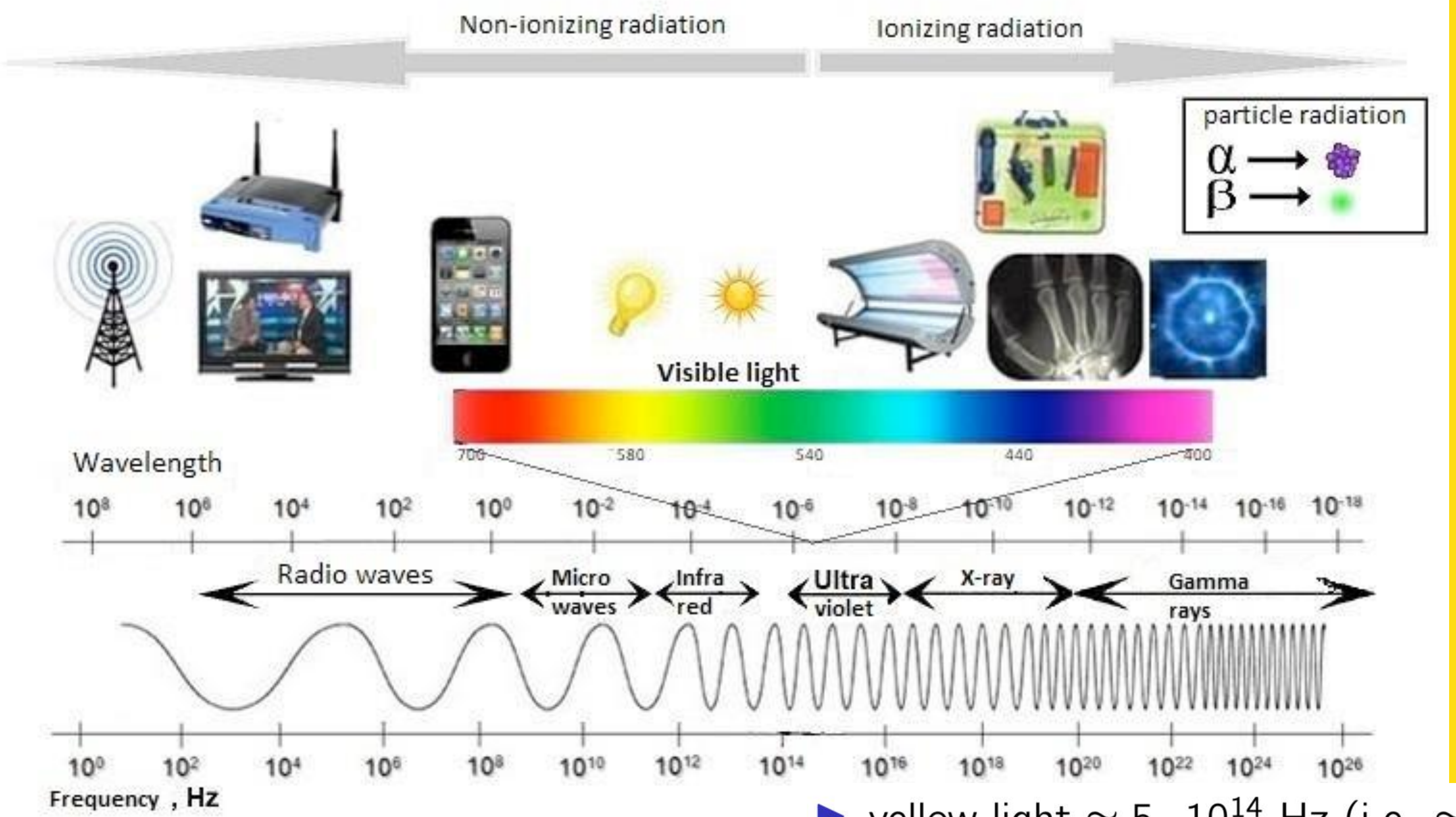
$$c = \frac{\omega}{|k|} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Eo, Bo – constant vectors, the amplitude of the wave

λ = **2π/k** - the wavelength of the wave

Short wavelength →high frequency → high energy

The electromagnetic spectrum



- yellow light $\approx 5 \cdot 10^{14}$ Hz (i.e. ≈ 2 eV !)
- LEP (SR) $\leq 2 \cdot 10^{20}$ Hz (i.e. ≈ 0.8 MeV !)
- gamma rays $< 3 \cdot 10^{21}$ Hz (i.e. < 12 MeV !)

WAVE FUNCTION. CONSTRAINS.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

→

$$\vec{k} \times \vec{E}_0 = \omega \vec{B}_0$$

$$\vec{k} \times \vec{B}_0 = -\frac{\omega}{c^2}\vec{E}_0$$

$$\vec{k} \cdot \vec{E}_0 = 0$$

$$\vec{k} \cdot \vec{B}_0 = 0$$

WAVE FUNCTION. CONSTRAINS.

- **Eo, Bo, and k** are mutually perpendicular;
- The field amplitudes are related by $\frac{E_0}{B_0} = c$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

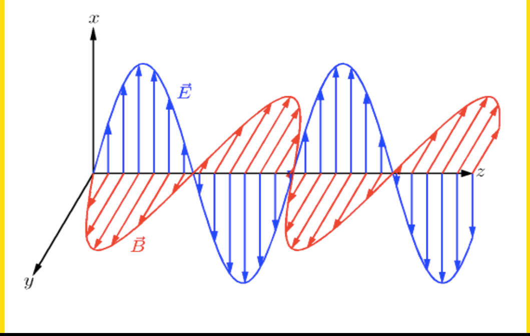
→

$$\vec{k} \times \vec{E}_0 = \omega \vec{B}_0$$

$$\vec{k} \times \vec{B}_0 = -\frac{\omega}{c^2}\vec{E}_0$$

$$\vec{k} \cdot \vec{E}_0 = 0$$

$$\vec{k} \cdot \vec{B}_0 = 0$$



Magnetic and electric fields are transverse to direction of propagation:

$$\vec{E} \perp \vec{B} \perp \vec{k}$$

ELECTROMAGNETISM: NON-STATIC CASE

- Faraday's Law of Induction
- Wave Function
- **PROPAGATION OF ELECTROMAGNETIC WAVES**

Propagation of electromagnetic waves in a conductor

OHMIC CONDUCTOR

$$\mathbf{J} = \sigma \mathbf{E}$$

One significant difference is that the electric field in the wave drives a flow of electric current in the conductor: this leads to ohmic energy losses

The constant σ is the conductivity of the material

REWRITE THE CONTINUITY EQUATION,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

$$\nabla \cdot \mathbf{J} = \sigma \nabla \cdot \mathbf{E} = \frac{\sigma}{\epsilon_0}\rho$$

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon_0}\rho = 0$$

$$\rho(t) = \rho_0 \exp -\frac{t}{\tau}$$

$$\tau = \frac{\epsilon_0}{\sigma}$$

relaxation time

Propagation of electromagnetic waves in a conductor

$\tau = \frac{\epsilon_0}{\sigma}$ relaxation time

- **PERFECT CONDUCTOR**

$\sigma \rightarrow \infty$

Relaxation time is vanishing

- **GOOD, BUT NOT PERFECT CONDUCTOR**

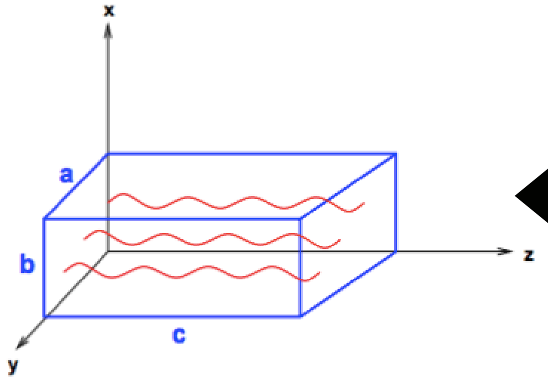
charges move almost instantly to the surface of the conductor

$\frac{\sigma}{\epsilon} \approx 10^{14} \text{sec}^{-1}$

- **ISOLATOR**

$\sigma = 0$

the solution of the wave equation is reduces to an ordinary plane wave

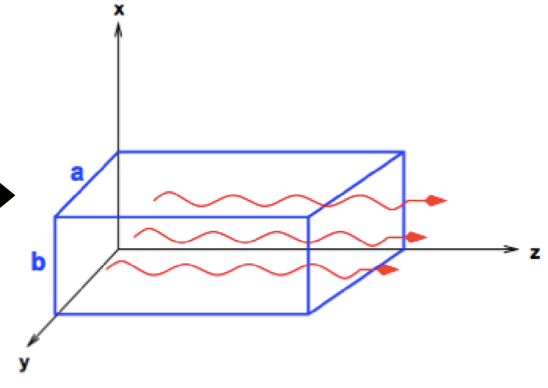


RF CAVITIES

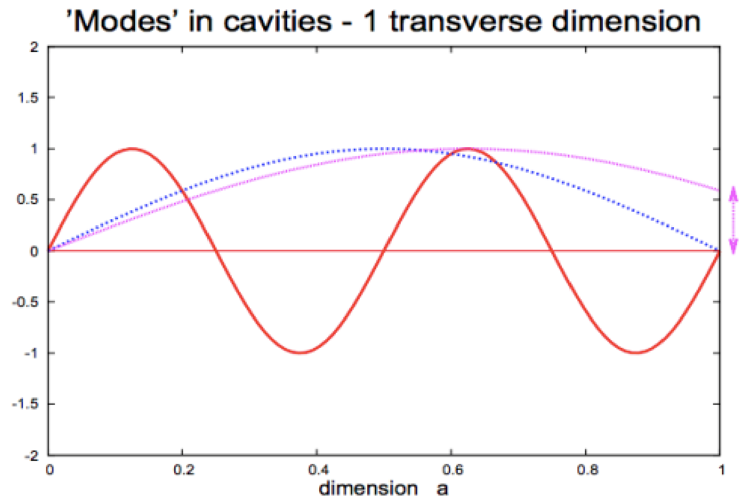
Field can persist and be stored

WAVEGUIDES

Plane waves can propagate along waveguides



Example: Fields in RF cavities



No electric field at boundaries, wave must have "nodes" = zero fields at the boundaries

Only modes which 'fit' into the cavity are allowed

In the example: $\frac{\lambda}{2} = \frac{a}{4}$, $\frac{\lambda}{2} = \frac{a}{1}$, $\frac{\lambda}{2} = \frac{a}{0.8}$

(then either "sin" or "cos" is 0)

Consequences: RF cavities

Field must be zero at conductor boundary, only possible if:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

and for k_x, k_y, k_z we can write, (then they all fit):

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b}, \qquad k_z = \frac{m_z \pi}{c},$$

The integer numbers m_x, m_y, m_z are called mode numbers, important for design of cavity !

$\rightarrow$ half wave length $\lambda/2$ must always fit exactly the size of the cavity.

(For cylindrical cavities: use cylindrical coordinates)

Consequences: wave guides

Similar considerations as for cavities, no field at boundary.
We must satisfy again the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

This leads to modes like (no boundaries in direction of propagation z):

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b},$$

The numbers m_x, m_y are called mode numbers for planar waves in wave guides !

In z direction: No Boundary - No Boundary Condition ...

Consequences: wave guides

Re-writing the condition as:

$$k_z^2 = \frac{\omega^2}{c^2} - k_x^2 - k_y^2 \quad \rightarrow \quad k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$$

Propagation without losses requires k_z to be real, i.e.:

$$\frac{\omega^2}{c^2} > k_x^2 + k_y^2 = \left(\frac{m_x \pi}{a}\right)^2 + \left(\frac{m_y \pi}{b}\right)^2$$

which defines a cut-off frequency ω_c. For lowest order mode:

$$\omega_c = \frac{\pi \cdot c}{a}$$

- Above cut-off frequency: propagation without loss
- At cut-off frequency: standing wave
- Below cut-off frequency: attenuated wave (it does not "fit in").

There is a very easy way to show that very high frequencies easily propagate

RECAP: ELECTROMAGNETISM: NON-STATIC CASE

- **Faraday's Law of Induction**

  The electromotive force around a closed path is equal to the negative of the time rate of change of the magnetic flux enclosed by the path.

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

- **Wave Function**

Wave-function number

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0 \qquad \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0 \qquad c = \frac{\omega}{|k|} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$\mathbf{E} = \mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \quad \text{and} \quad \mathbf{B} = \mathbf{B}_0\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

- **Propagation of electromagnetic waves**

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

$$k_x = \frac{m_x \pi}{a}, \quad k_y = \frac{m_y \pi}{b}, \quad k_z = \frac{m_z \pi}{c},$$

The integer numbers m_x, m_y, m_z are called mode numbers, important for design of cavity !

THE END

National Research Tomsk State University

Thank you for your attention!

I would like to thank my colleagues who gave the EM course previously (and I could profit from it while preparing the lecture (A. Latina, A. Wolski, P. Skowronski, W. Herr)

CONTACT INFORMATION

irina.shreyber@cern.ch

https://www.linkedin.com/in/ishreyber/