Off-lightcone Wilson-line operators in gradient flow

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Outlines

Motivation: Spin-dependent potentials

- Relativistic corrections of QCD static potentials can be defined in terms of chromomagnetic and chromoelectric field insertions into the Wilson loops.

  See N. Brambilla, A. Pineda, J. Soto and A. Vairo, Rev. Mod. Phys. 77 (2005) 1423 for a review.

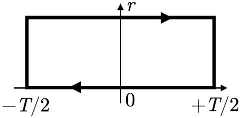
- Spin-dependent potentials, such as

$$V^{(1,1)}_{L_2 S_1}(r) \quad = -i \tfrac{c_F}{r^2} \mathbf{r} \cdot \lim_{T\to\infty} \int_0^T dt\, t \langle\!\langle g\mathbf{B}_1(t) \times g\mathbf{E}_2(0) \rangle\!\rangle, \qquad (1)$$

where $\langle\!\langle \ldots \rangle\!\rangle \equiv \langle \ldots W_\square \rangle / \langle W_\square \rangle$,

- $W_\square$ is the Wilson-loop (time-like) defined as

$$W_\square \equiv \mathrm{P} \exp \left\{ -ig \oint_{r\times T_W} dz^\mu A_\mu(z) \right\}. \qquad (2)$$



Motivation: Quasi-PDFs

- Direct lattice calculation of lightcone parton-distribution functions (PDFs) is difficult. One usually calculates the moments of PDFs on lattice. However, it has been difficult to calculate higher moments, due to power divergences and rapid decay in signals.
- Quasi-PDFs defined as the hadronic matrix elements of the Wilson-line operators (space-like, instead of light-like in PDFs),

$$\mathcal{O}_\Gamma(zv) = \bar{\psi}(zv)\Gamma W(zv,0)\psi(0), \quad \mathrm{quark\,quasi-PDF} \tag{3}$$

$$\mathcal{O}^{\mu\nu\alpha\beta}(zv) = g^2 F^{\mu\nu}(zv)W(zv,0)F^{\alpha\beta}(0), \quad \mathrm{gluon\,quasi-PDF} \tag{4}$$

  where Γ is a Dirac matrix and the Wilson-line

$$W(zv,0) \equiv \mathrm{P}\exp\left(ig\int_0^z ds\, v\cdot A(sv)\right). \tag{5}$$

- Quasi-PDFs ("lattice cross sections", can be directly calculated on lattice) are related to usual lightcone PDFs, through matching in the framework of Large momentum effective theory (LaMET).

See X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and Y. Zhao, Rev. Mod. Phys. 93 (2021) 035005 for a review.

Motivation: P-wave quarkonium decay & Heavy quark diffusion coefficient

- Vacuum expectation value of $g^2 F^{\mu\nu}(zv)W(zv,0)F^{\alpha\beta}(0)$ is related to P-wave quarkonium decay in the framework of pNRQCD (time-like v, $zv=\tau$, chromelectric component of $F^{\mu\nu}, F^{\alpha\beta}$)

  See N. Brambilla, D. Eiras, A. Pineda, J. Soto and A. Vairo, PRL 88 (2002) 012003, hep-ph/0109130

$$\mathcal{E}_3 = \frac{T_F}{N_c}\int_0^\infty d\tau\,\tau^3 \langle 0|gE(\tau,\mathbf{0})W(\tau,0)gE(0,\mathbf{0})|0\rangle, \tag{6}$$

- and heavy quark diffusion coefficient,

  See the talk by Peter Petreczky and the talk by Saumen Datta this afternoon.

$$G_E \sim \langle 0|gE(\tau,\mathbf{0})W(\tau,0)gE(0,\mathbf{0})|0\rangle, \tag{7}$$

$$G_B \sim \langle 0|gB(\tau,\mathbf{0})W(\tau,0)gB(0,\mathbf{0})|0\rangle. \tag{8}$$

Highlights

- Purpose of this work: Matching for off-lightcone Wilson-line operators from gradient-flow scheme to $\overline{\mathrm{MS}}$ scheme – building a bridge between $\overline{\mathrm{MS}}$ renormalized results and lattice results.
- The non-local off-lightcone Wilson-line operators were proved to be multiplicatively renormalizible (local renormalization) in the framework of one-dimensional auxiliary-field formalism.

  See H. Dorn, Fortsch. Phys. 34 (1986) 11 for a review of one-dimensional auxiliary-field formalism.
- Highlight of this work: Using the renormalization properties of off-lightcone Wilson-line operators to simplify the matching calculations in the small flow-time limit, which has not been realized in previous studies.

  1710.04607, 1705.11193, Ph.D. thesis (2021) of A. M. Eller and 2311.01525.

One-dimensional auxiliary-field formalism – Generalized HQET

- The one-dimensional auxiliary-field formalism is defined by enlarging the QCD Lagrangian to include an extra term

$$\mathcal{L}_{h_v} = \bar{h}_{v,0}(iv \cdot D)h_{v,0}, \tag{9}$$

where $h_{v,0}$ is an auxiliary "heavy" Grassmann (or complex) scalar field in either fundamental or adjoint representations of the SU(3) gauge group.

- $\mathcal{L}_{h_v}$ is equivalent to the leading-order HQET Lagrangian if v is time-like, for instance, $v = (1, 0, 0, 0)$.
- The extra term $\mathcal{L}_{h_v}$ is renormalizible,

$$\mathcal{L}_{h_v} = Z_{h_v}\bar{h}_v(iv \cdot \partial - i\delta m)h_v + gZ_g Z_A^{\frac{1}{2}} Z_{h_v}\bar{h}_v v \cdot A^a T^a h_v, \tag{10}$$

where

$$h_{v,0} = Z_{h_v}^{\frac{1}{2}} h_v, \quad g_0 = Z_g g, \quad A_0 = Z_A^{\frac{1}{2}} A, \tag{11}$$

and the "mass correction" $i\delta m$ is linearly divergent.

Wilson-line operators in one-dimensional auxiliary-field formalism

- Based on the one-dimensional auxiliary-field formalism, we can relate the off-lightcone Wilson line to the “heavy” field h_v

$$\langle h_{v,0}(x)\bar{h}_{v,0}(0)\rangle_{h_v} = W\left(\frac{v\cdot x}{v^2},0\right)\theta\left(\frac{v\cdot x}{v^2}\right)\delta^{(d-1)}(x_\perp), \qquad (12)$$

where $\langle ...\rangle_{h_v}$ stands for integrating out the “heavy” field h_v.

- In this way, the off-lightcone Wilson-line operators can be substituted with products of local current operators, for instance,

$$\begin{aligned}\mathcal{O}^{\mathrm{B}}_{\Gamma}(zv) &= \bar{\psi}_0(zv)\Gamma W(zv,0)\psi_0(0) = \int d^dx\,\delta\left(\frac{v\cdot x}{v^2}-z\right) \\ &\quad\times\langle\bar{\psi}_0(x)h_{v,0}(x)\Gamma\bar{h}_{v,0}(0)\psi_0(0)\rangle_{h_v}, \qquad (13)\end{aligned}$$

$$\begin{aligned}\mathcal{O}^{\mu\nu\alpha\beta,\mathrm{B}}(zv) &= g^2F_0^{\mu\nu}(zv)W(zv,0)F_0^{\alpha\beta}(0) = \int d^dx\,\delta\left(\frac{v\cdot x}{v^2}-z\right) \\ &\quad\times\langle g^2F_0^{\mu\nu}(x)h_{v,0}(x)\bar{h}_{v,0}(0)F_0^{\alpha\beta}(0)\rangle_{h_v}, \qquad (14)\end{aligned}$$

where the superscript “B” indicates bare composite operators.

E-E and B-B gluonic Wilson-line operators

- For convenience, we introduce the following projectors,

$$g_{\parallel}^{\mu\nu} = \frac{v_\mu v_\nu}{v^2}, \quad g_{\perp}^{\mu\nu} = g^{\mu\nu} - \frac{v_\mu v_\nu}{v^2}, \tag{15}$$

to project out the parallel and the transverse components of the field strength tensor $F^{\mu\nu}$ by defining,

$$F_{\parallel\perp}^{\mu\nu} = g_{\parallel}^{\mu\alpha} g_{\perp}^{\nu\beta} F^{\alpha\beta}, \tag{16}$$

$$F_{\perp\perp}^{\mu\nu} = g_{\perp}^{\mu\alpha} g_{\perp}^{\nu\beta} F^{\alpha\beta}. \tag{17}$$

- When $v = (1, 0, 0, 0)$ and $\mu \neq \nu$, $F_{\parallel\perp}^{\mu\nu}, F_{\perp\perp}^{\mu\nu}$ are proportional to the chromoelectric field **E** and the chromomagnetic field **B**.
- Define the following gluonic Wilson-line operators,

$$\mathcal{O}_{\parallel\perp}^{\mu\nu\alpha\beta}(zv) = g^2 F_{\parallel\perp}^{\mu\nu}(zv) W(zv, 0) F_{\parallel\perp}^{\alpha\beta}(0), \quad (E - E) \tag{18}$$

$$\mathcal{O}_{\perp\perp}^{\mu\nu\alpha\beta}(zv) = g^2 F_{\perp\perp}^{\mu\nu}(zv) W(zv, 0) F_{\perp\perp}^{\alpha\beta}(0). \quad (B - B) \tag{19}$$

Renormalization of local current operators

- Defining the local “heavy-to-light” and “heavy-to-gluon” current operators

$$J_q(x) = \bar{\psi}(x) h_v(x), \quad (20)$$
$$J^{\mu\nu}_{\parallel\perp}(x) = g F^{\mu\nu}_{\parallel\perp}(x) h_v(x), \quad (21)$$
$$J^{\mu\nu}_{\perp\perp}(x) = g F^{\mu\nu}_{\perp\perp}(x) h_v(x). \quad (22)$$

- The renormalization formulas for the above local current operators are given by,

$$J^{\mathrm{B}}_q(x,\Lambda) = Z_{J_q}(\Lambda,\mu) J^{\mathrm{R}}_q(x,\mu), \quad (23)$$
$$J^{\mu\nu,\,\mathrm{B}}_{\parallel\perp}(x,\Lambda) = Z_{J,\,\parallel\perp}(\Lambda,\mu) J^{\mu\nu,\,\mathrm{R}}_{\parallel\perp}(x,\mu), \quad (24)$$
$$J^{\mu\nu,\,\mathrm{B}}_{\perp\perp}(x,\Lambda) = Z_{J,\,\perp\perp}(\Lambda,\mu) J^{\mu\nu,\,\mathrm{R}}_{\perp\perp}(x,\mu), \quad (25)$$

which are well known in HQET.

Multiplicative renormalizability

- Key insight: The renormalizations of the non-local Wilson-line operators are simplified into the renormalization of two local "heavy-to-light" or "heavy-to-gluon" currents, after subtracting linear divergences related to the "mass correction" δm.

  That is,

$$\mathcal{O}_\Gamma^{\mathrm{B}}(zv,\Lambda) = Z_q(\Lambda,\mu)e^{\delta m(\Lambda)z}\mathcal{O}_\Gamma^{\mathrm{R}}(zv,\mu), \tag{26}$$

$$\mathcal{O}_{\parallel\perp}^{\mu\nu\alpha\beta,\,\mathrm{B}}(zv,\Lambda) = Z_{\parallel\perp}(\Lambda,\mu)e^{\delta m(\Lambda)z}\mathcal{O}_{\parallel\perp}^{\mu\nu\alpha\beta,\,\mathrm{R}}(zv,\mu), \tag{27}$$

$$\mathcal{O}_{\perp\perp}^{\mu\nu\alpha\beta,\,\mathrm{B}}(zv,\Lambda) = Z_{\perp\perp}(\Lambda,\mu)e^{\delta m(\Lambda)z}\mathcal{O}_{\parallel\perp}^{\mu\nu\alpha\beta,\,\mathrm{R}}(zv,\mu), \tag{28}$$

  in which,

$$Z_q(\Lambda,\mu) = Z_{J_q}^2(\Lambda,\mu), \tag{29}$$

$$Z_{\parallel\perp}(\Lambda,\mu) = Z_{J,\,\parallel\perp}^2(\Lambda,\mu), \tag{30}$$

$$Z_{\perp\perp}(\Lambda,\mu) = Z_{J,\,\perp\perp}^2(\Lambda,\mu). \tag{31}$$

See X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and Y. Zhao, Rev. Mod. Phys. 93 (2021) 035005 for a review of the renormalization of the quasi-PDF operators.

Multiplicative renormalizability

- For the spin dependent potential, such as

$$V^{(1,1)}_{L_2S_1}(r) \quad = -i\frac{c_F}{r^2}\mathbf{r}\cdot\lim_{T\to\infty}\int_0^T dt\, t\langle\langle g\mathbf{B}_1(t)\times g\mathbf{E}_2(0)\rangle\rangle, \quad (32)$$

- Each insertion of a chromomagnetic field into a Wilson loop can be related to the following local current operator as,

$$J^{\mu\nu}_F(x) \quad = \quad \bar{h}_v(x)gF^{\mu\nu}_{\perp\perp}(x)h_v(x), \quad (33)$$

where the renormalization formula for this operator is ,

$$J^{\mu\nu,\,\mathrm{B}}_F(x) \quad = \quad Z_F J^{\mu\nu,\,\mathrm{R}}_F(x) = Z_A^{1/2}Z_{h_v}Z_F^V J^{\mu\nu,\,\mathrm{R}}_F(x). \quad (34)$$

Gradient flow formalism

- In the gradient-flow formalism, the flow time (mass dimension of t is -2) dependent gluon field $B^a_\mu(x,t)$ and quark field $\chi^i_\alpha(x,t)$ are determined by the flow equations, M. Lüscher, JHEP 08 (2010) 071

$$\partial_t B_\mu = \mathcal{D}_\nu G_{\nu\mu} + \kappa \mathcal{D}_\mu \partial_\nu B_\nu, \tag{35}$$

$$\partial \chi = \mathcal{D}_\mu \mathcal{D}_\mu \chi - \kappa(\partial_\mu B_\mu)\chi, \tag{36}$$

$$\partial \bar{\chi} = \bar{\chi} \overleftarrow{\mathcal{D}}_\mu \overleftarrow{\mathcal{D}}_\mu + \kappa \bar{\chi} \partial_\mu B_\mu, \tag{37}$$

where κ is a gauge parameter , and

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]\,, \tag{38}$$

$$\mathcal{D}_\mu = \partial_\mu + [B_\mu, .]\,, \tag{39}$$

with the boundary conditions

$$B_\mu(x; t=0) = g A_\mu(x), \tag{40}$$

$$\chi(x; t=0) = \psi(x), \tag{41}$$

$$\bar{\chi}(x; t=0) = \bar{\psi}(x). \tag{42}$$

Advantages of gradient-flow formalism

- It has been expected that gradient flow can improve signal to noise ratio in lattice computations – reducing noise.
- No composite operator renormalization besides fields and QCD parameters renormalization – UV regulator.
- Perturbative gradient flow calculation (matching from gradient flow to $\overline{\mathrm{MS}}$) is much easier than lattice perturbative calculations – dimensional regularization can be used in perturbative gradient flow.

Small flow time expansion for the flowed current operators

- The relation between renormalized flowed current operators and the $\overline{\mathrm{MS}}$ renormalized un-flowed current operators is given by

$$\mathcal{O}^{\mathrm{R}}(t) = c_{\mathcal{O}}(t,\mu)\mathcal{O}^{\overline{\mathrm{MS}}}(\mu) + O(t), \tag{43}$$

  which is small-distance (t) operator-product-expansion (OPE).

- For the four local current operators $\bar{\psi}h_v$, $gF^{\mu\nu}_{\parallel\perp}h_v$, $gF^{\mu\nu}_{\perp\perp}h_v$, and $g\bar{h}_v F^{\mu\nu}_{\perp\perp}h_v$, we can express the matching coefficient $c_{\mathcal{O}}(t,\mu)$ as

$$c_{\psi h_v}(t,\mu) = \mathring{\zeta}_{\psi}(t,\mu)\zeta^{F}_{h_v}(t,\mu)\zeta_{\psi h_v}(t,\mu), \tag{44a}$$

$$c_{\parallel\perp}(t,\mu) = \zeta_{A}(t,\mu)\zeta^{A}_{h_v}(t,\mu)\zeta_{\parallel\perp}(t,\mu), \tag{44b}$$

$$c_{\perp\perp}(t,\mu) = \zeta_{A}(t,\mu)\zeta^{A}_{h_v}(t,\mu)\zeta_{\perp\perp}(t,\mu), \tag{44c}$$

$$c_{F}(t,\mu) = \zeta_{A}(t,\mu)(\zeta^{F}_{h_v}(t,\mu))^2\zeta_{F}(t,\mu). \tag{44d}$$

Matching for the Wilson-line operators

- Key insights from the multiplicative renormalization of the off-lightcone Wilson-line operators: In the small flow-time limit, besides the linear divergences, we just need to do matching for the local current operators, no additional matching for the combined current operators in different space-time. That is, we have

$$\mathcal{O}_{\Gamma}^{\mathrm{R}}(zv,t) = \mathcal{C}_{\psi}(t,\mu)e^{\delta m z}\mathcal{O}_{\Gamma}^{\overline{\mathrm{MS}}}(zv)+O(t), \tag{45}$$

$$\mathcal{O}_{\parallel\perp}^{\mathrm{R}}(zv,t) = \mathcal{C}_{\parallel\perp}(t,\mu)e^{\delta m z}\mathcal{O}_{\parallel\perp}^{\overline{\mathrm{MS}}}(zv)+O(t), \tag{46}$$

$$\mathcal{O}_{\parallel\perp}^{\mathrm{R}}(zv,t) = \mathcal{C}_{\perp\perp}(t,\mu)e^{\delta m z}\mathcal{O}_{\perp\perp}^{\overline{\mathrm{MS}}}(zv)+O(t), \tag{47}$$

in which,

$$\mathcal{C}_{\psi}(t,\mu) = c_{\psi h_v}^2(t,\mu), \tag{48}$$

$$\mathcal{C}_{\parallel\perp}(t,\mu) = c_{\parallel\perp}^2(t,\mu), \tag{49}$$

$$\mathcal{C}_{\perp\perp}(t,\mu) = c_{\perp\perp}^2(t,\mu). \tag{50}$$

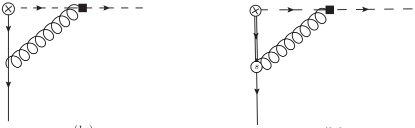
which are independent from the distance z.

One-loop matching calculations

We need to calculate

- δm – linear divergence coming from Wilson-line self-energy
- ζ_{h_v} – matching coefficient of the heavy field h_v
- $\mathring{\zeta}_\psi$ – matching coefficient of the quark field ψ
- $\zeta_{\psi h_v}$ – matching coefficient of the vertex $\bar{\psi} h_v$
- ζ_A – matching coefficient of the gluon field A
- $\zeta_{\parallel\perp}$ – matching coefficient of the vertex $gF^{\mu\nu}_{\parallel\perp} h_v$
- $\zeta_{\perp\perp}$ – matching coefficient of the vertex $gF^{\mu\nu}_{\perp\perp} h_v$
- ζ_F – matching coefficient of the vertex $\bar{h}_v g F^{\mu\nu}_{\perp\perp} h_v$

Example: one-loop calculation of $\zeta_{\psi h_v}$



(b_1) (b_2)

- To obtain the matching coefficient, we calculate the one-loop corrections for both the flowed and un-flowed vertex $\zeta_{\psi h_v}$ and compare the results (diagram b_2 does not contribute with our choice $\kappa = 1$).

$$
\begin{aligned}
\mathcal{M}_{(b_1)} &= -g^2 \tilde{\mu}^{2\epsilon} C_F \int \frac{d^d k}{(2\pi)^d} \frac{(\gamma \cdot v)\, \not{k}}{(-k \cdot v - i0)\, \left(k^2\right)^2} e^{-2tk^2} \\
&= g^2 \tilde{\mu}^{2\epsilon} C_F \int \frac{d^d k}{(2\pi)^d} \frac{1}{\left(k^2\right)^2} e^{-2tk^2} \\
&= \frac{\alpha_s C_R}{4\pi} \left[-\frac{1}{\epsilon_{\mathrm{IR}}} - \log\left(2\mu^2 t e^{\gamma_E}\right) - 1 \right]. \qquad (51)
\end{aligned}
$$

Example: one-loop calculation of $\zeta_{\psi h_v}$

- The one-loop correction of the un-flow vertex $\zeta_{\psi h_v}$ leads to scaleless integral that gives results

$$\begin{aligned} \mathcal{M}(t=0) &= -g^2\tilde{\mu}^{2\epsilon}C_F\int\frac{d^dk}{(2\pi)^d}\frac{(\gamma\cdot v)\,\not{k}}{(-k\cdot v-i0)\,(k^2)^2} \\ &= g^2\tilde{\mu}^{2\epsilon}C_F\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^2} \\ &= \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}}-\frac{1}{\epsilon_{\mathrm{IR}}}\right]. \end{aligned} \tag{52}$$

- The UV divergence is removed by renormalization, the IR divergence is removed through matching, which leads to the result of matching coefficient

$$\zeta_{\psi h_v} = 1-\frac{\alpha_s C_R}{4\pi}\left[\log\left(2\mu^2 t e^{\gamma_E}\right)+1\right]+O(\alpha_s^2). \tag{53}$$

One-loop results of matching coefficients

At one-loop level, we obtain

$$\delta m = -\frac{\alpha_s}{4\pi} C_R \frac{\sqrt{2\pi}}{\sqrt{t}} + O(\alpha_s^2), \quad (54)$$

$$\zeta_{h_v} = 1 - \frac{\alpha_s C_R}{4\pi} \log\left(2\mu^2 t e^{\gamma_E}\right) + O(\alpha_s^2), \quad (55)$$

$$\mathring{\zeta}_{\psi} = 1 + \frac{1}{2} \times \frac{\alpha_s}{4\pi} C_F \left[\log\left(2\mu^2 t e^{\gamma_E}\right) - \log(432)\right] + O(\alpha_s^2), \quad (56)$$

$$\zeta_{\psi h_v} = 1 - \frac{\alpha_s C_R}{4\pi} \left[\log\left(2\mu^2 t e^{\gamma_E}\right) + 1\right] + O(\alpha_s^2), \quad (57)$$

$$\zeta_{A} = 1 + \frac{\alpha_s}{4\pi} C_A \left[\log\left(2\mu^2 t e^{\gamma_E}\right) + \frac{1}{2}\right] + O(\alpha_s^2), \quad (58)$$

$$\zeta_{\perp\perp} = 1 + \frac{\alpha_s C_A}{4\pi} \left[\log\left(2\mu^2 t e^{\gamma_E}\right) - \frac{1}{2}\right] + O(\alpha_s^2), \quad (59)$$

$$\zeta_{\parallel\perp} = 1 - \frac{1}{2}\frac{\alpha_s}{4\pi} C_A + O(\alpha_s^2), \quad (60)$$

$$\zeta^{F}_{\perp\perp} = 1 + \frac{\alpha_s}{4\pi} \left[(C_A + 2C_F)\log\left(2\mu^2 t e^{\gamma_E}\right) + \frac{3}{8} C_A\right] + O(\alpha_s^2). \quad (61)$$

One-loop results of matching coefficients

- Combining the one-loop results shown in previously slide, gives

$$c_{\psi h_v}(t,\mu) = 1 - \frac{\alpha_s}{4\pi} C_F \left[\frac{3}{2} \log\left(2\mu^2 t e^{\gamma_E}\right) + \frac{\log(432)}{2} + 1 \right] + O(\alpha_s^2), \quad (62)$$

$$c_{\parallel\perp}(t,\mu) = 1 + O(\alpha_s^2), \quad (63)$$

$$c_{\perp\perp}(t,\mu) = 1 + \frac{\alpha_s}{4\pi} C_A \times \log\left(2\mu^2 t e^{\gamma_E}\right) + O(\alpha_s^2), \quad (64)$$

$$c_F(t,\mu) = 1 + \frac{\alpha_s}{4\pi} C_A \left[2\log\left(2\mu^2 t e^{\gamma_E}\right) + \frac{7}{8} \right] + O(\alpha_s^2). \quad (65)$$

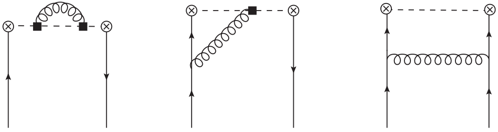
- And keep in mind the relations

$$\mathcal{C}_\psi(t,\mu) = c^2_{\psi h_v}(t,\mu), \quad (66)$$

$$\mathcal{C}_{\parallel\perp}(t,\mu) = c^2_{\parallel\perp}(t,\mu), \quad (67)$$

$$\mathcal{C}_{\perp\perp}(t,\mu) = c^2_{\perp\perp}(t,\mu). \quad (68)$$

One-loop quark quasi-PDF with full flow time dependence



- After renormalization and subtracting the linear divergent term,

$$\mathcal{M}_q = \frac{\alpha_s}{4\pi} C_F \Big[a_\Gamma + 3\log\left(\bar{z}^2 e^{\gamma_E}\right) - \log(432) - 4e^{-\bar{z}^2} - 3\text{Ei}(-\bar{z}^2) - 4\sqrt{\pi}\bar{z}\text{erf}(\bar{z}) + 4\sqrt{\pi}\bar{z} \Big], \tag{69}$$

where $\bar{z} = \frac{z}{r_F}$ with $r_F = \sqrt{8t}$ representing the flow radius, and

$$a_{\gamma \cdot v} = 5 + \frac{3}{\bar{z}^4}\left(1 - e^{-\bar{z}^2}\right) - \frac{1}{\bar{z}^2}\left(2 + e^{-\bar{z}^2}\right), \tag{70}$$

$$a_{\gamma_\perp^\alpha} = 3 - \frac{1}{\bar{z}^4}\left(1 - e^{-\bar{z}^2}\right) + \frac{1}{\bar{z}^2}\left(2 - e^{-\bar{z}^2}\right). \tag{71}$$

One-loop quark quasi-PDF with full flow time dependence

- The un-flowed result is

$$\mathcal{M}_q^{\overline{\text{MS}}} = \frac{\alpha_s}{4\pi} C_F \left[a'_\Gamma + 3\log\left(\frac{z^2\mu^2}{4}\right) + 6\gamma_E \right], \quad (72)$$

with $a'_{\gamma\cdot v} = 7$, $a'_{\gamma_\perp^\alpha} = 5$.

- By comparing $\mathcal{M}_q$ with $\mathcal{M}_q^{\overline{\text{MS}}}$, we obtain the matching coefficient

$$\begin{aligned} \mathcal{C}_q(t,\mu,z) \quad = \quad & 1 + \frac{\alpha_s}{4\pi} C_F \Big[a_\Gamma - a'_\Gamma - 3\log\left(2\mu^2 t e^{\gamma_E}\right) - \log(432) \\ & -4e^{-\bar{z}^2} - 3\text{Ei}(-\bar{z}^2) - 4\sqrt{\pi}\bar{z}\,\text{erf}(\bar{z}) + 4\sqrt{\pi}\bar{z} \Big]. \quad (73) \end{aligned}$$

- Taking the limit $t \to 0$, we have

$$\mathcal{C}_{q,\,t\to 0}(t,\mu) = 1 - \frac{\alpha_s}{4\pi} C_F \left[3\log\left(2\mu^2 t e^{\gamma_E}\right) + 2 + \log(432) \right], \quad (74)$$

which is independent of z and equivalent to $c^2_{\psi h_v}$ as expected.

Finite flow time effect

To see how large the finite flow time effect will be, we plot

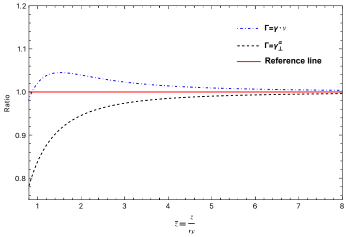


Figure: $\text{Ratio} = \frac{\mathcal{C}_q(t,\mu,z)-1}{\mathcal{C}_{q,\ t\to 0}(t,\mu)-1}$. The renormalization scale μ is chosen so that $\log\left(2t\mu^2 e^{\gamma_E}\right) = 0$.

- The finite flow time effect is very small as long as the as long as the flow radius $r_F = \sqrt{8t}$ is smaller than the distance z ($\bar{z} > 1$).
- $\mathcal{C}_q(t,\mu,z)$ was also calculated in 1710.04607, in which the small flow time limit of the result was not independent from z. Thus the result in 1710.04607 is incorrect.

Advantages of our approach

- The renormalization properties of the off-lightcone Wilson-line operators were never used to simplify the matching calculations.
- For instance, the matching coefficient of the vacuum expectation value of $g^2 F^{\mu\nu}_{\|\perp}(zv)W(zv,0)F^{\alpha\beta}_{\|\perp}(0)$ (related to heavy quark diffusion coefficient) was calculated in the Ph.D. thesis (2021) of A. M. Eller, in which the result of $\mathcal{C}_{|\perp}(t,\mu)$ was obtained in a rather complicated way (more than 60 pages).
- In our approach, the calculation is greatly simplified (within 2 pages) because we have reduced the two-scale (z and t) problem into single-scale (t) problem in the small flow time limit.
- Notably, our approach is also applicable for matching from lattice regularization schemes to the $\overline{\mathrm{MS}}$ scheme in the small lattice spacing limit.

Applications of our results

- $c_F(t,\mu)$ – spin-dependent potentials
- $c_{\psi h_v}(t,\mu)$ – matrix elements of $\bar{\psi}(zv)\Gamma W(zv,0)\psi(0)$, such as the quark quasi-PDFs...
- $c_{\parallel\perp}(t,\mu)$ – matrix elements of $g^2 F^{\mu\nu}_{\parallel\perp}(zv)W(zv,0)F^{\alpha\beta}_{\parallel\perp}(0)$, such as gluon quasi-PDFs, masses of gluelumps, heavy quark diffusion coefficient, P-wave quarkonium decay long distance matrix element in the framework of pNRQCD,

$$\mathcal{E}_3 = \frac{T_F}{N_c}\int_0^\infty d\tau\,\tau^3 \langle 0|gE(\tau,\mathbf{0})W(\tau,0)gE(0,\mathbf{0})|0\rangle. \tag{75}$$

- $c_{\perp\perp}(t,\mu)$ – matrix elements of $g^2 F^{\mu\nu}_{\perp\perp}(zv)W(zv,0)F^{\alpha\beta}_{\perp\perp}(0)$, such as gluon quasi-PDFs, masses of gluelumps, heavy quark diffusion coefficient...

Summary and outlook

- We have developed a systematic approach for matching from the gradient-flow scheme to the $\overline{\mathrm{MS}}$ scheme for Wilson-line operators in the small flow-time limit.
- The matching of Wilson-line operators is reduced into the matching of local current operators, which greatly simplifies the matching calculations (two scales to single scale).
- Our results have various applications such as lattice calculations of quasi-PDFs, spin-dependent potentials, heavy quark diffusion coefficient, gluonic correelators appear in P-wave quarkonium decay in the framework of pNRQCD and so on.
- We are looking forward to extend the matching calculations at two-loop level, which will be more helpful in lattice computations.