Pentaquark spectrum from Fermi statistics

Davide Germani

Based on:

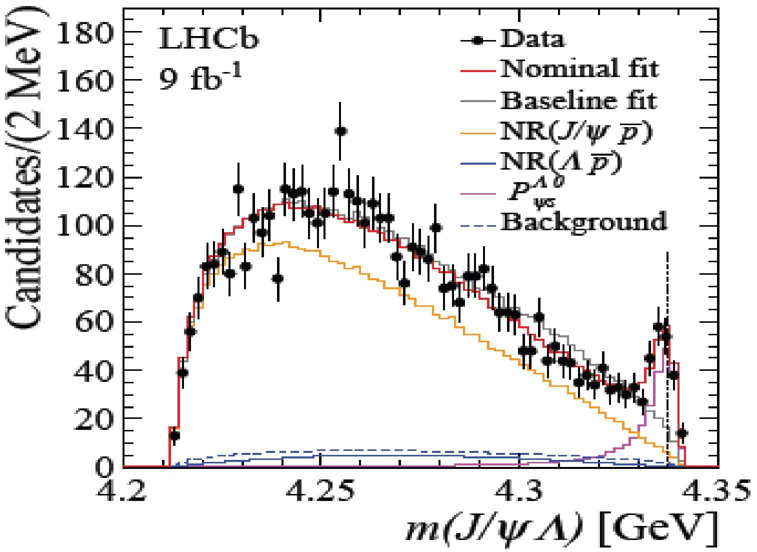
L. Maiani, A. D. Polosa, V. Riquer, The Pentaquark Spectrum from Fermi Statistics arXiv:2303.04056 [hep-ph]

D. Germani, A. D. Polosa, F. Niliani, A Simple Model of Pentaquarks (almost there)

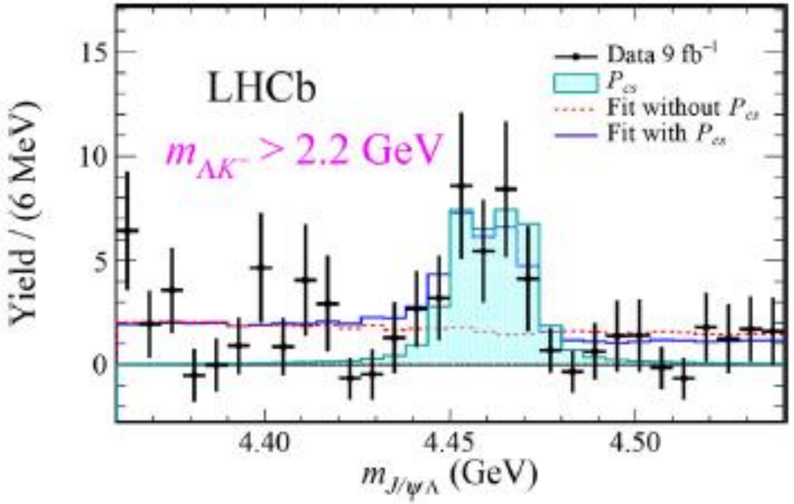
Outline

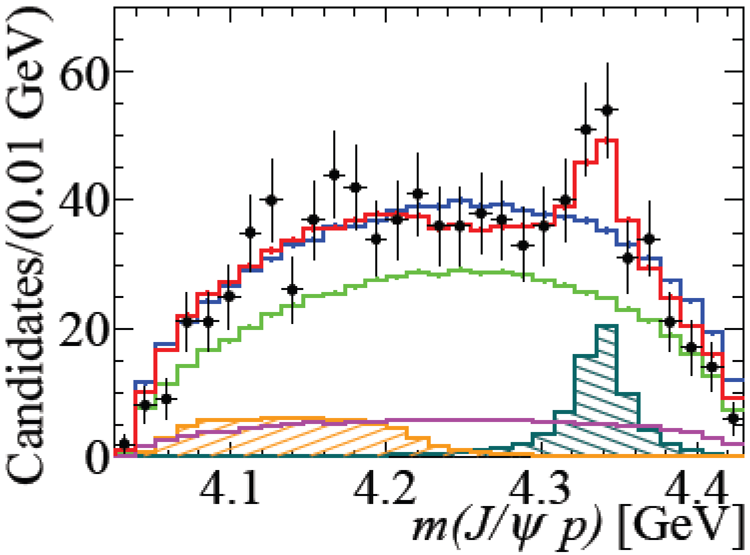
- Overview of the current situation;
- Fermi Statistics and BO approximation: what do they tell us?
  - QCD couplings and Fierz Relations
  - BO approximation: Consistency Condition
  - Some results
- A simple model for pentaquarks
  - A different point of view
  - Our ‘Hadro-Charmonia’ model: the role of the exchange interaction
  - Some results (part two)

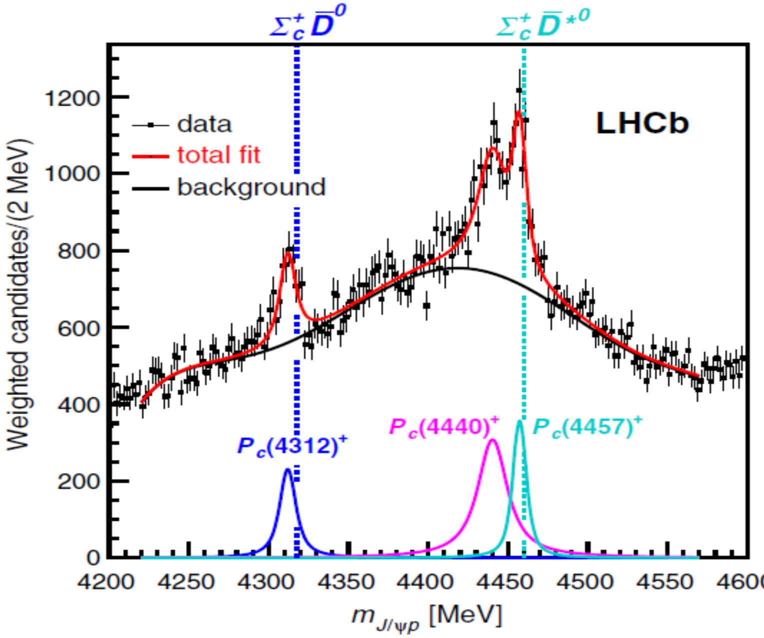
Current situation



State	Mass [MeV]	Width [MeV]	Observed Process	Year
$P_c(4312)$	$4311.9 \pm 0.7^{+6.8}_{-0.6}$	$9.8 \pm 2.7^{+3.7}_{-4.5}$	$\Lambda_b^0 \to (J/\psi\, p)\, K^-$	2019
$\widetilde{P}_c(4337)$	$4337^{+7}_{-4}{}^{+2}_{-2}$	$29^{+26}_{-12}{}^{+14}_{-14}$	$B_s^0 \to (J/\psi p)\, \overline{p}$	2022
$P_c(4440)$	$4440.3 \pm 1.3^{+4.1}_{-4.7}$	$20.6 \pm 4.9^{+8.7}_{-10.1}$	$\Lambda_b^0 \to (J/\psi\, p)\, K^-$	2019
$P_c(4457)$	$4457.3 \pm 0.6^{+4.1}_{-1.7}$	$6.4 \pm 2.0^{+5.7}_{-1.9}$	$\Lambda_b^0 \to (J/\psi\, p)\, K^-$	2019
$\widetilde{P}_{cs}(4338)^{\frac{1}{2}^-}$	$4338.2 \pm 0.7 \pm 0.4$	$7.0 \pm 1.2 \pm 1.3$	$B^- \to (J/\psi\, \Lambda)\, \overline{p}$	2022
$P_{cs}(4459)$	$4458.9 \pm 2.9^{+4.7}_{-1.1}$	$17.3 \pm 6.5^{8.0}_{-5.7}$	$\Xi_b^- \to (J/\psi\, \Lambda)\, K^-$	2021







R. Aaij et al. arXiv:1904.03947 [hep-ex] (2019), R. Aaij et al. arXiv:2012.10380v2 [hep-ex] (2021), R. Aaij et al. arXiv:2210.10346 [hep-ex] (2021), R. Aaij et al. arXiv:2108.04720 [hep-ex] (2022)

Starting Point

We will consider a Pentaquark made of $c\bar{c}uud$. The natural choice is to assume the heavy quark-antiquark pair to be in a color octet, coupled to the light quarks to make an overall color singlet

$$\mathcal{P}_{BO} = [\bar{c}(\boldsymbol{x}_{\bar{c}})\lambda^A c(\boldsymbol{x}_c)] B^A$$

$$P^A(u_1, d|u_2) = \bar{\theta}(u_1 d)\lambda^A u_2 = u_1^a d^b \,(\epsilon_{abc}\, \lambda^A_{cd})\, u_2^d$$

$$\bar{\theta}(qq')_a = \epsilon_{abc}\, q^b q'^c$$

(Color part)

$$\Phi^A(u_1, d|u_2) = \phi^{ad}\,(\epsilon_{abc}\,\lambda^A_{cd})\, u_2^b$$

$$\phi^{ab}(u, d) = (u^a d^b + u^b d^a)$$

$-$	$\|(u_1, u_2)_{\bar{\mathbf{3}}}, d\rangle_{\mathbf{8}}$	$\|(u_1, u_2)_{\mathbf{6}}, d\rangle_{\mathbf{8}}$	$\|(u_2, d)_{\bar{\mathbf{3}}}, u_1\rangle_{\mathbf{8}}$	$\|(u_2, d)_{\mathbf{6}}, u_1\rangle_{\mathbf{8}}$
$\|(u_1, d)_{\bar{\mathbf{3}}}, u_2\rangle_{\mathbf{8}}$	$+1/2$	$+\sqrt{3}/2$	$+1/2$	$-\sqrt{3}/2$
$\|(u_1, d)_{\mathbf{6}}, u_2\rangle_{\mathbf{8}}$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}/2$	$-1/2$

This table can be derived from the group S_3 of the six permutations of three elements

QCD couplings and Fierz Relations

$$T = a\,|(cq)_{\boldsymbol{R}_1}\dots\rangle_{\mathbf{1}} + b\,|(cq)_{\boldsymbol{R}_2}\dots\rangle_{\mathbf{1}} + \dots \quad \Longrightarrow \quad \lambda_{cq} = a^2\lambda_{cq}(\boldsymbol{R}_1) + b^2\lambda_{cq}(\boldsymbol{R}_2) + \dots.$$

$$g_{cq}^2 = \alpha_s \lambda_{cq}$$

$$\lambda_{cq} = \frac{1}{2}(C_2(\boldsymbol{R}) - 8/3)$$

$$T \;=\; |(\bar{Q}Q)_{\mathbf{8}}(\bar{\theta}q)_{\mathbf{8}}\rangle_{\mathbf{1}} = \sqrt{\frac{2}{3}}|(Qq)_{\bar{\mathbf{3}}}(\bar{Q}\bar{\theta})_{\mathbf{3}}\rangle_1 - \frac{1}{\sqrt{3}}|(Qq)_{\mathbf{6}}(\bar{Q}\bar{\theta})_{\bar{\mathbf{6}}}\rangle_1 \;=\; \sqrt{\frac{8}{9}}|(\bar{Q}q)_{\mathbf{1}}(\bar{\theta}Q)_{\mathbf{1}}\rangle - \frac{1}{\sqrt{9}}|(\bar{Q}q)_{\mathbf{8}}(\bar{\theta}Q)_{\mathbf{8}}\rangle_{\mathbf{1}}$$

$$T = |(\bar{Q}Q)_{\mathbf{8}}(\phi q)_{\mathbf{8}}\rangle_{\mathbf{1}} = |(\bar{Q}q)_{\mathbf{8}}(\phi Q)_{\mathbf{8}}\rangle_{\mathbf{1}} = |(Qq)_{\bar{\mathbf{3}}}(\phi\bar{Q})_{\mathbf{3}}\rangle_{\mathbf{1}}$$

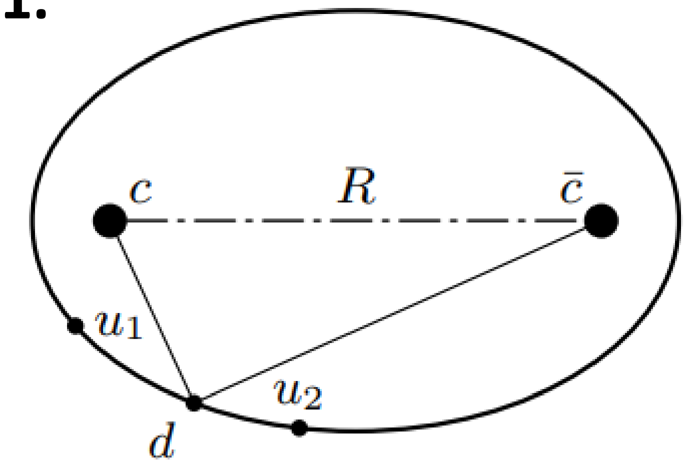
G. S. Bali QCD forces and heavy quark bound states arXiv:hep-ph/0001312

L. Maiani, A.D. Polosa, V. Riquer The Hydrogen Bond of QCD arXiv:1903.10253v2 [hep-ph]

L. Maiani, A. Pilloni, A.D. Polosa, V. Riquer Doubly Heavy Tetraquarks in the Born-Oppenheimer approximation arXiv:2208.02730v2 [hep-ph]

BO Approximation

1.

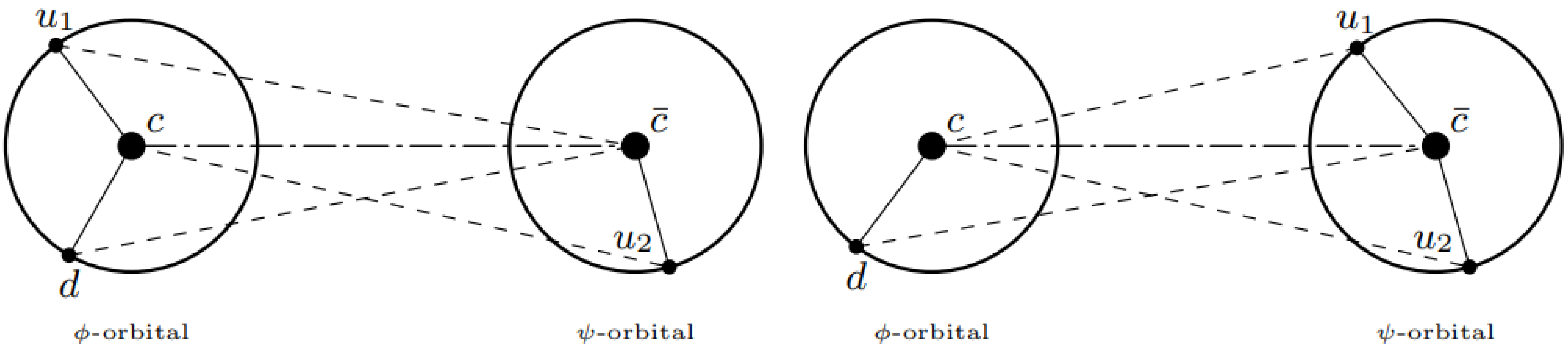


Consistency Condition

Quarks of different flavor in the same orbital must share the same QCD coupling to the heavy quarks at the center of the orbital

$$\lambda_{cq} = \lambda_{cq'} \qquad \lambda_{\bar{c}q} = \lambda_{\bar{c}q'}$$

2.



Where is the Fermi statistics?

The light quarks must respect the Fermi statistics

$$\mathrm{SU}(6) \supset \mathrm{SU}(3)_f \otimes \mathrm{SU}(2)_{\mathrm{spin}}$$

Flavor+ Spin

Colour	Coordinates	$SU(6)$	Notes
M	S	$\mathbf{70}$ (M)	Colour $\otimes SU(6)$: A
M	M	$\mathbf{56}$ (S)	Colour $\otimes$ Coordinates: A
M	M	$\mathbf{20}$ (A)	Colour $\otimes$ Coordinates: S
M	M	$\mathbf{70}$ (M)	Colour $\otimes$ Coordinates: M

$$S = \frac{1}{\sqrt{2}}(M_1^\rho M_2^\rho + M_1^\lambda M_2^\lambda) \qquad A = \frac{1}{\sqrt{2}}(M_1^\rho M_2^\lambda - M_1^\lambda M_2^\rho)$$

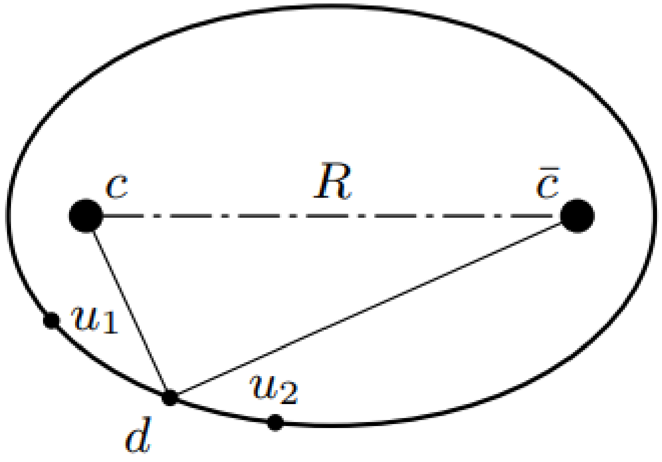
$$M^\rho = \frac{1}{\sqrt{2}}(M_1^\rho M_2^\lambda + M_1^\lambda M_2^\rho) \qquad M^\lambda = \frac{1}{\sqrt{2}}(M_1^\rho M_2^\rho - M_1^\lambda M_2^\lambda)$$

$$M^\lambda = |(q_1, q_2)_{\mathbf{6}}\, q_3\rangle_{\mathbf{8}}$$
$$M^\rho = |(q_1, q_2)_{\bar{\mathbf{3}}}\, q_3\rangle_{\mathbf{8}}$$
$$F^\rho = \frac{1}{\sqrt{2}}(u_1 d_2 - d_1 u_2) u_3$$
$$F^\lambda = -\frac{1}{\sqrt{6}}\left[(u_1 d_2 + d_1 u_2) u_3 - 2 u_1 u_2 d_3\right] \text{ (octet)}$$
$$F^S = \frac{1}{\sqrt{3}}(d_1 u_2 u_3 + u_1 d_2 u_3 + u_1 u_2 d_3) \text{ (decuplet)}$$
$$N^\rho = n\, \frac{1}{\sqrt{2}}\, [\phi(1)\psi(2) - \phi(2)\psi(1)]\, \phi(3)$$
$$N^\lambda = -n\, \frac{1}{\sqrt{6}}\, \{[\phi(1)\psi(2) + \phi(2)\psi(1)]\phi(3) - 2\phi(1)\phi(2)\psi(3)\}$$
$$n = \frac{1}{\sqrt{1 - S^2}} \quad \text{and} \quad S = \int d1\, \phi(1)\psi(1) < 1,\ (\phi,\ \psi = \text{ real})$$

S. Capstick and W. Roberts arXiv:nucl-th/0008028 [nucl-th].

One Orbital Case

Colour	Coordinates	$SU(6)$	Notes
M	S	$\mathbf{70}\ (M)$	Colour $\otimes SU(6)$: A



$$\mathbf{70}: (\mathbf{1}, 1/2) \oplus (\mathbf{8}, 3/2) \oplus (\mathbf{8}, 1/2) \oplus (\mathbf{10}, 1/2)$$

NOT POSSIBLE

$$\mathcal{P}^{\mathbf{70}}_{\mathbf{8,3/2,1\text{-orbit}}} = \frac{1}{\sqrt{2}}[M^\rho F^\lambda - M^\lambda F^\rho] \otimes \chi^S = -\frac{1}{2}\left\{\frac{1}{\sqrt{3}}([u_1, d_2]u_3 + [d_1, u_2]u_3 - 2[u_1, u_2]d_3) + ((u_1, d_2)u_3 - (d_1, u_2)u_3)\right\}$$

'Gauge Fixing': $d_3 = d$

$$\mathcal{P}^{\mathbf{70}}_{\mathbf{8,3/2,1\text{-orbit}}}(d_3 = d) = [u_1, u_2]d = +\frac{1}{2}[d, u_2]u_1 - \frac{\sqrt{3}}{2}(d, u_2)u_1$$

$$\lambda_{dc} = -\frac{2}{6} \neq \lambda_{u_1 c} = \frac{1}{4}\left[\frac{2}{3}\left(-\frac{2}{3}\right) + \frac{1}{3}\left(+\frac{1}{3}\right)\right] + \frac{3}{4}\left(-\frac{2}{3}\right) = -\frac{7}{12}$$

$$\lambda_{d\bar{c}} = -\frac{7}{6} \neq \lambda_{u_1 \bar{c}} = \frac{1}{4}\left[\frac{8}{9}\left(-\frac{4}{3}\right) + \frac{1}{9}\left(+\frac{1}{6}\right)\right] + \frac{3}{4}\left(+\frac{1}{6}\right) = -\frac{2}{12}$$

$$|(\bar{Q}Q)_{\mathbf{8}}(\bar{\theta}q)_{\mathbf{8}}\rangle_{\mathbf{1}} = \sqrt{\frac{2}{3}}|(Qq)_{\bar{\mathbf{3}}}(\bar{Q}\bar{\theta})_{\mathbf{3}}\rangle_{\mathbf{1}} - \frac{1}{\sqrt{3}}|(Qq)_{\mathbf{6}}(\bar{Q}\bar{\theta})_{\bar{\mathbf{6}}}\rangle_{\mathbf{1}}$$

$$|(\bar{Q}Q)_{\mathbf{8}}(\bar{\theta}q)_{\mathbf{8}}\rangle_{\mathbf{1}} = \sqrt{\frac{8}{9}}|(\bar{Q}q)_{\mathbf{1}}(\bar{\theta}Q)_{\mathbf{1}}\rangle - \frac{1}{\sqrt{9}}|(\bar{Q}q)_{\mathbf{8}}(\bar{\theta}Q)_{\mathbf{8}}\rangle_{\mathbf{1}}$$

$$|(\bar{Q}Q)_{\mathbf{8}}(\phi q)_{\mathbf{8}}\rangle_{\mathbf{1}} = |(\bar{Q}q)_{\mathbf{8}}(\phi Q)_{\mathbf{8}}\rangle_{\mathbf{1}} = |(Qq)_{\bar{\mathbf{3}}}(\phi\bar{Q})_{\mathbf{3}}\rangle_{\mathbf{1}}$$

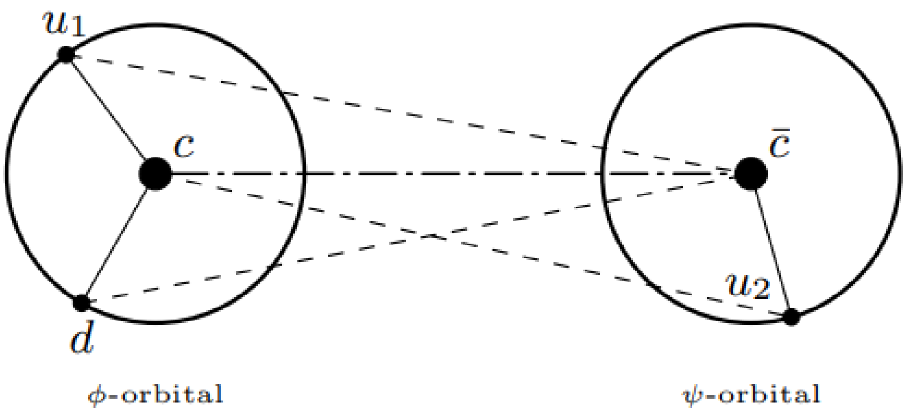
Two Orbital Case

We can demonstrate that there are few possible combinations.

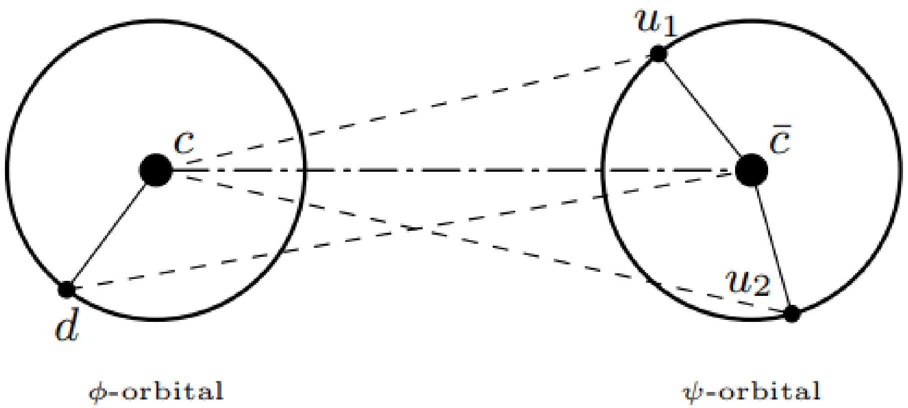
Colour	Coordinates	$SU(6)$	Notes
M	S	**70** (M)	Colour $\otimes SU(6)$: A
M	M	**56** (S)	Colour $\otimes$ Coordinates: A
M	M	**20** (A)	Colour $\otimes$ Coordinates: S
M	M	**70** (M)	Colour $\otimes$ Coordinates: M

Assuming that the lower state has $S_{qqq} = 1/2$

$\mathbf{56}: (\mathbf{8}, 1/2)$



$\mathbf{56}: (\mathbf{8}, 1/2)$ & $\mathbf{20}: (\mathbf{8}, 1/2)$



Conclusion (1)

1. What I have shown applies to a proton-like pentaquark, but one can easily introduce strangeness by simply considering the substitution $u_2 \to s$ and using the rising (or lowering) operator of Vspin.

2. Combining the spin 1/2 of light quarks with $c\bar{c}$ total spin (0, 1) we obtain **three** pentaquarks octets with spin compositions: $2(\mathbf{8}, 1/2) + (\mathbf{8}, 3/2)$. We expect therefore three pentaquark lines for both S = 0 (strangeness) and S = −1, with

$$\mathcal{P}_{(S=0)} \to J/\psi + p \qquad \mathcal{P}_{(\Lambda,S=-1)} \to J/\psi + \Lambda$$

3. The two alternatives, **56** or **20**, are distinguished by presence or absence of pentaquarks decaying into spin 3/2 resonances, e.g.

$$\mathcal{P}^{++}_{(\Delta,S=0)} \to J/\psi + \Delta^{++} \to J/\psi + p + \pi^{+}$$

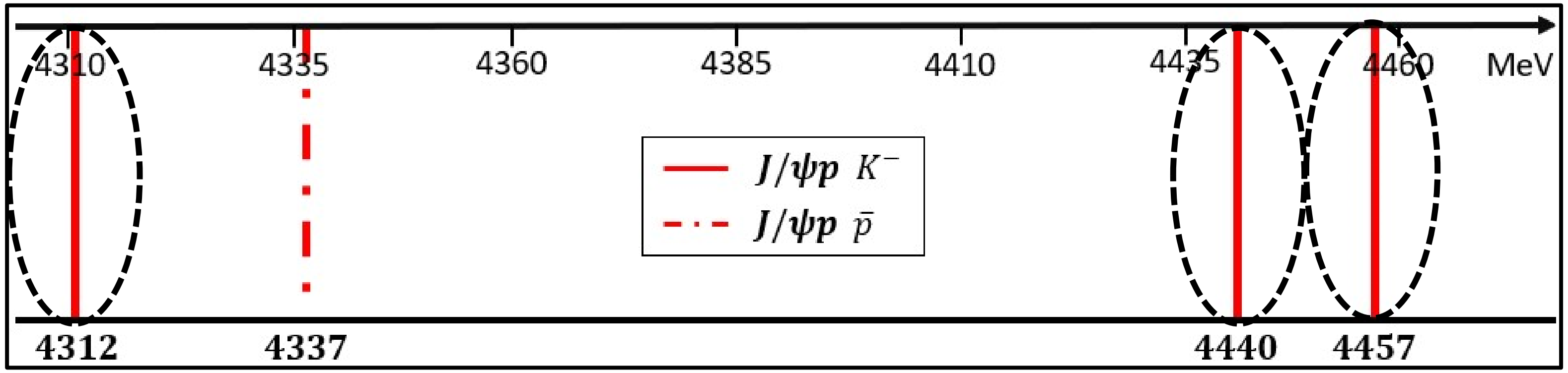
4. Further study can be conducted by applying the BO approximation to compute the mass differences with respect to the ground state, i.e., those due to the hyperfine interactions.

A different point of view

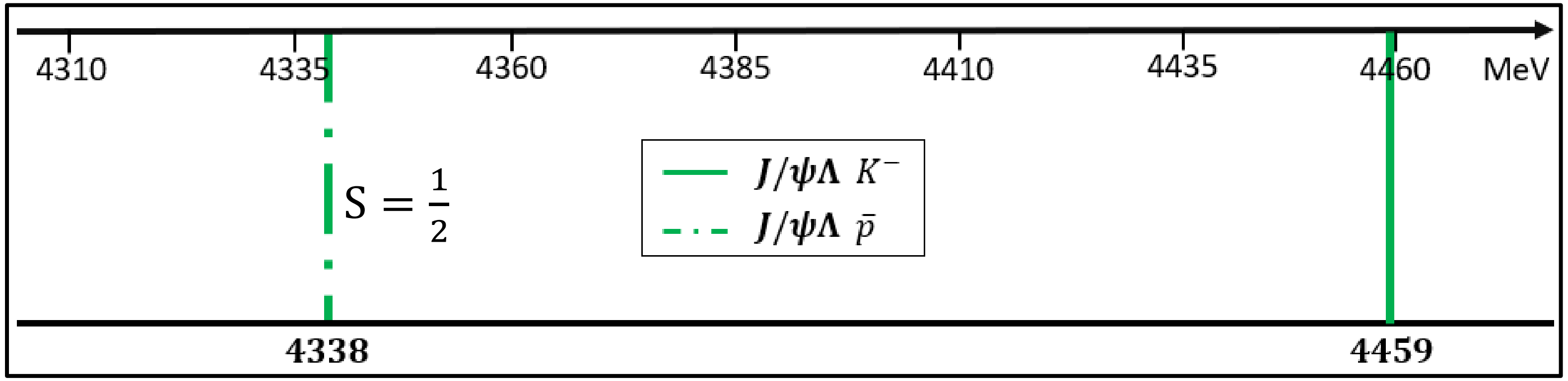
- We divide the spectrum according to strangeness content.
- Data suggests two different type of production: in association with a K^- or with an $\bar{p}$.
- Pentaquark seems to appear in triplet

Can we build a model to account for these properties?

$P_c^+ \ [c\bar{c}uud]$



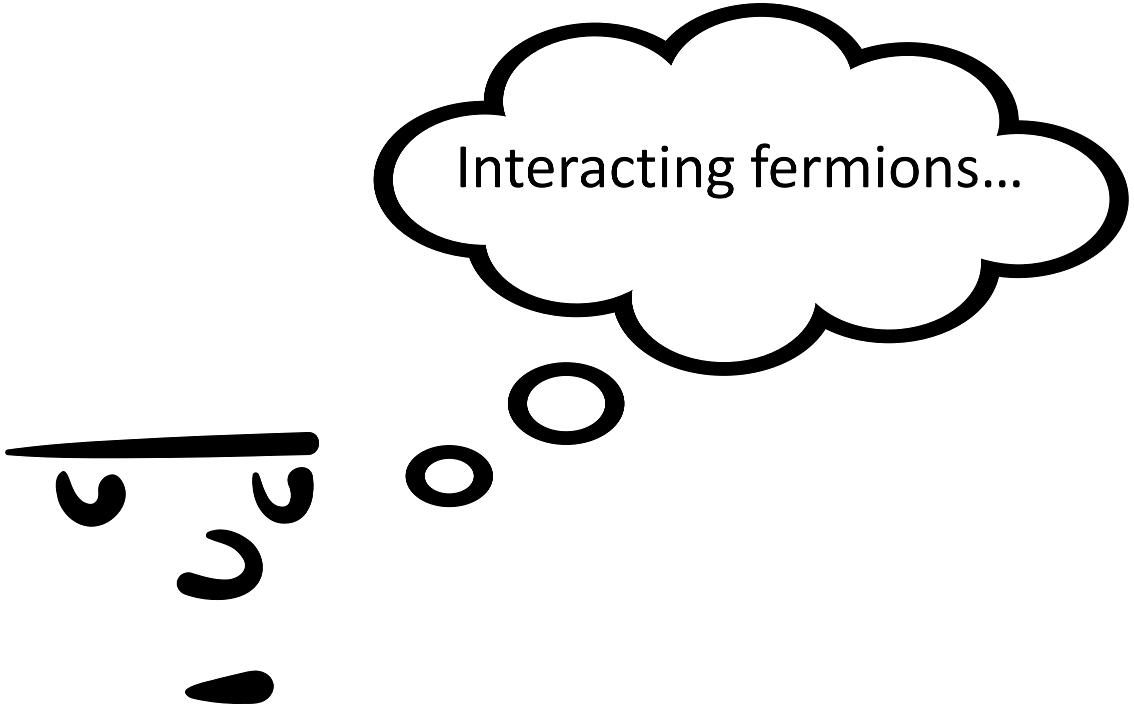
$P_{cs}^0 \ [c\bar{c}uds]$

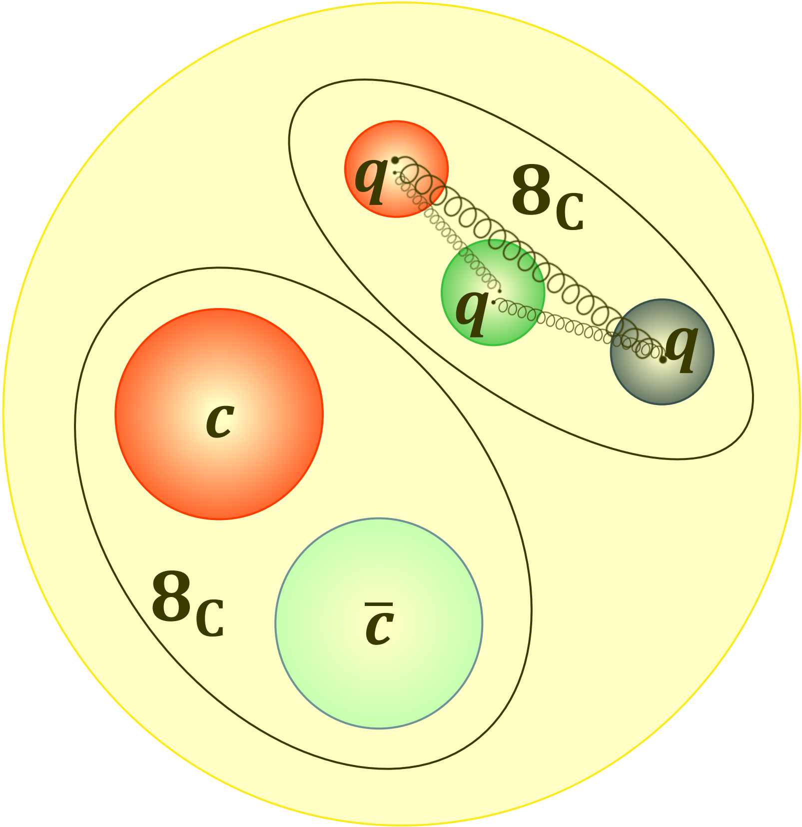


An ‘Hadro-Charmonia’ Model

- The starting point is more or less the same of the Born-Oppenheimer approach. We consider the $c\bar{c}$ system in a color octet as well as the three light quarks sysyem;

- **Fermi statistics for ligth quarks;**





Exchange Interaction! Three Fermions Case

$$V = -\sum_{\text{pairs}} J_{ab}\left(\frac{1}{2} + 2\mathbf{S}_a \cdot \mathbf{S}_b\right)$$

Generalization of the well known two fermions case

$$J_{ab} \equiv \int [\phi_a(\mathbf{r}_1)\phi_b(\mathbf{r}_2)]^* \, U(\mathbf{r}_1 - \mathbf{r}_2) \, [\phi_b(\mathbf{r}_1)\phi_a(\mathbf{r}_2)] \, d^3\mathbf{r}_1 \, d^3\mathbf{r}_2$$

The potential has **three** distinct eigenvalues

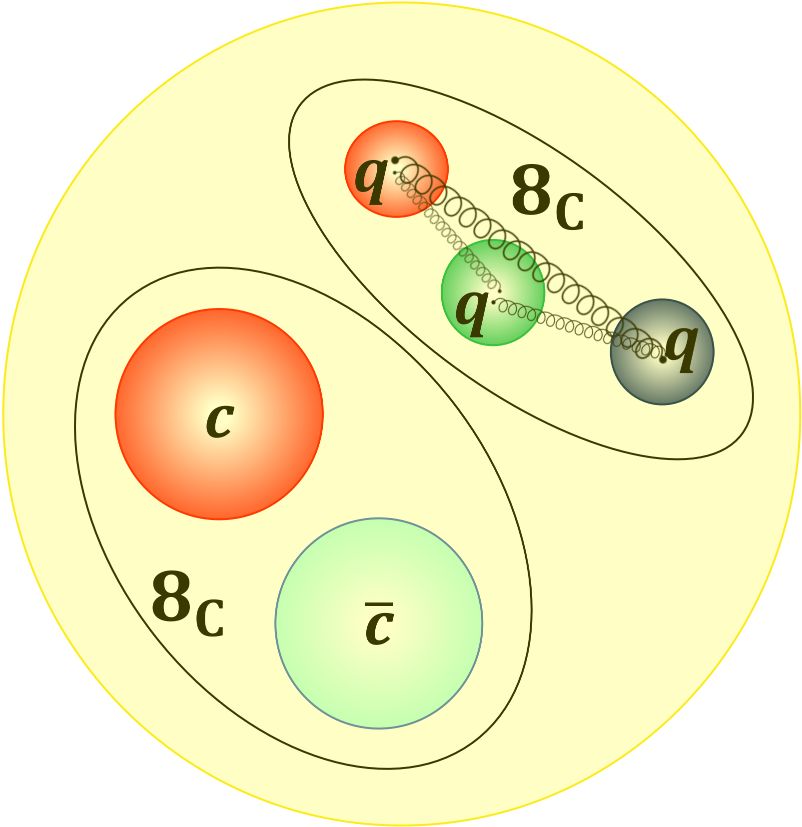
$$\Delta E_{1/2} = \pm\sqrt{J_{12}^2 + J_{13}^2 + J_{23}^2 - J_{12}J_{13} - J_{12}J_{23} - J_{13}J_{23}}$$

$$\Delta E_{3/2} = -J_{12} - J_{13} - J_{23}$$

An ‘Hadro-Charmonia’ Model

- The starting point is more or less the same of the Born-Oppenheimer approach. We consider the $c\bar{c}$ system in a color octet as well as the three light quarks sysyem;
- Fermi statistics for ligth quarks;
- **Exchange Interaction**. But to use to use the exchange interaction, we need to have a precise symmetry on the spin-orbital part which means a complete symmetry in the **color-flavor** sector

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8}' \oplus \mathbf{10}$$ (flavor)



Symmetrc and Antisymmetric Combination

1. $$\mathbf{S}^{abc}_{ijk} = 6\left(\eta_i^{[a}\eta_j^{b]}\eta_k^c - \eta_j^{[a}\eta_k^{b]}\eta_i^c\right) = 6\left(\eta^a_{[i}\eta^b_{j]}\eta^c_k - \eta^a_{[j}\eta^b_{k]}\eta^c_i\right) \qquad \eta^{a,i}\eta^{b,j} = \eta^{b,j}\eta^{a,i}$$

2. $$\mathbf{A}^{abc}_{ijk} = 6\left(\psi_i^{[a}\psi_j^{b]}\psi_k^c - \psi_j^{[a}\psi_k^{b]}\psi_i^c\right) = 6\left(\psi^a_{(i}\psi^b_{j)}\psi^c_k - \psi^a_{(j}\psi^b_{k)}\psi^c_i\right) \qquad \psi^{a,i}\psi^{b,j} = -\psi^{b,j}\psi^{a,i}$$

Baryon Octet Matrix

We consider the greek letters to be spin indices and the latin letters to be flavor ones

$$\mathbf{B}^{abc}_{\alpha\beta\gamma} = \epsilon_{ijk}\left(\mathbf{S}^{abc}_{\alpha\beta\gamma}\right)^{ijk} \Longrightarrow \mathbf{B}^{abc}_{\alpha\beta\gamma} = \epsilon_{ijk}\left(\psi^a_{[\alpha}\psi^b_{\beta]}\psi^c_{\gamma} - \psi^a_{[\beta}\psi^b_{\gamma]}\psi^c_{\alpha}\right)^{ijk}$$

Baryon Operator

$$\mathcal{B}^a_b \equiv \frac{1}{2}\epsilon_{bcd}\mathbf{B}^{cda} \Longrightarrow \mathcal{B}^a_b = \begin{pmatrix} \frac{\Lambda^0}{\sqrt{6}} + \frac{\Sigma^0}{\sqrt{2}} & \Sigma^+ & p \\ \Sigma^- & \frac{\Lambda^0}{\sqrt{6}} - \frac{\Sigma^0}{\sqrt{2}} & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda^0 \end{pmatrix}$$

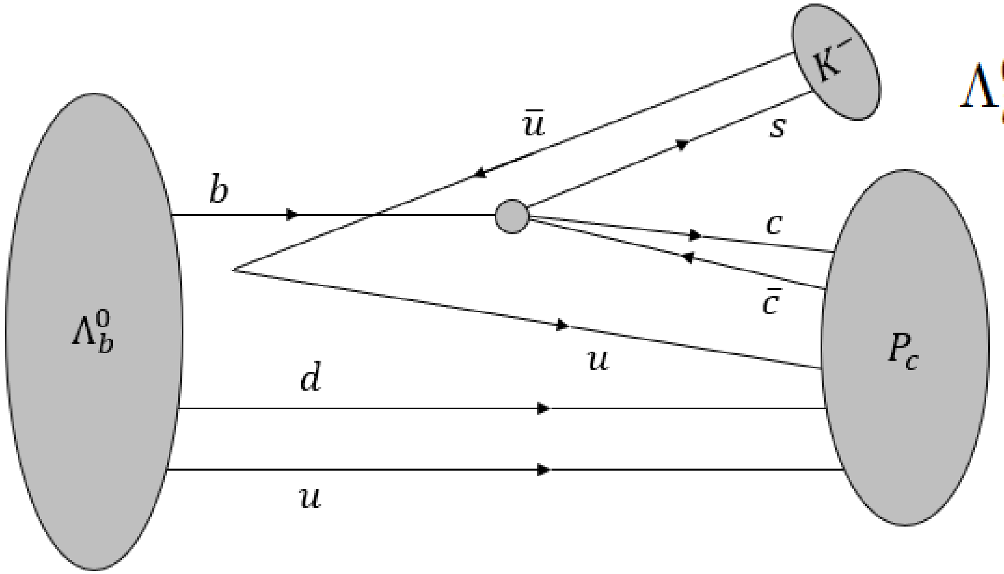
$$|p, +\frac{1}{2}\rangle \propto u^{\uparrow}u^{\uparrow}d^{\downarrow} - u^{\uparrow}u^{\downarrow}d^{\uparrow} \qquad \left[\mathbf{B}^{duu}_{\downarrow\uparrow\uparrow}\right]_{SF} = d_{[\downarrow}u_{\uparrow]}u_{\uparrow} = u_{\uparrow}u_{\uparrow}d_{\downarrow} - u_{\uparrow}u_{\downarrow}d_{\uparrow}$$

H. Georgi, Lie algebras in particle physics. From isospin to unified theories

Symmetric and Antisymmetric Combination

1. $$\mathbf{S}^{abc}_{ijk} = 6\left(\eta_i^{[a}\eta_j^{b]}\eta_k^c - \eta_j^{[a}\eta_k^{b]}\eta_i^c\right) = 6\left(\eta^a_{[i}\eta^b_{j]}\eta^c_k - \eta^a_{[j}\eta^b_{k]}\eta^c_i\right)$$ P $\qquad \eta^{a,i}\eta^{b,j} = \eta^{b,j}\eta^{a,i}$

2. $$\mathbf{A}^{abc}_{ijk} = 6\left(\psi_i^{[a}\psi_j^{b]}\psi_k^c - \psi_j^{[a}\psi_k^{b]}\psi_i^c\right) = 6\left(\psi^a_{(i}\psi^b_{j)}\psi^c_k - \psi^a_{(j}\psi^b_{k)}\psi^c_i\right)$$ $\tilde{P}$ $\qquad \psi^{a,i}\psi^{b,j} = -\psi^{b,j}\psi^{a,i}$



$$\Lambda_b^0 \to (J/\psi\, p)\, K^-$$

The diquark ud has initially $[ud]^{\bar{3}_c}_{3_F\,0_S}$ (good diquark) so it means that is in a symmetric configuration wrt color-flavor. We assume that the color-flavor symmetry is preserved in the process so we choose the S tensor for the pentaquarks produced in association with K^-.

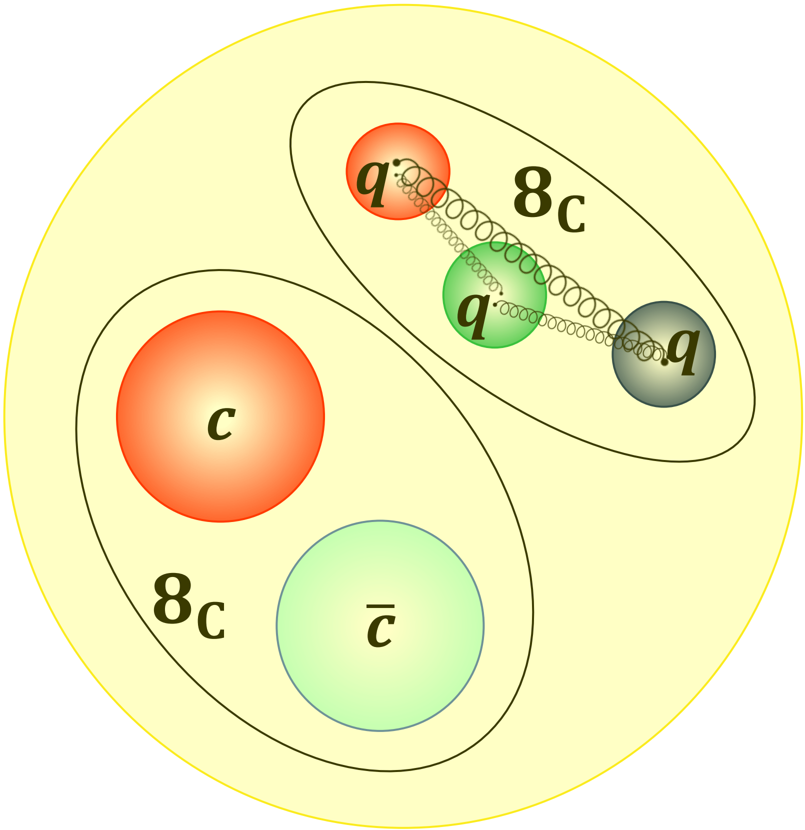
Our 'Hadro-Charmonia' model

- **Fermi statistics for ligth quarks**

Color-Flavour	+	Spin-Orbital
S		A
A		S

- **Exchange Interaction**

$$V = -\sum_{\text{pairs}} J_{ab}\left(\frac{1}{2} + 2\mathbf{S}_a \cdot \mathbf{S}_b\right)$$



- **J couplings:**

Experimental Data + qq interactions in one gluon exchange approximation + Spin assignment

Conclusion (2)

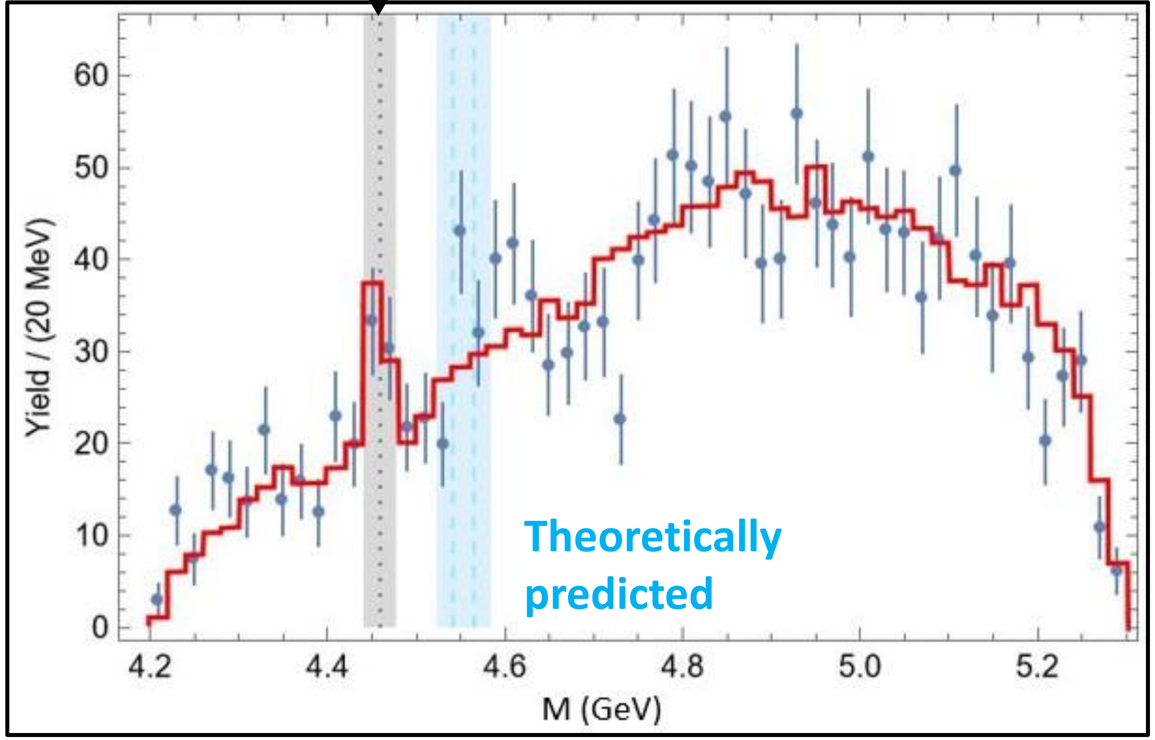
	Mass [MeV]		Mass [MeV]
$P_c(4312)$	$(4311.9^{+7}_{-0.9})$	$P_{cs}(4459)$	$(4458.8^{+6}_{-3.1})$
$P_c(4440)$	(4440.0^{+4}_{-5})	$\mathbf{P}'_{cs}$	4541 ± 6
$P_c(4457)$	$(4457.3^{+7}_{-1.8})$	$\mathbf{P}''_{cs}$	4565 ± 6
$\widetilde{\mathbf{P}}''_{\mathbf{c}}$	4187 ± 7	$\widetilde{P}_{cs}(4338)$	(4338.2 ± 0.8)
$\widetilde{\mathbf{P}}'_{\mathbf{c}}$	4276 ± 12	$\widetilde{\mathbf{P}}'_{\mathbf{cs}}$	4387 ± 4
$\widetilde{P}_c(4337)$	4332 ± 7 $(4337^{+7}_{-4}{}^{+2}_{-2})$	$\widetilde{\mathbf{P}}''_{\mathbf{cs}}$	4435 ± 4

We have the spin prediction for these particles. Each triplet is ordered from top to bottom with $S = 1/2, 3/2, 1/2$.

	Symmetric [MeV]	Antysimmetric [MeV]	No symmetry [MeV]
J^{qq}	$29.9^{+2.5}_{-2.8}$	$-42.8^{+2.4}_{-1.6}$	-
J^{qs}	17.9 ± 2	-25.7 ± 2	-3.9 ± 2

R. Aaij et al. arXiv:2012.10380v2 [hep−ex]

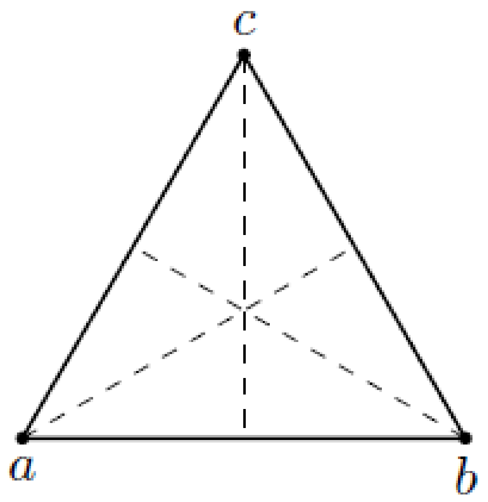
$P^0_{cs}(4459)$ $\Xi_b^- \to (J/\psi\Lambda)K^-$



Please be kind with the questions

Backup

The S_3 2d representation



$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad D(\sigma_1) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \qquad D(\sigma_2) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$$

$$D(\tau_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad D(\tau_2) = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \qquad D(\tau_3) = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

$a \leftrightarrows b$ $\qquad$ $a \leftrightarrows c$ $\qquad$ $b \leftrightarrows c$

$$D(\tau_1)M^\lambda = M^\lambda$$
$$D(\tau_1)M^\rho = -M^\rho$$

$\Rightarrow$

$$D(\tau_2)M^\lambda = -\frac{\sqrt{3}}{2}M^\rho - \frac{1}{2}M^\lambda; \quad D(\tau_3)M^\lambda = \frac{\sqrt{3}}{2}M^\rho - \frac{1}{2}M^\lambda$$
$$D(\tau_2)M^\rho = \frac{1}{2}M^\rho - \frac{\sqrt{3}}{2}M^\lambda; \quad D(\tau_3)M^\rho = \frac{1}{2}M^\rho + \frac{\sqrt{3}}{2}M^\lambda$$

$-$	$\lvert(u_1,u_2)_{\bar{\mathbf{3}}},d\rangle_{\mathbf{8}}$	$\lvert(u_1,u_2)_{\mathbf{6}},d\rangle_{\mathbf{8}}$	$\lvert(u_2,d)_{\bar{\mathbf{3}}},u_1\rangle_{\mathbf{8}}$	$\lvert(u_2,d)_{\mathbf{6}},u_1\rangle_{\mathbf{8}}$
$\lvert(u_1,d)_{\bar{\mathbf{3}}},u_2\rangle_{\mathbf{8}}$	$+1/2$	$+\sqrt{3}/2$	$+1/2$	$-\sqrt{3}/2$
$\lvert(u_1,d)_{\mathbf{6}},u_2\rangle_{\mathbf{8}}$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}/2$	$-1/2$

The spin 1/2 octet of 56

$$\mathcal{P} = (\bar{c}c) \times \frac{-n}{\sqrt{2}} \left\{ \frac{1}{\sqrt{2}} \left((q_1^\phi, q_2^\psi) q_3^\phi - (q_1^\psi, q_2^\phi) q_3^\phi \right) + \frac{1}{\sqrt{6}} \left([q_1^\phi, q_2^\psi] q_3^\phi + [q_1^\psi, q_2^\phi] q_3^\phi - 2[q_1^\phi, q_2^\phi] q_3^\psi \right) \right\}$$

⇓ It can be recast

$$\mathcal{P} = (\bar{c}c) \times \frac{n}{\sqrt{3}} \left\{ [q_1^\phi, q_2^\phi] q_3^\psi + \text{cyclic permutations in 1,2,3} \right\}$$

This tensor has to be combined with $S(\mathbf{56}) = \frac{1}{\sqrt{2}}[F^\rho \chi^\rho + F^\lambda \chi^\lambda]$

$-$	$\lvert(u_1,u_2)_{\bar{\mathbf{3}}},d\rangle_{\mathbf{8}}$	$\lvert(u_1,u_2)_{\mathbf{6}},d\rangle_{\mathbf{8}}$	$\lvert(u_2,d)_{\bar{\mathbf{3}}},u_1\rangle_{\mathbf{8}}$	$\lvert(u_2,d)_{\mathbf{6}},u_1\rangle_{\mathbf{8}}$
$\lvert(u_1,d)_{\bar{\mathbf{3}}},u_2\rangle_{\mathbf{8}}$	$+1/2$	$+\sqrt{3}/2$	$+1/2$	$-\sqrt{3}/2$
$\lvert(u_1,d)_{\mathbf{6}},u_2\rangle_{\mathbf{8}}$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}/2$	$-1/2$

We may bring the d_i in the same position. One obtains the superposition of three replicas which are cyclical permutations of the assignments of u and d flavours to quarks 1, 2, 3. In practical calculations, we may decide that the d flavour is in quark q_3

$$\mathcal{P}^{\mathbf{56}}_{\mathbf{8},1/2} = \sqrt{3}\,\mathcal{P} \otimes S(\mathbf{56}) = (\bar{c}c) \times \frac{n}{\sqrt{2}} \left\{ \frac{1}{\sqrt{2}} \left((u_1^\phi, u_2^\psi)_1 d^\phi - (u_1^\psi, u_2^\phi)_1 d^\phi \right) + \frac{1}{\sqrt{6}} \left([u_1^\phi, u_2^\psi]_1 d^\phi + [u_1^\psi, u_2^\phi]_1 d^\phi - 2[u_1^\phi, u_2^\phi]_1 d^\psi \right) \right\}$$

Young Tableau

$$M^{abc} \equiv \left(T^{abc}, \begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \cline{1-1} \end{array} \right) = T^{abc} + T^{bac} - T^{cba} - T^{bca}$$

$$\widetilde{M}^{abc} \equiv \left(T^{abc}, \begin{array}{|c|c|}\hline a & c \\ \hline b & \\ \cline{1-1} \end{array} \right) = T^{abc} + T^{cba} - T^{bac} - T^{cab}$$

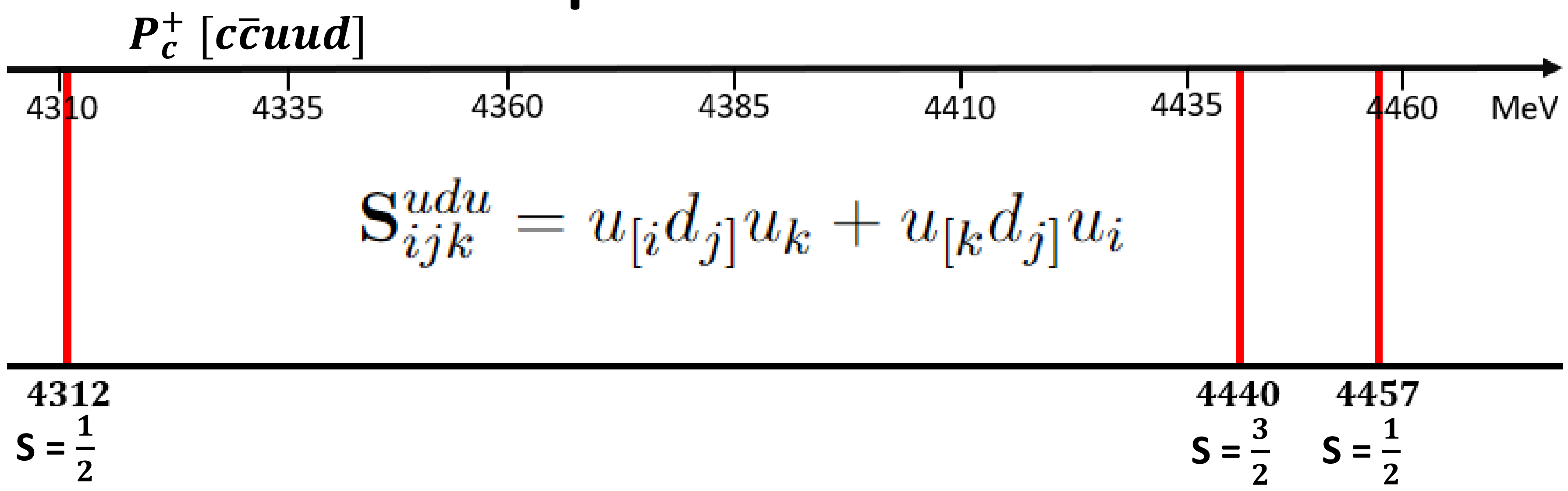
$$\overline{M}^{abc} \equiv \left(T^{abc}, \overline{\begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \cline{1-1} \end{array}} \right) = T^{abc} - T^{bac} + T^{cba} - T^{bca}$$

$$S^{abc}_{ijk} \equiv \overline{M}^{ijk}_2 \widetilde{M}^{abc}_1 + \overline{M}^{ikj}_2 M^{acb}_1$$

$$A^{abc}_{ijk} \equiv \widetilde{M}^{ijk}_2 M^{abc}_1 - M^{ikj}_2 \widetilde{M}^{acb}_1$$

(Anti-) Symmetric under the exchange of any pair e.g. $(\boldsymbol{i}, \boldsymbol{a}) \leftrightarrow (\boldsymbol{b}, \boldsymbol{j})$

J parameters: how do we fit



1. $J_S^{uu} = J_S^{qq} > 0, \qquad J_A^{ud} = J_A^{qq} < 0$

2. $$\begin{cases} M_{P_c^+}(4457) - M_{P_c^+}(4312) = 2|J_S^{qq} - J_A^{qq}| \\ M_{P_c^+}(4440) - \frac{M_{P_c^+}(4457)+M_{P_c^+}(4312)}{2} = -2J_A^{qq} - J_S^{qq} \end{cases}$$

3. $J_S = -\frac{1}{2}J_A \longrightarrow 2(C(\mathbf{6}) - 2C(\mathbf{3})) = -(C(\bar{\mathbf{3}}) - 2C(3))$

We search for a solution of the sistem that respects the constraints on the sign an ratio of the couplings.

$$\mathbf{S}_{ijk}^{uds} = u_{[i}d_{j]}s_k + u_{[k}d_{j]}s_i \quad \left[\begin{array}{l} J_A^{qs} = k\, J_S^{qs} \\ J^{ds} = J^{qs} = \dfrac{J_S^{qs} + J_A^{qs}}{2} = \dfrac{1+k}{2} J_S^{qs} \\ k_\kappa = \dfrac{\kappa_A^{qs}}{\kappa_A^{qq}} \approx 0.60 \longrightarrow J_A^{qs} = k_\kappa\, J_A^{qq} \end{array}\right.$$

	Symmetric [MeV]	Antysimmetric [MeV]	No symmetry [MeV]
J^{qq}	$29.9^{+2.5}_{-2.8}$	$-42.8^{+2.4}_{-1.6}$	-
J^{qs}	17.9 ± 2	-25.7 ± 2	-3.9 ± 2