The static force from the lattice with gradient flow for $N_f = 0$
arXiv:2312.17231

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QWG meeting Mohali, India, Feb 28 2024

Motivation

- Static force relies on the static potential which is well known
- Static potential/force well known perturbatively and on the lattice
- Useful for testing new methodologies
  Here: Force from generalized Wilson loops, Gradient flow
  $\rightarrow$ preparation for similar objects needed in NREFTs

- Static potential/force is used for scale setting
- Static potential/force can be used to extract $\Lambda_0 = \Lambda_{\overline{\mathrm{MS}}}^{n_f=0}$
- Λ_0 got more attention recently with new methods

Motivation

- Lattice: we use stochastic methods to solve the Euclidean path integral non-perturbatively:

$$\langle O(t)O(0)\rangle = \frac{1}{Z}\int \mathcal{D}[U]e^{-S_E[U]}O(t)O(0) \approx \frac{1}{N}\sum_{p(U)\propto e^{-S_E}} O(t)O(0)$$

  on a discretized space-time grid (lattice), with lattice spacing/lattice regulator a
- We are interested in long-t correlation to obtain spectra
- Comparing of non-perturbative lattice results and PT expressions $\rightarrow$ extract α_S
- Several issues:
  - large t $\rightarrow$ larger statistical error/bad signal-to-noise ratio
  - discretization errors $\rightarrow$ continuum limit
    But: Not trivial in our case

$\Rightarrow$ We use gradient flow to tackle both problems

Methodological background

- Gradient flow: smearing technique

(Narayanan, JHEP 03,064 (2006)) (Lüscher, Commun.Math.Phys. 293,899-919 (2010)) (Lüscher, JHEP 08,071 (2010))

$$\dot{B}_\mu(t_F, x) = -g_0^2 \frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu(t_F, x)} \qquad B_\mu^{LO}(t_F, x) = g_0 \int d^4 y K_{t_F}(x-y) A_\mu(y)$$

$$B_\mu(t_F = 0, x) = A_\mu(x) \qquad K_{\tau_F}(z) = \frac{e^{-z^2/4t_F}}{(4\pi t_F)^2}$$

- Introduces new scale: $t_F \; / \; \sqrt{8t_F}$ which regularizes/renormalizes observables

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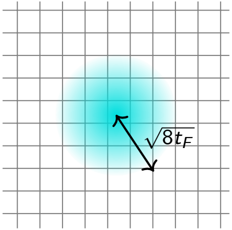

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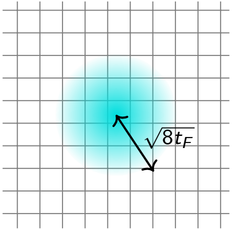


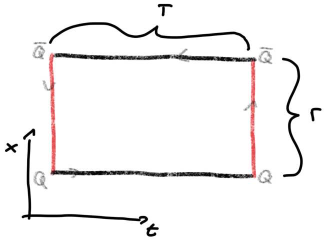
- Introduces new scale: t_F / $\sqrt{8t_F}$ which regularizes/renormalizes observables
- ALERT: $t_F \neq 0$ is not our physical world $\to$ need to perform a $t_F \to 0$ limit:
  - Performing $a \to 0$ (continuum limit) while keeping t_F/a^2 fixed
  - Performing $a \to 0$ while keeping t_F fixed in physical units
    $\to$ perform $t_F \to 0$ limit in the continuum
- Gradient flow perturbatively treatable $\to$ guides the $t_F \to 0$ limit

Physical Background: Static Potential

- Static potential $V(r)$ encoded in the spectrum of Wilson loops at fixed r:

$$\langle W_{r\times T}\rangle \overset{\mathrm{large}\,T}{\propto} e^{-aV(r)}$$

- In continuum PT: renormalon ambiguity
- On the lattice: $1/a$ - divergence (r-independent)
- Lattice and PT should agree for $r\Lambda_{\mathrm{QCD}} \ll 1$



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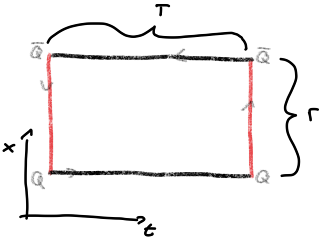
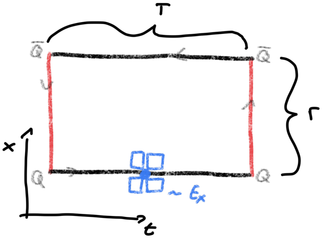
- On the lattice: $1/a$ - divergence (r-independent)
- Lattice and PT should agree for $r\Lambda_{\text{QCD}} \ll 1$
- Derivative: $\partial_r V(r) \equiv F(r)$
  $[V] = \text{fm}^{-1}$, $[F] = \text{fm}^{-2} \rightarrow [rV] = 1 = [r^2 F]$
- numerical derivative of $V(r)$ introduces systematic uncertainties



➡ We use an alternative definition of F

Physical Background: Static Force

- $F_E(r) = \lim_{T \to 0} \frac{\langle WE_{r\times T} \rangle}{\langle W_{r\times T} \rangle}$
  (Vairo, EPJ Web Conf. 126, 02031 (2016)) (Vairo, Mod.Phys.Lett.A 31, 1630039 (2016)) (Brambilla, Phys. Rev. D 63, 014023 (2001))
- Chromo E-field insertion in one of the temporal Wilson lines
- It is the same quantity as $\partial_r V(r)$
- Discretization of E causes a non-trivial behavior to the continuum



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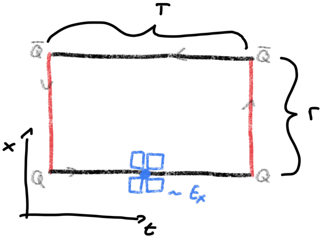
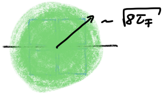
- Chromo E-field insertion in one of the temporal Wilson lines
- It is the same quantity as $\partial_r V(r)$
- Discretization of E causes a non-trivial behavior to the continuum
- may be absorbed into a renormalization constant:

$$Z_E F_E(r) = \partial_r V(r) \qquad\qquad Z_E \to 1 \text{ for } a \to 0$$

- F_E and $\partial_r V(r)$ are clearly defined on the lattice
- use gradient flow to see the impact on Z_E

➡ We use GF to renormalize E-field insertion, and to improve the signal-to-noise ratio



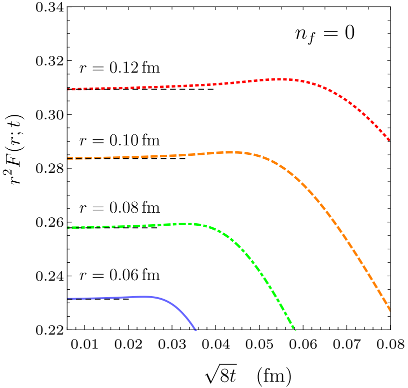


Remark on the continuum force

- Static force/potential is known up to N^{3}LL, natural scale: $\mu = 1/r$
- at finite flow time: known up to NLO
  (Brambilla, JHEP 01,184 (2022))
  new scales: $1/\sqrt{8t_F}$, $1/\sqrt{r^2+8t_F}$
- We use $1/\sqrt{r^2+8bt_F}$ with $-0.5 \leq b \leq 1$, default: $b=0$
  $\sim$ freedom to choose the scale
- t_F-expansion:

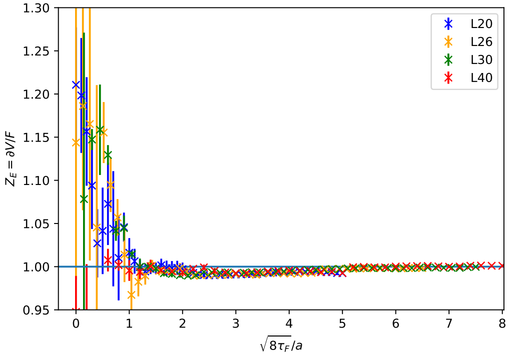
$$r^2F(r,t_F) \overset{t_F \text{ small}}{\approx} r^2F(r,t_F=0) + \underbrace{const}_{\propto n_f}\frac{t_F}{r^2}$$

$\Rightarrow$ constant at small t_F for $n_f = 0$



Renormalization effect of gradient flow

- $Z_E = \frac{\partial_r V}{F_E}$ with little r-dependence
  (Brambilla, Phys.Rev.D105, 054514 (2022))
- Z_E approaches 1 for $\sqrt{8t_F} > a$, for all lattice sizes (flow time scale dominates over lattice regulator scale)
  $\rightarrow$ renormalizes E-field insertion



> Perform continuum limit at fixed t_F for $\sqrt{8t_F} > a$

Lattice & Scaling details

Simulation parameters:

N_S	N_T	β	a [fm]	t_0/a^2	N_{conf}	Label
20^3	40	6.284	0.060	7.868(8)	6000	L20
26^3	52	6.481	0.046	13.62(3)	6000	L26
30^3	60	6.594	0.040	18.10(5)	6000	L30
40^3	80	6.816	0.030	32.45(7)	3300	L40

- find reference scale t_0/a^2 for continuum limit
- find r_0/a, r_1/a
- common to set $r_0 = 0.5\,\mathrm{fm}$ for pure gauge
- t_0-scale is the natural scale for gradient flow studies

$$\frac{\sqrt{8t_0}}{r_0} = 0.9569(66)$$

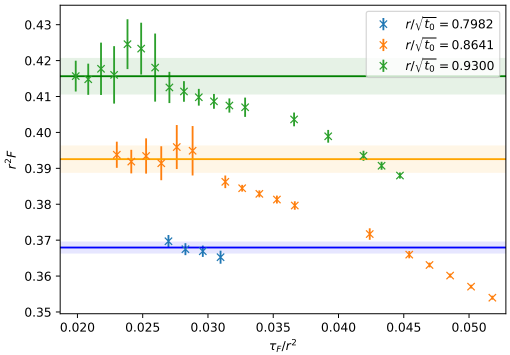
$$\frac{\sqrt{8t_0}}{r_1} = 1.325(13)$$

$$\frac{r_0}{r_1} = 1.380(14)$$

in continuum and after $t_F \to 0$

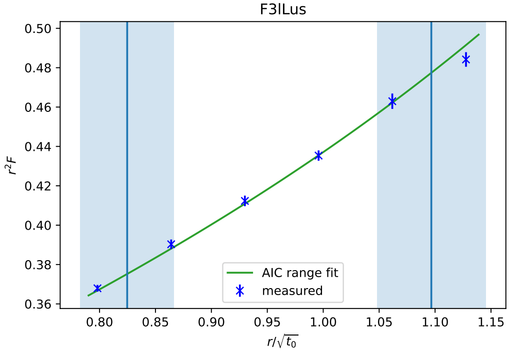
$t_F \to 0$ limit: $F(t_F \to 0)$, Λ_0 from small r

- 1-loop constant at small t_F
- constant $t_F \to 0$ limit
- obtain $F(t_F = 0)$



$t_F \to 0$ limit: $F(t_F \to 0)$, Λ_0 from small r

- 1-loop constant at small t_F
- constant $t_F \to 0$ limit
- obtain $F(t_F = 0)$
- Fit NLO, N^2LO, N^2LL, $N^3LO(+u.s.)$ where Λ_0 is the fit parameter

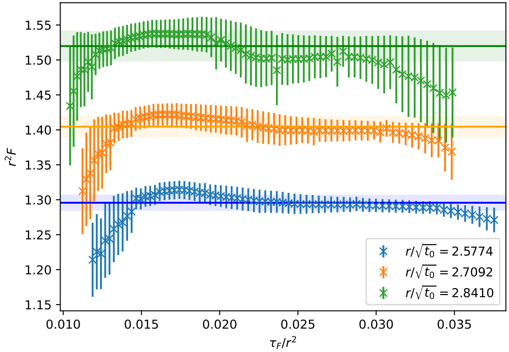


➡ Λ_0 extraction works, but only at larger r

$t_F \to 0$ limit: $F(t_F \to 0)$, large r

- Perform constant zero flow time limit at large r
- Perform Cornell fit:

$$r^2 F(r) = A + \sigma r^2$$



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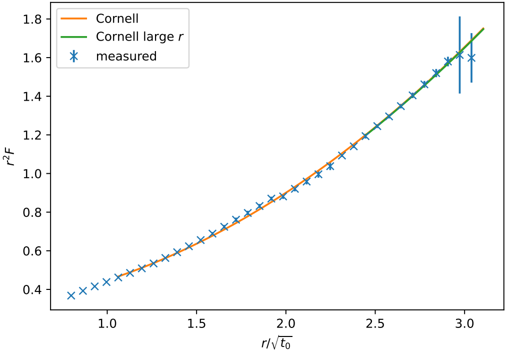
$$r^2 F(r) = A + \sigma r^2$$

- Fit result:

$$A = 0.268(33)$$
$$\sigma t_0 = 0.154(6)$$

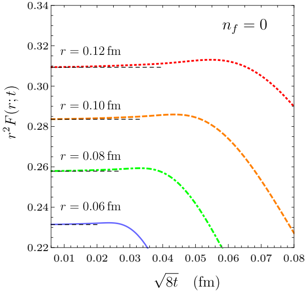


➡ $\sigma t_0 \in [0.143, 0.159]$ see Knechtli's talk

$t_F \to 0$ limit: Λ_0 from $F(t_F)$

- Fit perturbative $F(t_F)$ directly to $F(t_F)$ obtained from the lattice
- PT $F(t_F)$ determined by Λ_0
- Higher order crucial for reliable Λ_0 extraction but only up to NLO is known at finite t_F
- Combined model function:

$$F^{\text{model}} = \begin{cases} F(r) \text{ at any order} & t_F = 0 \\ \text{flowtime part from NLO} & t_F \neq 0 \end{cases}$$

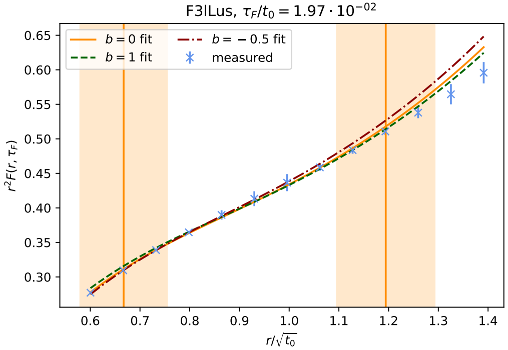


$t_F \to 0$ limit: Λ_0 from $F(t_F)$, r scale

- Fit at fixed t_F, along r
- scale: $\mu = \frac{1}{\sqrt{r^2+8bt_F}}$, $-0.5 \leq b \leq 1$

Obtainings:

- fit range depends on b, slope of fitted function is stable



different Λ_0 at different fixed t_F, but should not depend too much on that

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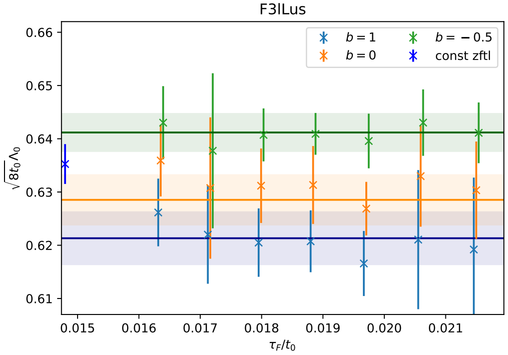
➡ works at smaller r and along the physical relevant scale



Uncertainties of Λ_0

- Statistical error: Jackknife sampling
- Fit errors for different fit ranges: Akaike information criterion
- Perturbation errors:
  - $\mu = \frac{1}{\sqrt{r^2+8bt_F}}$ not unique
  - at finite t_F: variation $-0.5 \leq b \leq 1$
  - at $t_F = 0$: $\mu = \frac{s}{r}$ with $\frac{1}{\sqrt{2}} \leq s \leq \sqrt{2}$
- Errors are independent of each other
  $\rightarrow$ sum in quadrature

⇨ Final result includes statistical and perturbative uncertainties

Results

- All methods agree within their errors
- We state fit at fixed t_F along r at N^{3}LO+u.s. as final result:

$$\sqrt{8t_0}\Lambda_0 = 0.629^{+22}_{-26}$$
$$\delta(\sqrt{8t_0}\Lambda_0) = (4)^{\mathrm{lattice}}(^{+18}_{-25})^{\mathrm{s-scale}}(^{+13}_{-7})^{\mathrm{b-scale}}$$
$$r_0\Lambda_0 = 0.657^{+23}_{-28}$$
$$\Lambda_0 = 259^{+9}_{-11}\,\mathrm{MeV},\; r_0 = 0.5\,\mathrm{fm}$$

- Cornell fit:

$$r^2F(r) = A + \sigma r^2$$
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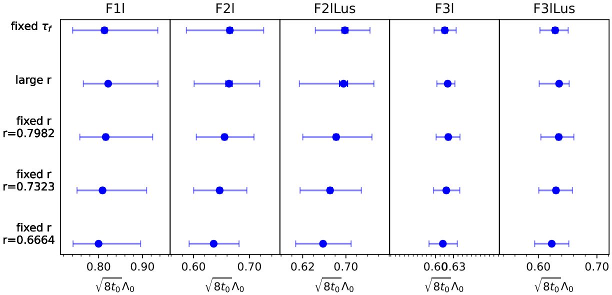

Summary

- With GF direct force measurement possible
- GF renormalizes E-field insertions
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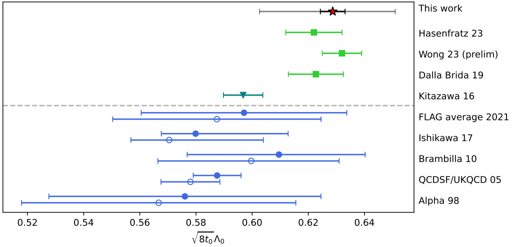
- Λ_0 extraction in several ways
- Λ_0 compatible to recent GF studies
- Λ_0 with GF is systematically larger even in our study
- GF applicable with fermions



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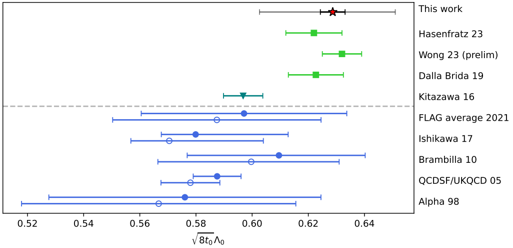


Thank you for your attention!

Backup: Continuum limit details

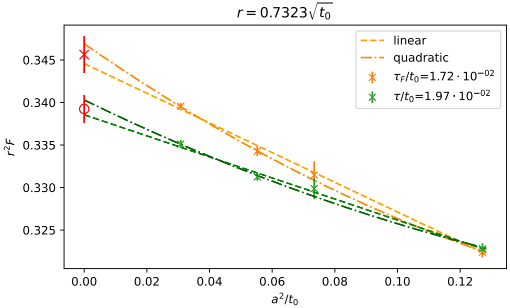
- Polynomial interpolations
- Tree level improvement at any fixed r and t_F:

$$F^{\text{impr latt}} = \frac{F^{\text{tree cont}}}{F^{\text{tree latt}}} F^{\text{latt}}$$

- Trivial continuum limit:

$$F^{\text{impr latt}} = \text{Polynomial}(a^2) = F^{\text{cont}} + \mathcal{O}(a^2)$$

where $\sqrt{8t_F} > a$ for the coarsest lattice

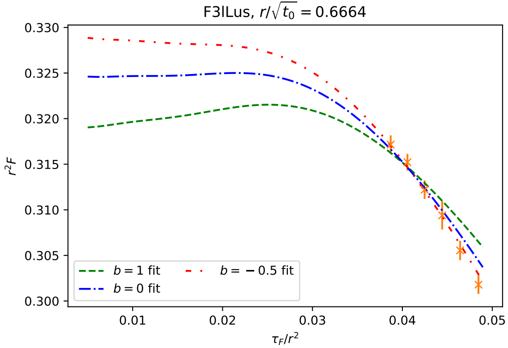


Backup: $t_F \to 0$ limit: Λ_0 from $F(t_F)$, t_F scale

- Fit at fixed r, along t_F
- scale: $\mu = \frac{1}{\sqrt{r^2+8bt_F}}$, $-0.5 \leq b \leq 1$

Obtainings:

- slope of fitted function highly depends on b
- works less good at larger r



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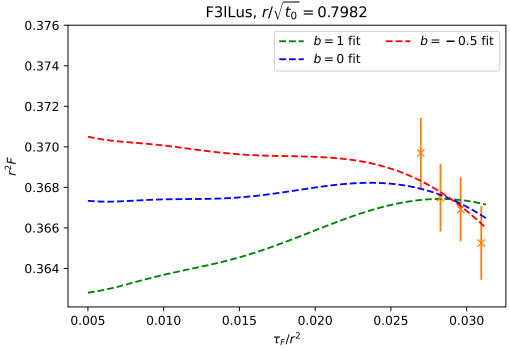

Fit works, but t_F is not the physical scale

Sample frame title

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Knock Knock!! Who's there!?