Born-Oppenheimer Effective Theory (BOEFT): Hybrids, Tetraquarks, Pentaquarks

QWG workshop 2024, IISER-Mohali

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Exotic Hadron

- Dozens of XYZ mesons discovered since 2003.

- Multiple Models for Exotics:

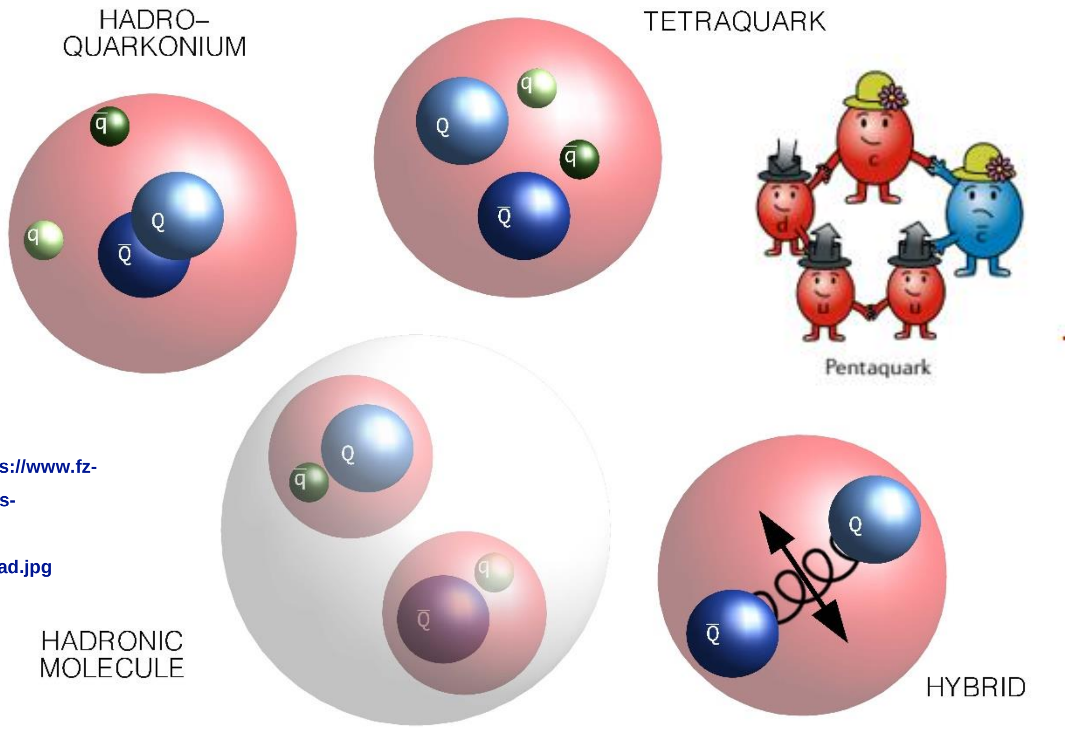
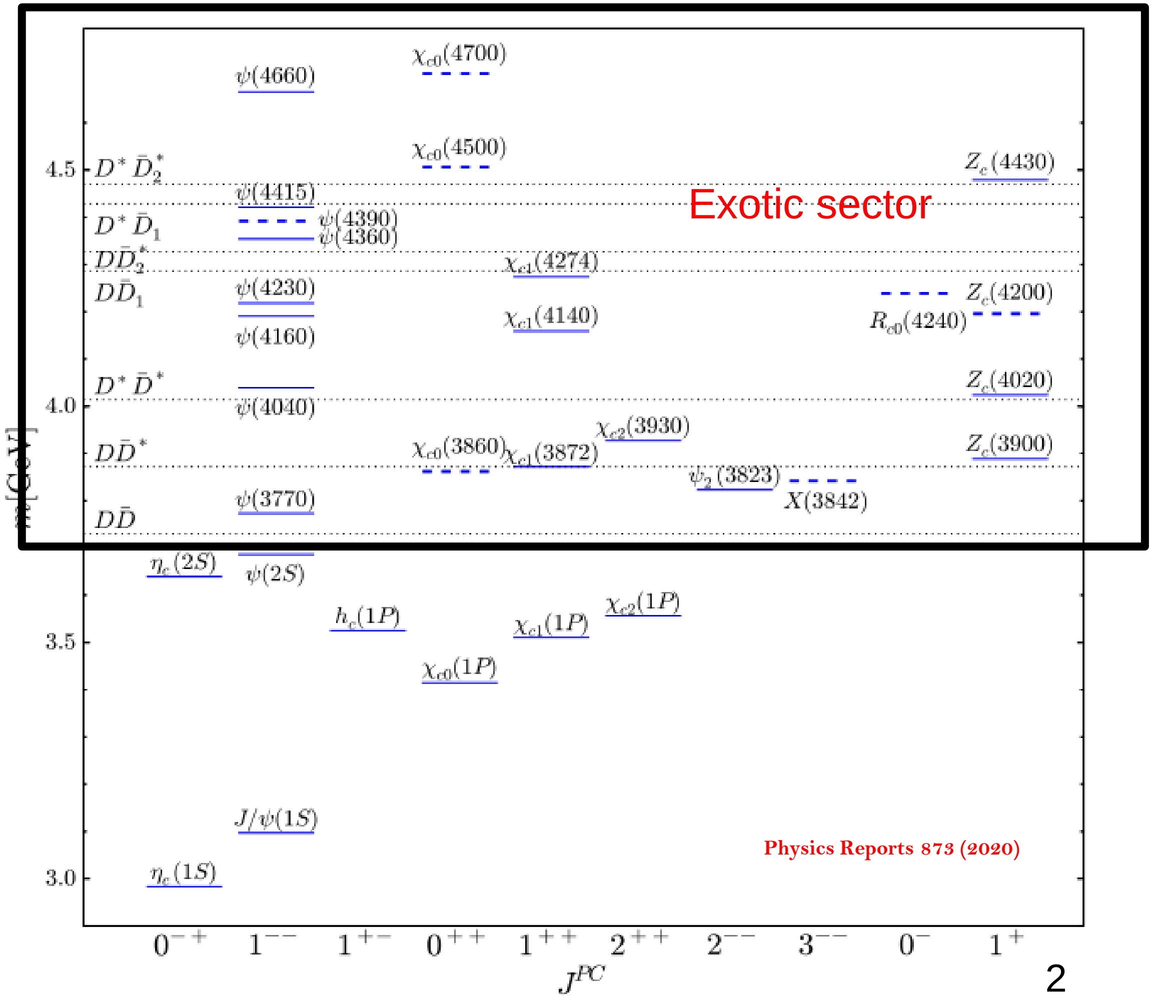


Figure from https://www.fz-juelich.de/en/ias/ias-4/research/exotic-hadrons/exotics_pad.jpg



- Individual success in describing some XYZ hadrons. No success in revealing general pattern.

Exotic Hadron

- Multiple Models for Exotics:

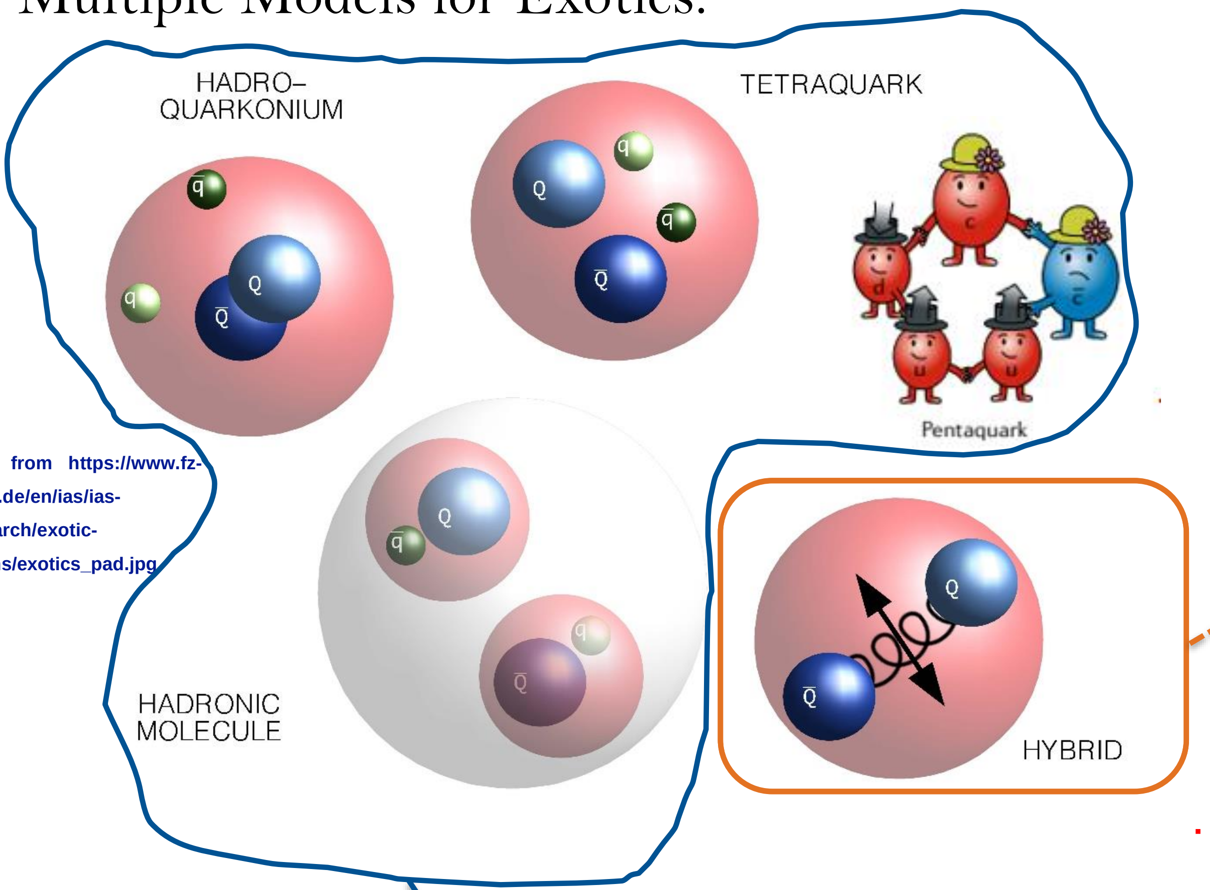


Figure from https://www.fz-juelich.de/en/ias/ias-4/research/exotic-hadrons/exotics_pad.jpg

Hybrids ($Q\bar{Q}g$): Isospin scalar exotic state.

Use EFT + lattice for describing hybrid

Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015), 114019

……….

Non-zero isospin states. Can be described in the EFT.
However, lack of lattice inputs on the static energies for these states

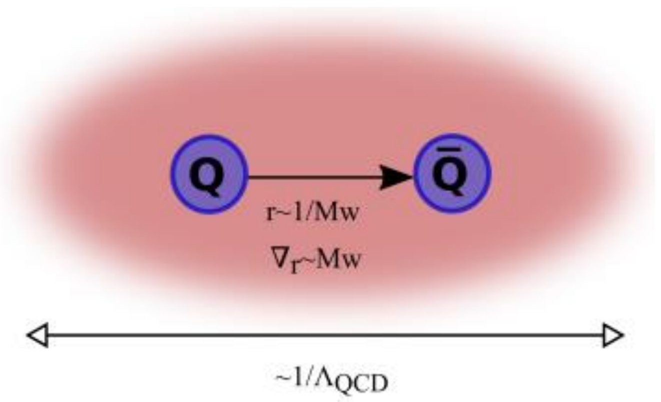
Brambilla, AM, Vairo arXiv 2402.xxxxx

BOEFT: Exotic Hadron

- Exotic hadron ($\boldsymbol{Q\bar{Q}X}, \boldsymbol{QQX}, \ldots\ldots$), $\boldsymbol{X}$: any combination of light quark and gluons to obtain color singlet hadron

:

Example: Quarkonium hybrids $\boldsymbol{Q\bar{Q}g}$, Tetraquarks $\boldsymbol{Q\bar{Q}q\bar{q}}$



- Hierarchy of scales:

$$m \gg mv \gtrsim \Lambda_{\mathrm{QCD}} \gg mv^2$$

BOEFT: Effective theory based on Born-Oppenheimer Approximation

BOEFT: EFT focused at energy scale mv^2

Heavy quark: slow-degrees of freedom $\boldsymbol{X}$: fast-degrees of freedom

$$\mathrm{QCD} \rightarrow \mathrm{NRQCD} \rightarrow \mathrm{pNRQCD/BOEFT}$$

- Time-scale for dynamics of $\mathrm{Q\bar{Q}}$: $\sim \dfrac{1}{\mathrm{mv}^2} \gg \dfrac{1}{\Lambda_{\mathrm{QCD}}}$

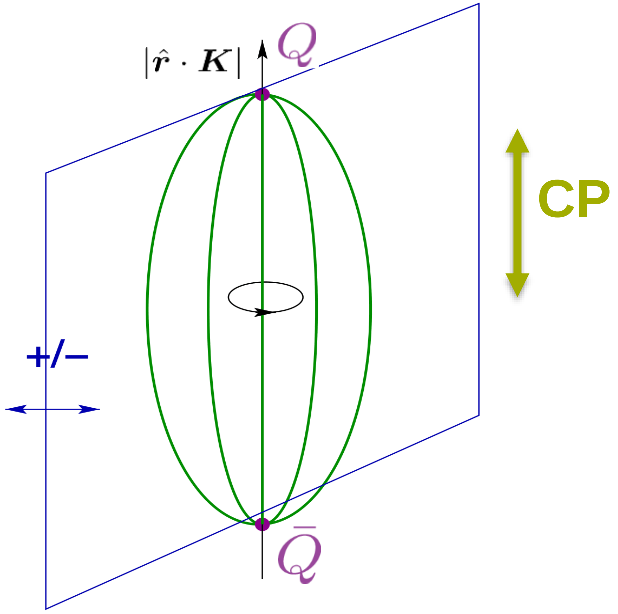
Born-Oppenheimer (BO) Approximation

Castellà , Soto Phys. Rev. D. 102, (2020)

Brambilla, AM, Vairo arXiv 2402.xxxxx

Braaten, Langmack, Smith Phys. Rev. D. 90, 014044 (2014)

BOEFT: Quantum #'s



- **BOEFT potentials ($V_\Gamma(r)$):** LDF (light quarks, gluons) **static energies.**
  Potential between Q & $\bar{Q}$

- $V_\Gamma(r)$: **Γ** labelled by cylindrical symmetry ($D_{\infty h}$) representation (diatomic molecules):

  ✓ Absolute value of component of angular momentum of light d.o.f

  $$|r \cdot K_{\text{light}}| \equiv \Lambda = 0, 1, 2, \ldots \ldots (\text{or } \Sigma, \Pi, \Delta, \Phi, \ldots)$$

  ✓ Product of charge conjugation and parity (CP):

  $\eta = +1$ (g), -1 (u)

  ✓ σ: Eigenvalue of reflection about a plane containing static sources.

  $\sigma = P\,(-1)^{K_{\text{light}}} = \pm 1$

$$\Gamma \equiv \Lambda_\eta^\sigma$$

Braaten, Langmack, Smith Phys. Rev. D. 90, 014044 (2014)

- r→ 0: **Spherical symmetry restored**: Labelled by gluon quantum #'s $\kappa = K^{PC}$.

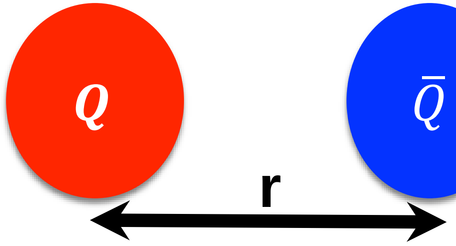
Brambilla, Pineda , Soto, and Vairo Nucl. Phys. B566 (2000) Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015), 114019

BOEFT: Exotic

Brambilla, AM, Vairo
arXiv 2402.xxxxx

- Exotic hadron $(Q\bar{Q}X, QQX, \ldots\ldots)$, X is light d.o.f.

$Q\bar{Q}X$



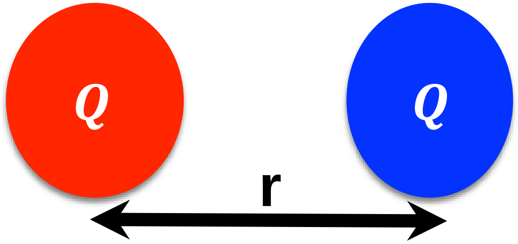
color: $3 \otimes \bar{3} = 1 \oplus 8$

X= gluon $\rightarrow$ **Hybrid**

X= $q\bar{q}$ $\rightarrow$ **Tetraquark / Molecule**

X= qqq $\rightarrow$ **Pentaquark / Molecule** and so on

QQX



color: $3 \otimes \bar{3} = \bar{3} \oplus 6$

X= q $\rightarrow$ **Double heavy baryon**

X= $\bar{q}\bar{q}$ $\rightarrow$ **Tetraquark**

X= $q\bar{q}q$ $\rightarrow$ **Pentaquark** and so on

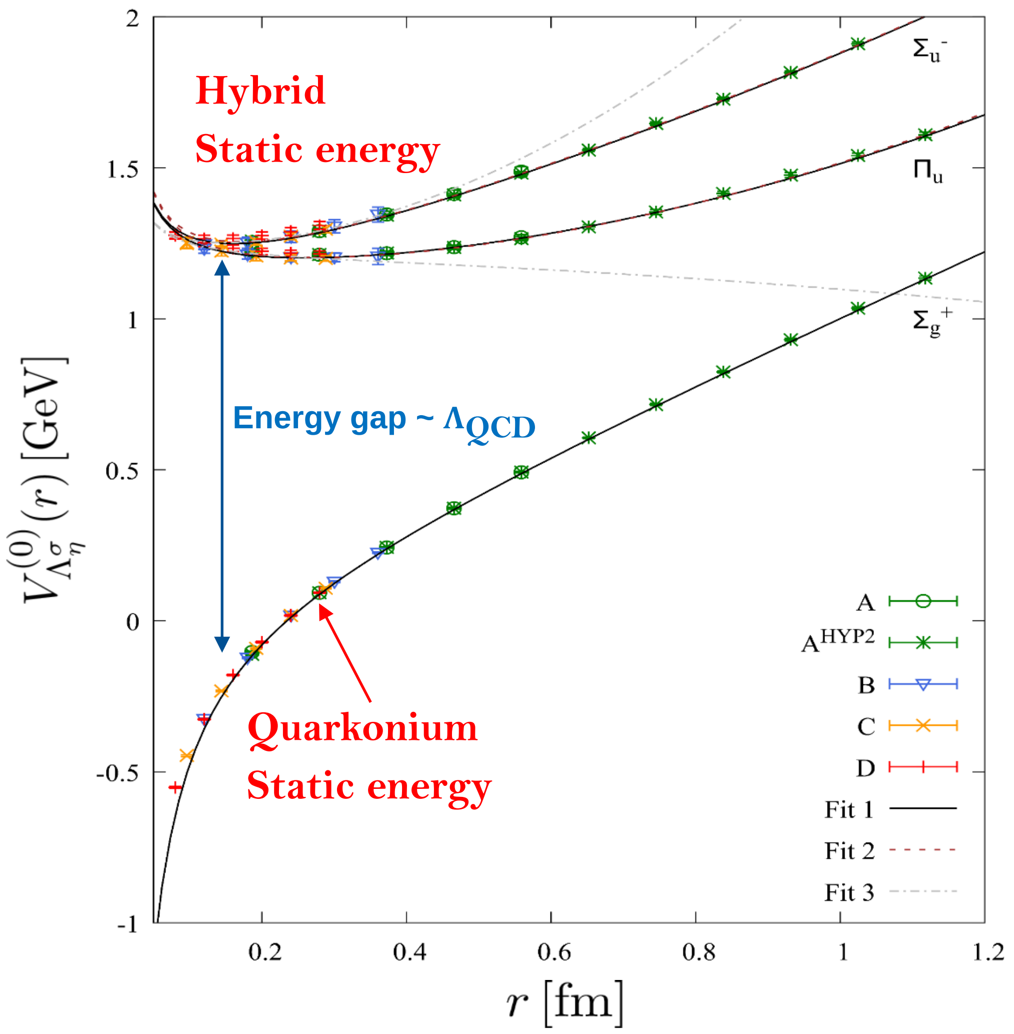
BOEFT can address all these states with inputs from Lattice QCD on BOEFT (static) potentials

BOEFT

Brambilla, AM, Vairo arXiv 2402.xxxxx

- BOEFT Lagrangian: $$L_{\text{BOEFT}} = L_{Q\bar{Q}} + L_{Q\bar{Q}g} + L_{Q\bar{Q}q\bar{q}} + L_{\text{mixing}} + \cdots$$

Castellà , Soto Phys. Rev. D. 102, (2020)



- Gap of order Λ_{QCD} allows us to focus individually on low-lying states corresponding to quarkonium, hybrid, tetraquark etc.

- L_{mixing}: Mixing between different states with similar masses and same quantum-numbers.

Ex: Hybrid-quarkonium mixing, Tetraquark-hybrid & Tetraquark-quarkonium mixing etc.

BOEFT

Brambilla, AM, Vairo
arXiv 2402.xxxxx

- BOEFT Lagrangian:

$$L_{\text{BOEFT}} = \int d^3\boldsymbol{R} \int d^3\boldsymbol{r} \sum_{\kappa\lambda\lambda'} \text{Tr}\Big\{ \Psi^{\dagger}_{\kappa\lambda}(\mathbf{r},\, \mathbf{R},\, t) \Big[i\partial_t\, \delta_{\lambda\lambda'} - V_{\kappa\lambda\lambda'}(r) + P^{i\dagger}_{\kappa\lambda}(\theta,\phi)\, \frac{\boldsymbol{\nabla}^2_r}{m_Q} P^{i}_{\kappa\lambda'}(\theta,\phi) \Big] \Psi_{\kappa\lambda'}(\mathbf{r},\, \mathbf{R},\, t) \Big\}$$

$\kappa = \boldsymbol{K}^{PC}$: Light d.o.f quantum number $\qquad \lambda = -\Lambda, \cdots, \Lambda$

Castellà , Soto
Phys. Rev. D. 102, (2020)

BOEFT potential: $V_{\kappa\lambda\lambda'}(r) = \boxed{V^{(0)}_{\kappa\lambda}(r)}\,\delta_{\lambda\lambda'} + \frac{V^{(1)}_{\kappa\lambda\lambda'}(r)}{m_Q} + \ldots ,$

Static potential

- Matching procedure: $\mathcal{O}_K(t,\, \mathbf{r},\, \mathbf{R}) \longrightarrow Z_{\Psi_K}(r,\, \Lambda_{\text{QCD}})\Psi_K(t,\, \mathbf{r},\, \mathbf{R})$

NRQCD $\qquad$ BOEFT

exotic hadron $\boldsymbol{Q\bar{Q}X}$: $\mathcal{O}_K(t,\, \mathbf{r},\, \mathbf{0}) = \chi^{\dagger}(\mathbf{t},\, \mathbf{r}/\mathbf{2})\phi(\mathbf{t},\, \mathbf{r}/\mathbf{2}, \mathbf{0})\mathbf{H}_{\mathbf{K}}(\mathbf{t},\, \mathbf{0})\phi(\mathbf{t},\, \mathbf{0},\, -\mathbf{r}/\mathbf{2})\psi(\mathbf{t},\, -\mathbf{r}/\mathbf{2})$

$H_K(t,\, \mathbf{0})$: Operator that characterizes the light d.o.f ***X*** corresponding to quantum # **K**, isospin, color etc..

Brambilla, AM, Vairo arXiv 2312.xxxxx

Light quark operators characterized by K^{PC} essential for determinig BO-potentials $V_{\Lambda_\eta^\sigma}(r)$

Gluonic operators $\mathbf{H}_{\mathbf{K}^{\mathbf{PC}}}$ in lattice characterizing Hybrids **$Q\bar{Q}g$**

$$\mathbf{H}_{1+-}(t,\boldsymbol{x}) = \boldsymbol{B}(t,\boldsymbol{x})$$
$$\mathbf{H}_{1--}(t,\boldsymbol{x}) = \boldsymbol{E}(t,\boldsymbol{x})$$

Light quark operators $\mathbf{H}_{\mathbf{K}^{\mathbf{PC}}}$ relevant for lattice computation of static energies for tetraquarks **$Q\bar{Q}q\bar{q}$**

$$\mathbf{H}_{0++}(t,\boldsymbol{x}) = [\bar{q}(t,\boldsymbol{x})T^a q(t,\boldsymbol{x})]\, T^a$$
$$\mathbf{H}_{0-+}(t,\boldsymbol{x}) = \left[\bar{q}(t,\boldsymbol{x})\gamma^5 T^a q(t,\boldsymbol{x})\right] T^a$$
$$\mathbf{H}_{1++}(t,\boldsymbol{x}) = \left[\bar{q}(t,\boldsymbol{x})\boldsymbol{\gamma}\gamma^5 T^a q(t,\boldsymbol{x})\right] T^a$$
$$\mathbf{H}_{1--}(t,\boldsymbol{x}) = [\bar{q}(t,\boldsymbol{x})\boldsymbol{\gamma} T^a q(t,\boldsymbol{x})]\, T^a$$
$$\mathbf{H}_{1+-}(t,\boldsymbol{x}) = [\bar{q}(t,\boldsymbol{x})\,(\boldsymbol{\gamma}\times\boldsymbol{\gamma})\, T^a q(t,\boldsymbol{x})]\, T^a$$

Light quark operators $\mathbf{H}_{\mathbf{K}^{\mathbf{PC}}}$ relevant for lattice computation of static energies for pentaquarks **$Q\bar{Q}qqq$**

Castellà , Soto Phys. Rev. D. 102, (2020)

$$H^{\alpha}_{I_3,(1/2)^+}(t,\boldsymbol{x}) =$$
$$\Big[\left(\delta_{\alpha\beta_1}\sigma^2_{\beta_2\beta_3} + \delta_{\alpha\beta_2}\sigma^2_{\beta_1\beta_3} + \delta_{\alpha\beta_3}\sigma^2_{\beta_1\beta_2}\right)\left(\delta_{I_3 f_1}\tau^2_{f_2 f_3} + \delta_{I_3 f_2}\tau^2_{f_1 f_3} + \delta_{I_3 f_3}\tau^2_{f_1 f_2}\right)(T_2)^a_{l_1,l_2,l_3}$$
$$+ \left(\delta_{\alpha\beta_1}\sigma^2_{\beta_2\beta_3} + \delta_{\alpha\beta_2}\sigma^2_{\beta_3\beta_1} + \delta_{\alpha\beta_3}\sigma^2_{\beta_2\beta_1}\right)\left(\delta_{I_3 f_1}\tau^2_{f_2 f_3} + \delta_{I_3 f_2}\tau^2_{f_3 f_1} + \delta_{I_3 f_3}\tau^2_{f_2 f_1}\right)(T_3)^a_{l_1,l_2,l_3}$$
$$+ \left(\delta_{\alpha\beta_1}\sigma^2_{\beta_3\beta_2} + \delta_{\alpha\beta_2}\sigma^2_{\beta_3\beta_1} + \delta_{\alpha\beta_3}\sigma^2_{\beta_1\beta_2}\right)\left(\delta_{I_3 f_1}\tau^2_{f_3 f_2} + \delta_{I_3 f_2}\tau^2_{f_3 f_1} + \delta_{I_3 f_3}\tau^2_{f_1 f_2}\right)(T_1)^a_{l_1,l_2,l_3}\Big]$$
$$(P_+ q_{l_1 f_1}(t,\boldsymbol{x}))^{\beta_1}\,(P_+ q_{l_2 f_2}(t,\boldsymbol{x}))^{\beta_2}\,(P_+ q_{l_3 f_3}(t,\boldsymbol{x}))^{\beta_3}\, T^a\,. \quad (53)$$

Similar operator list can be written for Doubly heavy tetraquark **$QQ\bar{q}\bar{q}$** and Pentaquark states **$QQqq\bar{q}$**. List of operators will be addressed in Brambilla, AM, Vairo arXiv 2402.xxxxx

BOEFT

Brambilla, AM, Vairo
arXiv 2402.xxxxx

- Adiabatic Radial Schrödinger equation: Mixing different static energies at short-distances:

$$\sum_{\lambda}\left[-\frac{1}{m_Q r^2}\partial_r\, r^2\partial_r + \frac{1}{m_Q r^2} M_{\lambda'\lambda} + V^{(0)}_{\kappa\lambda'\lambda}\right]\psi^{(N)}_{\kappa\lambda}(r) = \mathcal{E}_N\,\psi^{(N)}_{\kappa\lambda'}(r)\,,$$

Mixing term from angular momentum piece:
Coupling different static energies Λ_η^σ at short-distance

- General expression of $M_{\lambda'\lambda}$ (matrix in $\lambda' - \lambda$ basis) : $\lambda, \lambda' = -\Lambda, \cdots, \Lambda$

$$\begin{aligned}M_{\lambda'\lambda} &= \langle l,m;k,\lambda'|\boldsymbol{L}_Q^2|l,m;k,\lambda\rangle \\ &= \left(l(l+1) - 2\lambda^2 + k(k+1)\right)\delta^{\lambda'\lambda} - \sqrt{k(k+1)-\lambda(\lambda+1)}\sqrt{l(l+1)-\lambda(\lambda+1)}\delta^{\lambda'\lambda+1} \\ &\quad - \sqrt{k(k+1)-\lambda(\lambda-1)}\sqrt{l(l+1)-\lambda(\lambda-1)}\delta^{\lambda'\lambda-1}\end{aligned}$$

$\boldsymbol{L} = \boldsymbol{L}_Q + \boldsymbol{K}$

$\boldsymbol{K}$: angular-momentum of light d.o.f

$\boldsymbol{L_Q}$: orbital-angular momentum of QQ or $Q\bar{Q}$ pair.

Angular wave-function:

$$|l,m;k,\lambda\rangle = \int \frac{d\Omega}{\sqrt{2\pi}}|\theta,\phi\rangle\,|k,\lambda\rangle\, D^{\lambda}_{lm}(\psi,\theta,\varphi)$$

BOEFT

Brambilla, AM, Vairo
arXiv 2402.xxxxx

- Lastly construct parity eigenstates (specifically for $\lambda > 0$) : $M_{\lambda'\lambda}$ in block-diagonal form.

- Example of (adiabatic) mixing matrices $M_{\lambda'\lambda}$ (accounting for parity) :

$$M_{l1\sigma_P} = \begin{pmatrix} l(l+1)+2 & 2\sqrt{l(l+1)} \\ 2\sqrt{l(l+1)} & l(l+1) \end{pmatrix}$$

$$M_{l2\sigma_P} = \begin{pmatrix} l(l+1)+6 & -2\sqrt{3l(l+1)} & 0 \\ -2\sqrt{3l(l+1)} & l(l+1)+4 & -2\sqrt{(l-1)(l+2)} \\ 0 & -2\sqrt{(l-1)(l+2)} & (l-1)(l+2) \end{pmatrix}$$

Mixing matrix for hybrid with $K^{PC} = 1^{+-}$.

$$M_{l3/2\epsilon} = \begin{pmatrix} (l-\epsilon\sigma_P)(l-\epsilon\sigma_P+1)+\frac{9}{4} & -\sqrt{3\left(l-\frac{1}{2}\right)\left(l+\frac{3}{2}\right)} \\ -\sqrt{3\left(l-\frac{1}{2}\right)\left(l+\frac{3}{2}\right)} & \left(l-\frac{1}{2}\right)\left(l+\frac{3}{2}\right) \end{pmatrix}$$

Hybrids

BOEFT: Hybrids

- BOEFT Lagrangian:

$$L_{\mathrm{BOEFT}} = L_\Psi + L_{\Psi_{\kappa\lambda}} + L_{\mathrm{mixing}},$$

Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)

Brambilla, Krein, Castellà , Vairo Phys. Rev. D. 97, (2018)

Quarkonium:

$$L_\Psi = \int d^3\mathbf{R} \int d^3\mathbf{r}\, \mathrm{Tr}\left[\Psi^\dagger(\mathbf{r},\,\mathbf{R},\,t)\left(i\partial_t + \frac{\nabla_r^2}{m_Q} - V_\Psi(r)\right)\Psi(\mathbf{r},\,\mathbf{R},\,t)\right]$$

Trace over spin indices.

Hybrid:

$$L_{\Psi_{\kappa\lambda}} = \int d^3\mathbf{R} \int d^3\mathbf{r} \sum_{\kappa\lambda\lambda'} \mathrm{Tr}\left\{\Psi^\dagger_{\kappa\lambda}(\mathbf{r},\,\mathbf{R},\,t)\left[i\partial_t - V_{\kappa\lambda\lambda'}(r) + P^{i\dagger}_{\kappa\lambda}\frac{\nabla_r^2}{m_Q}P^i_{\kappa\lambda'}\right]\Psi_{\kappa\lambda'}(\mathbf{r},\,\mathbf{R},\,t)\right\}$$

r: relative coordinate

R: COM coordinate

Hybrid potential:

$$V_{\kappa\lambda\lambda'}(r) \equiv P^{i\dagger}_{\kappa\lambda}V^{ij}_\kappa(r)P^j_{\kappa\lambda'} = V^{(0)}_{\kappa\lambda}(r)\delta_{\lambda\lambda'} + \frac{V^{(1)}_{\kappa\lambda\lambda'}(r)}{m_Q} + \ldots$$

Static potential

Spin-dependent potential

Brambilla, Lai, Segovia, Castellà, Vairo Phys. Rev. D. 101, (2020)

Brambilla, Lai, Segovia, Castellà, Vairo Phys. Rev. D. 99, (2019)

Soto, Valls, arXiv 2302.01765

Hybrid-Quarkonium mixing:

$$L_{\mathrm{mixing}} = -\int d^3\mathbf{R} \int d^3\mathbf{r} \sum_{\kappa\lambda} \mathrm{Tr}\left[\Psi^\dagger\, V^{\mathrm{mix}}_{\kappa\lambda}\,\Psi_{\kappa\lambda} + \mathrm{h.c.}\right]$$

R. Oncala, J. Soto,
Phys. Rev. D96, 014004 (2017).

- No lattice calculations on mixing potential. Current work, ignore mixing, $V^{\mathrm{mix}}_{\kappa\lambda} = 0$

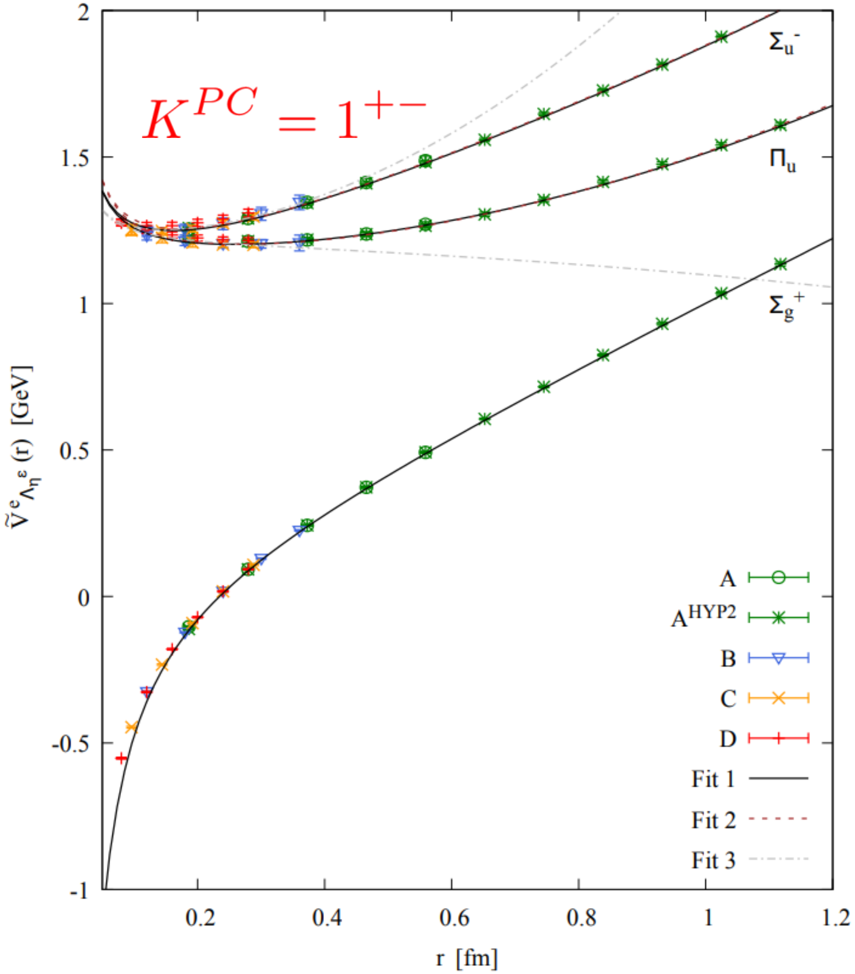
BOEFT: Hybrids

- Degeneracy at $r \to 0$, mixes Σ_u^- and Π_u potential
- Coupled Schrödinger Eq:

$$\left[-\frac{1}{m_Q r^2}\partial_r r^2 \partial_r + \frac{1}{m_Q r^2}\begin{pmatrix} l(l+1)+2 & 2\sqrt{l(l+1)} \\ 2\sqrt{l(l+1)} & l(l+1)\end{pmatrix} + \begin{pmatrix} E_{\Sigma_u^-} & 0 \\ 0 & E_{\Pi_u}\end{pmatrix}\right]\begin{pmatrix}\psi_\Sigma^{(m)} \\ \psi_{-\Pi}^{(m)}\end{pmatrix} = E_m^{Q\bar{Q}g}\begin{pmatrix}\psi_\Sigma^{(m)} \\ \psi_{-\Pi}^{(m)}\end{pmatrix}$$

$$\left[-\frac{1}{m_Q r^2}\partial_r r^2 \partial_r + \frac{l(l+1)}{m_Q r^2} + E_{\Pi_u}\right]\psi_{+\Pi}^{(m)} = E_m^{Q\bar{Q}g}\psi_{+\Pi}^{(m)}$$

$\lambda = 0, \pm 1$



Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)

Hybrid Spectrum:

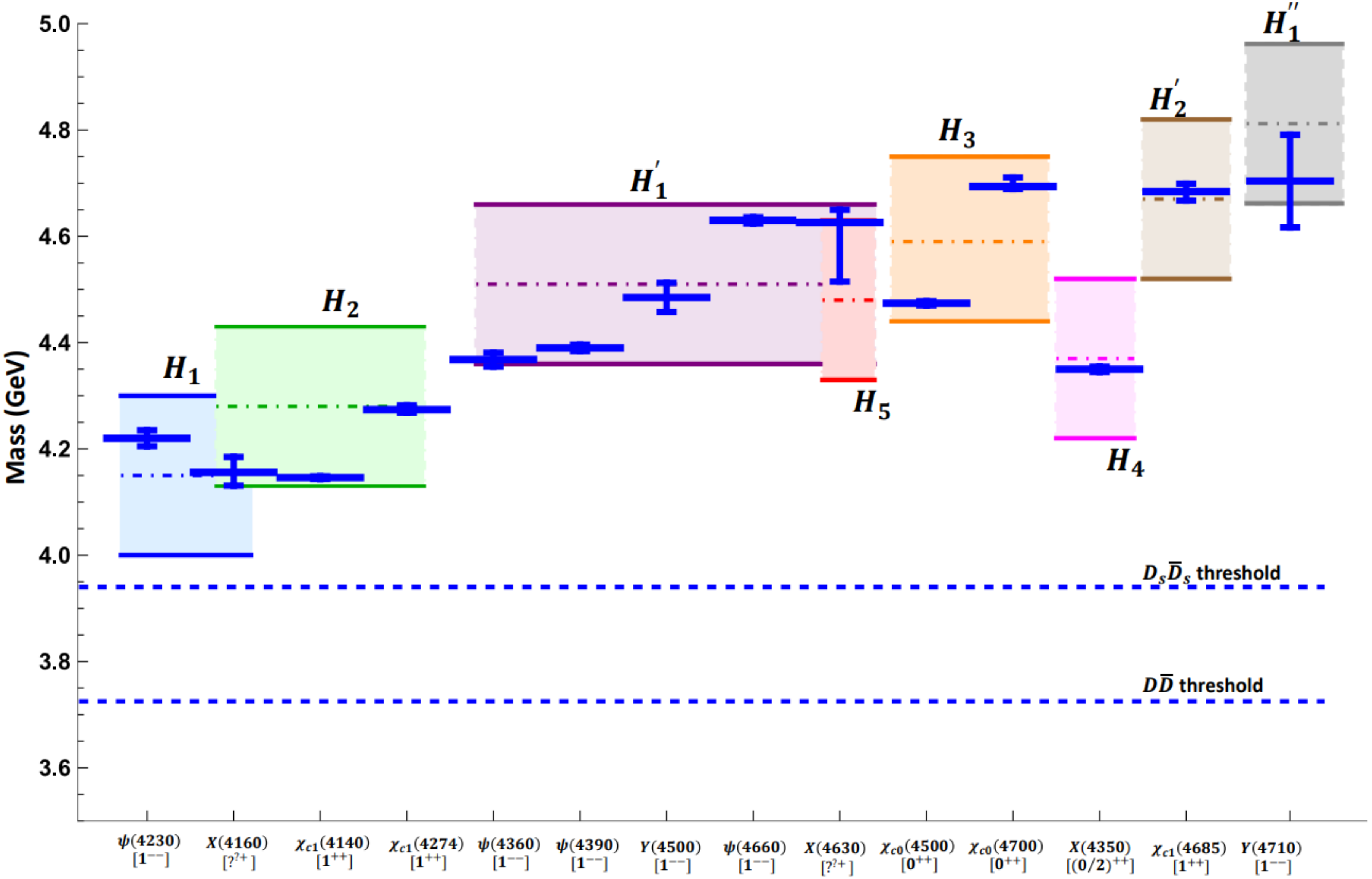
Multiplet	J^{PC}	$M_{c\bar{c}g}$	$M_{b\bar{b}g}$
H_1	$\{1^{--},(0,1,2)^{-+}\}$	4155	10786
H_1'		4507	10976
H_1''		4812	11172
H_2	$\{1^{++},(0,1,2)^{+-}\}$	4286	10846
H_2'		4667	11060
H_2''		5035	11270
H_3	$\{0^{++},1^{+-}\}$	4590	11065
H_3'		5054	11352
H_3''		5473	11616
H_4	$\{2^{++},(1,2,3)^{+-}\}$	4367	10897
H_5	$\{2^{--},(1,2,3)^{-+}\}$	4476	10948

Λ- doubling: opposite parity states non-degenerate.

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

BOEFT: Hybrids

- **Charmonium hybrids**: comparison with experimental results:



	l	$J^{PC}\{s=0, s=1\}$	$E_n^{(0)}$
H_1	1	$\{1^{--}, (0,1,2)^{-+}\}$	Σ_u^-, Π_u
H_2	1	$\{1^{++}, (0,1,2)^{+-}\}$	Π_u
H_3	0	$\{0^{++}, 1^{+-}\}$	Σ_u^-
H_4	2	$\{2^{++}, (1,2,3)^{+-}\}$	Σ_u^-, Π_u
H_5	2	$\{2^{--}, (1,2,3)^{-+}\}$	Π_u

PDG 2022

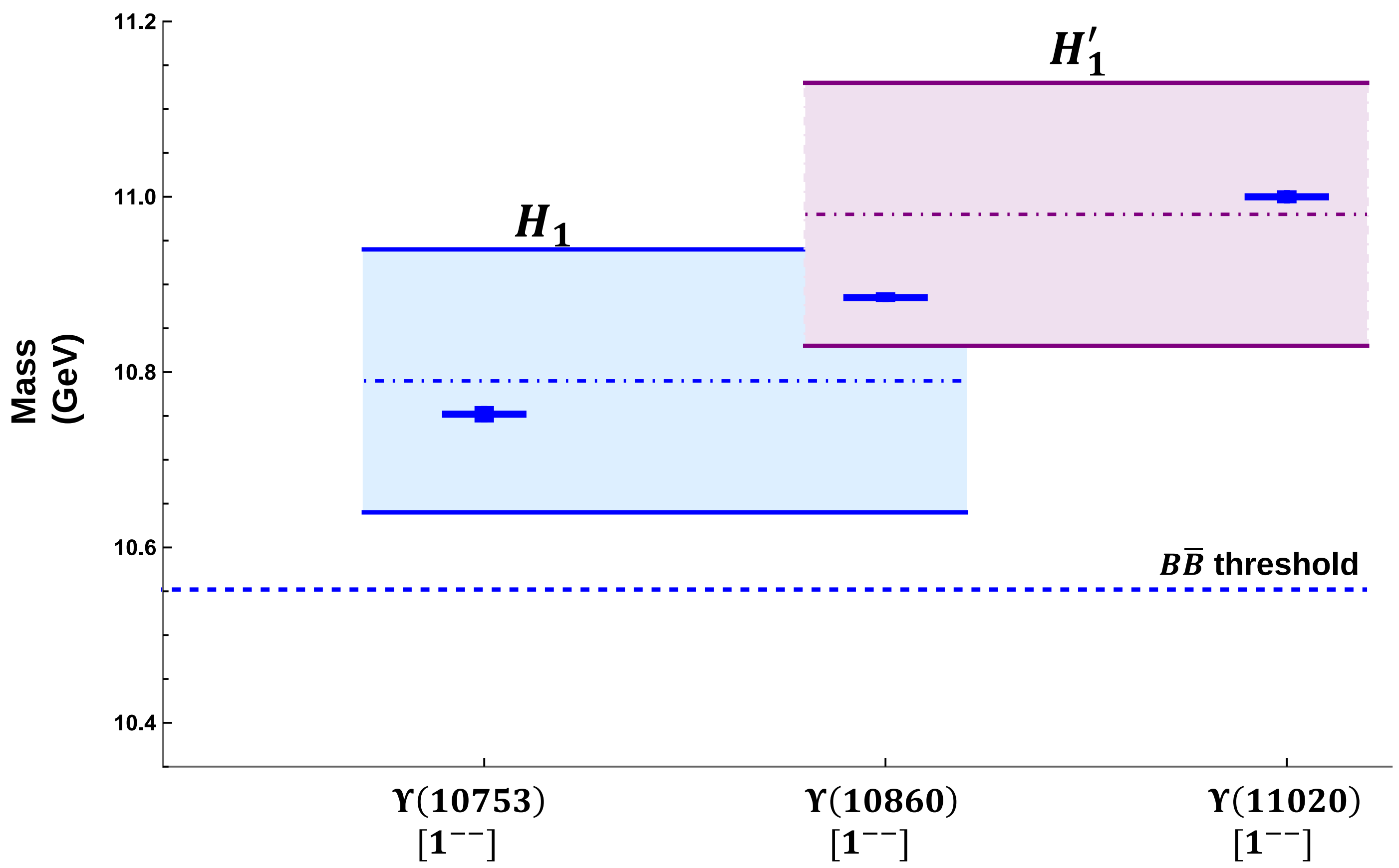
Brambilla, Lai, AM, Vairo

Phys. Rev. D 107, 054034 (2023)

Spectrum with spin-dependence potential: see J. Soto talk !!!

BOEFT: Hybrids

- **Bottomonium hybrids**: comparison with experimental results:



PDG 2022

	l	$J^{PC}\{s=0, s=1\}$	$E_n^{(0)}$
H_1	1	$\{1^{--}, (0,1,2)^{-+}\}$	Σ_u^-, Π_u
H_2	1	$\{1^{++}, (0,1,2)^{+-}\}$	Π_u
H_3	0	$\{0^{++}, 1^{+-}\}$	Σ_u^-
H_4	2	$\{2^{++}, (1,2,3)^{+-}\}$	Σ_u^-, Π_u
H_5	2	$\{2^{--}, (1,2,3)^{-+}\}$	Π_u

Spectrum with spin-dependence potential: see J. Soto talk !!!

Brambilla, Lai, AM, Vairo arXiv:2212.09187

Hybrid Decays

Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)

- BOEFT can describe decays of hybrids to quarkonium.
- Semi-inclusive process: $H_m \to Q_n + X$; Q_n: low-lying quarkonium (states below threshold) & X: light hadrons.

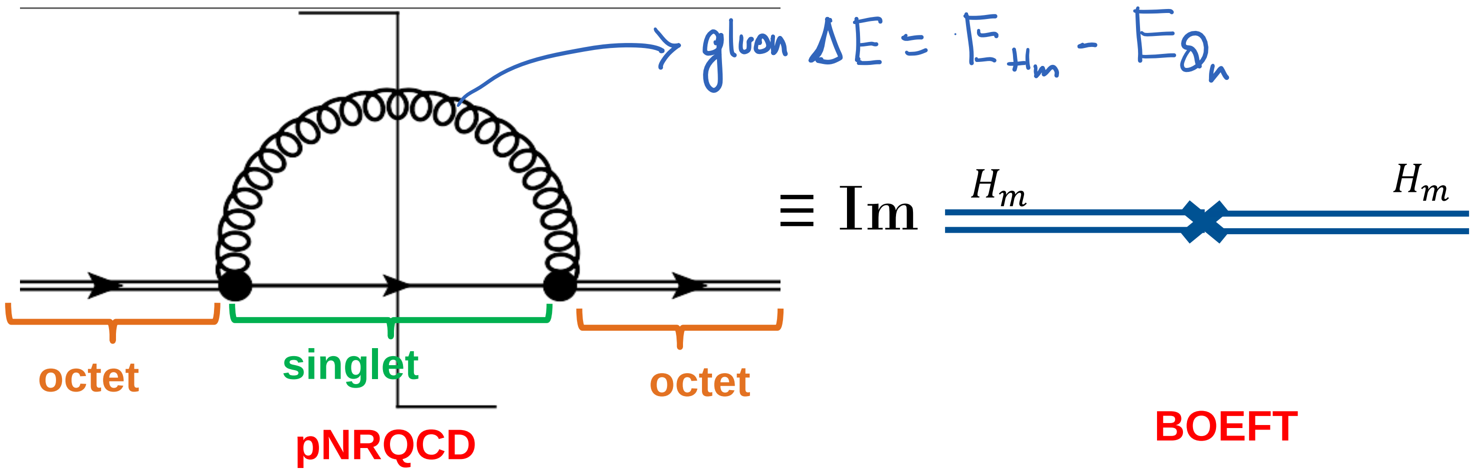
  ✓ ΔE: Large energy difference ⇒ $\Delta E \equiv E_{H_m} - E_{Q_n} \gtrsim 1\,\text{GeV}$.

  ✓ Hierarchy of scales: $\Delta E \gg \Lambda_{QCD} \gg mv^2$

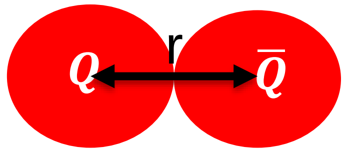
  Perturbative computation

  ✓ Constituent gluon of the hybrid is a spectator.

matching **pNRQCD** and **BOEFT:**



Virtual gluon resolves color structure of $Q\bar{Q}$ pair ($\mathbf{r} \to \mathbf{0}$) in quarkonium and hybrid in short-distance limit



Quarkonium ---> Singlet

Hybrid ---> Octet

- Decays are computed from local imaginary terms in the hybrid potential (BOEFT potential).

Optical theorem: $\sum_n \Gamma(H_m \to Q_n) = -2\,\text{Im}\,\langle H_m|V|H_m\rangle$

DISCLAIMER!!!
Decay to open-flavor threshold states not accounted here.

Hybrid Decays

Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)

- Spin-conserving decay due to $\boldsymbol{r}\cdot\boldsymbol{E}$ term :

$$\Gamma(H_m \to Q_n) = \frac{4\alpha_s\,(\Delta E)\,T_F}{3N_c}\, T^{ij}\,(T^{ij})^\dagger\,\Delta E^3$$

DISCLAIMER!!!
Decay to open-flavor threshold states not accounted here.

$|S_H = 1 > ---\to |S_Q = 1 >$
$|S_H = 0 > ---\to |S_Q = 0 >$

$$T^{ij} \equiv \langle H_m | r^j | Q_n \rangle = \int d^3\mathbf{r}\, \Psi^{i\dagger}_{(m)}(\mathbf{r})\, r^j\, \Phi^{Q\bar{Q}}_{(n)}(\mathbf{r})$$

$\Psi^i_{(m)}$: Hybrid wf
Φ^Q_n : Quarkonium wf

R. Oncala, J. Soto, Phys. Rev. D96, 014004 (2017).

J. Castellà, E. Passemar, Phys. Rev. D104, 034019 (2021)

$$\langle H_m | \boldsymbol{r} | Q_n \rangle = \sqrt{T^{ij}(T^{ij})^\dagger}$$

- Spin-flipping decay due to $\boldsymbol{S}.\boldsymbol{B}$ term:

$|\chi_H\rangle$: Hybrid spin wf
$|\chi_Q\rangle$: Quarkonium spin wf

$|S_H = 1 > ---\to |S_Q = 0 >$
$|S_H = 0 > ---\to |S_Q = 1 >$

$$T^{ij} \equiv \langle H_m | \left(S_1^j - S_2^j\right) | Q_n \rangle = \left[\int d^3\mathbf{r}\, \Psi^{i\dagger}_{(m)}(\mathbf{r})\, \Phi^Q_{(n)}(\mathbf{r})\right] \langle \chi_H | \left(S_1^j - S_2^j\right) | \chi_Q \rangle$$

Depends on overlap of quarkonium and hybrid wavefunctions.

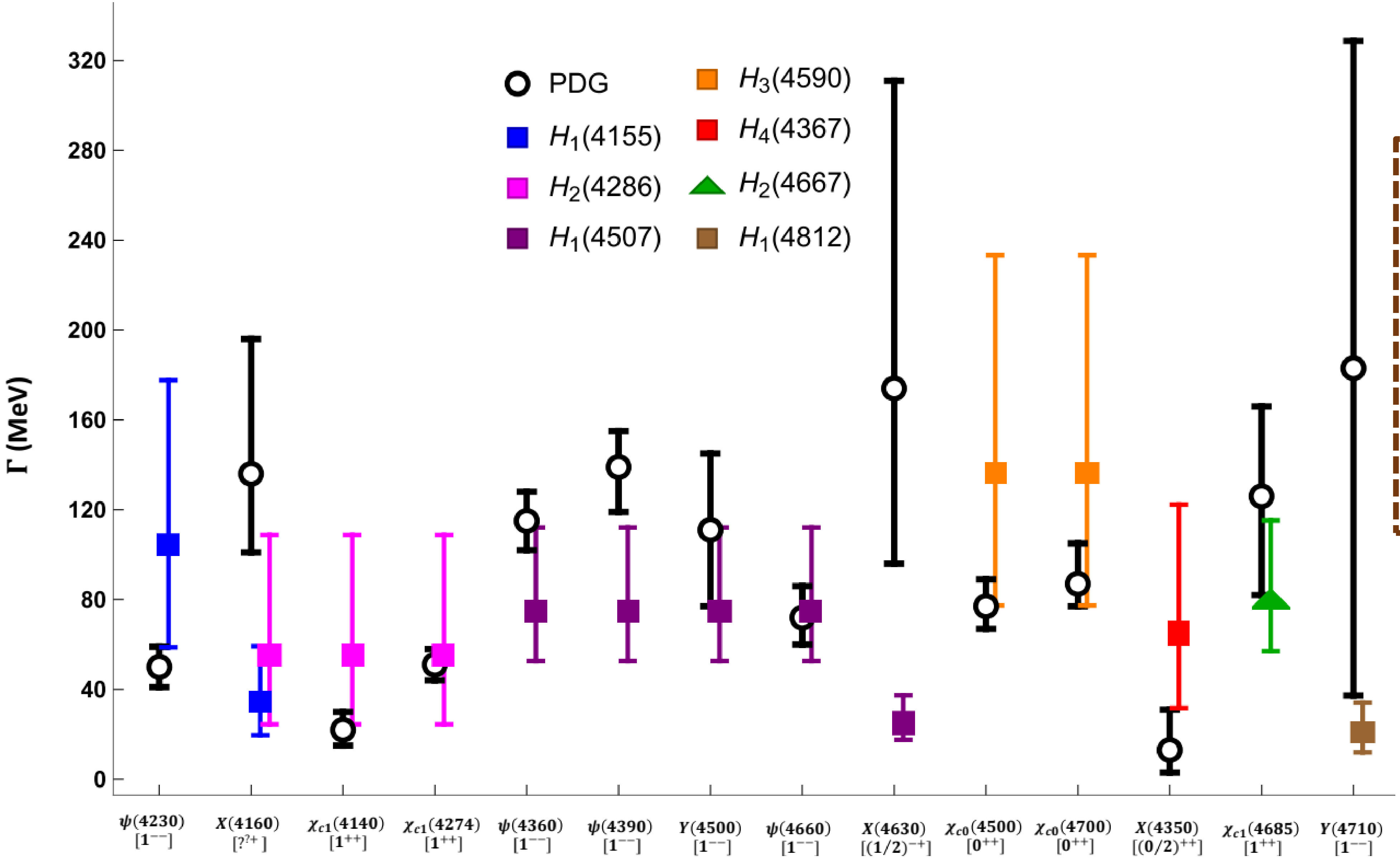
Hybrid-to-Quarkonium transition decay rate
= spin-conserving + spin-flipping decay rates.

Our estimate of decay rate are lower-bounds for the **total width of hybrids**

Results

Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)

- Comparison: charm exotic states with corresponding charmonium hybrid state:



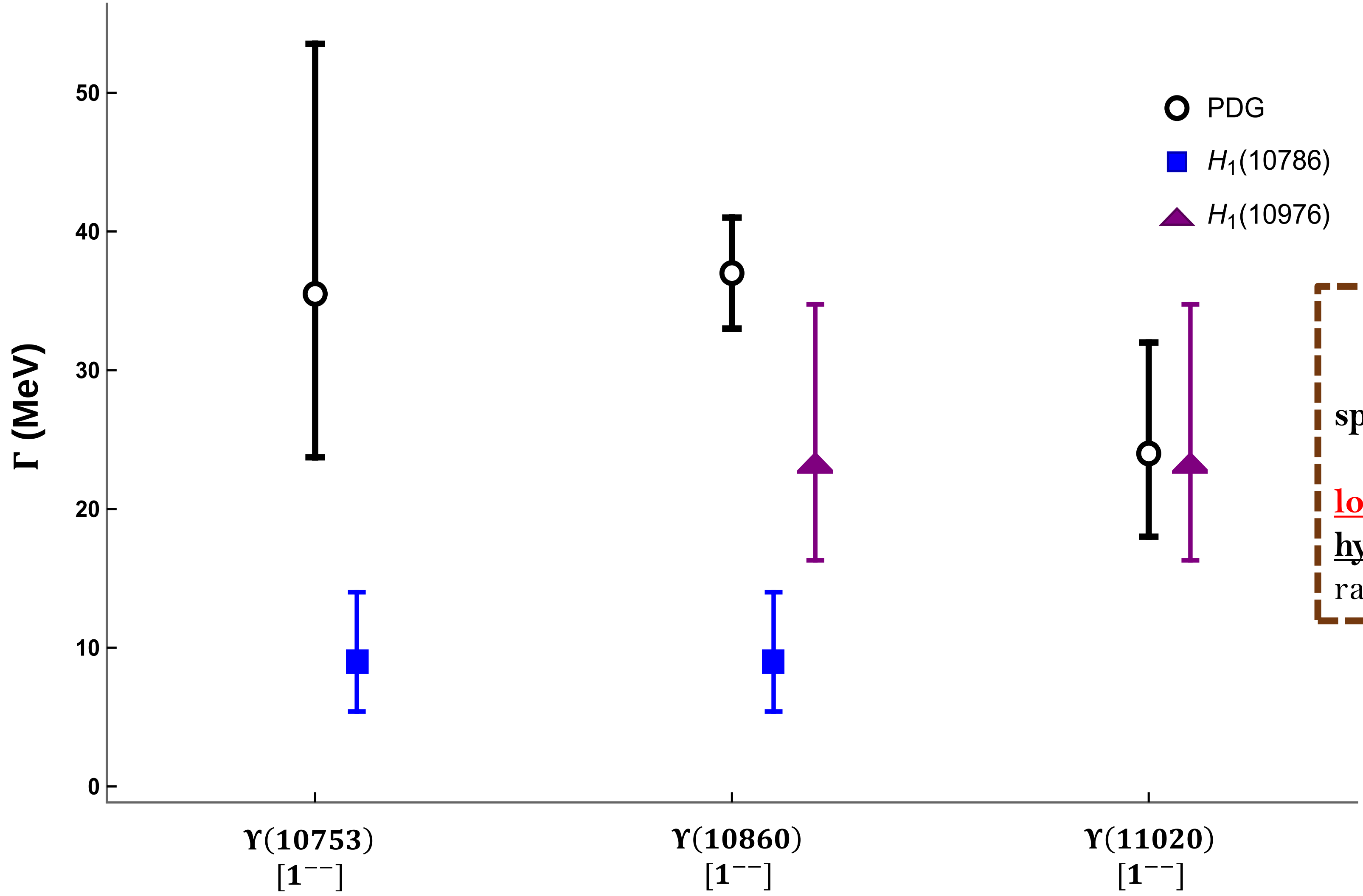
$H_m \to Q_n + X$

spin-conserving + spin-flipping decays
=
lower bound on the total decay widths of hybrids which is compared with inclusive rate of physical states in PDG.

Results

Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)

- Comparison: bottom exotic states with corresponding bottomonium hybrid state:



$$H_m \to Q_n + X$$

spin-conserving + **spin-flipping decays**

=

lower bound on the total decay widths of hybrids which is compared with inclusive rate of physical states in PDG.

Hybrid: Mixing with heavy-light

- **Hybrid decays to meson-pair threshold states:** $\Delta E \lesssim \Lambda_{QCD}$

Conventional Wisdom: Hybrid decays to two S-wave mesons forbidden! $H_m \nrightarrow D^{(*)}\bar{D}^{(*)}$

Kou & Pene, Phys Lett B 631 (2005) Page, Phys Lett B 407 (1997) Farina, Tecocoatzi, Giachino, Santopinto & Swanson, Phys Rev D 102 (2020)

Born Oppenheimer quantum numbers for hybrids and ground state meson pair **does allow for decay to two s-wave mesons**. Bruschini 2306.17120

	l	$J^{PC}\{s=0, s=1\}$	$E_n^{(0)}$
H_1	1	$\{1^{--}, (0,1,2)^{-+}\}$	Σ_u^-, Π_u
H_2	1	$\{1^{++}, (0,1,2)^{+-}\}$	Π_u
H_3	0	$\{0^{++}, 1^{+-}\}$	Σ_u^-
H_4	2	$\{2^{++}, (1,2,3)^{+-}\}$	Σ_u^-, Π_u
H_5	2	$\{2^{--}, (1,2,3)^{-+}\}$	Π_u

Most quarkonium hybrids can decay into pair of s-wave mesons !!!

see R. Bruschini talk !!!

forbidden for decay into pair of s-wave mesons

Recent lattice computation for $c\bar{c}$ hybrid 1^{-+} **decay** to

$D_1\bar{D}$: 258(133) MeV

$D^*\bar{D}$: 88(18) MeV

$D^*\bar{D}^*$: 150(118) MeV

Shi et al 2306.12884

Computing these decays of hybrid to threshold states in BOEFT framework ??

Hybrid: Mixing with heavy-light

Brambilla, AM, Vairo
arXiv 2402.xxxxx

- **Hybrid decays to meson-pair threshold states:** $\Delta E \lesssim \Lambda_{QCD}$ $\quad H_m \nrightarrow D^{(*)}\bar{D}^{(*)}$

Conventional Wisdom: Hybrid decays to two S-wave mesons forbidden!

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BOEFT: Mixing allowed if Λ_η^σ (BO-quantum numbers) are same.

Hybrid

Light spin K^{PC}	Static energies $D_{\infty h}$	l	J^{PC} $\{S_Q=0, S_Q=1\}$	Multiplets
1^{+-}	$\{\Sigma_u^-, \Pi_u\}$	1	$\{1^{--}, (0,1,2)^{-+}\}$	H_1
	$\{\Pi_u\}$	1	$\{1^{++}, (0,1,2)^{+-}\}$	H_2
	$\{\Sigma_u^-\}$	0	$\{0^{++}, 1^{+-}\}$	H_3
	$\{\Sigma_u^-, \Pi_u\}$	2	$\{2^{++}, (1,2,3)^{+-}\}$	H_4
	$\{\Pi_u\}$	2	$\{2^{--}, (1,2,3)^{-+}\}$	H_5

Meson-antimeson threshold

$K_{\bar{q}}^P \otimes K_q^P$	K^{PC}	Static energies $D_{\infty h}$	
$(1/2)^- \otimes (1/2)^+$	0^{-+}	$\{\Sigma_u^-\}$	s-wave+s-wave Ex. $\mathbf{D\bar{D}}$ threshold
	1^{--}	$\{\Sigma_g^+, \Pi_g\}$	
$(1/2)^- \otimes (1/2)^-$	0^{++}	$\{\Sigma_g^+\}$	s-wave+p-wave Ex. $\mathbf{D_1\bar{D}}$ threshold
	1^{+-}	$\{\Sigma_u^-, \Pi_u\}$	
$(1/2)^- \otimes (3/2)^-$	1^{+-}	$\{\Sigma_u^-, \Pi_u\}$	
	2^{++}	$\{\Sigma_g^+, \Pi_g, \Delta_g\}$	

Σ_u^- component in hybrids mix with Σ_u^- component in s-wave+s-wave threshold!!!!

Bruschini 2306.17120 Bruschini & Gonzalez, 1912.07337 J. Castella 2401.13393

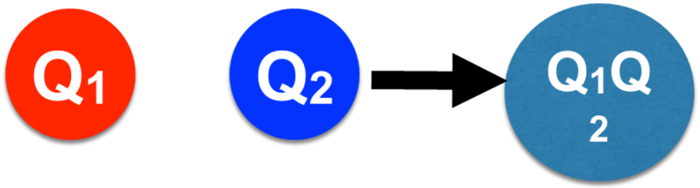
Tetraquarks & Pentaquarks

BOEFT: $QQ\bar{q}\bar{q}$ multiplets

Brambilla, AM, Vairo arXiv 2402.xxxx

doubly heavy core

spin: $1/2 \otimes 1/2 = 0 \oplus 1$



color: $3 \otimes 3 = 6 \oplus 3^*$

flavor: $\{Q_1Q_2\}$ $[Q_1Q_2]$

J^P: 1^+ 0^+

light antiquarks

q q'

$\{qq'\}$, 1^+ $[qq']$, 0^+

Defines the Born-Oppenheimer static potentials $\Sigma_g^+, \{\Sigma_g^-, \Pi_g\}$

doubly heavy tetraquarks

QQ color state	Light spin K^{PC}	Static energies	Isospin I	l	J^P $S_Q = 0$	J^P $S_Q = 1$
anti-triplet $\bar{3}$	0^+	$\{\Sigma_g^+\}$	0	0	—	1^+
				1	1^-	—
	1^+	$\{\Sigma_g^-, \Pi_g\}$	1	0	0^-	—
				1	1^-	$(0,1,2)^+$

J^P for T_{cc}^+

Limited lattice inputs available on Born-Oppenheimer static potentials $\Sigma_g^+, \{\Sigma_g^-, \Pi_g\}$

Bicudo, Cichy, Peters, & Wagner PRD 93, 034501 (2016)

BOEFT: $Q\bar{Q}q\bar{q}$ multiplets

Brambilla, AM, Vairo arXiv 2402.xxxx

$Q\bar{Q}$ color state	Light spin K^{PC}	Static energies	l	J^{PC} $\{S_Q = 0, S_Q = 1\}$	Multiplets
Octet	0^{-+}	$\{\Sigma_u^-\}$	0	$\{0^{++}, 1^{+-}\}$	T_1^0
			1	$\{1^{--}, (0,1,2)^{-+}\}$	T_2^0
			2	$\{2^{++}, (1,2,3)^{+-}\}$	T_3^0
	1^{--}	$\{\Sigma_g^{+\prime}, \Pi_g\}$	1	$\{1^{+-}, (0,1,2)^{++}\}$	T_1^1
		$\{\Sigma_g^{+\prime}\}$	0	$\{0^{-+}, 1^{--}\}$	T_2^1
		$\{\Pi_g\}$	1	$\{1^{-+}, (0,1,2)^{--}\}$	T_3^1
		$\{\Sigma_g^{+\prime}, \Pi_g\}$	2	$\{2^{-+}, (1,2,3)^{--}\}$	T_4^1

J^{PC} for neutral partner of Z_c, Z_b states. Probably mixing between both channels required ?

J^{PC} for $X(3872)$

Limited lattice inputs available on Born-Oppenheimer static potentials $\Sigma_u^-, \{\Sigma_g^+, \Pi_g\}$

Prelovsek, Bahtiyar, & Petkovic
Phys. Lett. B 805 (2020)

Mixing of BO-potentials with pair of heavy-light states relevant for states near threshold. More on this mixing see Brambilla, AM, Vairo arXiv 2402.xxxx

BOEFT: Pentaquark multiplets

Brambilla, AM, Vairo arXiv 2402.xxxx

$Q\bar{Q}qqq$

$Q\bar{Q}$ color state	Light spin $\boldsymbol{K}^P$	Static energies	l	J^P $\{S_Q=0, S_Q=1\}$
Octet	$(1/2)^+$	$(1/2)_g$	$1/2$	$\{1/2^-, (1/2, 3/2)^-\}$
	$(3/2)^+$	$(3/2)_g$	$3/2$	$\{3/2^-, (1/2, 3/2, 5/2)^-\}$

No lattice inputs available on Born-Oppenheimer static potentials for pentaquarks

$QQqq\bar{q}$

QQ color state	Light spin $\boldsymbol{K}^P$	heavy spin $S_Q=0$	heavy spin $S_Q=1$
sextet	$(1/2)^-$	$\{(1/2)^-\}$	$\{(1/2, 3/2)^+, (1/2, 3/2, 5/2)^+\}$
	$(3/2)^-$	$\{(3/2)^-\}$	$\{(1/2, 3/2)^+\}, \{(1/2, 3/2, 5/2)^+\}, \{(3/2, 5/2, 7/2)^+\}$
antitriplet	$(1/2)^-$	$\{(1/2)^+, (3/2)^+\}$	$\{(1/2, 3/2)^-\}$
	$(3/2)^-$	$\{(1/2)^+\}, \{(3/2)^+\}, \{(5/2)^+\}$	$\{(1/2, 3/2, 5/2)^-\}$

Coupled Schrödinger equation for these pentaquark states derived in Brambilla, AM, Vairo arXiv 2402.xxxxx.

Takeaway message

Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)
Brambilla, AM, Vairo arXiv 2402.xxxxx

- BOEFT provides a model-independent & systematic way to study heavy quark exotics.

- BOEFT results for hybrid-to-quarkonium transition widths

  Hybrid-to-Quarkonium transition decay rate = **spin-conserving** + **spin-flipping** decay rates.

- Our analysis disfavors: $\psi(4230)$, $\chi_{c1}(4140)$, $\chi_{c0}(4500)$, $\chi_{c0}(4700)$, and $X(4350)$ as pure hybrid states.

- Our analysis suggests:
  - $\boldsymbol{X(4160)}$: could be **charm hybrid** $\boldsymbol{H_1[2^{-+}](4155)}$.
  - $\mathbf{X(4630)}$: could be **charm hybrid** $\boldsymbol{H_1[(1/2^{-+})](4507)}$.
  - $\boldsymbol{\Upsilon(10753)}$: could be **bottom hybrid** $\boldsymbol{H_1[(1^{--})](10786)}$.
  - $\boldsymbol{\psi(4390)}$: could be **charm hybrid** $\boldsymbol{H_1[1^{--}](4507)}$.
  - $\boldsymbol{\psi(4710)}$: could be **charm hybrid** $\boldsymbol{H_1[(1^{--})](4812)}$.

DISCLAIMER!!!
All the above interpretation can differ accounting for decays to meson-pair threshold states and hybrid-quarkonium mixing.

- Obtained new results for tetraquark and pentaquark multiplets based on BOEFT.

- BOEFT: describes static potential (BO-potential) mixing with heavy-light states .

Thank you!!

Backup Slides

Virial theorem: Nonrelativistic Bound State

- Non relativistic because $v \sim \alpha \ll 1$.

Def. p and v are the particle momentum and velocity in the center of mass reference frame.

One can see this from the virial theorem applied to a central potential:

$$\begin{cases} \left\langle \frac{p^2}{2m_{red}} \right\rangle = \frac{1}{2}\langle r V'(r)\rangle = \frac{1}{2}\left\langle \frac{\alpha}{r}\right\rangle \\ \quad \text{for } V(r) = -\frac{\alpha}{r} \text{ (Coulomb potential)} \\ E_n = -\frac{m_{red}\alpha^2}{2n^2} = \left\langle \frac{p^2}{2m_{red}} \right\rangle - \left\langle \frac{\alpha}{r}\right\rangle = -\frac{1}{2}\left\langle \frac{\alpha}{r}\right\rangle \end{cases} \Rightarrow \quad \left\langle \frac{\alpha}{r}\right\rangle = \frac{m_{red}\alpha^2}{n^2}, \quad \left\langle \frac{p^2}{2m_{red}} \right\rangle = \frac{m_{red}\alpha^2}{2n^2}$$

$m_{red} \equiv$ reduced mass $\equiv m_1 m_2/(m_1+m_2)$

In particular this implies

$$\left\langle \frac{1}{r}\right\rangle = \frac{m_{red}\,\alpha}{n^2} \quad \text{and} \quad \sqrt{\langle p^2\rangle} = \frac{m_{red}\,\alpha}{n}$$

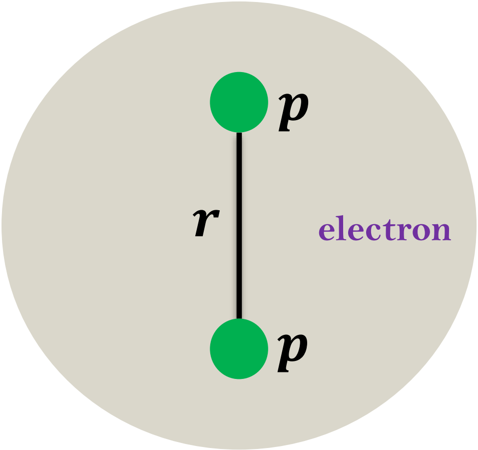
- if $m_1 = m_2 = m$, $m_{red} = \frac{m}{2}$; $m v^2 \equiv \left\langle \frac{p^2}{m}\right\rangle = \frac{m\alpha^2}{4n^2} \Rightarrow v = \frac{\alpha}{2n}$
- if $m_1 = \infty$, $m_2 = m$, $m_{red} = m$; $\frac{1}{2} m v^2 \equiv \left\langle \frac{p^2}{2m}\right\rangle = \frac{m\alpha^2}{2n^2} \Rightarrow v = \frac{\alpha}{n}$

EFT lectures by A. Vairo

Born-Oppenheimer Philosophy

- Sharp difference between time or energy scales of heavy & light degrees of freedom.

Ex. H_2^+ molecule: 2 protons & 1 electron. $m_p \sim 1\,\mathrm{GeV} \gg m_e \sim 0.5\,\mathrm{MeV}$



Protons (nuclei) move very slowly compared to electrons and can be considered **static** (fixed) when considering the motion of the electrons

Electrons instantaneously adjust as $\mathbf{r}$ changes

1. Solve electron Schrödinger eq. for fixed $\mathbf{r}$

$$H_{\mathrm{el}}(\mathbf{r})\,|\psi_{\mathrm{el}}^i;\mathbf{r}\rangle = E_{\mathrm{el}}^i(r)\,|\psi_{\mathrm{el}}^i;\mathbf{r}\rangle$$

2. Solve nuclei (proton) Schrödinger eq. with $\boldsymbol{E_{\mathrm{el}}^i(r)}$ as **potential**.

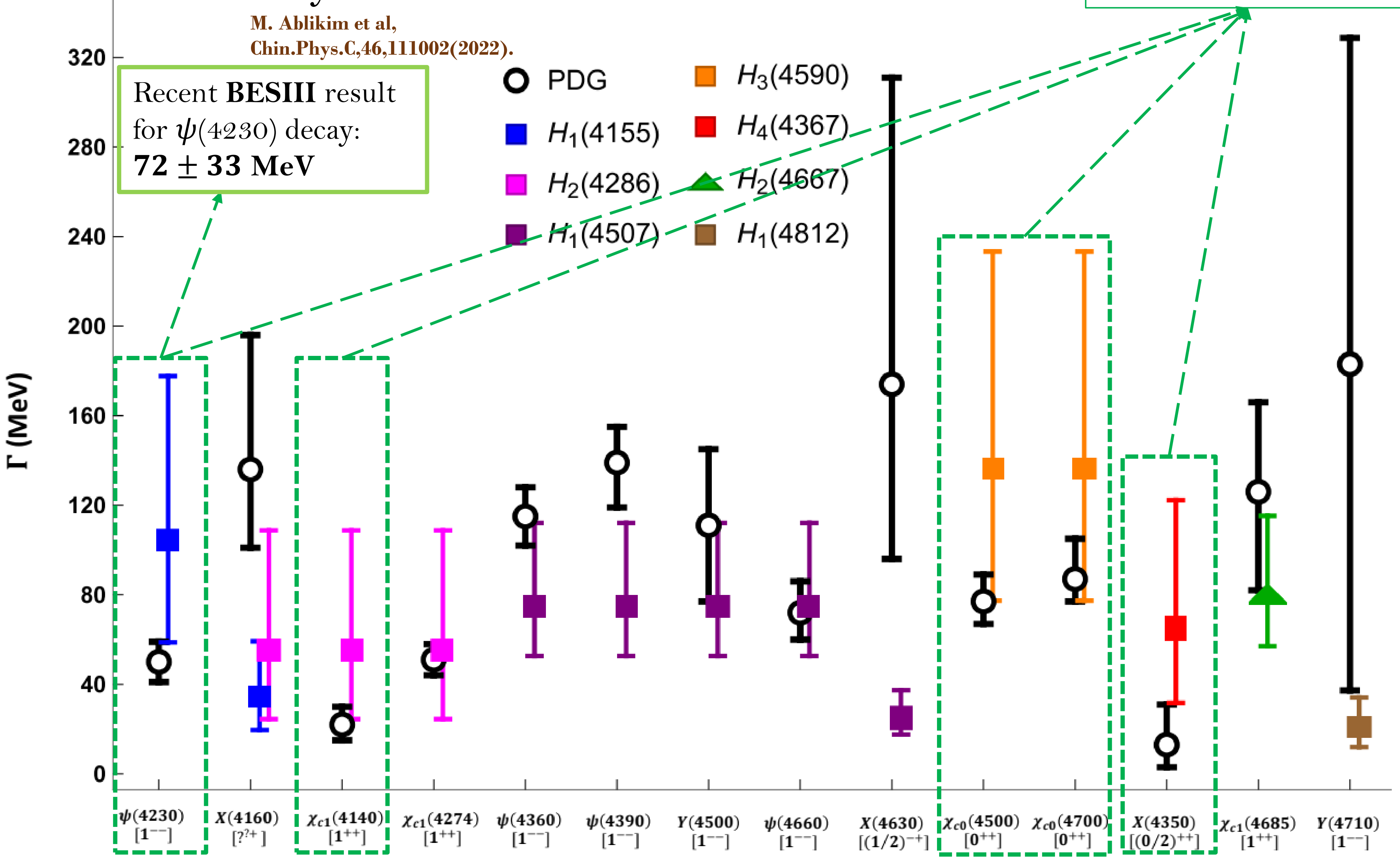
QCD states with 2-heavy quarks (XYZ mesons): analogous of molecules in atomic systems !!!

Heavy quarks ↔ nuclei **Gluons & light quarks ↔ electrons**

Results

- Comparison: charm exotic states with corresponding charmonium hybrid state:

Disfavors hybrid interpretation !



	l	$J^{PC}\{s=0, s=1\}$	$E_n^{(0)}$
H_1	1	$\{1^{--}, (0,1,2)^{-+}\}$	Σ_u^-, Π_u
H_2	1	$\{1^{++}, (0,1,2)^{+-}\}$	Π_u
H_3	0	$\{0^{++}, 1^{+-}\}$	Σ_u^-
H_4	2	$\{2^{++}, (1,2,3)^{+-}\}$	Σ_u^-, Π_u
H_5	2	$\{2^{--}, (1,2,3)^{-+}\}$	Π_u

$\chi_{c0}(4500)$ & $\chi_{c0}(4700)$

Decay to $\chi_c(1P)$ not seen !!!. Major contribution to theoretical estimate from this decay channel.

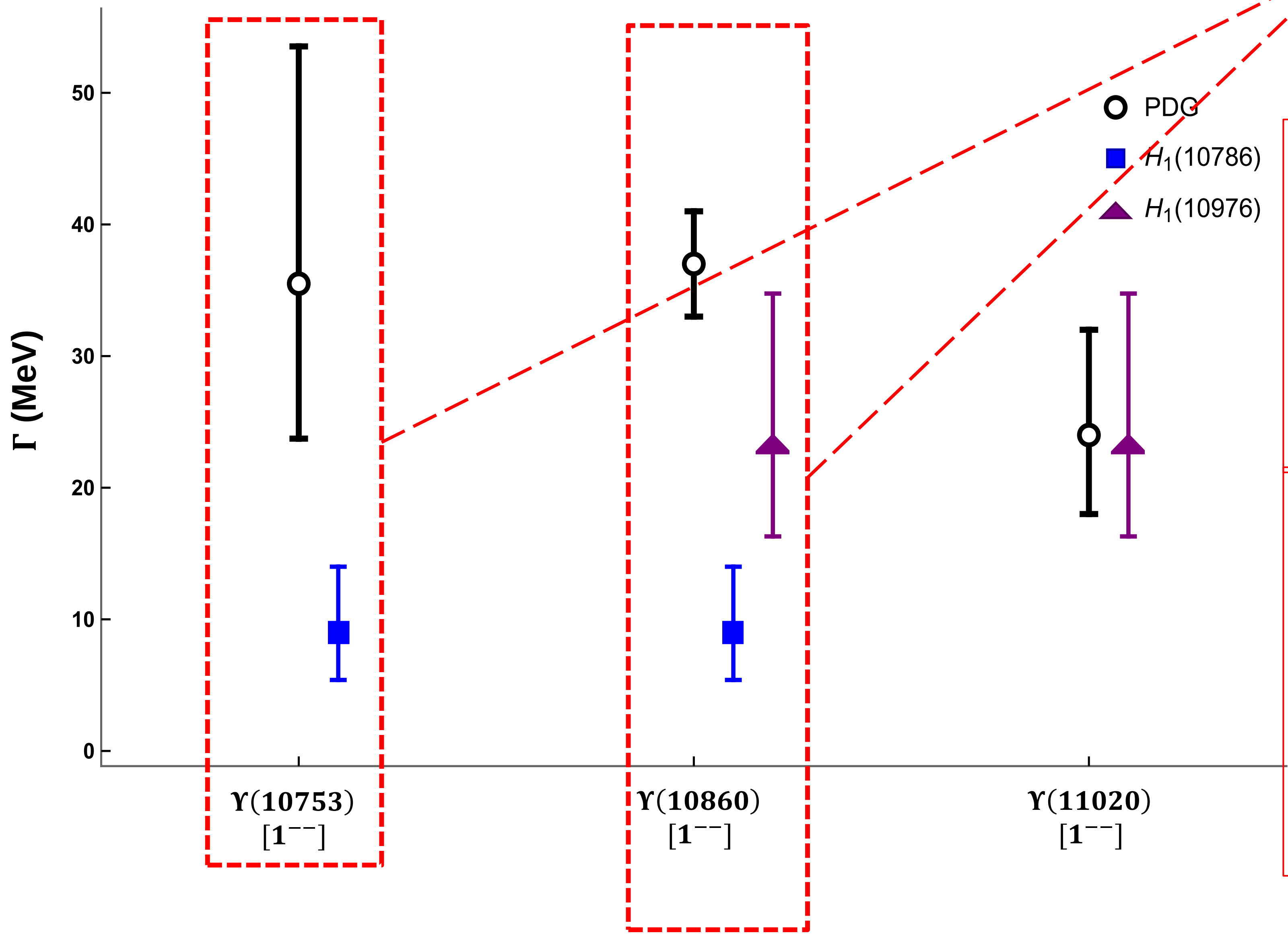
Brambilla, Lai, AM, Vairo

Phys. Rev. D 107, 054034 (2023)

Results

Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)

- Hybrid-to-quarkoniunium transition widths:



Could have a significant hybrid component.

ϒ(10860)

- ✓ Inclusive rate: **37 ± 4** MeV. **Branching fraction for decay to open-bottom mesons**: **76.6%**. Decay rate to quarkonium: $\mathbf{8.8^{+2.6}_{-1.8}}$ MeV. Good agreement with our lower-bound estimate for $H_1[1^{--}](10786)$.
- ✓ Candidate for $\Upsilon(5S)$ state

$\Upsilon(11020)$ & $\Upsilon(10753)$

Large branching fraction for decays to open-bottom mesons expected. Different quark model calculations predict large branching fraction. Need experimental input on branching fraction. See Hüsken, Mitchell & Swanson, Phys Rev D 106 (2022).

$\Upsilon(10753)$: Candidate for $\Upsilon(3D)$ state.

pNRQCD

$$S_{pNRQCD} = \int d^3\mathbf{x}_1\, d^3\mathbf{x}_2\, dt\, \mathrm{tr}\Bigg(\Psi^\dagger(\mathbf{x}_1, \mathbf{x}_2, t)$$

$$\left\{ iD_0 + \frac{\mathbf{D}^2_{\mathbf{x}_1}}{2m} + \frac{\mathbf{D}^2_{\mathbf{x}_2}}{2m} \right\} \Psi(\mathbf{x}_1, \mathbf{x}_2, t) \Bigg)$$

$$+ \frac{\alpha_s}{|\mathbf{x}_1 - \mathbf{x}_2|} \mathrm{tr}\Bigg(T^a \Psi(\mathbf{x}_1, \mathbf{x}_2, t) T^a \Psi^\dagger(\mathbf{x}_1, \mathbf{x}_2, t) \Bigg)$$

Upon making the local field redefinition,

$$\Psi(\mathbf{x}_1, \mathbf{x}_2, t) = P\left[e^{ig\int_{\mathbf{x}_2}^{\mathbf{x}_1} \mathbf{A}d\mathbf{x}}\right] S(\mathbf{x}, \mathbf{X}, t)$$

$$+ P\left[e^{ig\int_{\mathbf{X}}^{\mathbf{x}_1} \mathbf{A}d\mathbf{x}}\right] O(\mathbf{x}, \mathbf{X}, t) P\left[e^{ig\int_{\mathbf{x}_2}^{\mathbf{X}} \mathbf{A}d\mathbf{x}}\right]$$

$$\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2, \;\; \mathbf{X} = (\mathbf{x}_1 + \mathbf{x}_2)/2$$

pNRQCD

Under gauge transformations,

$$\Psi(\mathbf{x}_1, \mathbf{x}_2, t) \to g(\mathbf{x}_1, t)\Psi(\mathbf{x}_1, \mathbf{x}_2, t)g^{-1}(\mathbf{x}_2, t)$$

$$S(\mathbf{x}, \mathbf{X}, t) \to S(\mathbf{x}, \mathbf{X}, t)$$

$$O(\mathbf{x}, \mathbf{X}, t) \to g(\mathbf{X}, t)O(\mathbf{x}, \mathbf{X}, t)g^{-1}(\mathbf{X}, t)$$

Upon multipole expanding (up to O($\mathbf{x}^2$)),

$$\mathcal{L}_{pNRQCD} = \int d^3\mathbf{x}\; tr\Big\{ S^\dagger \Big\{ i\partial_0 - \frac{\mathbf{p}^2}{m} + \frac{C_f \alpha_s}{|\mathbf{x}|} \Big\} S + O^\dagger \Big\{ iD_0 - \frac{\mathbf{p}^2}{m} - \frac{1}{2N_c}\frac{\alpha_s}{|\mathbf{x}|} \Big\} O$$

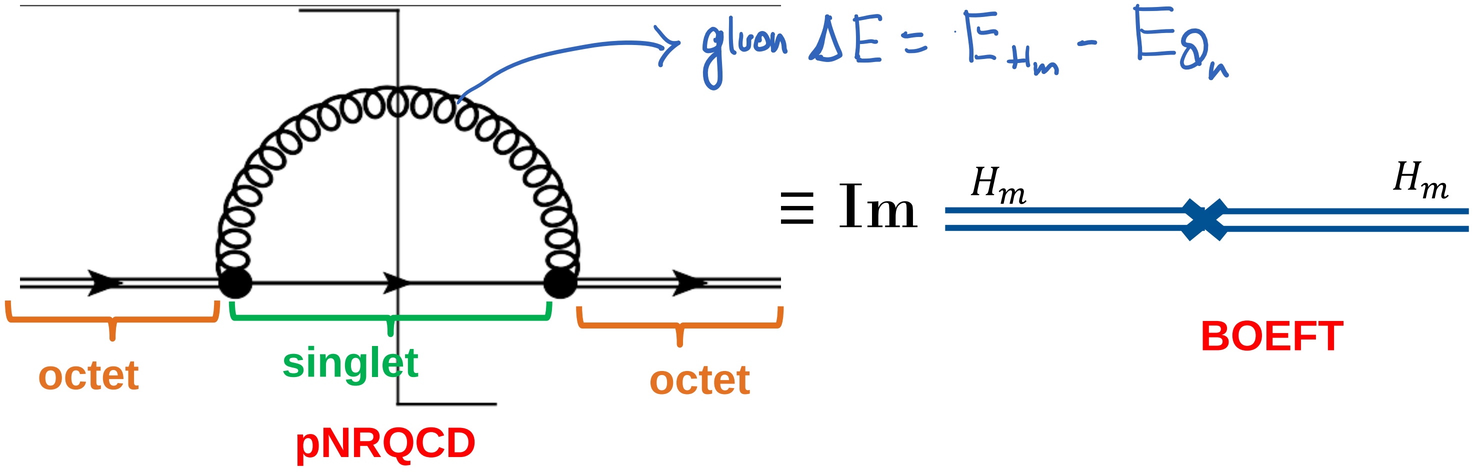
$$+ g\mathbf{x} O \mathbf{E}(\mathbf{X}, t) S^\dagger + g\mathbf{x} O^\dagger \mathbf{E}(\mathbf{X}, t) S + \frac{g}{2}\mathbf{x} O O^\dagger \mathbf{E}(\mathbf{X}, t) + \frac{g}{2}\mathbf{x} O^\dagger O \mathbf{E}(\mathbf{X}, t) \Big\}$$

Hybrid Decays

Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)

✓ Hierarchy of scales: $\Delta E \gg \Lambda_{QCD} \gg mv^2$. Integrate out the scale ΔE perutbatively.

matching **pNRQCD** and **BOEFT:**



Virtual gluon resolves color structure of $Q\bar{Q}$ pair ($\mathbf{r} \to \mathbf{0}$) in quarkonium and hybrid in short-distance limit

Quarkonium ---> Singlet

Hybrid ---> Octet

Weakly-coupled pNRQCD Lagrangian

$$L_{\mathrm{pNRQCD}} = \int d^3R \Big\{ \int d^3r \Big(\mathrm{Tr}\left[\mathrm{S}^\dagger \left(i\partial_0 - h_s\right) \mathrm{S} + \mathrm{O}^\dagger \left(iD_0 - h_o\right) \mathrm{O} \right]$$

$$+ g\mathrm{Tr}\left[\mathrm{S}^\dagger \boldsymbol{r}\cdot\boldsymbol{E}\,\mathrm{O} + \mathrm{O}^\dagger \boldsymbol{r}\cdot\boldsymbol{E}\,\mathrm{S} + \frac{1}{2}\mathrm{O}^\dagger \boldsymbol{r}\cdot\{\boldsymbol{E},\mathrm{O}\} \right] + \frac{g}{4m}\mathrm{Tr}\left[\mathrm{O}^\dagger \boldsymbol{L}_{Q\bar{Q}}\cdot[\boldsymbol{B},\mathrm{O}] \right]$$

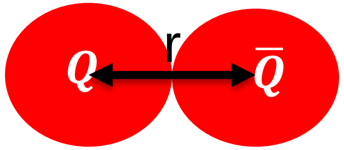
$$+ \frac{gc_F}{m}\mathrm{Tr}\left[\mathrm{S}^\dagger (\boldsymbol{S}_1 - \boldsymbol{S}_2)\cdot\boldsymbol{B}\,\mathrm{O} + \mathrm{O}^\dagger (\boldsymbol{S}_1 - \boldsymbol{S}_2)\cdot\boldsymbol{B}\,\mathrm{S} + \mathrm{O}^\dagger \boldsymbol{S}_1\cdot\boldsymbol{B}\,\mathrm{O} - \mathrm{O}^\dagger \boldsymbol{S}_2 \mathrm{O}\cdot\boldsymbol{B} \right] - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu\, a} \Big\}$$

- **Spin preserving decays $[O(r^2)]$**
- **Spin flipping decays $[O(1/m^2)]$**

Hybrid Decays

Brambilla, Lai, AM, Vairo Phys. Rev. D 107, 054034 (2023)

- Color configuration of $Q\bar{Q}$ pair ($\mathbf{r} \to \mathbf{0}$): quarkonium and hybrid in short-distance limit



Quarkonium -----> Singlet

Hybrid -----> Octet

- Quarkonium and Hybrid fields in short-distance limit $\mathbf{r} \to \mathbf{0}$ (matching condition)

Fields:

$$S\,(\mathbf{r},\mathbf{R},t) \to Z_{\Psi}^{1/2}(\mathbf{r})\,\Psi(\mathbf{r},\mathbf{R},t),$$

$$P^{i\dagger}_{\kappa\lambda}O^{a}\,(\mathbf{r},\mathbf{R},t)\,G^{ia}_{\kappa}(\mathbf{R},t) \to Z_{\kappa}^{1/2}(\mathbf{r})\,\Psi_{\kappa\lambda}(\mathbf{r},\mathbf{R},t)$$

singlet (S) and octet (O)

G^{ia}_{κ}: Gluon fields

Potentials:

$$E_{\Sigma_g^+}(r) = V_s(r) + b_{\Sigma_g^+}\,r^2 + \ldots,$$

$$E_{\Sigma_u^-,\Pi_u}(r) = V_o(r) + \Lambda + b_{\Sigma,\Pi}r^2 + \ldots$$

V_s & V_o: singlet and octet potential

Λ: gluelump mass

For decay rate computation, **start with effective theory of singlet and octet fields** and match to **BOEFT of quarkonium and hybrid fields**

BOEFT

Brambilla, Lai, AM, Vairo arXiv:2212.09187

- Quarkonium static potential: $V_{\Psi}(r) = E_{\Sigma_g^+}(r)$
- Hybrid static potential: $V_{10}(r) = E_{\Sigma_u^-}(r)\,,$ $V_{1\pm1}(r) = E_{\Pi_u}(r)$

Quarkonium Potential:

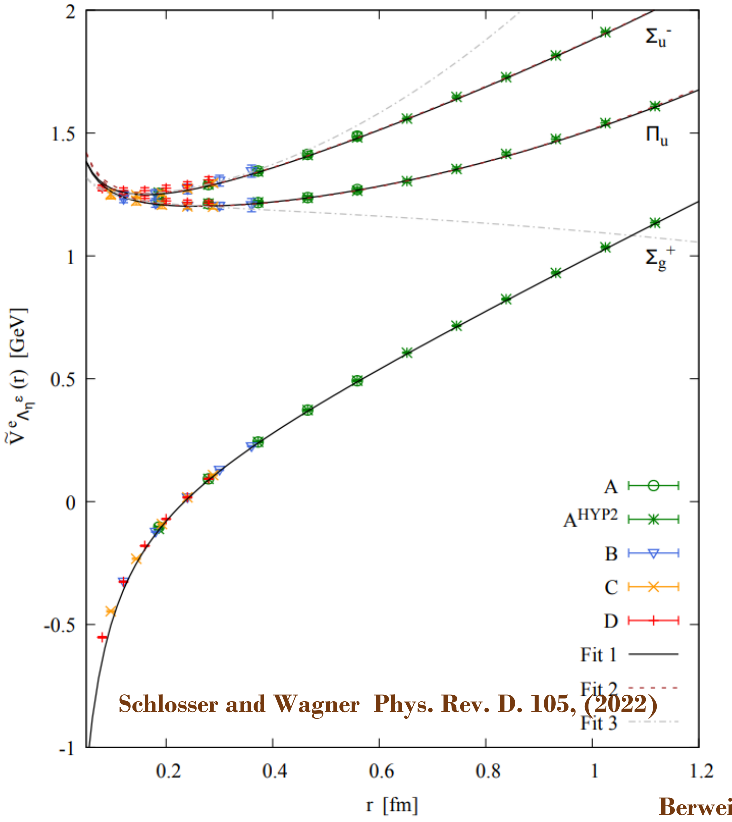
$$V_{\Sigma_g^+}(r) = -\frac{\kappa_g}{r} + \sigma_g r + E_g^{Q\bar{Q}}$$

$m_c^{RS} = 1.477\,(40)$ GeV

$m_b^{RS} = 4.863\,(55)$ GeV

$\kappa_g = 0.489\,,\quad \sigma_g = 0.187\,\mathrm{GeV}^2 \qquad E_g^{c\bar{c}} = -0.254\,\mathrm{GeV},\ E_g^{b\bar{b}} = -0.195\,\mathrm{GeV},$

Gluonic Static energies from lattice:



Schlosser and Wagner Phys. Rev. D. 105, (2022)

Hybrid Potential:

$$E_{\Sigma_u^-,\Pi_u}(r) = \begin{cases} V_o^{\mathrm{RS}}(\nu_f) + \Lambda_{\mathrm{RS}}(\nu_f) + b_{\Sigma,\Pi} r^2, & r < 0.25\ \mathrm{fm} \\ \frac{a_1^{\Sigma,\Pi}}{r} + \sqrt{a_2^{\Sigma,\Pi} r^2 + a_3^{\Sigma,\Pi}} + a_4^{\Sigma,\Pi}, & r > 0.25\ \mathrm{fm} \end{cases}.$$

$a_1^{\Sigma} = 0.000\,\mathrm{GeV fm}\,,\quad a_2^{\Sigma} = 1.543\,\mathrm{GeV}^2/\mathrm{fm}^2,\quad a_3^{\Sigma} = 0.599\,\mathrm{GeV}^2,\quad a_4^{\Sigma} = 0.154\,\mathrm{GeV}\,,$

$a_1^{\Pi} = 0.023\,\mathrm{GeV fm}\,,\quad a_2^{\Pi} = 2.716\,\mathrm{GeV}^2/\mathrm{fm}^2,\quad a_3^{\Pi} = 11.091\,\mathrm{GeV}^2,\quad a_4^{\Pi} = -2.536\,\mathrm{GeV}\,,$

$b_{\Sigma} = 1.246\,\mathrm{GeV}/\mathrm{fm}^2,\quad b_{\Pi} = 0.000\,\mathrm{GeV}/\mathrm{fm}^2$ $\quad \Lambda_{\mathrm{RS}}$: 0.87 (15) GeV

Gluelump mass definition:

$$\langle 0|G^{ia}_{1+-}(\mathbf{R}, T/2)\phi^{ab}(T/2, -T/2)G^{jb}_{1+-}(\mathbf{R}, -T/2)|0\rangle = \delta^{ij} e^{-i\Lambda T}$$

✓ Perturbative RS-scheme potentials V_o^{RS} upto order α_s^3.

Berwein, Brambilla, Castellà , Vairo Phys. Rev. D. 92, (2015)

Bali and Pineda Phys. Rev. D. 69, (2004)

Hybrid-Quarkonium mixing and impact on hybrid interpretation of Exotics ?

Brambilla, AM, Vairo, Wagner, Schlosser (in progress)

- Hybrid states in the same energy range and with same quantum #'s as quarkonium can mix.
- Mixing impacts spectrum and decay properties of hybrid. **Implications for exotic hadrons !!**.

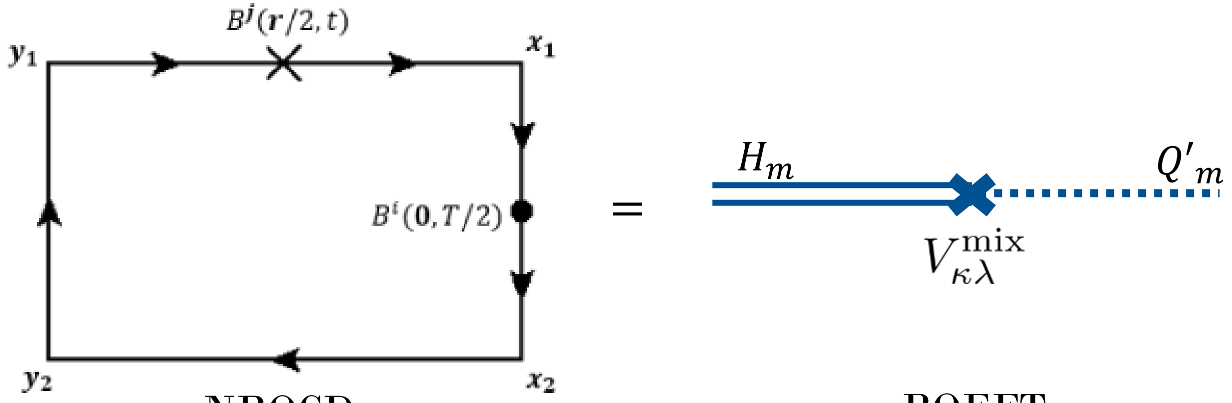
Oncala & Soto, PRD (2017).

Ex. $H_1\left[1^{--}\right](4155) \leftrightarrow c\bar{c}\left[1^{--}\right](3S)$ Effect on decay: $H_m \leftrightarrow Q_m^{'} \rightarrow (\eta_c, J/\psi, \cdots) + (\gamma, \cdots)$

- Hybrid-quarkonium mixing through heavy-quark spin dependent operator. Mixing potential at $O(1/m)$ in BOEFT.

$$L_{\text{BOEFT}} = L_{Q\bar{Q}} + L_{Q\bar{Q}g} + L_{\text{mixing}}, \qquad L_{\text{mixing}} = -\int d^3\mathbf{R} \int d^3\mathbf{r} \sum_{\kappa\lambda} \text{Tr}\left[\Psi^\dagger V_{\kappa\lambda}^{\text{mix}} \Psi_{\kappa\lambda} + \text{h.c.}\right]$$

Matching two-point correlators in **NRQCD** and **BOEFT:**



NRQCD BOEFT

Expression after matching:

$$V_{|\lambda|}^{\text{mix}} = -\frac{gc_F}{2m_Q}\, {}^{(0)}_{\lambda}\langle 1|B^j\left(\mathbf{r}/2,0\right)|0\rangle^{(0)} P_\lambda^j,$$

Above expression can be computed on lattice if we identify:

$$|0\rangle^{(0)} = |\Sigma_g^+\rangle$$

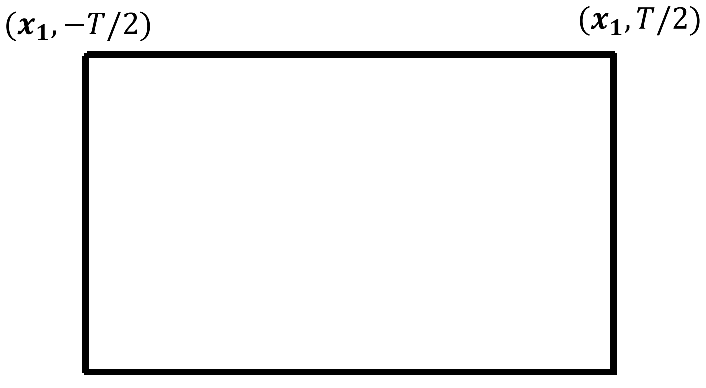
$$|1\rangle_{\lambda=0}^{(0)} = |\Sigma_u^-\rangle, |1\rangle_{|\lambda|=1}^{(0)} = |\Pi_u\rangle$$

pNRQCD/BOEFT: Potentials

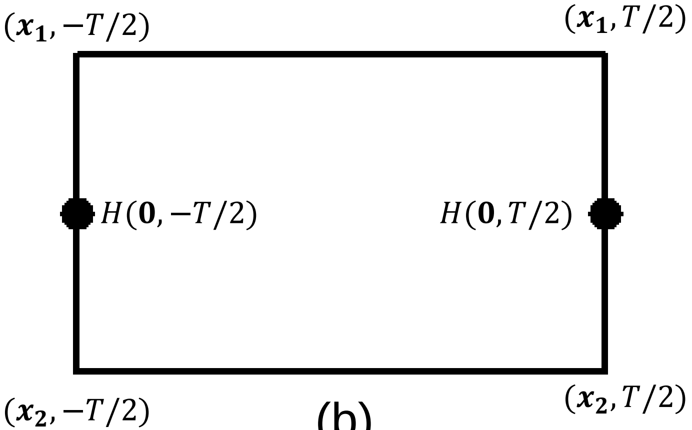
$$E_\kappa^{(0)}(r) = \lim_{T\to\infty} \frac{i}{T} \log\langle X_\kappa, T/2 | X_\kappa, -T/2\rangle$$

$\kappa = K^{PC}$ (Light d.o.f)

$$|X_\kappa; t\rangle = \mathcal{O}_\kappa^\dagger\,(t, \mathbf{r}, \mathbf{0})\,|\mathrm{vac}\rangle$$



(a)



(b)

$$W_\Box \equiv \langle 1\rangle_\Box$$

Wilson loop for quarkonium

$$\langle H\,(\mathbf{0}, T/2)\,H\,(\mathbf{0}, -T/2)\rangle_\Box$$

Wilson loop for exotics

Short-distance behavior of BO-Potentials:

$$E_{\Sigma_g^+}(r) = V_s(r) + b_{\Sigma_g^+}\, r^2 + \ldots,$$
$$E_{\Sigma_u^-,\Pi_u}(r) = V_o(r) + \Lambda + b_{\Sigma,\Pi} r^2 + \ldots$$

Long-distance behavior of BO-Potentials:

- String behavior (pure SU(3) gauge theory)

$$E_N(r) = \sqrt{\sigma^2 r^2 + 2\pi\sigma\,(N - 1/12)}$$

- Mixing with pair of heavy-light states based on BO-quantum numbers or Λ_η^σ representations

Brambilla, AM, Vairo arXiv 2402.xxxxx

K. Juge, J. Kuti, C. Morningstar, Phys. Rev. Lett. 90 (2003)

Bali et al (SESAM collaboration), Phys. Rev. D. 71 (2005)

Bulava et al, Phys. Lett. B. 793 (2019)