Fermilab

All-heavy Tetraquarks in QCD

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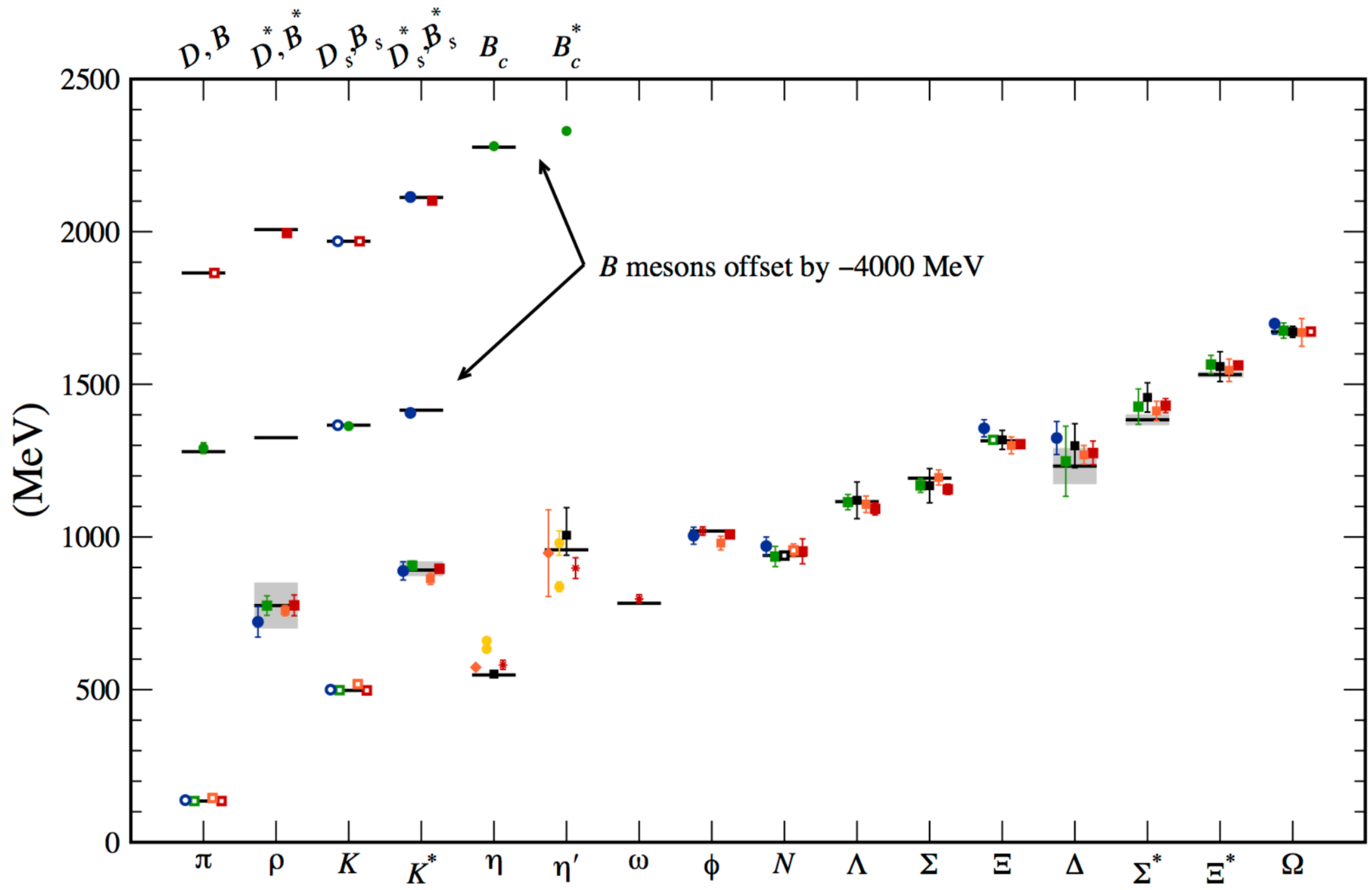
Hadrons and more

LQCD hadron spectrum: Kronfeld '12



LQCD: spectroscopy complete for most common hadrons

Exotic hadrons: recent experiments, Belle, LHCb etc. E.g. tetra/pentaquarks, dibaryons etc.

Exotics hard in LQCD, dense excited state spectrum, computationally inefficient

Kronfeld et al. 2207.07641

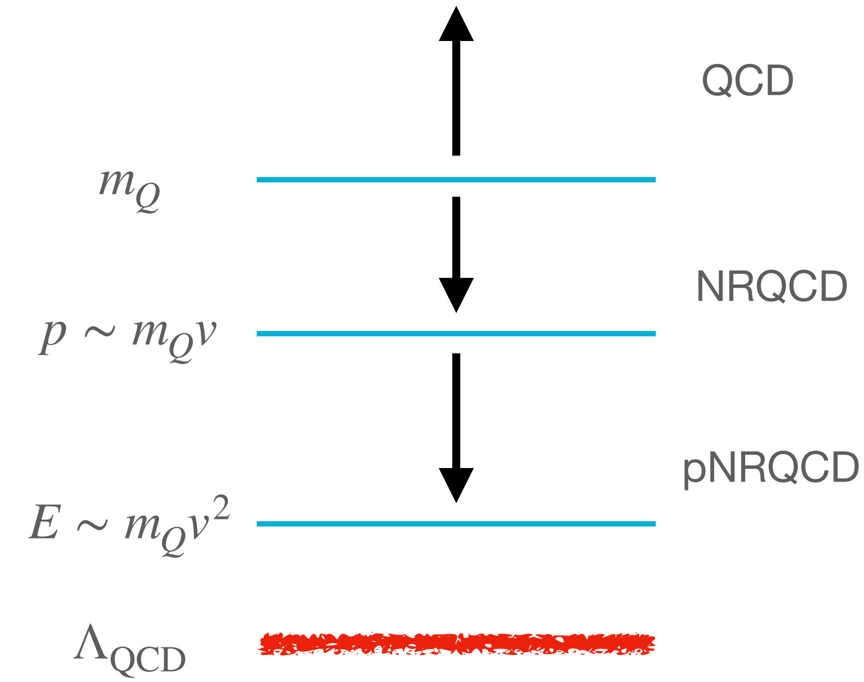
Can EFT help us understand all-heavy exotics and nuclear interactions at unphysical quark masses?

BA and Wagman, PRD 108

HQET and (p)NRQCD

Heavy quarks and their bound states call for **multi-scale** treatment

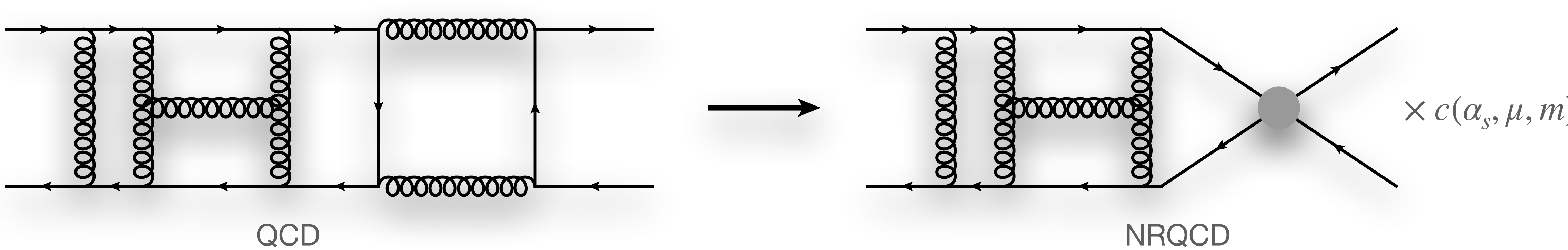
Expansion at the Lagrangian level in $m_Q \gg v \sim \alpha_s \Rightarrow$ **EFT description**



HQET & NRQCD: Operators and matching known to $D = 8$, NNLO and above for specific applications Gerlach et al. '19; Gunawardana et al. '17; BA, Kniehl, Soto '20

pNRQCD: Static $1/r$ potentials known to $\mathrm{N^3LO}$ and finite mass + spin-dependent terms to $m^2\mathrm{NNLO}$ Review: Pineda '11

Applications: $t, b, c, B, \eta_{b,c}, \Upsilon(1S), J/\psi, \ldots$ mass, decay, splitting, cross-sections etc.



Bodwin, Braaten, Brambilla, Caswell, Eichten, Georgi, LePage, Manohar, Pineda, Soto, Vairo, Wise,…;1987-2024

pNRQCD for (multi-)hadrons

Extending the pNRQCD Lagrangian $\Rightarrow$ **new operators**

Quarkonium operator:

$$L^{\text{pot}}_{\psi\chi} = -\int d^3\boldsymbol{r}_1 d^3\boldsymbol{r}_2\, \psi_i^\dagger(t,\boldsymbol{r}_1)\chi_j(t,\boldsymbol{r}_2)\chi_k^\dagger(t,\boldsymbol{r}_2)\psi_l(t,\boldsymbol{r}_1)\left[\delta_{ij}\delta_{kl}V_{\mathbf{1}}^{\psi\chi}(\boldsymbol{r},\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{S}_1,\boldsymbol{S}_2) + T^a_{ij}T^a_{kl}V_{\mathbf{8}}^{\psi\chi}(\boldsymbol{r},\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{S}_1,\boldsymbol{S}_2)\right]$$

Di-quark operator:

$$L^{\text{pot}}_{\psi\psi} = -\int d^3\boldsymbol{r}_1 d^3\boldsymbol{r}_2\, \psi_i^\dagger(t,\boldsymbol{r}_1)\psi_j^\dagger(t,\boldsymbol{r}_2)\psi_k(t,\boldsymbol{r}_2)\psi_l(t,\boldsymbol{r}_1)\left[\epsilon_{ijm}\epsilon_{klm}V_{\bar{\mathbf{3}}}^{\psi\psi}(\boldsymbol{r},\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{S}_1,\boldsymbol{S}_2) + \left(\delta_{il}\delta_{jk}+\delta_{jl}\delta_{ik}\right)V_{\mathbf{6}}^{\psi\psi}(\boldsymbol{r},\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{S}_1,\boldsymbol{S}_2)\right]$$

Tri-quark operator:

$$L^{\text{pot}}_{3\psi} = -\int d^3\boldsymbol{r}_1 d^3\boldsymbol{r}_2 d^3\boldsymbol{r}_3 \psi_i^\dagger(t,\boldsymbol{r}_1)\psi_j^\dagger(t,\boldsymbol{r}_2)\psi_k^\dagger(t,\boldsymbol{r}_3)\psi_l(t,\boldsymbol{r}_3)\psi_m(t,\boldsymbol{r}_2)\psi_n(t,\boldsymbol{r}_1)V_{ijk}^{3\psi}(\boldsymbol{r},\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{S}_1,\boldsymbol{S}_2)$$

$$V_{ijk}^{3\psi} = \epsilon_{ijk}\epsilon_{lmn}V_{\mathbf{1}}^{3\psi} + \left(\epsilon_{ijp}T^a_{pk}\epsilon_{lmo}T^a_{on}\right)V_{\mathbf{8}A}^{3\psi} + \left(\epsilon_{ikp}T^a_{jp}+\epsilon_{jkp}T^a_{ip}\right)\left(\epsilon_{lno}T^a_{mo}+\epsilon_{mno}T^a_{lo}\right)V_{\mathbf{8}S}^{3\psi} + \left(\delta_{il}\delta_{jm}\delta_{kn} + \text{perms}\right)V_{\mathbf{10}}^{3\psi}$$

For $N_q > 3$ N-body operators contribute at $\mathrm{NNLO}+$

Two-body potentials

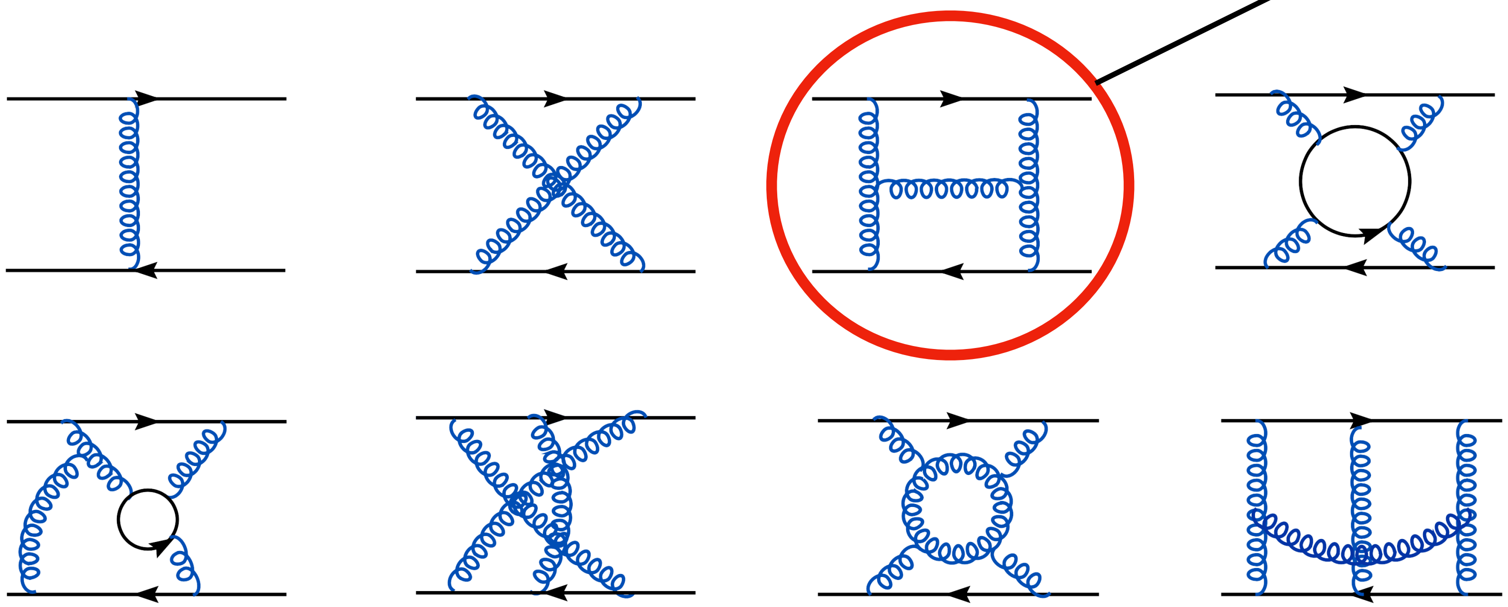
Potential of two heavy fermions under $\mathrm{SU(N_c)}$: $V_c(r) = \sum_{n,k} \frac{1}{m^n} V_c^{(n,k)}$

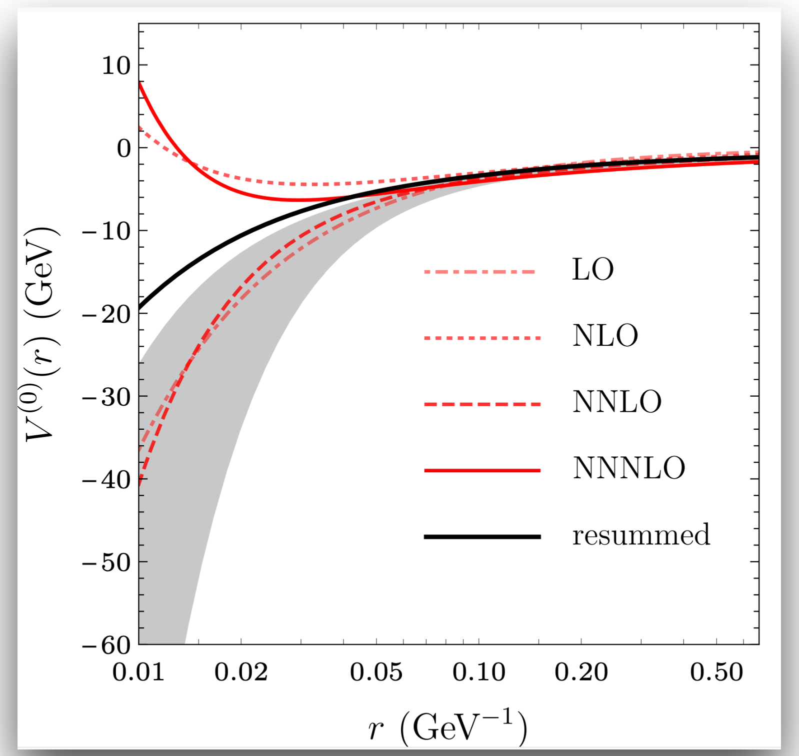
Known to $\mathcal{O}(1/m^2)$, NLO $\forall c$, N^{3}LO for $c = \mathbf{1}, \mathbf{8}$ Steinhauser et al. '98 - '18

Spin-dependent pieces arise in $V_c^{2,k}(r)$ at m^2LO Review: Pineda '11

Color relation (NNLO): $V_\rho(r) = -\mathscr{C}_\rho^{\text{tree}} \left(\frac{1}{C_F} V_{\mathbf{1}}^{\psi\chi}(r) - \frac{\alpha_s^3}{(4\pi)^2} \frac{\delta a_\rho}{r} \right)$

H.S. Chung '20





N-body potential

Three-body Lagrangian for general $N_c \mapsto$ **Hamiltonian** straightforward

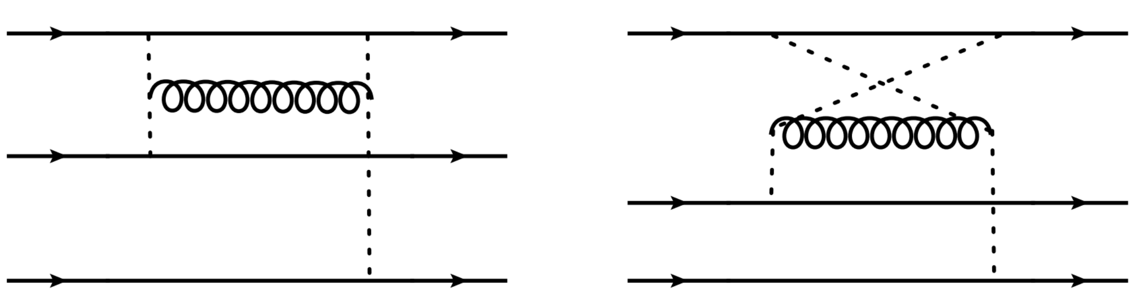
$$L_{3\psi}^{\text{pot}} = -\int d^3\boldsymbol{r}_1 d^3\boldsymbol{r}_2 d^3\boldsymbol{r}_3\, \psi_i^\dagger(t,\boldsymbol{r}_1)\psi_j^\dagger(t,\boldsymbol{r}_2)\psi_k^\dagger(t,\boldsymbol{r}_3)\psi_l(t,\boldsymbol{r}_3)\psi_m(t,\boldsymbol{r}_2)\psi_n(t,\boldsymbol{r}_1)$$

$$\times \left[\mathscr{F}^A_{ijk}\mathscr{F}^A_{lmn}V_A^{3\psi} + \mathscr{F}^S_{ijk}\mathscr{F}^S_{lmn}V_S^{3\psi} + \sum_{r,s} \mathscr{F}^{Mua}_{ijkq_1\ldots q_{N_c-3}} V^{3\psi}_{Muv} \mathscr{F}^{Mva}_{lmnq_1\ldots q_{N_c-3}} \right]$$

Three-body potential (NNLO)

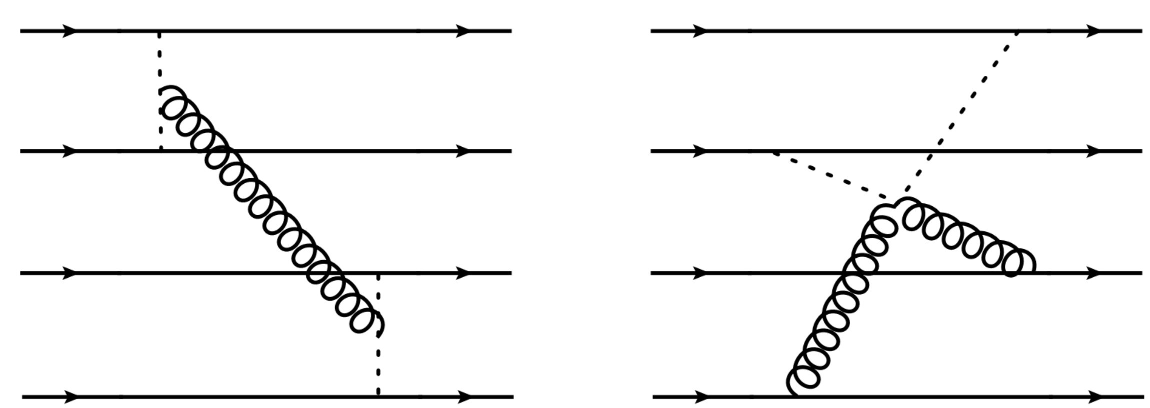
$$V_\rho^{3\psi} = \alpha \left(\frac{\alpha}{4\pi}\right)^2 \left[\mathscr{C}^{3\psi,1}_{\rho uv} v_3(\boldsymbol{r}_{12},\boldsymbol{r}_{13}) + \mathscr{C}^{3\psi,2}_{\rho uv} v_3(\boldsymbol{r}_{12},\boldsymbol{r}_{23}) + \mathscr{C}^{3\psi,3}_{\rho uv} v_3(\boldsymbol{r}_{13},\boldsymbol{r}_{23}) \right]$$

with v_3 from 3-body diagrams (**no spectators + transverse** g connector)



Non-exponentiating and non-zero 3-body diagram (NNLO)

Brambilla, Ghiglieri, Vairo '10



NNLO 4-body diagrams (vanishing for baryons)

Variational methods

Modern nuclear physics has many ways to solve Schrödinger equations

Problem: No a priori knowledge of eigenstates of system ⇒ **MC methods**

⇒ **Hamiltonian** of N-quark system and **trial wavefunction:**

$$\Psi_T(\boldsymbol{r}, \boldsymbol{\alpha}) = \sum_{i=1}^{M} \alpha_i \Phi_i(\boldsymbol{r}) \text{ with } \boldsymbol{r} \equiv \boldsymbol{r}_{1\ldots N} \text{ and } \textit{variational parameters } \boldsymbol{\alpha}$$

⇒ **Expectation value**:

$$E_0 \leq \langle \psi | H | \psi \rangle = \int d\mathbf{R}\, \psi^*(\mathbf{R}) \left[\sum_i \frac{\nabla_i^2}{2m_i} + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j) \right] \psi(\mathbf{R})$$

Idea: Vary $\boldsymbol{\alpha}$ with differential program that performs minimisation iteratively

Review: Carlson et al, Rev. Mod. Phys. 87 (2015)

Green's function Monte-Carlo

We now have Ψ_T and E_{min} but what if ∃ *lower-lying states*?

Limited in VMC by choice and size of basis ⇒ need new tool

GFMC: Given Ψ_T from VMC diffuse in time and reach **true ground-state energy**

$$|\Psi_T\rangle = \lim_{\tau\to\infty} e^{-\hat{H}\tau}|\psi_T\rangle$$

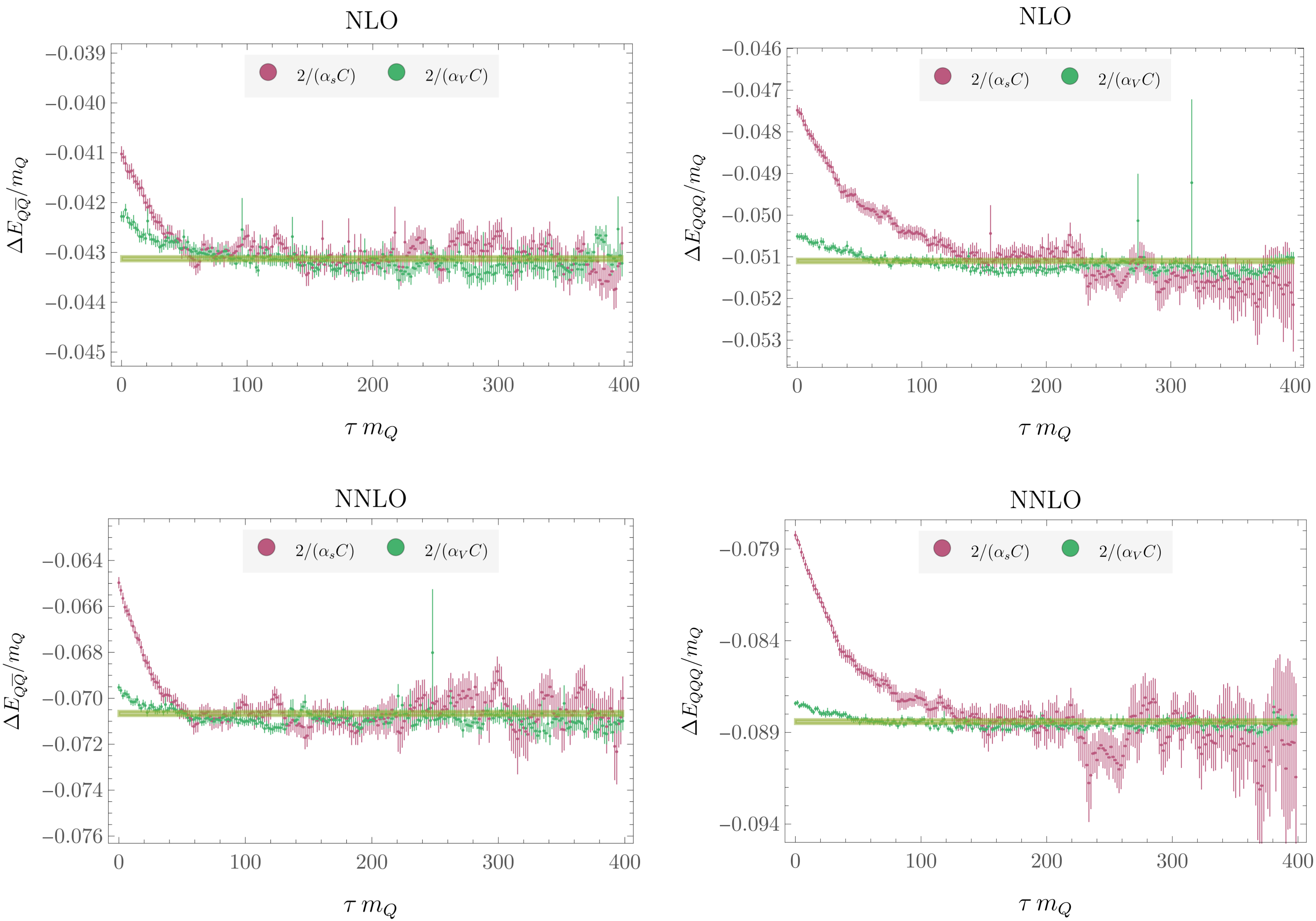
In practice: Take small time $\delta\tau = \tau/L$ for $L \gg 1$ and perform Trotter-Suzuki expansion

$$|0\rangle = \lim_{\tau\to\infty} e^{-H\tau}|\psi\rangle = \lim_{\tau\to\infty}\lim_{\delta\tau\to 0}\left(e^{-V\delta\tau/2}e^{-K\delta\tau}e^{-V\delta\tau/2}\right)^{\tau/\delta\tau}|\psi\rangle$$

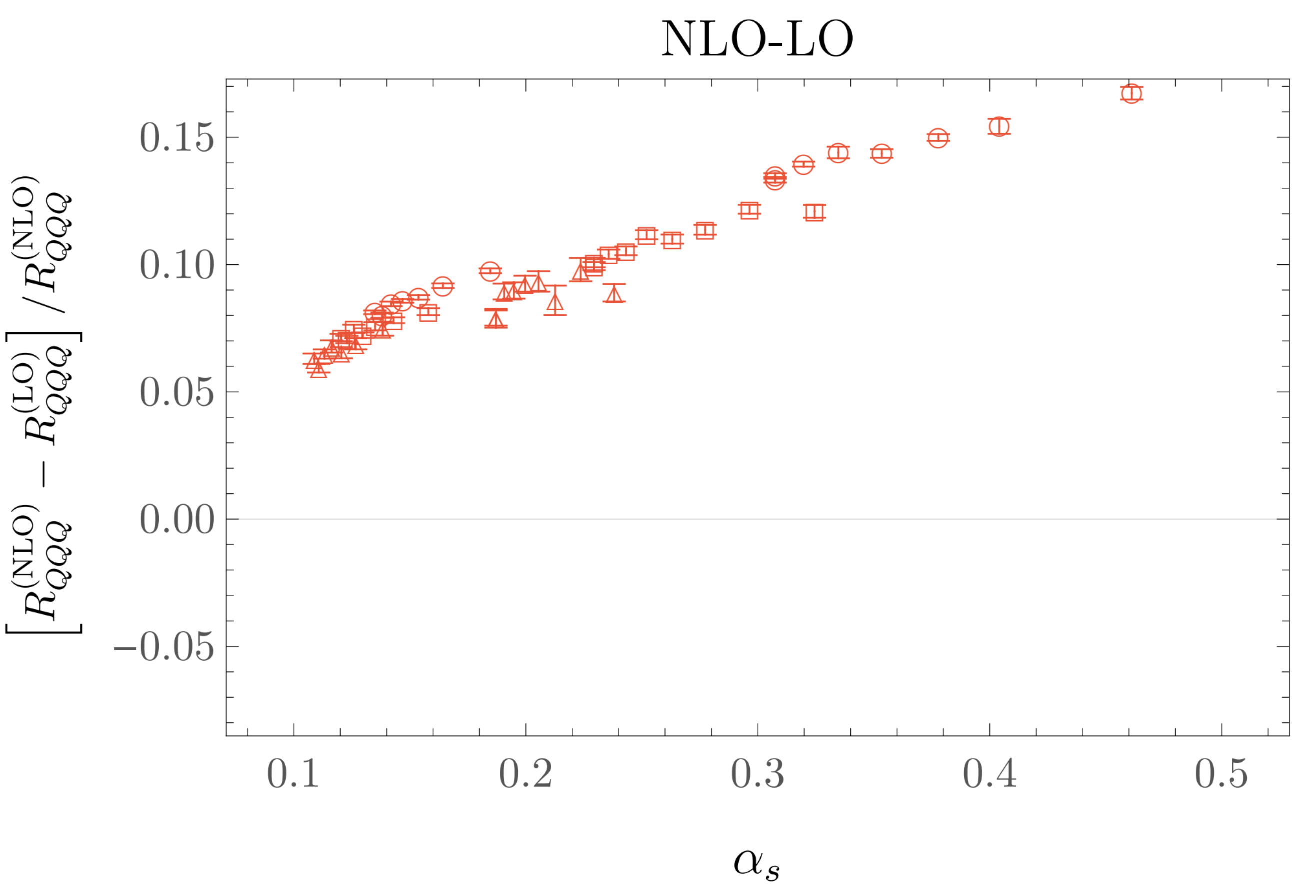
Review: Gandolfi et al. '20

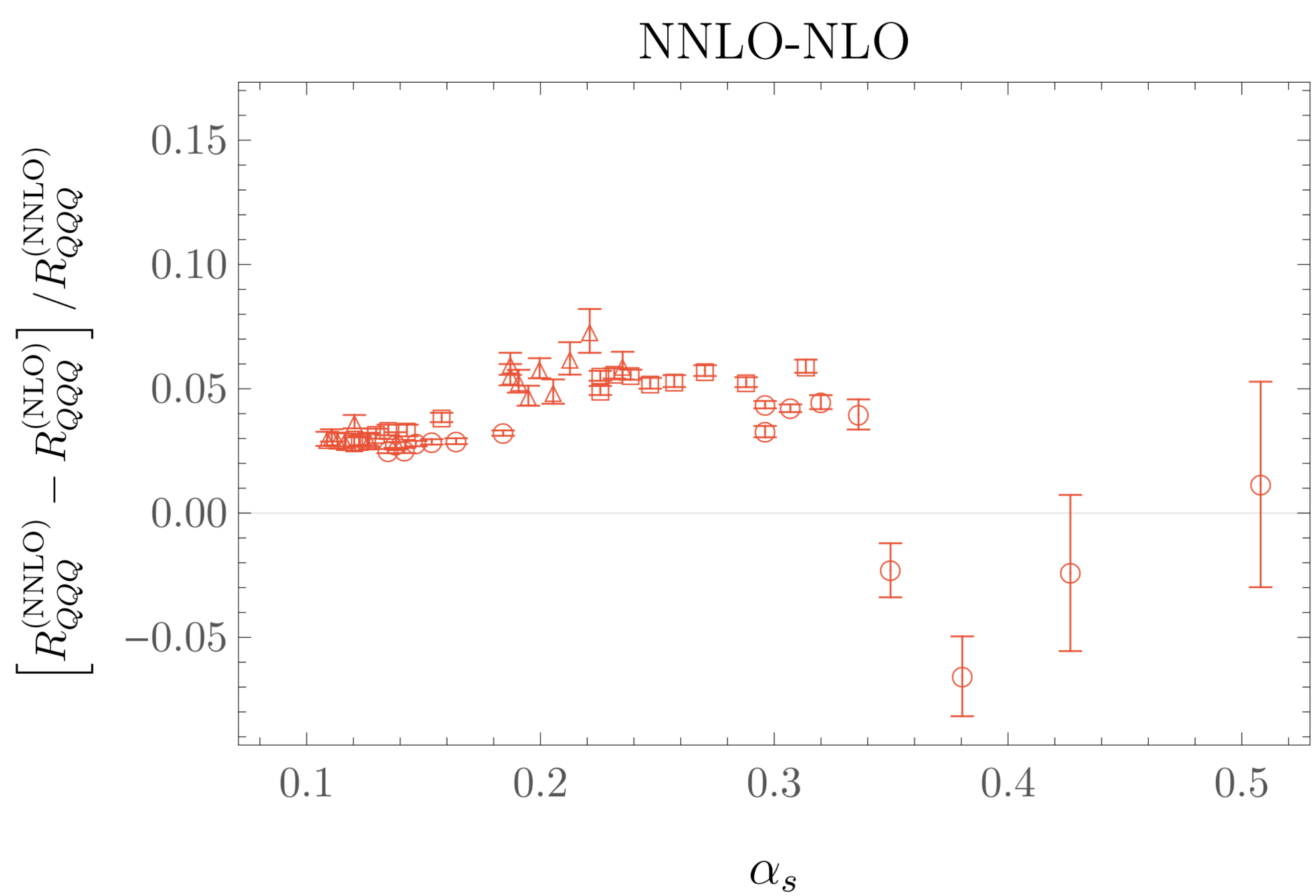
VMC vs. GFMC

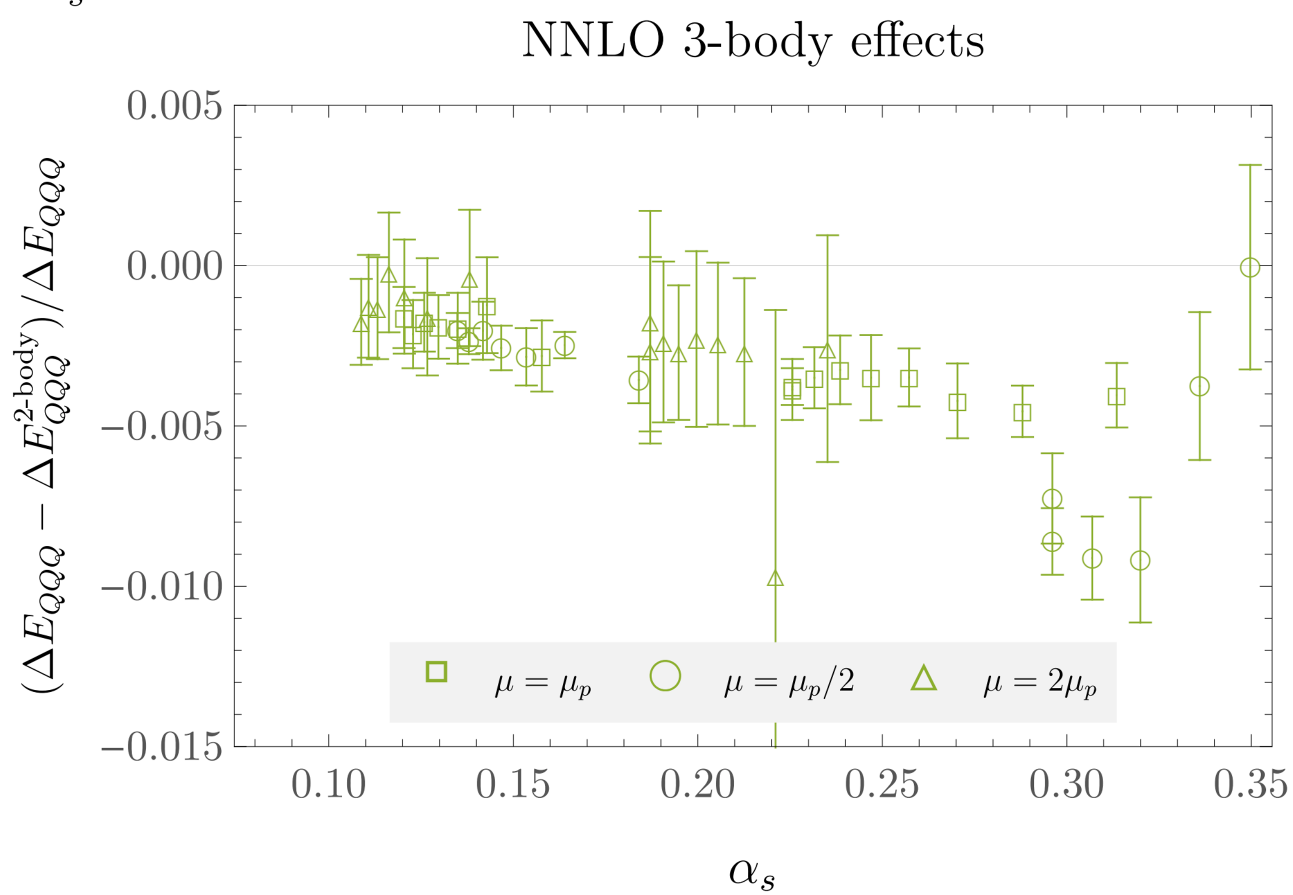
$\alpha_s = 0.2$



Order comparison







Baryons in pNRQCD

BA and Wagman, PRD 108

For $m_Q \gg \Lambda_{\rm QCD}$, potential can be computed as power series in $\alpha_s(\mu)$ where $\mu \propto m_Q$

Multiple color structure arise, e.g.

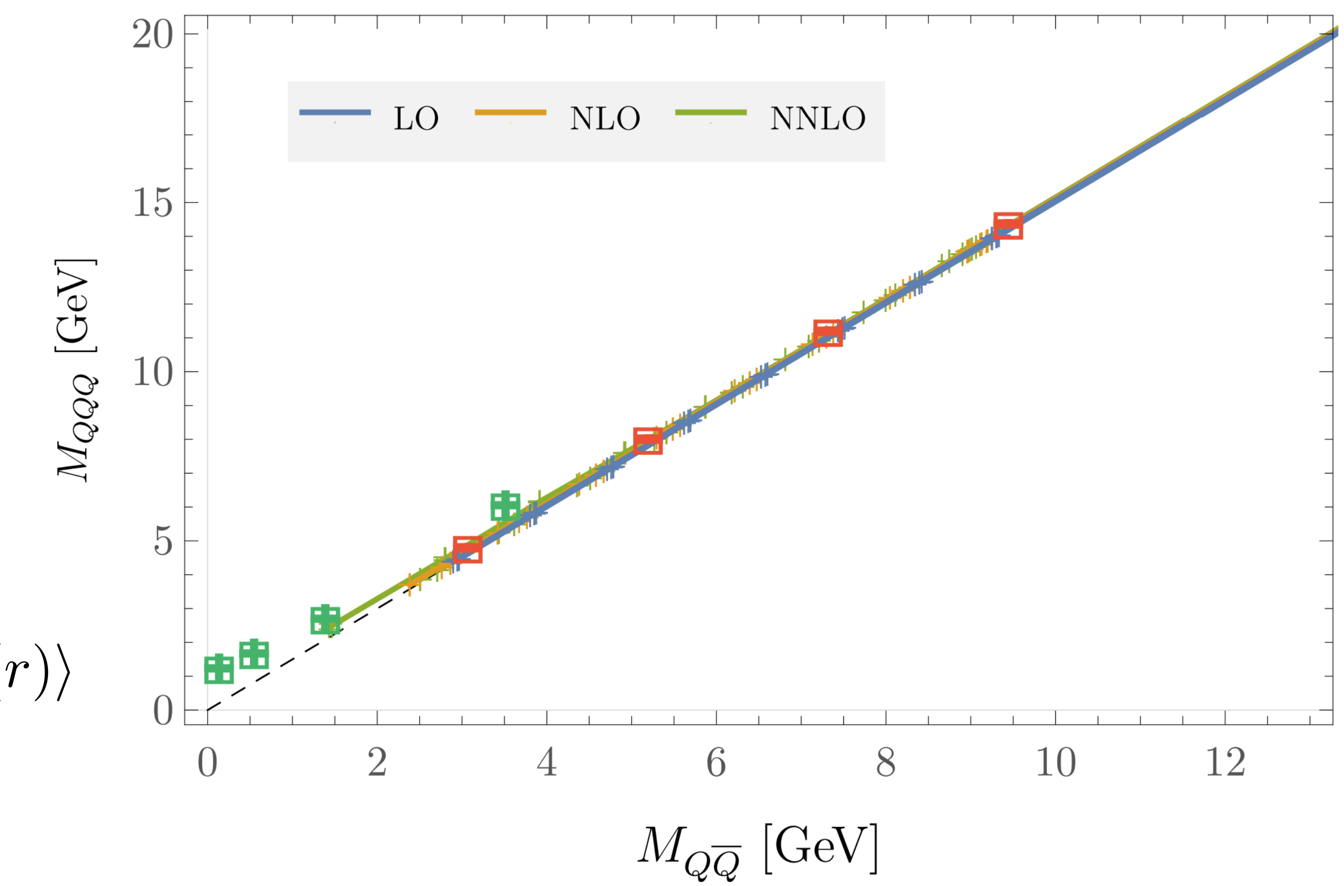
$$V\,|Q_i(r)Q_j(0)\rangle = \frac{1}{2}\left[(\delta_{il}\delta_{jk} + \delta_{jl}\delta_{ik})V_S(r) + \epsilon_{ijm}\epsilon_{klm}V_A(r)\right]|Q_k(r)Q_l(0) + Q_l(0)Q_k(r)\rangle$$

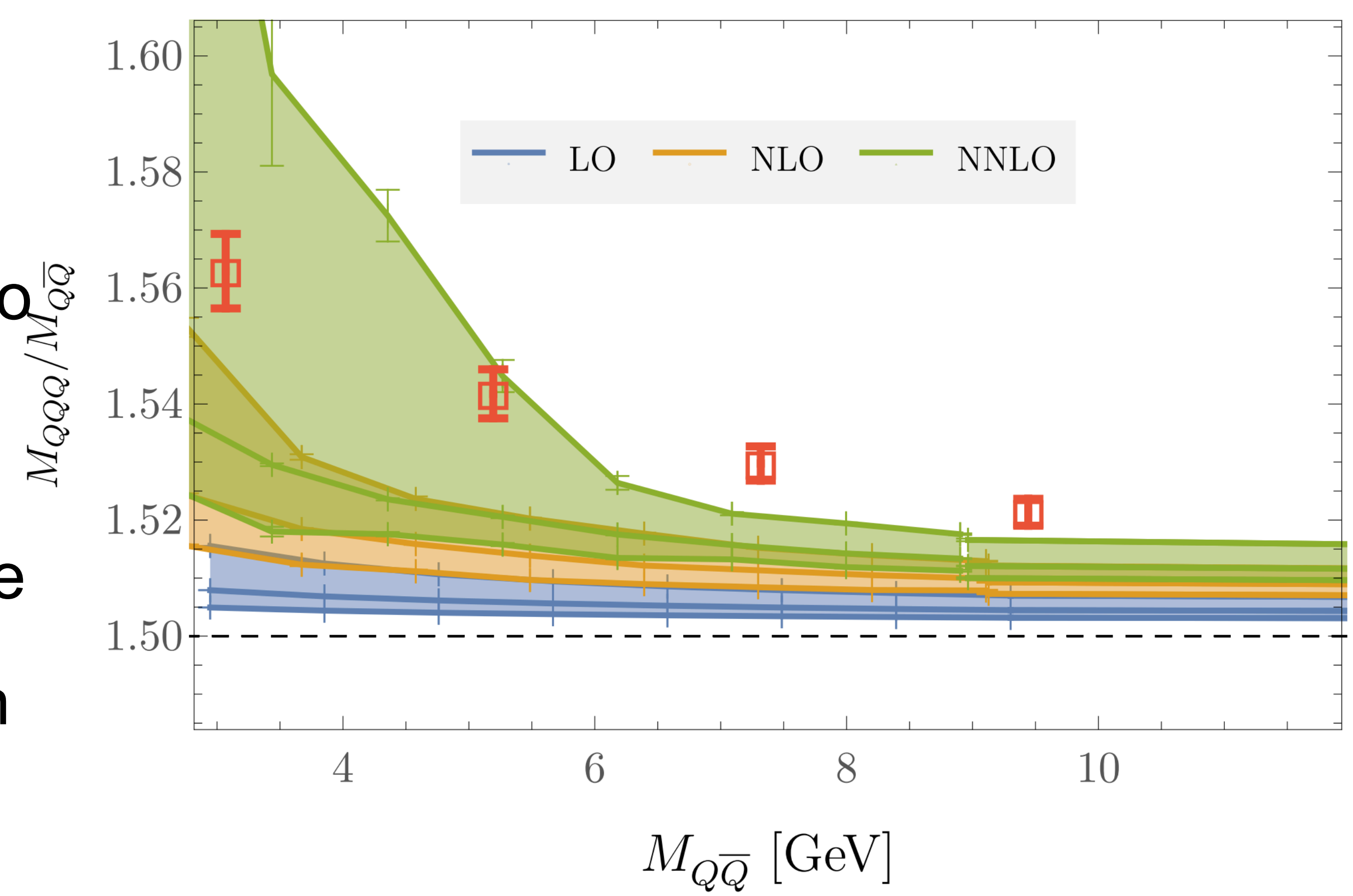
Triply-heavy baryon binding due to

$$V_A(r) = -\frac{2}{3}\left[\frac{\alpha_s(\mu)}{r} + O(\alpha_s(\mu)^2)\right]$$

Pole masses can be fixed by matching to observed quarkonium masses

Potential + masses completely determine pNRQCD Hamiltonian, enables predictions of e.g. triply-heavy baryon masses





QCD mass spectra

Renormalise m_Q with respect to corresponding spin-averaged $1S$ mass: $M_{Q\bar{Q}} = (3M_{S=1} + M_{S=0})/4$

$1S$ mesons	Order	$\alpha_s(\mu)$	m_Q	χ^2/dof	$M_{Q\bar{Q}}$	Measured $M_{Q\bar{Q}}$
(Υ, η_b)	LO (exact)	0.214850	4.77050	-	9.44295	9.44295(90)
(Υ, η_b)	NLO	0.227318	4.86886	1.09	9.44255(46)	9.44295(90)
(Υ, η_b)	NNLO	0.224750	4.96983	1.15	9.44386(40)	9.44295(90)
$(J/\psi, \eta_c)$	LO (exact)	0.282678	1.56206	-	3.06865	3.06865(10)
$(J/\psi, \eta_c)$	NLO	0.313613	1.65234	1.12	3.06870(33)	3.06865(10)
$(J/\psi, \eta_c)$	NNLO	0.297100	1.77092	0.59	3.06824(45)	3.06865(10)

Using $m_Q^{\text{ren}} \leftrightarrow \alpha_s$ to obtain BE $\Rightarrow$ triply-heavy baryon mass spectrum: $M_{Q_1Q_2Q_3} = \Delta E_{Q_1Q_2Q_3} + \sum_{i=1}^{3} m_{Q_i}$

Baryon	This work: M_{QQQ}	This work: χ^2/dof	Variational Methods	Lattice QCD
Ω_{ccc}	LO: 4.626723(8) NLO: 4.66561(42) NNLO: 4.69981(29)	LO: 1.0 NLO: 1.4 NNLO: 1.1	LO: 4.76(6) NNLO+mNLO: 4.97(20)	4.796(8)(18)
Ω_{ccb}	LO: 7.815284(14) NLO: 7.86032(34) NNLO: 7.91297(28)	LO: 1.3 NLO: 1.3 NNLO: 1.0	LO: 7.98(7) NNLO+mNLO: 8.20(15)	8.007(9)(20)
Ω_{cbb}	LO: 11.03611(1) NLO: 11.09206(28) NNLO: 11.110463(46)	LO: 1.5 NLO: 1.1 NNLO: 1.1	LO: 11.48(12) NNLO+mNLO: 11.34(26)	11.195(8)(20)
Ω_{bbb}	LO: 14.12990(3) NLO: 14.25397(4) NNLO: 14.25903(54)	LO: 1.2 NLO: 1.4 NNLO: 1.3	LO: 14.76(18) NNLO+mNLO: 14.57(25)	14.371(4)(12)

BA and Wagman, PRD 108 Jia '06, Llanes-Estrada '11 Meinel '10, Brown '14

Color scaling

Attained **precise** scaling relations for N_c mesons and baryons. E.g. for NLO barons:

$$\frac{\Delta E_{B_d}^{(\mathrm{NLO})}}{m_d \alpha_d^2 N_c^4} \approx -A_{B_d}^{(\mathrm{LO},0)} - \alpha_d A_{B_d}^{(\mathrm{NLO},1)} - \alpha_d^2 A_{B_d}^{(\mathrm{NLO},2)}$$

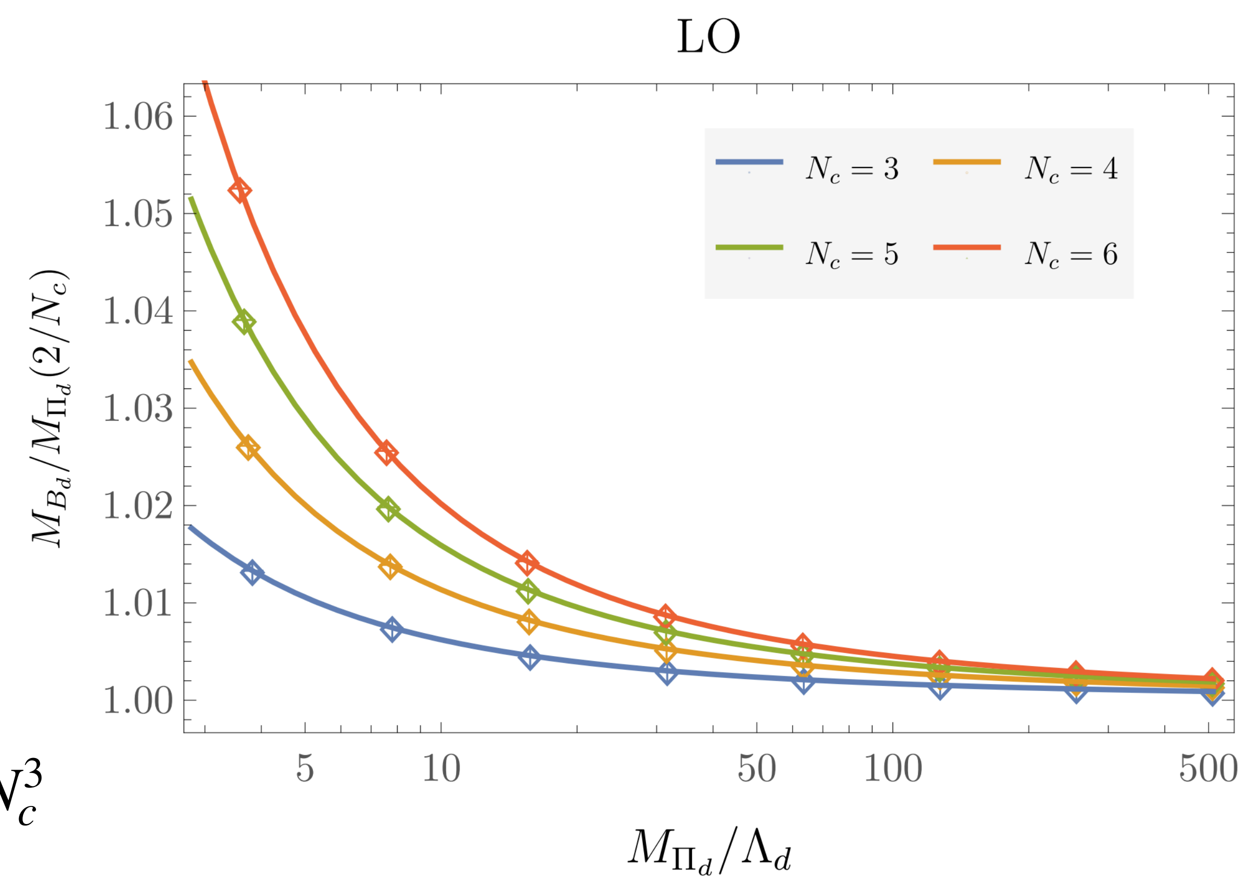
$$A_{B_d}^{(\mathrm{LO},0)} \approx \frac{0.013281(40)}{N_c} + \frac{0.02077(34)}{N_c^2} - \frac{0.0231(7)}{N_c^3}$$

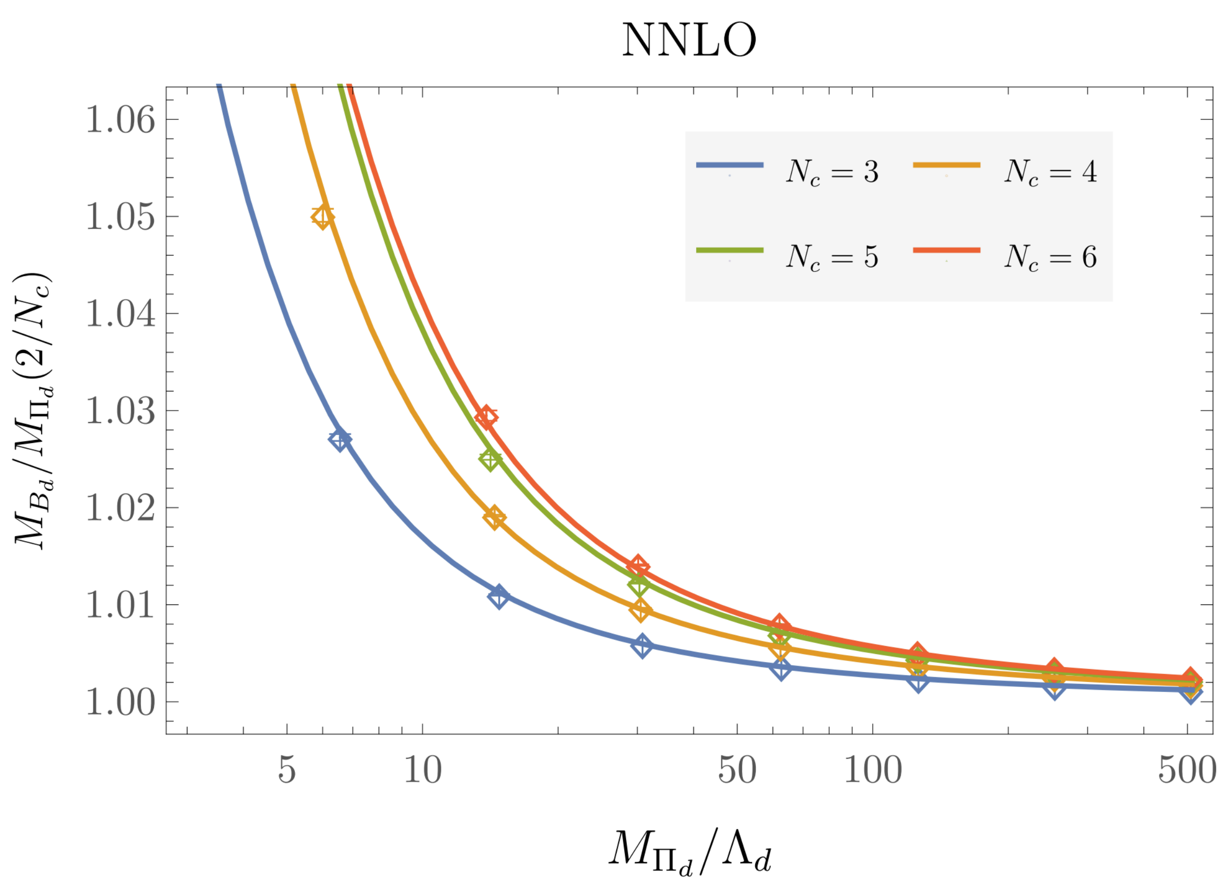
$$A_{B_d}^{(\mathrm{NLO},1)} \approx 0.0224(20) + \frac{0.179(18)}{N_c} - \frac{0.183(37)}{N_c^2}$$

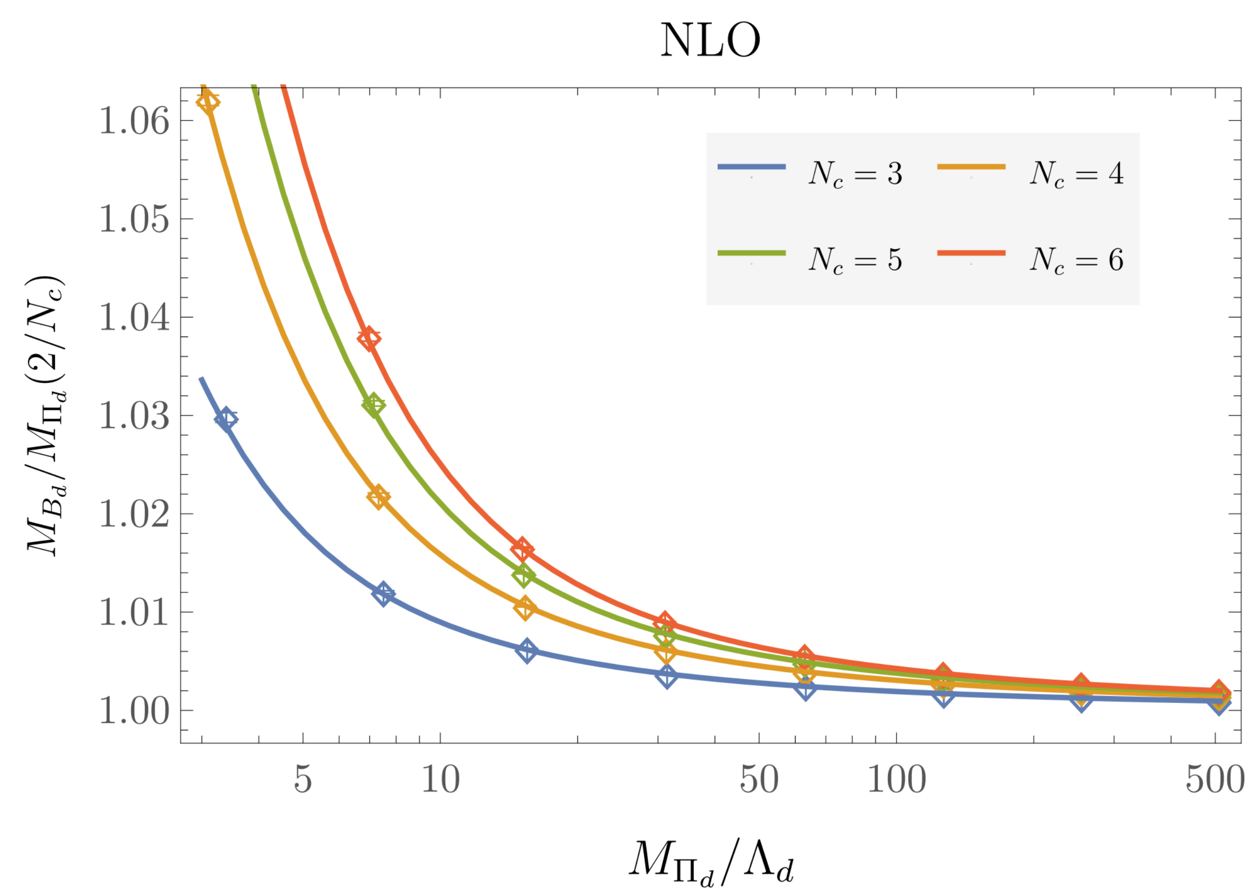
$$A_{B_d}^{(\mathrm{NLO},2)} \approx 0.0459(9)$$

Consistent with large N_c arguments $\Delta E_{B_d}/m_d \sim \alpha_d^2 N_c^3$

Witten '79





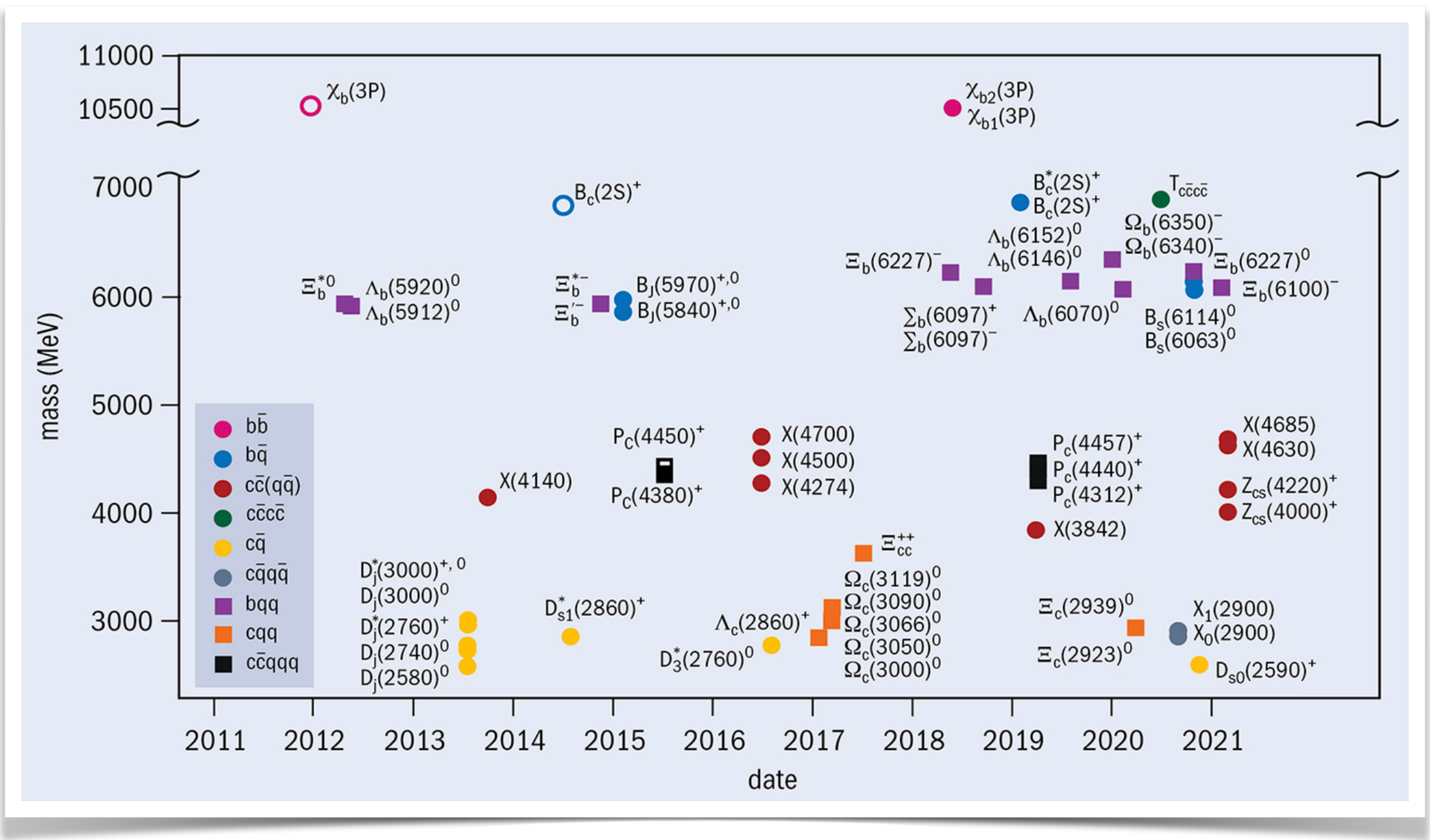


Tetraquarks

New exotics being observed — many are tetraquark candidates and first all-heavy candidate X(6900) observed…?

Heavy-light tetraquarks hard to study ⇒ instead clean all-heavy system with our formalism

Aim: Tackle all-heavy constituent exotics in QCD and beyond starting with tetraquarks



[CERN LHCb ’22]

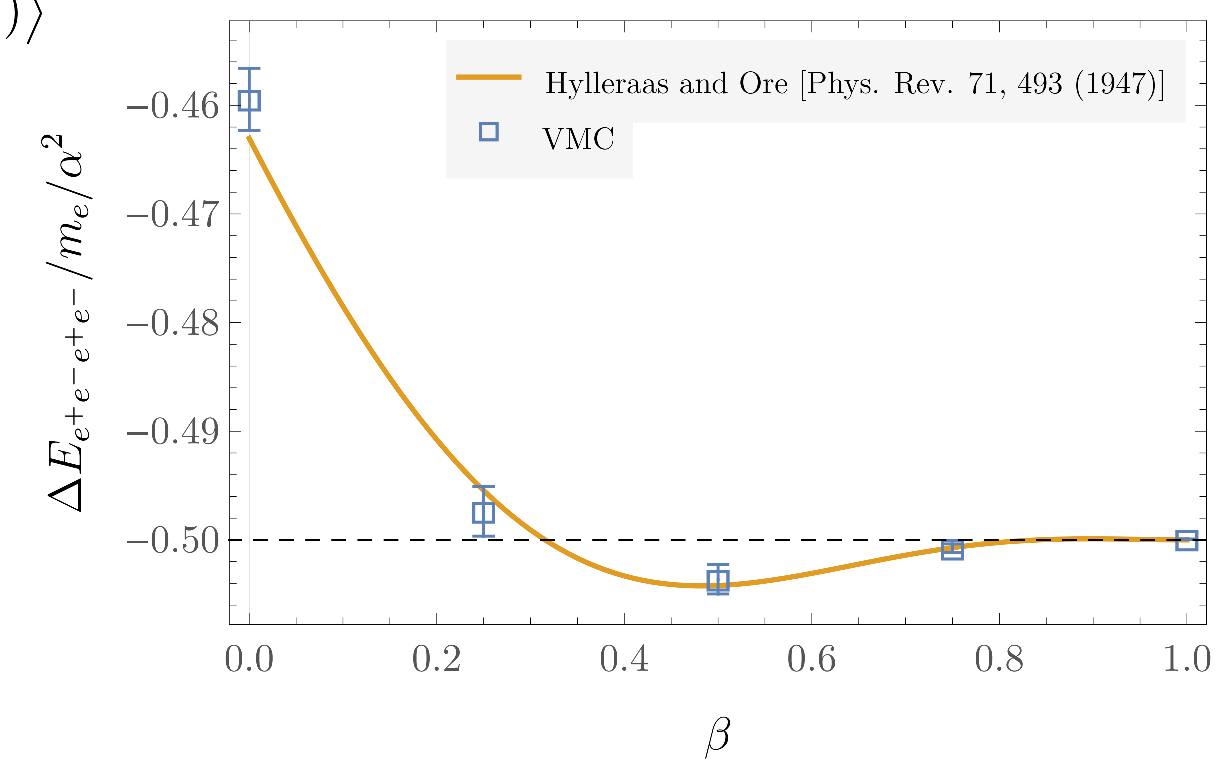
Positronium molecules in pNRQED

Same methods can be used to study "molecular" states in pNRQED such as Ps_2 the positronium molecule

$$\left\langle e^+(r_1)e^-(r_2)e^+(r_3)e^-(r_4)\right| V \left|e^+(r_1)e^-(r_2)e^+(r_3)e^-(r_4)\right\rangle$$

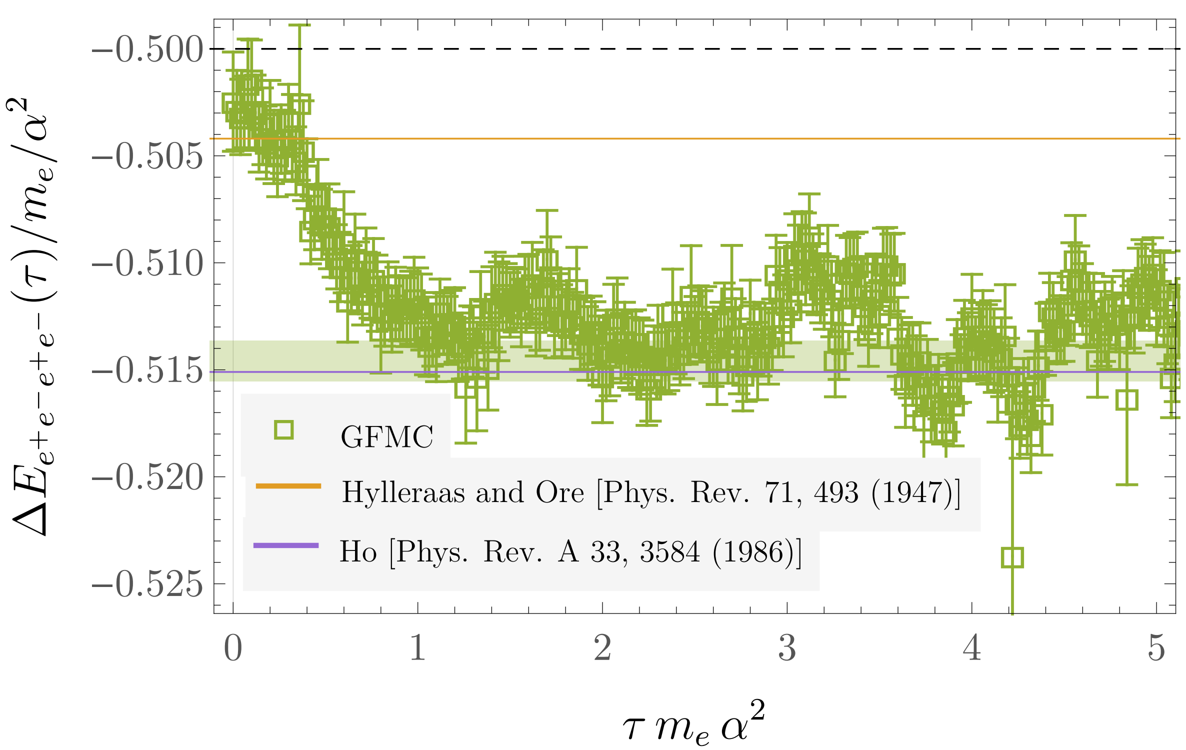
$$= \alpha\left(\frac{1}{r_{13}} + \frac{1}{r_{24}} - \frac{1}{r_{12}} - \frac{1}{r_{34}} - \frac{1}{r_{14}} - \frac{1}{r_{23}}\right)$$

BA and Wagman, *in preparation*



Hylleraas variational wavefunction demonstrates positronium molecule binds known from 1947

$$\psi \propto e^{-(r_{12}+r_{23}+r_{14}+r_{34})/2}$$
$$\times\cosh\left[\beta(r_{12} - r_{23} - r_{14} + r_{23})/2\right]$$



Applying GFMC to Hylleras wavefunctions reproduces state-of-the-art variational results for positronium molecule binding:

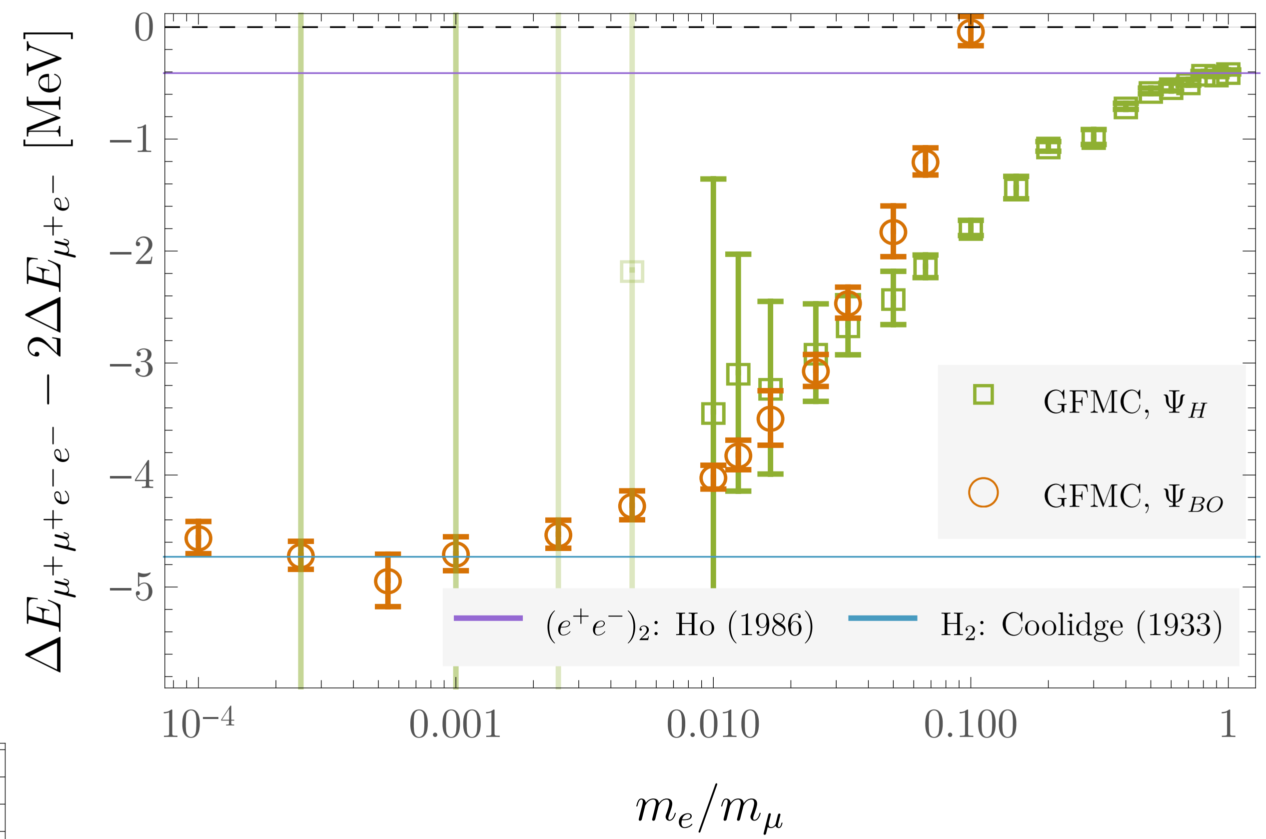
$$\Delta E_{\mu^+\mu^+e^-e^-} - 2\Delta E_{\mu^+e^-} = -0.40(3)\text{ eV}$$

“I like your paper on molecules”

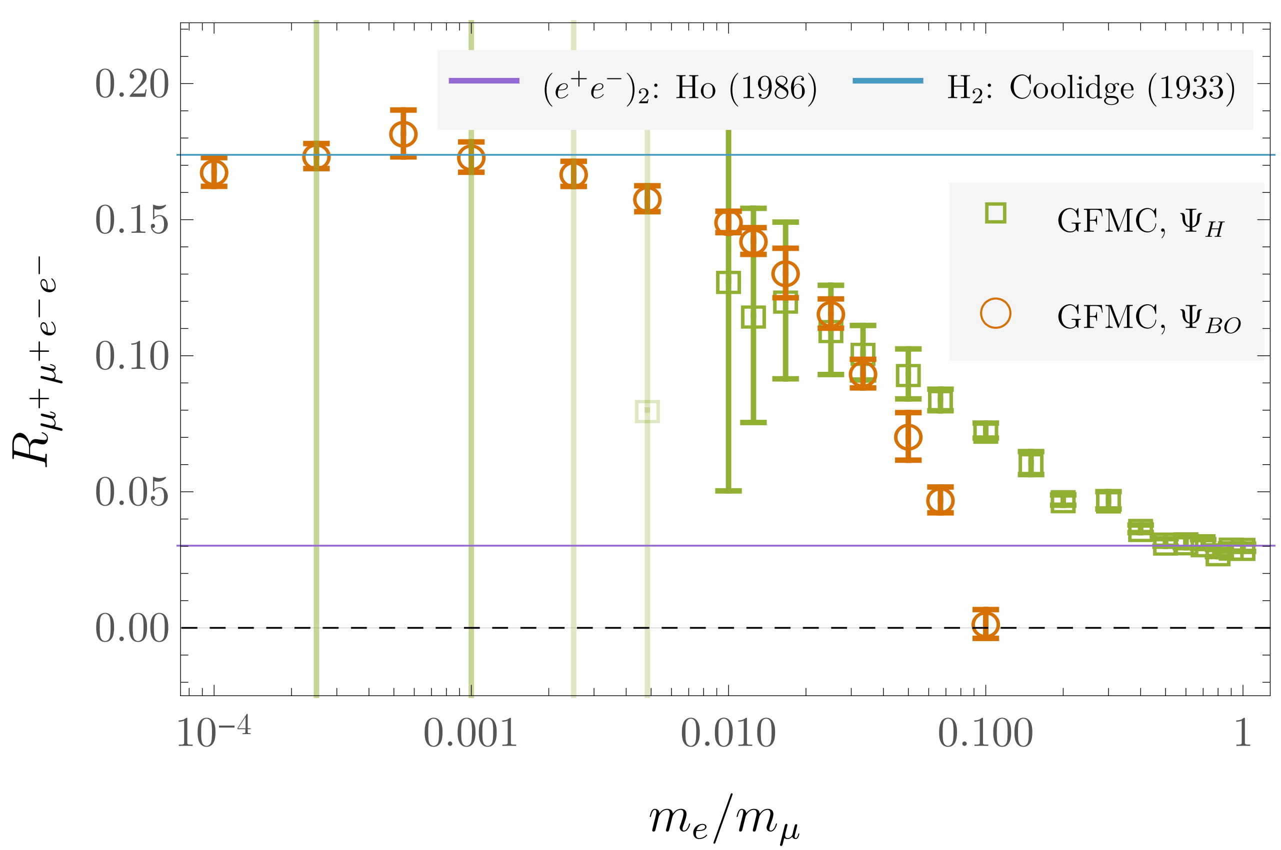
Same methods can be used to study the case of large unequal mass differences in order to understand transition to BO approximation in H_2

BO wavefunction is Hylleraas-like with fixed separation between heavy partners by delta-function

BA and Wagman, *in preparation*



Coolidge (1933)



BO trial state trial state are fixed distance $R = 1.4/(\alpha m_e)$ and applying GFMC to BO allows R to vary

$$\Delta E_{ppe^-e^-} - 2\Delta E_{pe^-} \sim -4.7 \text{ eV}$$

Meson interactions in pNRQCD

For multi-hadron states, interplay between different color tensors leads to complex dependence of potential on color states:

$$\left|\mathbf{1}\otimes\mathbf{1}\right\rangle \propto \left|Q_i\overline{Q}_j Q_k \overline{Q}_l\right\rangle \delta_{ij}\delta_{kl}$$

$$\left|\mathbf{3}\otimes\overline{\mathbf{3}}\right\rangle \propto \left|Q_i\overline{Q}_j Q_k \overline{Q}_l\right\rangle \epsilon_{ikm}\epsilon_{jlm}$$

$$\left|\mathbf{6}\otimes\overline{\mathbf{6}}\right\rangle \propto \left|Q_i\overline{Q}_j Q_k \overline{Q}_l\right\rangle (\delta_{ij}\delta_{kl}+\delta_{il}\delta_{kj})$$

Matrix elements of e.g. LO potential:

$$\left\langle\mathbf{6}\otimes\overline{\mathbf{6}}\right| V \left|\mathbf{6}\otimes\overline{\mathbf{6}}\right\rangle = \frac{\alpha_s}{3}\left(\frac{1}{r_{13}}+\frac{1}{r_{24}}\right) - \frac{5\alpha_s}{6}\left(\frac{1}{r_{12}}+\frac{1}{r_{34}}+\frac{1}{r_{14}}+\frac{1}{r_{23}}\right)$$

$$\left\langle\mathbf{3}\otimes\overline{\mathbf{3}}\right| V \left|\mathbf{3}\otimes\overline{\mathbf{3}}\right\rangle = -\frac{2\alpha_s}{3}\left(\frac{1}{r_{13}}+\frac{1}{r_{24}}\right) - \frac{\alpha_s}{3}\left(\frac{1}{r_{12}}+\frac{1}{r_{34}}+\frac{1}{r_{14}}+\frac{1}{r_{23}}\right)$$

$$\left\langle\mathbf{1}\otimes\mathbf{1}\right| V \left|\mathbf{1}\otimes\mathbf{1}\right\rangle = -\frac{4\alpha_s}{3}\left(\frac{1}{r_{12}}+\frac{1}{r_{34}}\right)$$

Product of two color-singlet mesons has 0 meson-meson potential at LO and NLO…

BA and Wagman, *in preparation*

Equal-mass tetraquarks

BA and Wagman, *in preparation*

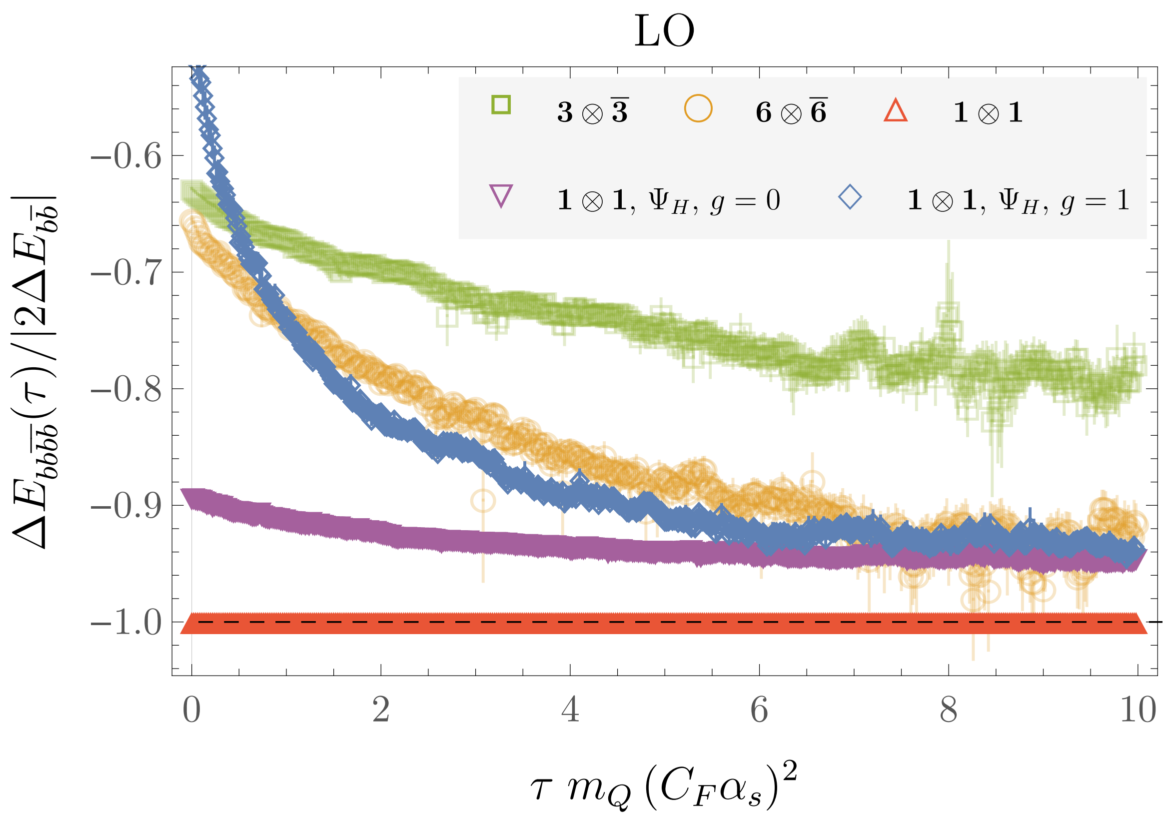
Applying GFMC to pNRQCD for equal-mass heavy quarks leads to energies above the 2-meson threshold

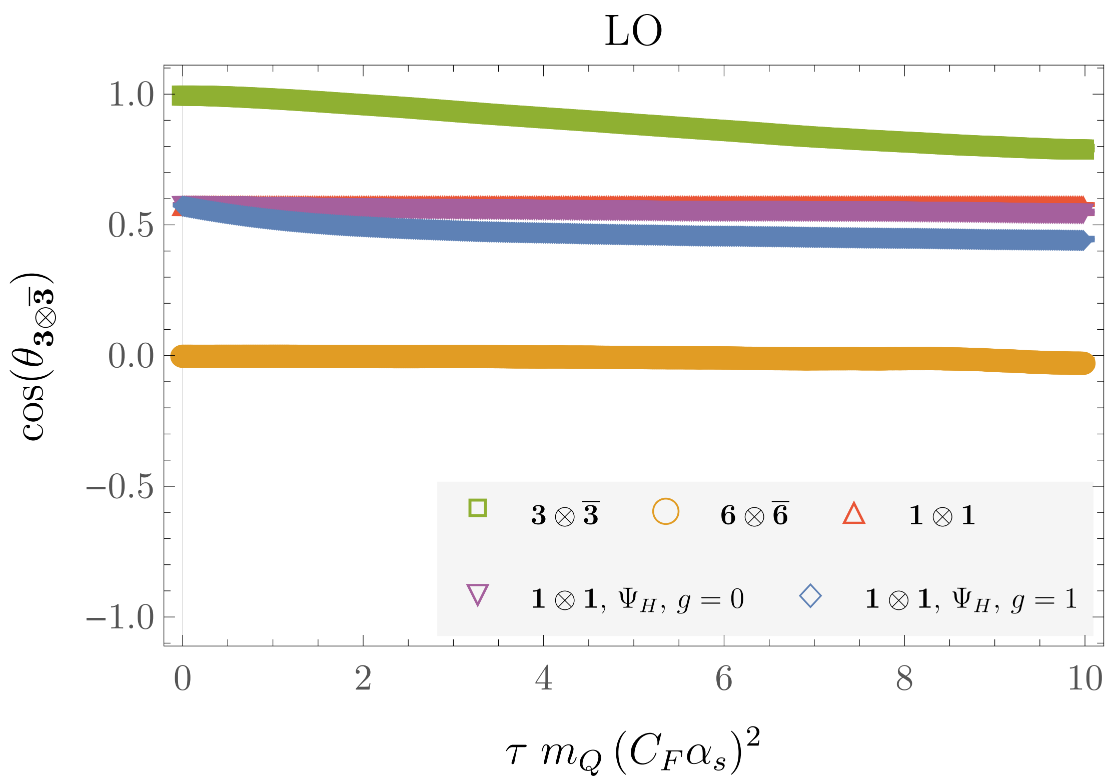
Consistent with Czarnecki, Long, and Voloshin, Phys. Lett. B 778 (2018); Eichten, Hughes PRD 97 (2018)

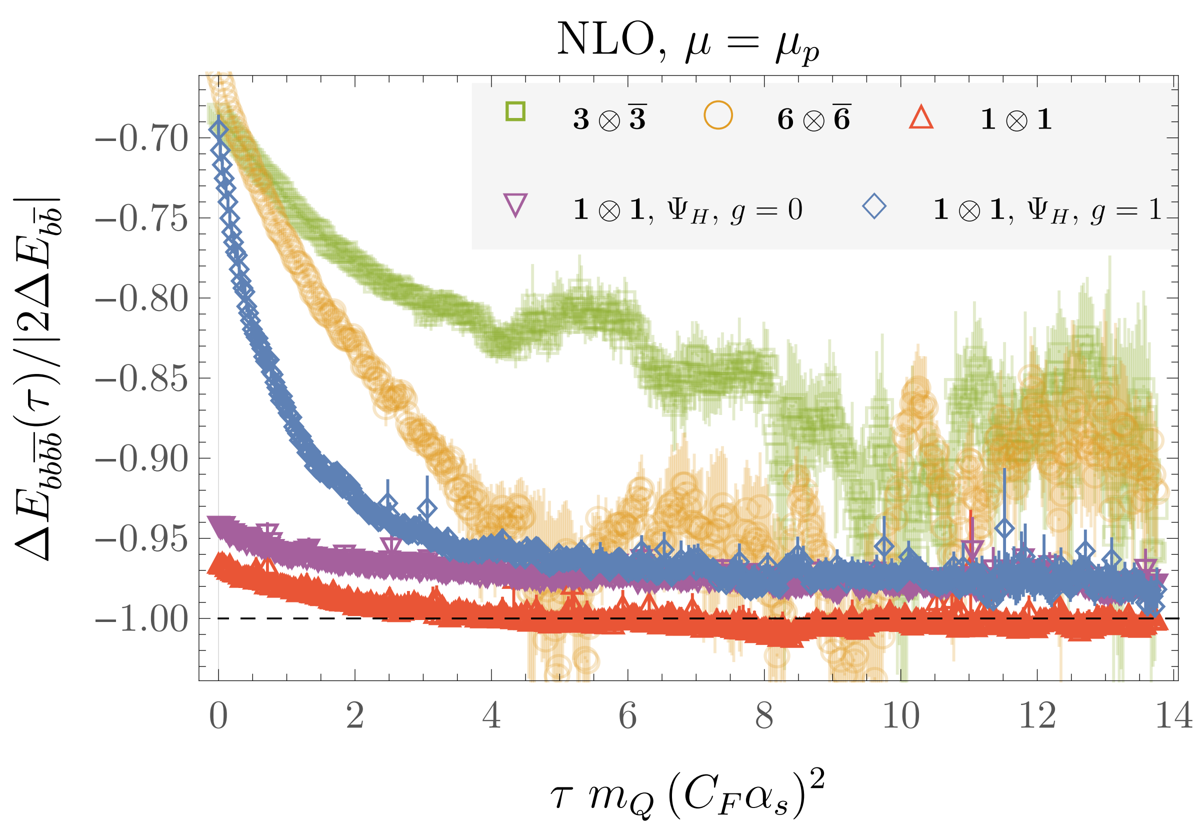
Inconsistent with Anwar et al, Eur. Phys. J. C. 78 (2018)

We investigate optimal spatial wavefunction for each color state

Hylleras wavefunctions with 1×1 color trial trial states lead to the strongest bounds







Unequal-mass tetraquarks

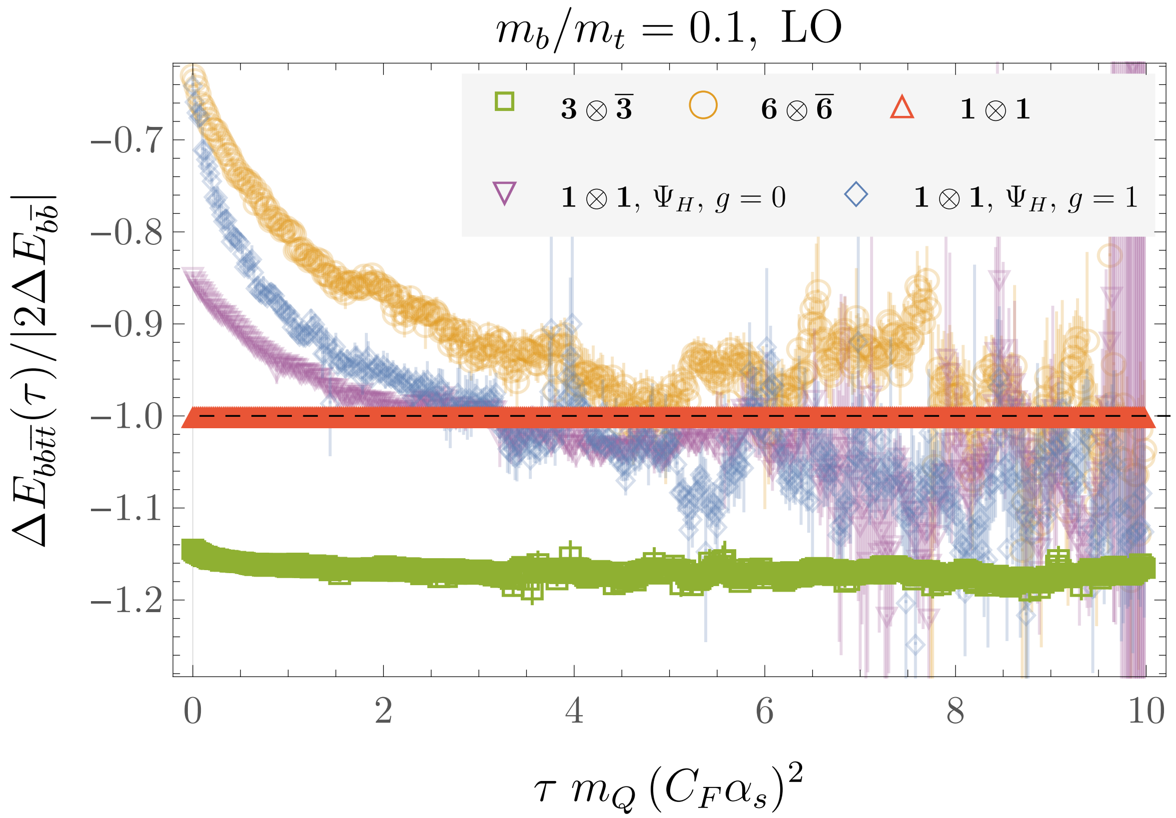
BA and Wagman, *in preparation*

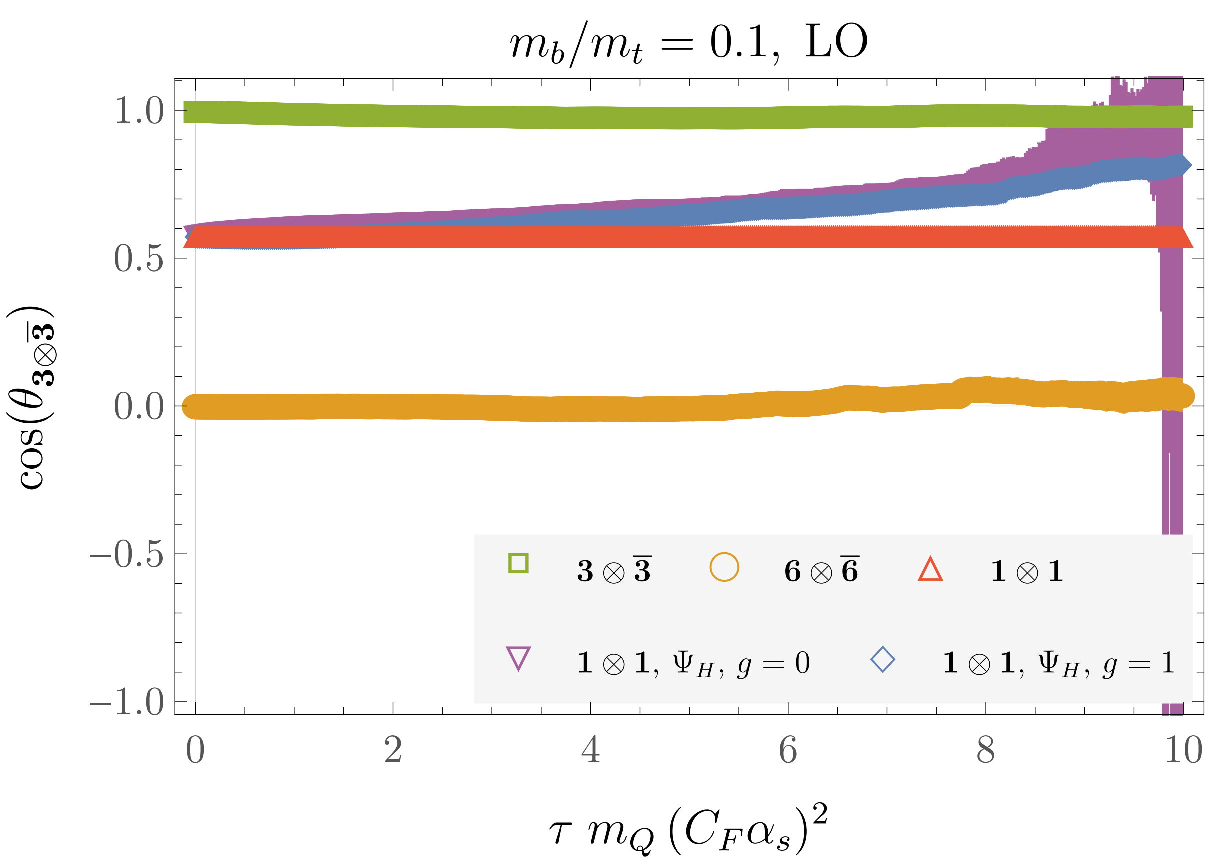
Sufficiently unequal heavy tetraquarks display clear signs of binding

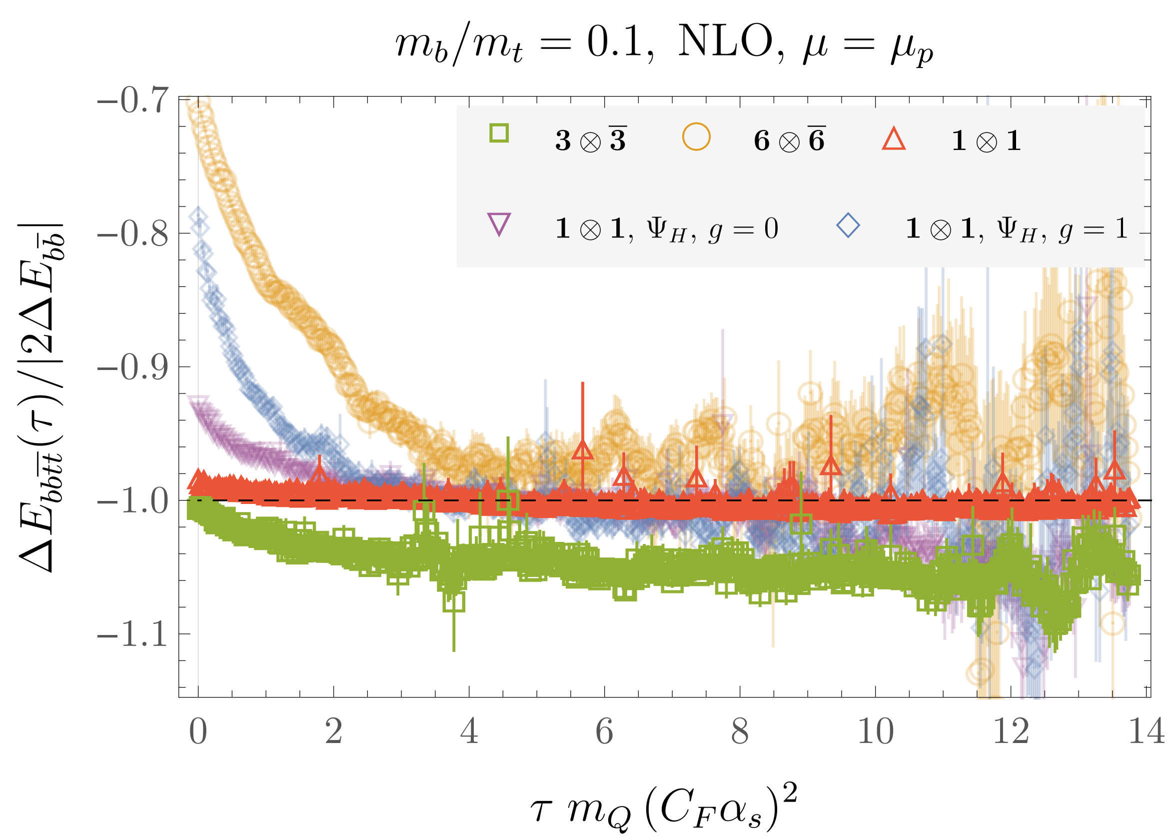
Consistent with Czarnecki, Long, and Voloshin, Phys. Lett. B 778 (2018)

Inconsistent with Anwar et al, Eur. Phys. J. C. 78 (2018)

NNLO requires 3- and 4-body potentials not yet derived





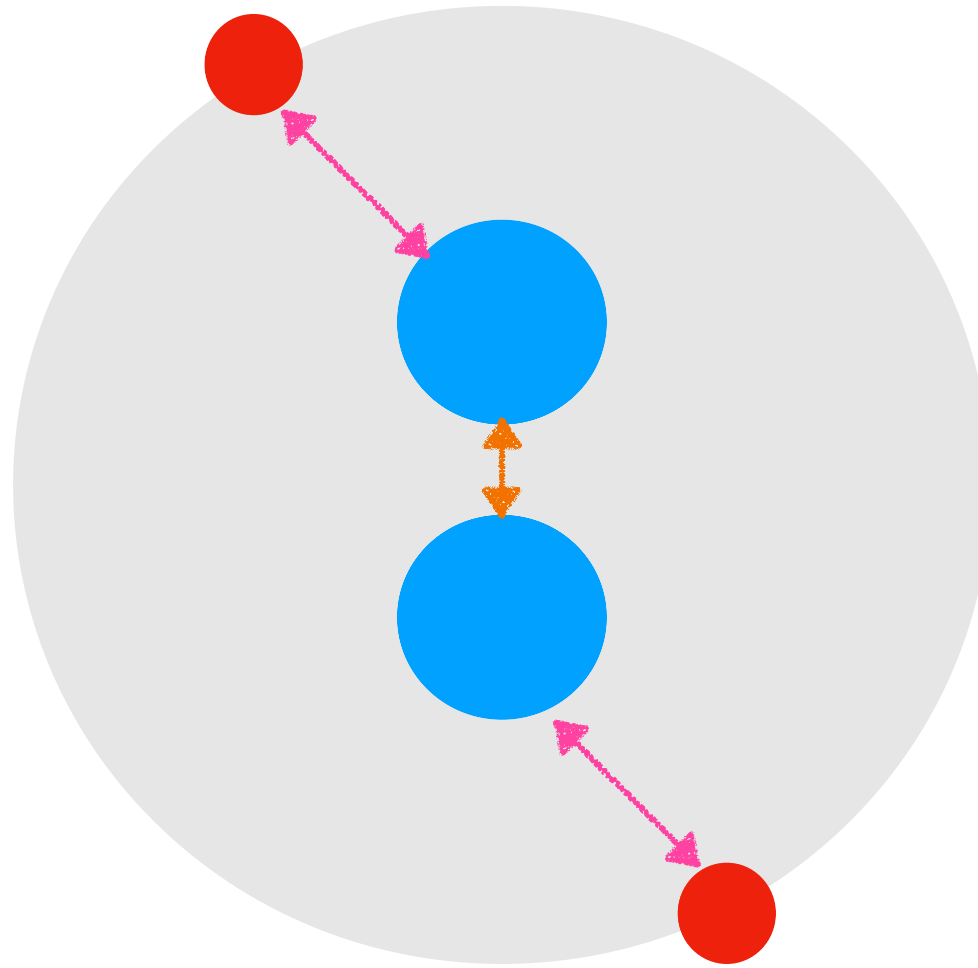


Unequal-mass tetraquark structure

Quarkonium wavefunction known exactly at LO

$$\Psi_T^{Q_i\overline{Q}_j}(\boldsymbol{R}; a) = \frac{\delta_{ij}}{\sqrt{N_c}} \frac{1}{\sqrt{\pi} a^{3/2}} e^{-|\boldsymbol{r}_{12}|/a_B}$$

Tetraquark wavefunction **not known**, we investigated large set of spatial wavefunctions for each color state



Optimal state: color antisymmetric Hyleraas-like wavefunction

$$\Psi_T^{Q_i\overline{Q}_jQ_k\overline{Q}_l}(\boldsymbol{R}; a, b) \propto \epsilon_{ijk}\epsilon_{klm} e^{-|\boldsymbol{r}_{13}|/(2a_{t\bar{t}})} e^{-|\boldsymbol{r}_{24}|/(2a_{b\bar{b}})} e^{-(|\boldsymbol{r}_{12}|+|\boldsymbol{r}_{14}|+|\boldsymbol{r}_{23}|+|\boldsymbol{r}_{34}|)/(4a_{t\bar{b}})}$$

Heavy **diquark core** with lighter diquarks twice mass ratio apart

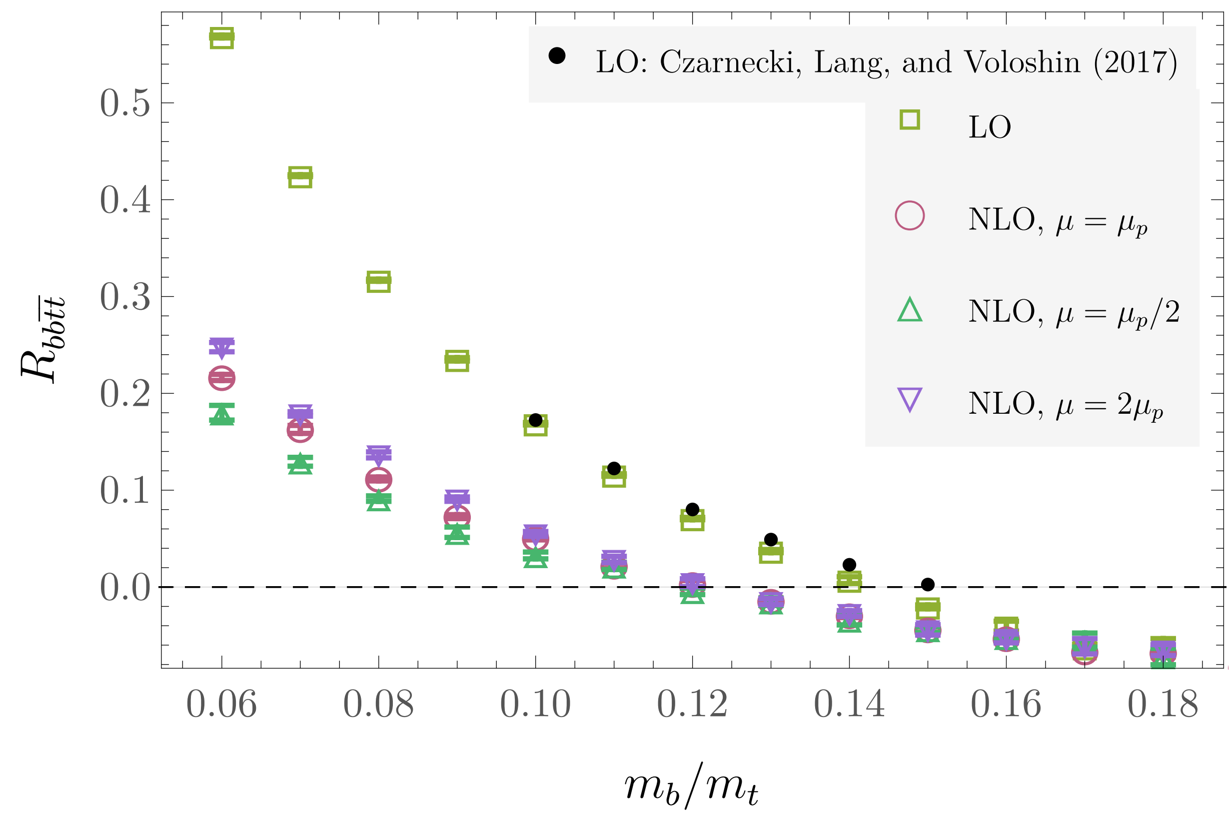
Unequal-mass tetraquarks

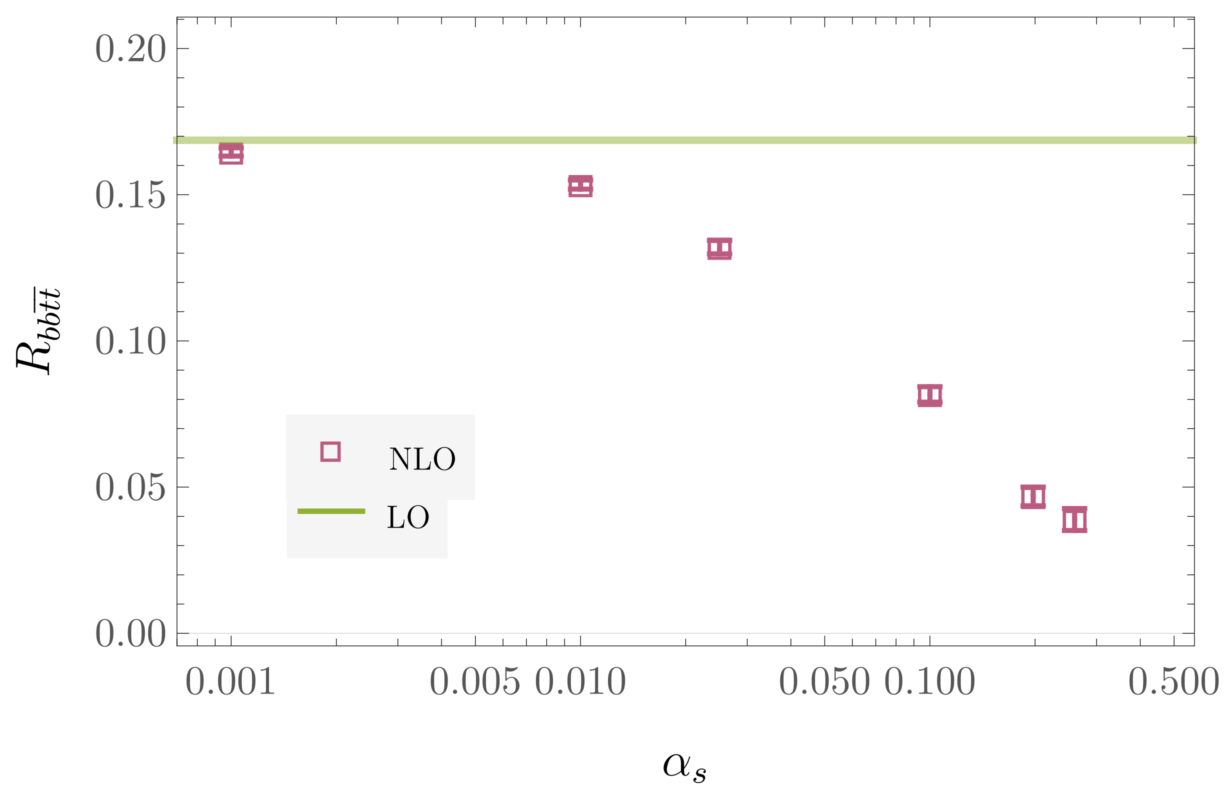
BA and Wagman, *in preparation*

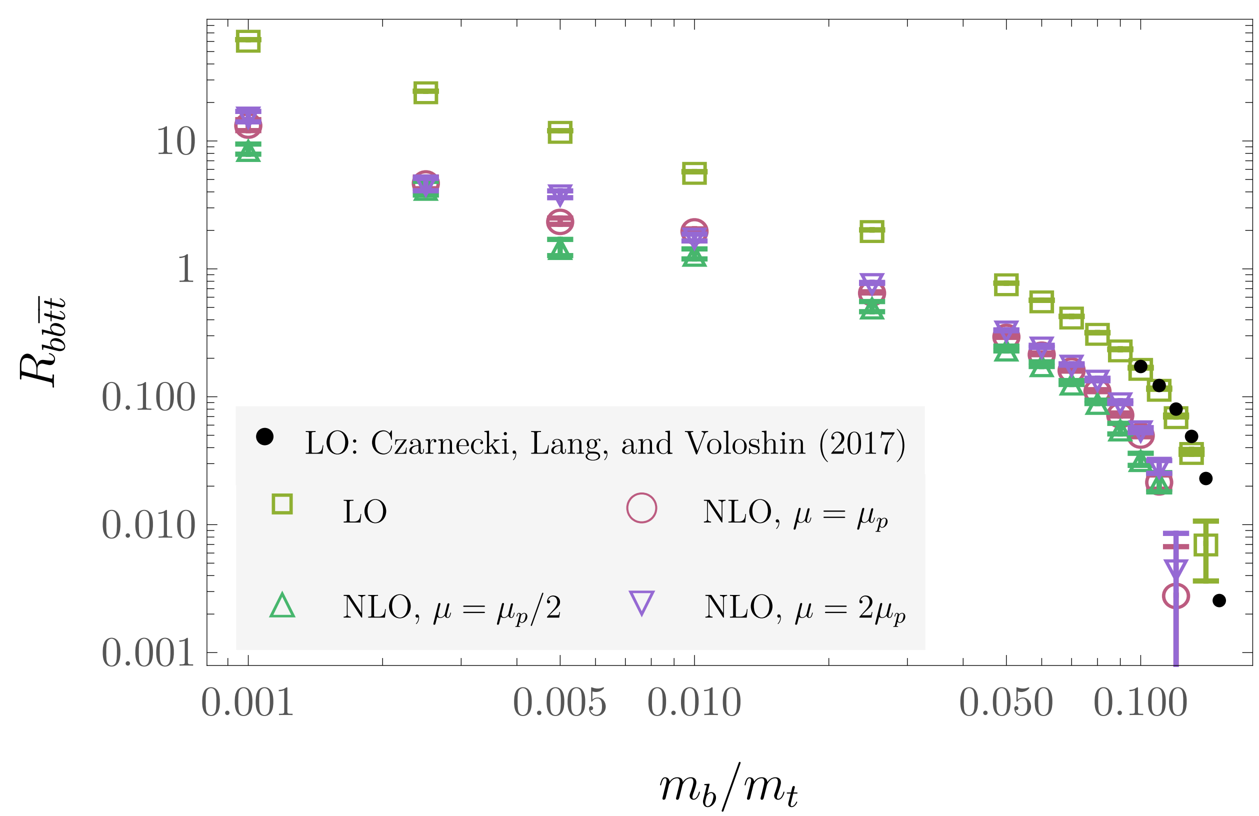
Sufficiently unequal heavy tetraquarks display clear signs of binding

Consistent with Czarnecki, Long, and Voloshin, Phys. Lett. B 778 (2018)

All-order convergence at infinite mass limit independent of unequal mass ratio







Summary

Developed and validated methods for using pNRQCD+QMC to study multi-hadron systems

In contrast to QED NR equal-mass tetraquarks show no signs of binding to NLO

Unequal mass exhibit clear signs of binding for $m_q/m_Q \sim 0.15$

Outlook

Study heavy dibaryons with these methods and compare with lattice prediction Mathur, Padmananth, Chakrabory PRL 130 (2023)

Investigate pheno relevant exotics, X(6900) as possible P-wave resonance BA, Brambilla, Mohapatra, Vairo, Wagman

Improve convergence by resummation, RS and SD pieces for mass-splittings BA, Kronfeld, Wagman and Vaiva, *in preparation*

BACK-UP Slides

Nuclear interactions in pNRQCD

Two-baryon systems comprised of heavy quarks can be studied similarly

$$|Q_i(r_1)Q_j(r_2)Q_k(r_3)Q_l(r_4)Q_m(r_5)Q_n(r_6)\rangle$$

Product of two color-singlet baryons has 0 inter-baryon potential at LO and NLO

$$\langle \mathbf{1}\otimes\mathbf{1}|\,V\,|\mathbf{1}\otimes\mathbf{1}\rangle = -\frac{2\alpha_s}{3}\left(\frac{1}{r_{12}}+\frac{1}{r_{23}}+\frac{1}{r_{23}}+\frac{1}{r_{45}}+\frac{1}{r_{46}}+\frac{1}{r_{56}}\right)$$

Other color structures lead to more interesting potentials:

Complete basis:

$$\mathbf{1}\subset\bar{\mathbf{3}}\otimes\bar{\mathbf{3}}\otimes\bar{\mathbf{3}},$$
$$\mathbf{1}\subset\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6},$$
$$\mathbf{1}\subset\bar{\mathbf{3}}\otimes\mathbf{6}\otimes\bar{\mathbf{3}},$$
$$\mathbf{1}\subset\bar{\mathbf{3}}\otimes\bar{\mathbf{3}}\otimes\mathbf{6},$$
$$\mathbf{1}\subset\mathbf{6}\otimes\bar{\mathbf{3}}\otimes\bar{\mathbf{3}}$$

Rao and Shrock, Phys. Lett. B 116 (1982)

$$(Q_iQ_j)(Q_kQ_l)(Q_mQ_n)T^{AAA}_{ijklmn}$$

$$\langle \bar{\mathbf{3}}\otimes\bar{\mathbf{3}}\otimes\bar{\mathbf{3}}|\,V\,|\bar{\mathbf{3}}\otimes\bar{\mathbf{3}}\otimes\bar{\mathbf{3}}\rangle = -\frac{2\alpha_s}{3}\left(\frac{1}{r_{12}}+\frac{1}{r_{34}}+\frac{1}{r_{56}}\right)-\frac{\alpha_s}{6}\left(\frac{1}{r_{13}}+\frac{1}{r_{14}}+\frac{1}{r_{15}}+\frac{1}{r_{16}}+\dots\right)$$

$$(Q_iQ_j)(Q_kQ_l)(Q_mQ_n)T^{SSS}_{ijklmn}$$

$$\langle \bar{\mathbf{6}}\otimes\bar{\mathbf{6}}\otimes\bar{\mathbf{6}}|\,V\,|\bar{\mathbf{6}}\otimes\bar{\mathbf{6}}\otimes\bar{\mathbf{6}}\rangle = \frac{\alpha_s}{3}\left(\frac{1}{r_{12}}+\frac{1}{r_{34}}+\frac{1}{r_{56}}\right)-\frac{5\alpha_s}{12}\left(\frac{1}{r_{13}}+\frac{1}{r_{14}}+\frac{1}{r_{15}}+\frac{1}{r_{16}}+\dots\right)$$

Could these color states describe bound dibaryons?

Stay tuned!

Harvey, Nucl. Phys. A 352 (1981)

BA and Wagman, *in preparation*

A simple trial wavefunction

Complete basis for Ψ_T and optimise for $\boldsymbol{\alpha} \Rightarrow$ require **functional form** of basis

Recall **Hydrogen**

$$H = -\frac{1}{2}\nabla_{\boldsymbol{r}} - \frac{b_0}{|\boldsymbol{r}|} \Rightarrow E_0 = -\frac{m}{2}b_0^2, \;\; \psi_{100}(r) = \sqrt{\frac{b_0^3}{\pi}}e^{-b_0 r}$$

Positronium and **quarkonium** at LO

$$H = -\frac{1}{2}(\nabla_{\boldsymbol{r}_1} + \nabla_{\boldsymbol{r}_2}) - \frac{b_0}{|\boldsymbol{r}_{12}|} \Rightarrow E_0 = -\frac{m}{4}b_0^2, \quad \Psi_{100}(\boldsymbol{r}_{1,2}) = \sqrt{\frac{b_0^3}{8\pi}}e^{-b_0|\boldsymbol{r}_{12}|/2}$$

Ansatz: N-particle **baryons** with only LO (pairwise) potential:

$$H = -\frac{1}{2}\sum_{i}^{N}\nabla_{\boldsymbol{r}_i} + V_{\bar{N}}^{\psi\psi}(\boldsymbol{r}_{1\ldots N}) \Rightarrow \Psi_{\min}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = \prod_{i<j}^{N}\psi_{100}(\boldsymbol{r}_{ij}, a_0 = 2/b_{N_c})$$

Upon analysis this simple product state gives $\sim E_{\min}$ for all N_c

Green's function Monte-Carlo

We now have Ψ_T and E_{min} but what if ∃ *lower-lying states*?

Limited in VMC by choice and size of basis ⇒ need new tool

GFMC: Given Ψ_T from VMC diffuse in time and reach **true ground-state energy**

$$|\Psi_T\rangle = \lim_{\tau\to\infty} e^{-\hat{H}\tau}|\psi_T\rangle$$

In practice: Take small time $\delta\tau = \tau/L$ for $L \gg 1$ and path integrate

$$\Psi(\tau, \boldsymbol{R}_N) = \int \prod_{i=0}^{N-1} d\boldsymbol{R}_i \langle \boldsymbol{R}_N | e^{-H\delta\tau} | \boldsymbol{R}_{N-1}\rangle \cdots \langle \boldsymbol{R}_1 | e^{-H\delta\tau} | \boldsymbol{R}_0\rangle\langle \boldsymbol{R}_0 | \Psi_T\rangle$$

with **Trotter-Suzuki expansion**

$$\langle \boldsymbol{R}_{i+1} | e^{-H\delta\tau} | \boldsymbol{R}_i\rangle = e^{-V(\boldsymbol{R}_{i+1})\delta\tau/2}\langle \boldsymbol{R}_{i+1} | e^{-T\delta\tau} | \boldsymbol{R}_i\rangle e^{-V(\boldsymbol{R}_i)\delta\tau/2} + \mathcal{O}(\delta\tau^3)$$

N.B. $\Psi(\tau, \boldsymbol{R}_N)$ is not the true wavefunction but a set of amplitudes

Dark Hadrons

Composite dark matter attractive candidate since implicit stability due to global flavour symmetry

$$\mathscr{L}_{\mathrm{dQCD}} = -\frac{1}{2}\mathrm{Tr}G_{\mu\nu}^2 + \bar{q}iDq + m_d\bar{q}q$$

New $SU(N_d)$ gauge sector **confines** at

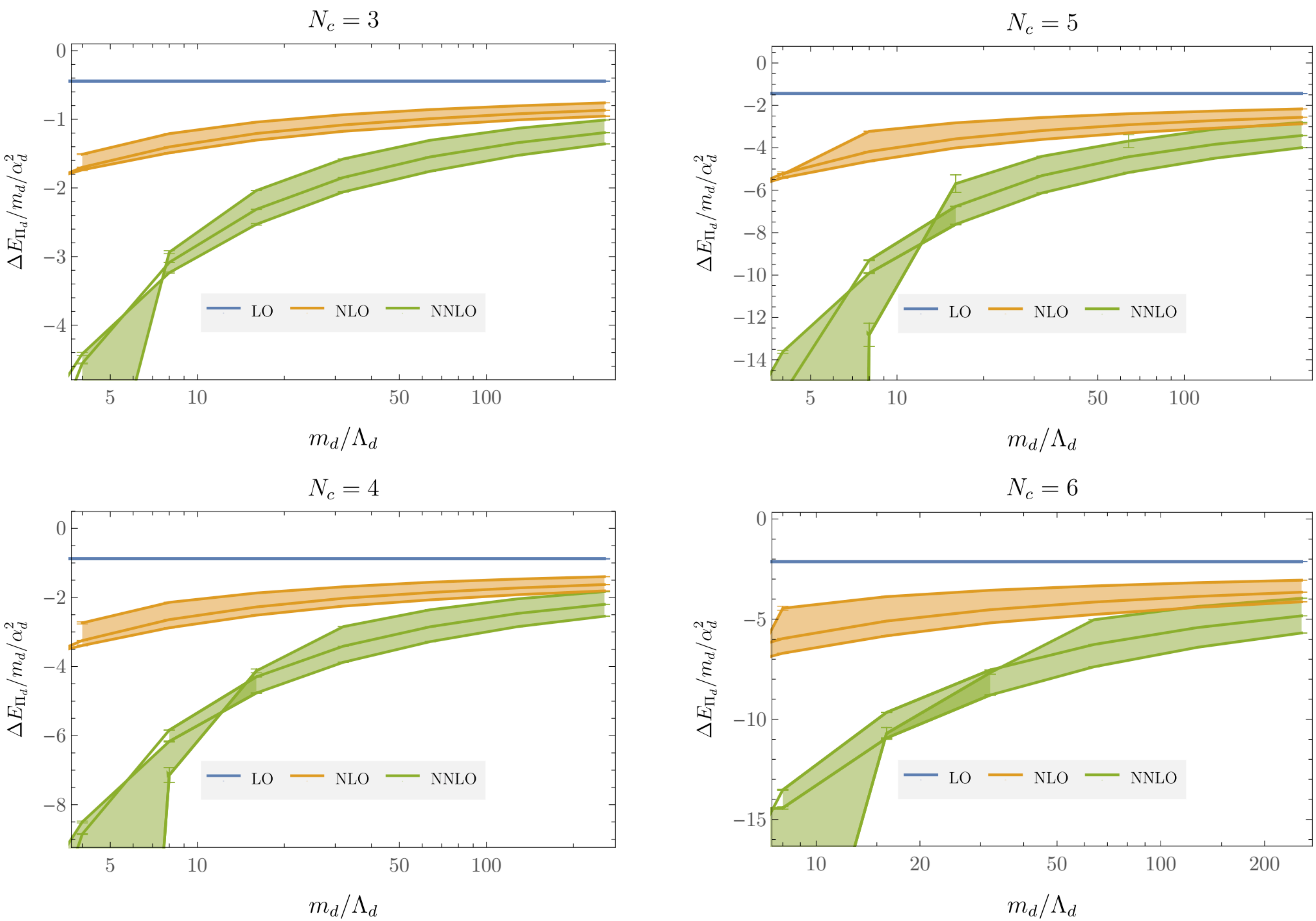
$$\Lambda_{\mathrm{dQCD}} \sim \exp\left(-\frac{2\pi}{\beta_0\alpha_d}\right)$$

DM **stability** is maintained beyond Gyrs $\sim$ proton stability in SM

SM-DM interactions **suppressed** in the EFT above confinement

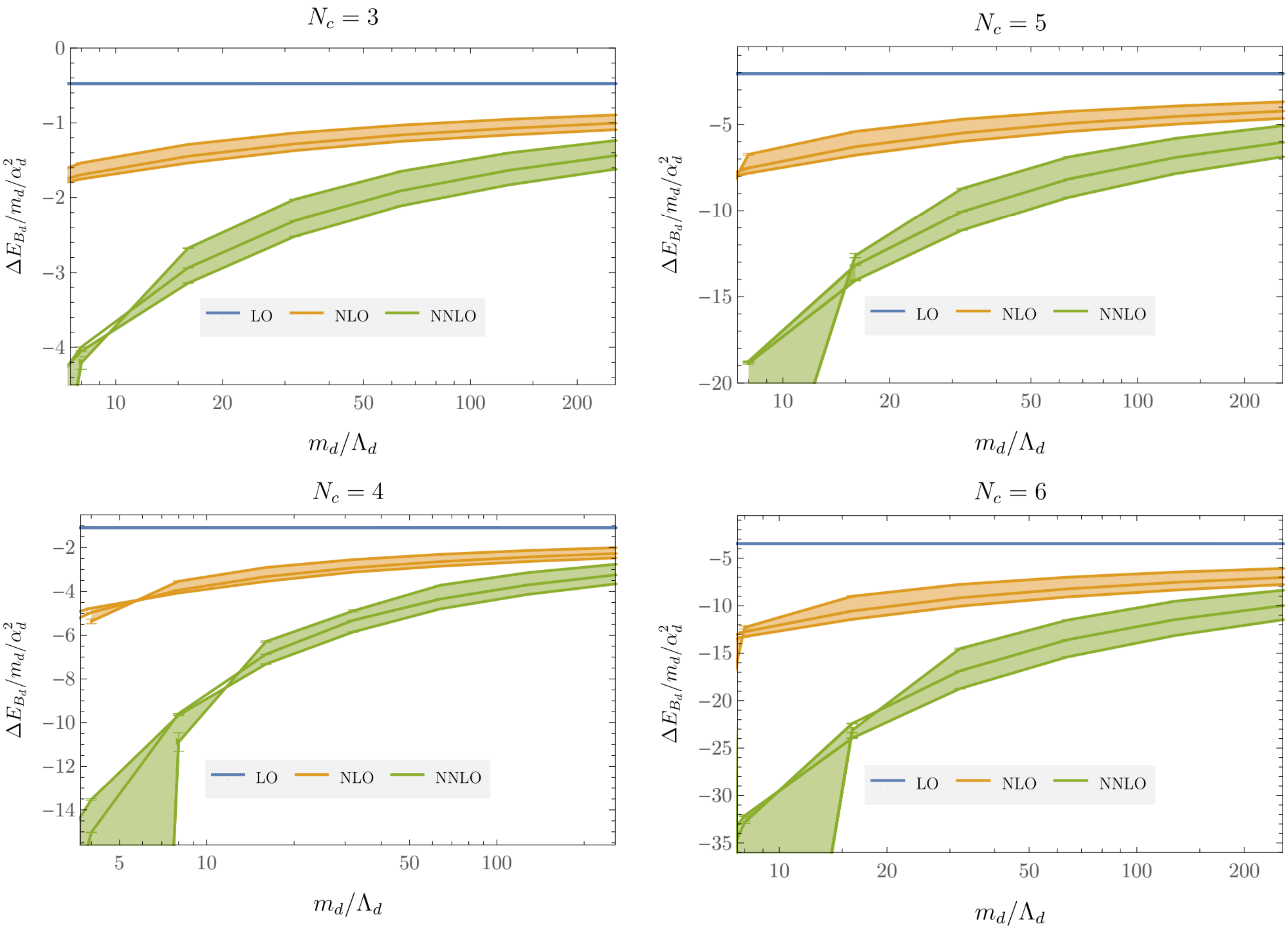
[Review: arXiv:1604.04627]

Dark Mesons



$N_c = 3$
0
−1
−2
−3
−4
5
10
50
100
$\Delta E_{\Pi_d}/m_d/\alpha_d^2$
m_d/Λ_d
LO
NLO
NNLO
$N_c = 5$
0
−2
−4
−6
−8
−10
−12
−14
5
10
50
100
$\Delta E_{\Pi_d}/m_d/\alpha_d^2$
m_d/Λ_d
LO
NLO
NNLO
$N_c = 4$
0
−2
−4
−6
−8
5
10
50
100
$\Delta E_{\Pi_d}/m_d/\alpha_d^2$
m_d/Λ_d
LO
NLO
NNLO
$N_c = 6$
0
−5
−10
−15
10
20
50
100
200
$\Delta E_{\Pi_d}/m_d/\alpha_d^2$
m_d/Λ_d
LO
NLO
NNLO

Dark baryons



Extra: pNRQCD Hamiltonian

Converting from Lagrangian to Hamiltonian formalism

Quarkonium and baryon **state** can be written

$$\left|Q\overline{Q}(\boldsymbol{r}_1,\boldsymbol{r}_2)\right\rangle = \frac{1}{\sqrt{N_c}}\left|\psi_m(\boldsymbol{r}_1),\chi_n^\dagger(\boldsymbol{r}_2)\right\rangle\delta_{mn} \qquad \left|B(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_{N_c})\right\rangle \equiv \frac{\epsilon_{i_1\ldots i_{N_c}}}{\sqrt{N_c!}}\psi_{i_i}^\dagger(\boldsymbol{r}_1)\cdots\psi_{i_{N_c}}^\dagger(\boldsymbol{r}_{N_c})\left|0\right\rangle$$

A non-relativistic potential **operator** then acts as

$$\begin{aligned}
&\hat{V}^{\psi\chi}\left|\psi_m(\boldsymbol{r}_1),\chi_n^\dagger(\boldsymbol{r}_2)\right\rangle \\
&= \int d^3\boldsymbol{s}_1 d^3\boldsymbol{s}_2\left[\frac{1}{N_c}\delta_{ij}\delta_{kl}V_{\boldsymbol{1}}^{\psi\chi}(\boldsymbol{s}_{12}) + \frac{1}{T_F}T_{ij}^a T_{kl}^a V_{\mathrm{Ad}}^{\psi\chi}(\boldsymbol{s}_{12})\right] \\
&\quad\times \psi_i^\dagger(t,\boldsymbol{s}_1)\chi_j(t,\boldsymbol{s}_2)\chi_k^\dagger(t,\boldsymbol{s}_2)\psi_l(t,\boldsymbol{s}_1)\left|\psi_m(\boldsymbol{r}_1),\chi_n^\dagger(\boldsymbol{r}_2)\right\rangle \\
&= \left[\frac{1}{N_c}\delta_{ij}\delta_{kl}V_{\boldsymbol{1}}^{\psi\chi}(\boldsymbol{r}_{12}) + \frac{1}{T_F}T_{ij}^a T_{kl}^a V_{\mathrm{Ad}}^{\psi\chi}(\boldsymbol{r}_{12})\right]\left|\psi_i(\boldsymbol{r}_1),\chi_j^\dagger(\boldsymbol{r}_2)\right\rangle\delta_{km}\delta_{ln}
\end{aligned}$$

More generally 3-quark potential acting on **arbitrary** state

$$\begin{aligned}
&\hat{V}^{3\psi}\left|\psi_1(\boldsymbol{r}_1),\ldots,\psi_{N_q}(\boldsymbol{r}_{N_q})\right\rangle \\
&= \sum_{\boldsymbol{\rho}\in\{\mathrm{A,S,MA,MS}\}}\int d^3\boldsymbol{s}_1 d^3\boldsymbol{s}_2 d^3\boldsymbol{s}_3 V_{\boldsymbol{\rho}}^{3\psi}(\boldsymbol{s}_{12},\boldsymbol{s}_{13},\boldsymbol{s}_{23})\left(\mathcal{F}_{ijk}^{\boldsymbol{\rho}}\right)^*\mathcal{F}_{lmn}^{\boldsymbol{\rho}}\psi_i^\dagger(t,\boldsymbol{s}_1)\psi_j^\dagger(t,\boldsymbol{s}_2)\psi_k^\dagger(t,\boldsymbol{s}_3) \\
&\quad\times\psi_l(t,\boldsymbol{s}_3)\psi_m(t,\boldsymbol{s}_2)\psi_n(t,\boldsymbol{s}_1)\left|\psi_{n_1}(\boldsymbol{r}_1),\ldots,\psi_{n_{N_q}}(\boldsymbol{r}_{N_q})\right\rangle \\
&= \sum_{I\neq J\neq K}\sum_{\boldsymbol{\rho}\in\{\mathrm{A,S,MA,MS}\}} V_{\boldsymbol{\rho}}^{3\psi}(\boldsymbol{r}_{IJ},\boldsymbol{r}_{IK},\boldsymbol{r}_{JK})\left(\mathcal{F}^{\boldsymbol{\rho}}{}_{m_I m_J m_K}\right)^*\mathcal{F}_{n_I n_J n_K}^{\boldsymbol{\rho}} \\
&\quad\times\left|\psi_{n_1}(\boldsymbol{r}_1),\ldots,\psi_{m_I}(\boldsymbol{r}_I),\ldots,\psi_{m_J}(\boldsymbol{r}_J),\ldots,\psi_{m_K}(\boldsymbol{r}_K),\ldots,\psi_{n_{N_q}}(\boldsymbol{r}_{N_q})\right\rangle.
\end{aligned}$$

Extra: Variational methods

Trial wave function PDF: $P(\boldsymbol{r}) = \dfrac{|\Psi_T(\boldsymbol{r},\boldsymbol{\alpha})|^2}{\int d\boldsymbol{r}\,|\Psi_T(\boldsymbol{r},\boldsymbol{\alpha})|^2}$

Two-step algorithm:

1) Pick initial $(\boldsymbol{r},\boldsymbol{\alpha}) \rightarrow$ compute $\langle H \rangle, |\Psi_T(\boldsymbol{r},\boldsymbol{\alpha})|^2$

2) $\boldsymbol{r}_p = \boldsymbol{r} + \boldsymbol{x} n_{\text{step}} : |\boldsymbol{x}| \in [0,1] \rightarrow$ **accept/reject** new $\boldsymbol{r}_p$ with ratio $P(\boldsymbol{r}_p)/P(\boldsymbol{r})$

So far we considered **two classes** of N-particle Hamiltonians:

Meson in $SU(N_c)$

$$\hat{H} = -\frac{1}{2}\sum_i^N \nabla_{\boldsymbol{r}_i} + \hat{V}^{\psi\chi}$$

N-heavy **baryon** $(N = N_c)$

$$\hat{H} = -\frac{1}{2}\sum_i^N \nabla_{\boldsymbol{r}_i} + \hat{V}^{\psi\psi} + \hat{V}^{3\psi}$$

Tetraquark state and potential

Action of quark-antiquark potential on four quark-antiquark state

$$\hat{V}^{\psi\chi}\left|\psi_{n_1}(\boldsymbol{r}_1),\chi^\dagger_{n_2}(\boldsymbol{r}_2)\psi_{n_3}(\boldsymbol{r}_3),\chi^\dagger_{n_4}(\boldsymbol{r}_4)\right\rangle$$

$$= \int d^3\boldsymbol{s}_1 d^3\boldsymbol{s}_2\left[\frac{1}{N_c}\delta_{ij}\delta_{kl}V^{\psi\chi}_{\mathbf{1}}(\boldsymbol{s}_{12}) + \frac{1}{T_F}T^a_{ij}T^a_{kl}V^{\psi\chi}_{\mathrm{Ad}}(\boldsymbol{s}_{12})\right]$$

$$\times\,\psi^\dagger_i(t,\boldsymbol{s}_1)\chi_j(t,\boldsymbol{s}_2)\chi^\dagger_k(t,\boldsymbol{s}_2)\psi_l(t,\boldsymbol{s}_1)\left|\psi_{n_1}(\boldsymbol{r}_1),\chi^\dagger_{n_2}(\boldsymbol{r}_2)\psi_{n_3}(\boldsymbol{r}_3),\chi^\dagger_{n_4}(\boldsymbol{r}_4)\right\rangle$$

$$= \left[\frac{1}{N_c}\delta_{ij}\delta_{n_1n_2}V^{\psi\chi}_{\mathbf{1}}(\boldsymbol{r}_{12}) + \frac{1}{T_F}T^a_{ij}T^a_{n_1n_2}V^{\psi\chi}_{\mathrm{Ad}}(\boldsymbol{r}_{12})\right]\left|\psi_i(\boldsymbol{r}_1),\chi^\dagger_j(\boldsymbol{r}_2)\psi_{n_3}(\boldsymbol{r}_3),\chi^\dagger_{n_4}(\boldsymbol{r}_4)\right\rangle$$

$$+\left[\frac{1}{N_c}\delta_{ij}\delta_{n_1n_4}V^{\psi\chi}_{\mathbf{1}}(\boldsymbol{r}_{14}) + \frac{1}{T_F}T^a_{ij}T^a_{n_1n_4}V^{\psi\chi}_{\mathrm{Ad}}(\boldsymbol{r}_{14})\right]\left|\psi_i(\boldsymbol{r}_1),\chi^\dagger_{n_2}(\boldsymbol{r}_2)\psi_{n_3}(\boldsymbol{r}_3),\chi^\dagger_j(\boldsymbol{r}_4)\right\rangle$$

$$+\left[\frac{1}{N_c}\delta_{ij}\delta_{n_1n_2}V^{\psi\chi}_{\mathbf{1}}(\boldsymbol{r}_{32}) + \frac{1}{T_F}T^a_{ij}T^a_{n_1n_2}V^{\psi\chi}_{\mathrm{Ad}}(\boldsymbol{r}_{32})\right]\left|\psi_{n_1}(\boldsymbol{r}_1),\chi^\dagger_j(\boldsymbol{r}_2)\psi_i(\boldsymbol{r}_3),\chi^\dagger_{n_4}(\boldsymbol{r}_4)\right\rangle$$

$$+\left[\frac{1}{N_c}\delta_{ij}\delta_{n_3n_4}V^{\psi\chi}_{\mathbf{1}}(\boldsymbol{r}_{34}) + \frac{1}{T_F}T^a_{ij}T^a_{n_3n_4}V^{\psi\chi}_{\mathrm{Ad}}(\boldsymbol{r}_{34})\right]\left|\psi_{n_1}(\boldsymbol{r}_1),\chi^\dagger_{n_2}(\boldsymbol{r}_2)\psi_i(\boldsymbol{r}_3),\chi^\dagger_j(\boldsymbol{r}_4)\right\rangle$$