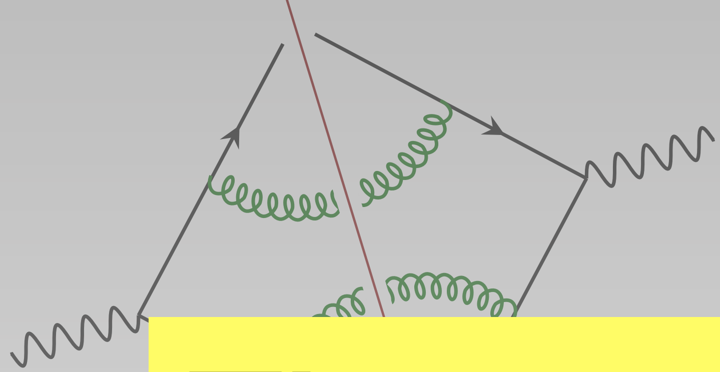
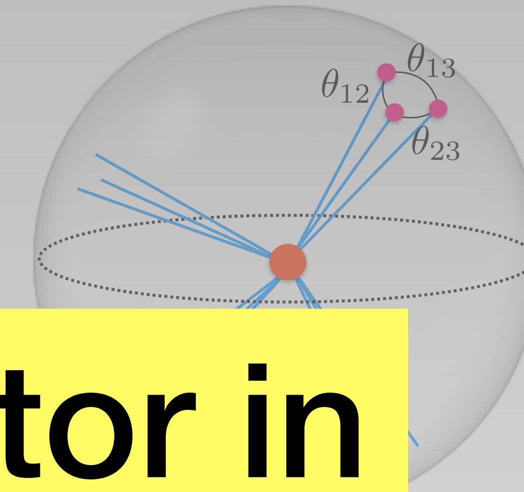




Three-point energy correlator in N=4 and QCD

$$\delta\left(x_1 - \frac{1-\cos\theta_{jk}}{2}\right)\delta\left(x_2 - \frac{1-\cos\theta_{ik}}{2}\right)\delta\left(x_3 - \frac{1-\cos\theta_{ij}}{2}\right)$$

Xiaoyuan Zhang
Harvard University

Based on:

- [Kai Yan and XYZ, *Phys.Rev.Lett.* 129 (2022) 2, 021602, arXiv: 2203.04349]
- [Tong-Zhi Yang and XYZ, *JHEP* 09 (2022) 006, arXiv: 2208.01051 and in progress work]

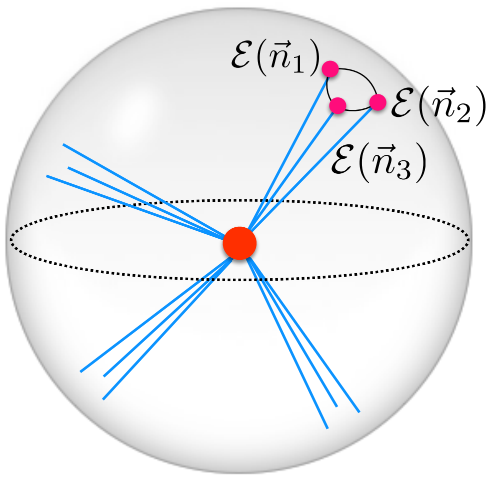
Also on:

- [Aurélien Dersy, Matthew Schwartz and XYZ, arXiv: 2206.04115]

Energy correlators

- Energy correlator is defined as the Wightman correlation function of the energy flow operators

$$\mathcal{E}(\vec{n}) = \int_{-\infty}^{\infty} d\tau \lim_{r\to\infty} r^2 n^i T_{0i}(t=\tau+r, r\vec{n})$$



N-point energy correlator ⇒ (N+2)-point correlation function

[Hofman, Maldacena, 0803.1467]

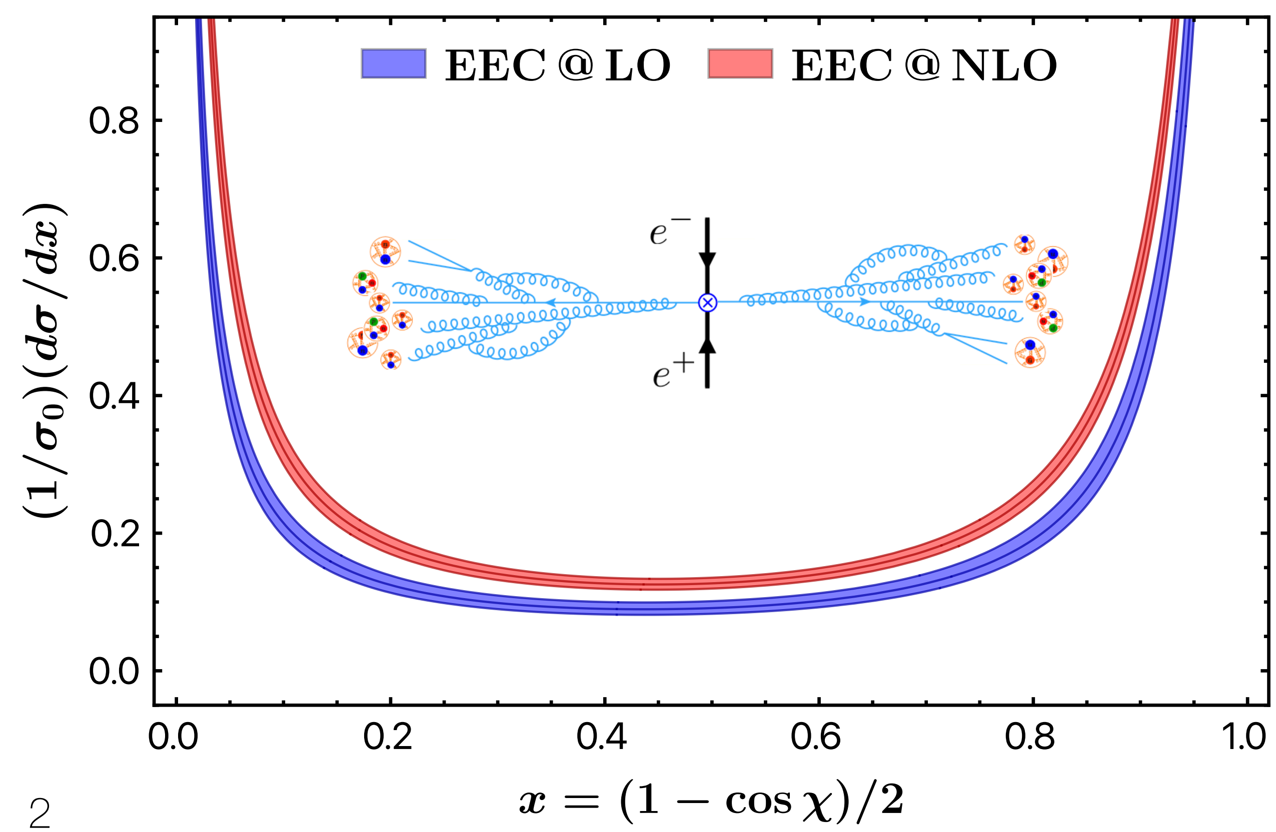
$$\int [d\Omega_{\vec{n}_i} \delta(\vec{n}_i \cdot \vec{n}_{i+1} - \cos\chi_i)] \times \int d^4x e^{iqx} \langle \Omega | J^\mu(x) \mathcal{E}(\vec{n}_1) \cdots \mathcal{E}(\vec{n}_N) J_\mu(0) | \Omega \rangle$$

- Perturbative: as a energy-weighted differential cross section,

  e.g. EEC (N=2)

$$\mathrm{EEC}(\chi) = \sum_{i,j} \int d\sigma \frac{E_i E_j}{Q^2} \delta(\vec{n}_i \cdot \vec{n}_j - \cos\chi)$$

[Basham, Brown, Ellis, Love, 1978]



Motivation

- Energy correlator is almost the simplest jet observables:
    - ✦ No jet algorithm needed;
    - ✦ Analytic measurement function;
    - ✦ Soft divergence suppressed;

$$\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\cdots\mathcal{E}(\vec{n}_N)\rangle$$

Analytic EEC

NNLO in N=4: [1311.6800; 1903.05314]
NLO in QCD: [1801.03219; 1903.07277]

Effective field theory

Collinear resummation: [Konishi et. al, 1978; Richards et.al, 1982; 1905.01444; 1905.01310]
Back-to-back resummation: [Collins, Soper, 1982; Kardos et. al, 2018; 1801.02627; 2012.07859; 2205.02242]

CFT

Spinning gluon: [2011.02492, 2104.00009]
Non-Gaussianities: [2205.02857]
Light-ray OPE: [1905.01311, 2010.04726, 2202.04090]
Power correction: [2301.03616]

Precision measurements

α_s determination: [1804.09146]
Dead cone effect: [2207.03511]
Top quark mass: [2201.08393]
Parton fragmentation: [2201.07800, 2210.10061]

Non-perturbative structure

Renormalon: [2305.19311]
Track function: [2108.01674, 2201.05166, 2210.10058]

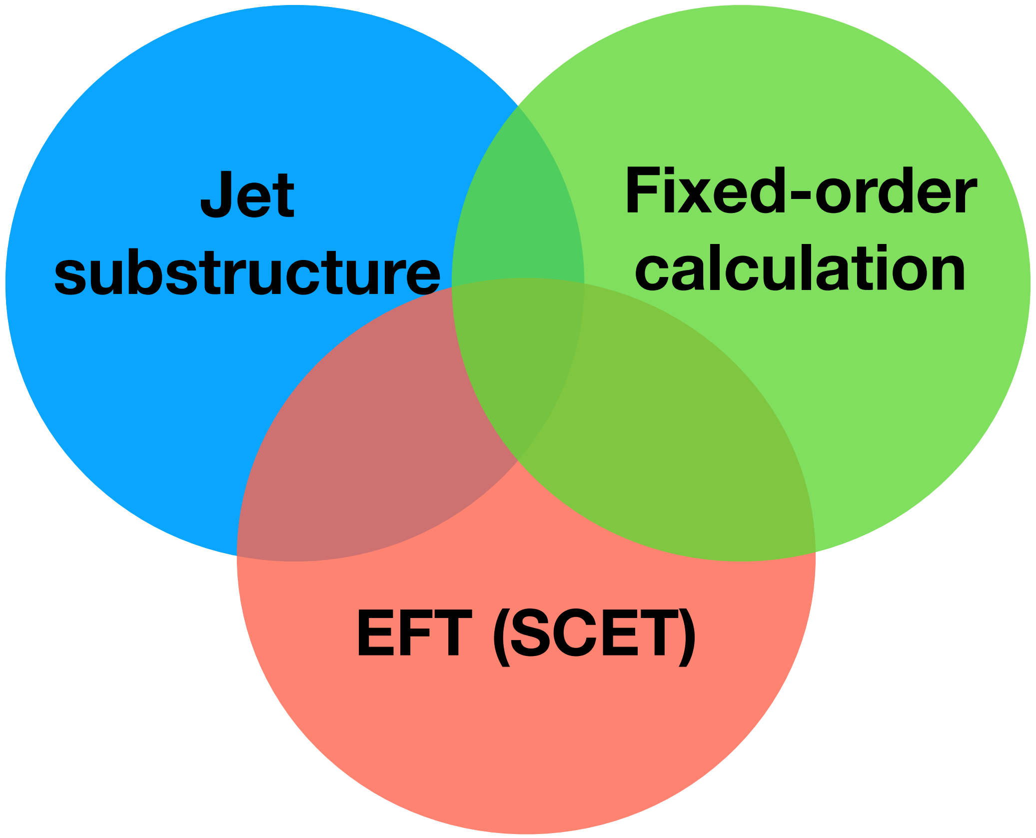
TMD physics

Transverse EEC: [1205.1689, 1901.04497 etc.]
DIS EEC: [2006.02437, 2102.05669]

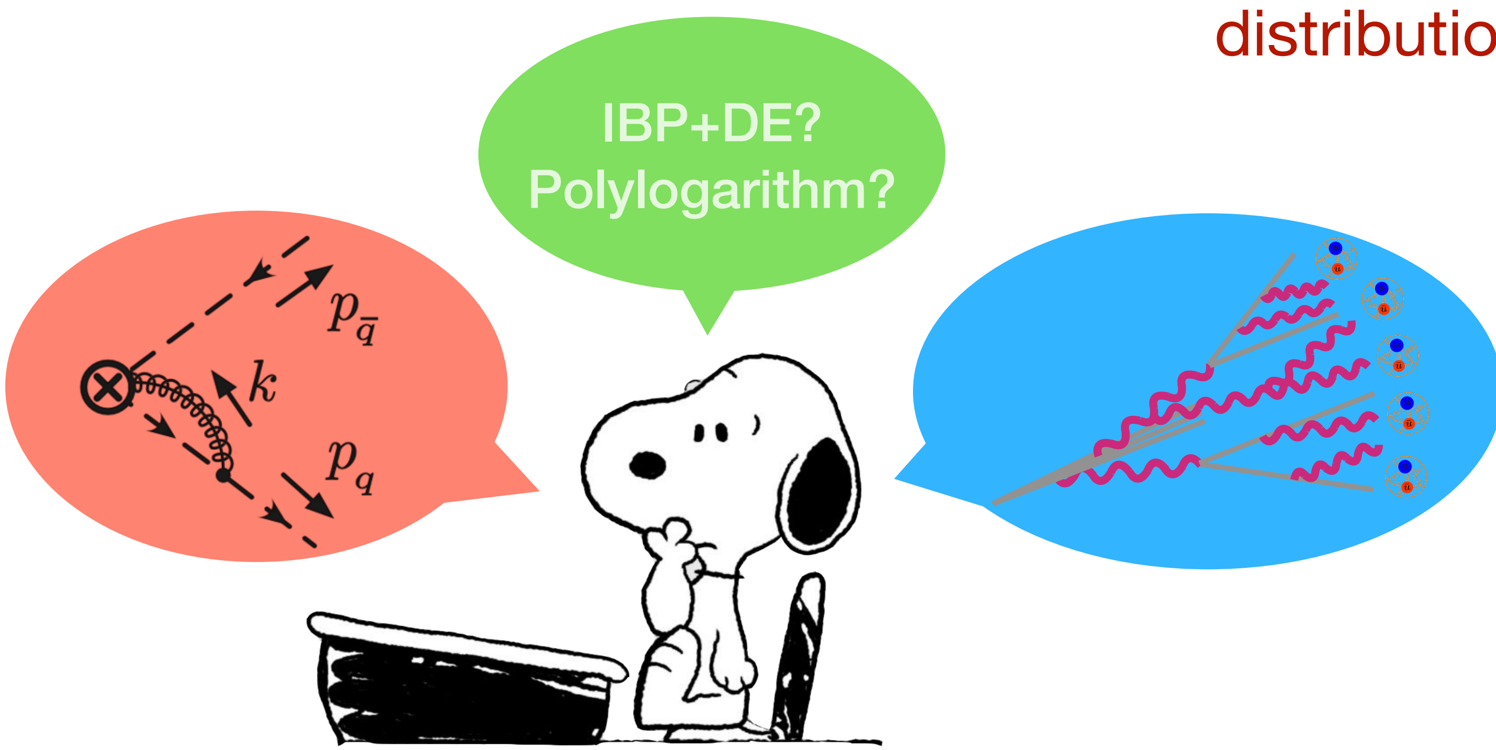
Nuclear theory

Nuclear EEC: [2209.02080, 2301.01788, 2303.01530]
QGP: [2209.11236, 2303.03413, 2303.08143]

Multi-point energy correlators



- N-point (multi-point) energy correlator is a perfect candidate for understand the internal structure of a jet, allowing approaches both from analytic fixed-order calculation and EFT factorization
- Compared to EEC, N-point gives rise to more interesting jet substructure (involving energy flows)
- How do we study a multi-dimensional distribution?



Jet substructure aspect

- Projected N-point energy correlator [Chen, Moult, **XYZ**, Zhu, 2004.11381]

projected to 1D

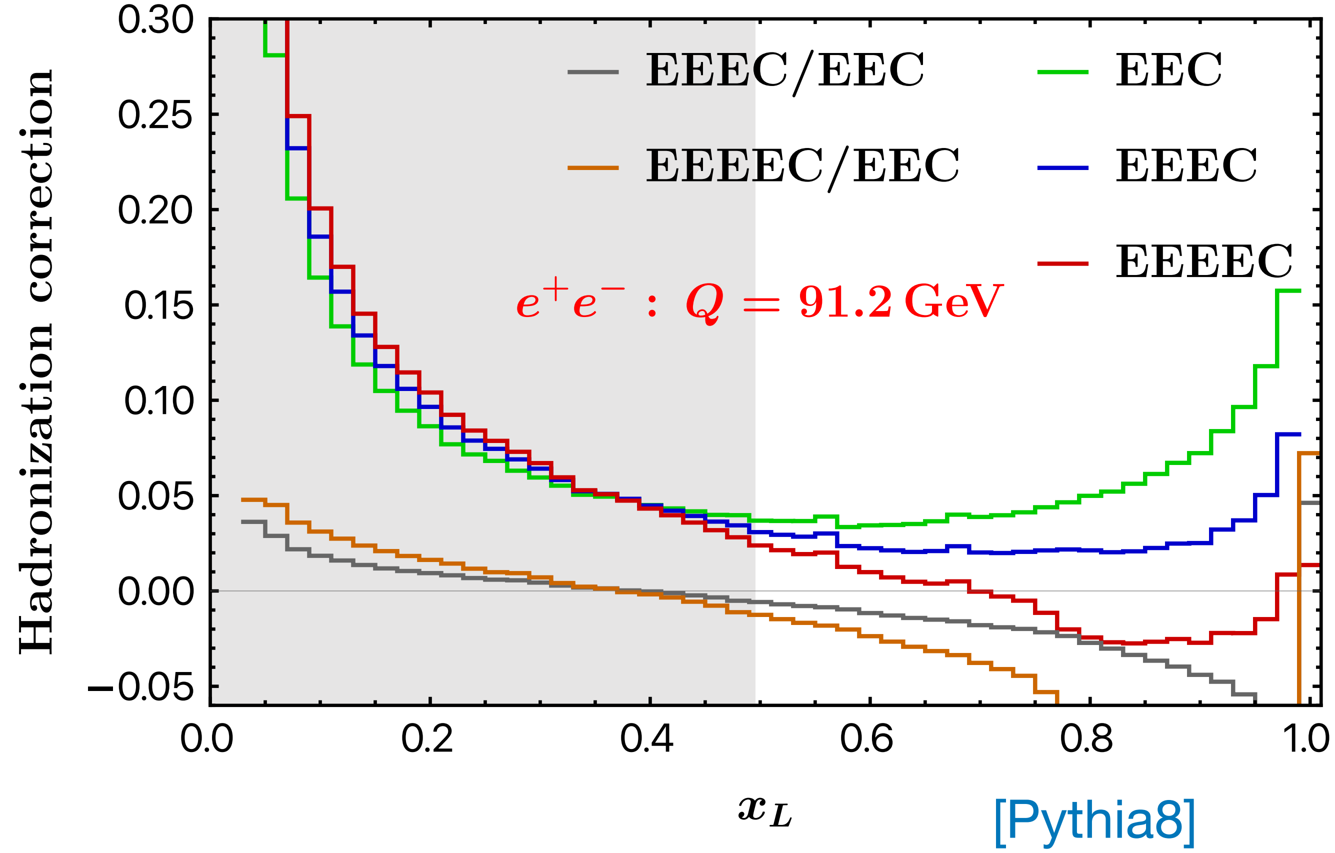
$$\frac{d\sigma^{[N]}}{dx_L} = \sum_n \sum_{1\le i_1,\cdots i_N\le n} \int d\sigma \frac{\prod_{a=1}^N E_{i_a}}{Q^N} \delta(x_L - \max\{x_{i_1,i_2}, x_{i_1,i_3}, \cdots x_{i_{N-1},i_N}\})$$

$$x_{i,j} \equiv (1-\cos\theta_{ij})/2$$

- N can also be extended to a complex number ν with $\mathrm{Re}[\nu] > 0$

- One of the main measurement at collider physics is the α_s determination

simple power correction compared to other event shapes (heavy jet mass, C parameter…)



[Pythia8]

Jet substructure aspect

- EEEC in the triple collinear limit $x_1,\, x_2,\, x_3 \sim \lambda^2$ [Chen, Luo, Moult, Yang, **XYZ**, Zhu, 1912.11050]

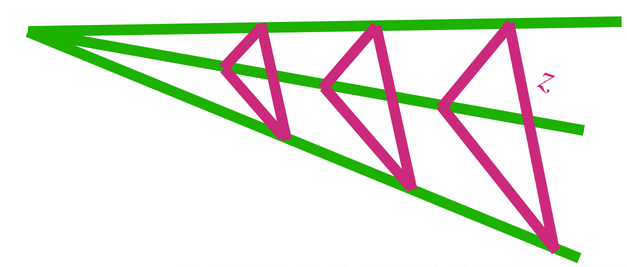
$$\frac{d^3\sigma}{dx_1 dx_2 dx_3} = \int \mathrm{dPS}\, |\mathcal{M}|^2\, \delta\left(x_3 - \frac{1-\cos\theta_{12}}{2}\right) \delta\left(x_2 - \frac{1-\cos\theta_{13}}{2}\right) \delta\left(x_1 - \frac{1-\cos\theta_{23}}{2}\right)$$

$$\overset{\text{coll}}{\approx} \frac{1}{x_L^3} \times G(z,\bar{z})$$

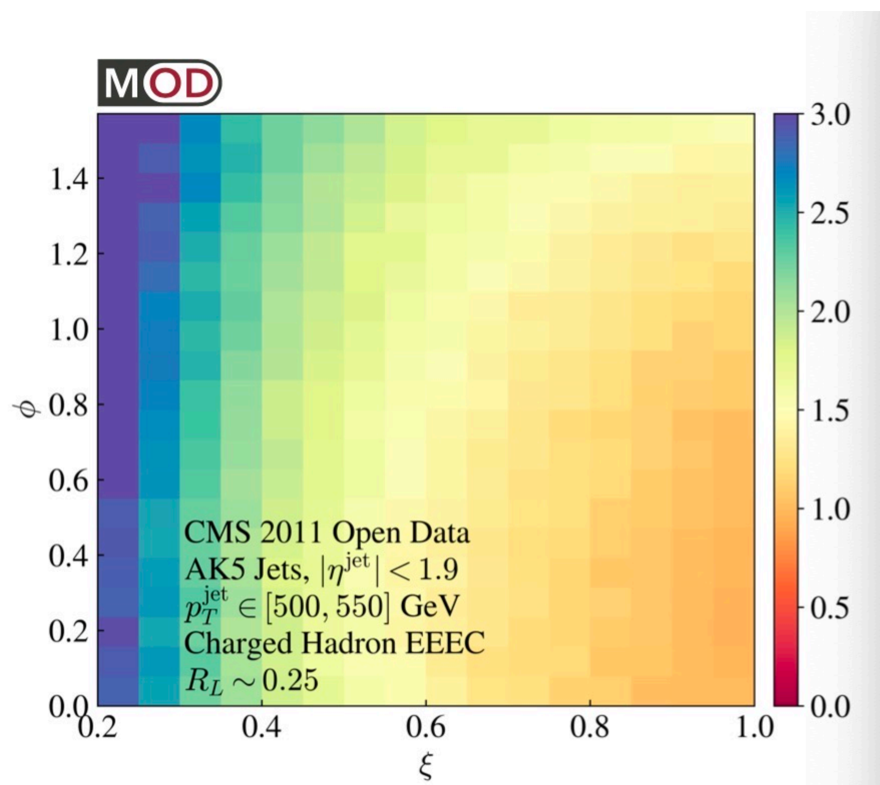
$\Rightarrow$ scaling $\times$ shape function

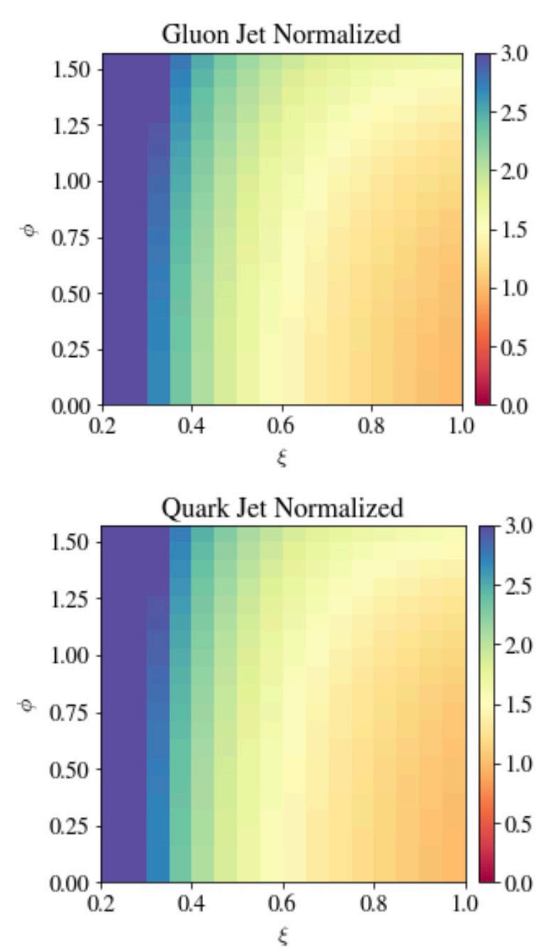
factorized to 2D

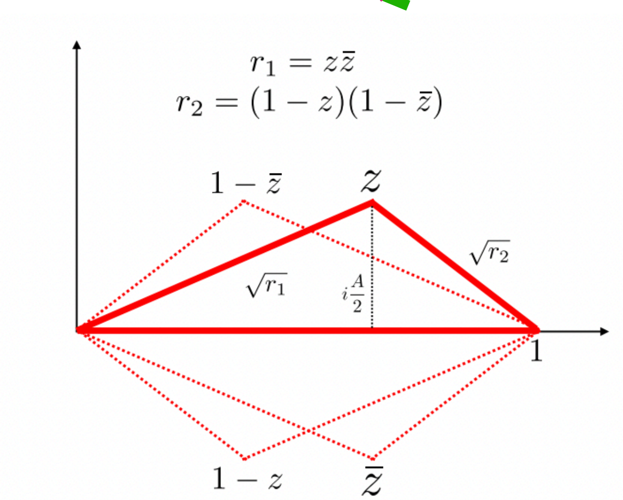
with $x_1 = x_L\, z\bar{z},\; x_2 = x_L\,(1-z)(1-\bar{z}),\; x_3 = x_L$



[Komiske, Moult, Thaler, Zhu, 2201.07800]







$$\xi = \frac{x_S}{x_M}, \quad \phi = \arcsin\sqrt{1 - \frac{(x_L - x_M)^2}{x_S^2}}$$

Figures from Chen & Zhu

EFT aspect: collinear limit

- Collinear factorization for projected N-point correlator: (N=2 is the EEC)

Cumulant: $$\Sigma^{[N]}\left(x_L, \ln\frac{Q^2}{\mu^2}\right) = \frac{1}{\sigma_{\rm tot}}\int_0^{x_L} dx'_L \, \frac{d\sigma^{[N]}}{dx'_L}\left(x'_L, \ln\frac{Q^2}{\mu^2}\right)$$

$$\Rightarrow \Sigma^{[N]}\left(x_L, \ln\frac{Q^2}{\mu^2}\right) = \int_0^1 dx\, x^N \vec{J}^{[N]}\left(\ln\frac{x_L x^2 Q^2}{\mu^2}\right) \cdot \vec{H}\left(x, \ln\frac{Q^2}{\mu^2}\right)$$

jet function hard function

- N=2 EEC for $N=4$ and e^+e^-: NNLL [Dixon, Moult, Zhu, 1905.01310]
- Projected N-/(ν-)point for e^+e^-: NLL [Chen, Moult, **XYZ**, Zhu, 2004.11381]
- Projected N=3-6-point for pp: NLL [Lee, Meçaj, Moult, 2205.03414]
- Projected 3-point for both e^+e^- and pp: NNLL [Chen, Gao, Li, Xu, **XYZ**, Zhu, to appear]

- It is conjectured that this functional form works for triple-collinear EEEC with a fixed shape (fixed z)
- The collinear squeezed factorization (collinear+ $z \to 0$) is also understood with light-ray OPE

EFT aspect: back-to-back limit

- Back-to-back factorization for EEC to all orders: [Moult, Zhu, 1801.02627]

$$\frac{d\sigma^{[2]}}{dz} = \frac{\hat{\sigma}_0}{2} H_{q\bar{q}}(Q,\mu) \int \frac{d^2\vec{b}_T d^2\vec{q}_T}{(2\pi)^2} e^{i\vec{q}_T\cdot\vec{b}_T} \delta\left(1 - z - \frac{q_T^2}{Q^2}\right)$$
$$\times J_q\left(b_T,\mu,\frac{\nu}{Q}\right) J_{\bar{q}}\left(b_T,\mu,\frac{\nu}{Q}\right) \tilde{S}_q(b_T,\mu,\nu)$$

TMD soft function

- NNLL: [Collins, Soper, 1981], [Ellis, Richards, Stirling, 1984], [Florian, Grazzini, 0407241]
- $\mathrm{N^3LL'}$ and $\mathrm{N^4LL}$: [Ebert, Mistlberger, Vita, 2012.07859], [Duhr, Mistlberger, Vita, 2205.02242]
- For three-point correlator (EEEC) and beyond, there is no unique back-to-back limit

  Instead, we have three back-to-back limits: $x_i \to 1, i = 1{,}2{,}3$ and a coplanar limit
- The fact that EEEC has multiple overlapping kinematic limits requires the factorization from multi-scale EFTs and the joint resummation in a 3D distribution
- How about the fixed-order side:

  This talk: first look at the fixed-order result (LO EEEC in N=4 and QCD)

EEEC in $N = 4$ SYM theory

[Yan, **XYZ**, 2203.04349]

- The leading order EEEC with arbitrary angle dependence:

$$\frac{1}{\sigma_0}\frac{d^3\sigma}{dx_1 dx_2 dx_3} = \sum_{i,j,k}\int \mathrm{dPS}_4\, |\mathcal{M}_{\mathcal{N}=4}|^2 \boxed{\frac{E_i E_j E_k}{Q^3}}$$

$$\times\, \delta\left(x_3 - \frac{1-\cos\theta_{12}}{2}\right)\delta\left(x_2 - \frac{1-\cos\theta_{13}}{2}\right)\delta\left(x_1 - \frac{1-\cos\theta_{23}}{2}\right)$$

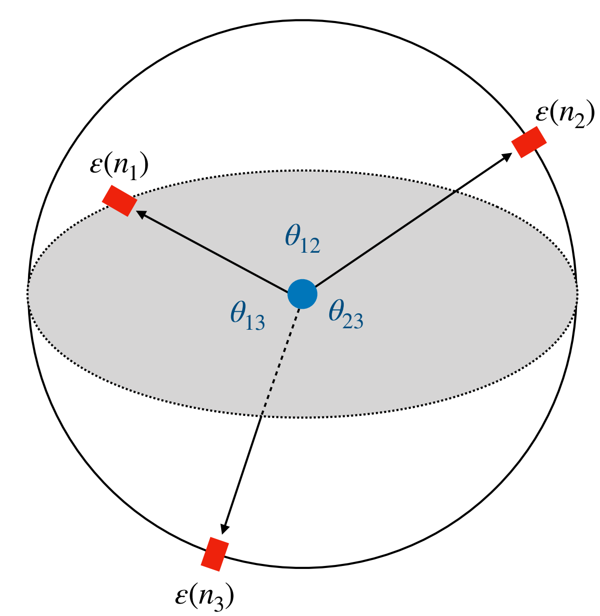
No soft divergence $\Rightarrow$ $d = 4$ finite

- The four-particle phase space:

$$\mathrm{dPS}_4 = (2\pi)^{4-3d}(Q^2)^{3-\frac{d}{2}}2^{1-2d}\prod_{i<j} ds_{ij}\, d\Omega_{d-1}\, d\Omega_{d-2}\, d\Omega_{d-3}$$

$$\times\, \delta\left(Q^2 - \sum_{i<j} s_{ij}\right)(-\Delta_4)^{\frac{d-5}{2}}\;\Theta\left(-\Delta_4\right)$$

3 detectors separated by angle θ_{ij}



Gram determinant:

$$\Delta_4 = \lambda(s_{12}s_{34},\, s_{13}s_{24},\, s_{14}s_{23}),$$

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2(xy + xz + yz)$$

- Matrix elements: tree-level $1 \to 4$ squared form factor $\left|\langle p_1 p_2 p_3 p_4 | \mathrm{tr}[\phi^2] | \rangle\right|^2$

$$|\mathcal{M}_{\mathcal{N}=4}|^2 \sim \frac{1}{24}\left[\frac{1}{s_{13}s_{14}s_{23}s_{24}} + \frac{4}{s_{123}s_{124}s_{13}s_{14}} + \frac{4}{s_{123}s_{124}s_{14}s_{23}} + \mathrm{perm}\right]$$

Calculation setup

- Attempt using traditional IBP+DE method: for each δ measurement function

Reverse unitarity: $$\delta\left(\mathcal{M}_{jk}(x_i)\right) = \frac{1}{2\pi i}\left(\frac{1}{\mathcal{M}_{jk}(x_i) - i\epsilon} - \frac{1}{\mathcal{M}_{jk}(x_i) + i\epsilon}\right)$$

$$\mathcal{M}_{jk}(x_i) \equiv (p_j \cdot Q)(p_k \cdot Q)(\vec{n}_j \cdot \vec{n}_k - (1 - 2x_i)) = (p_j \cdot Q)(p_k \cdot Q)(2x_i) - p_j \cdot p_k$$

Additional IBP equation: $$\left[(p_j \cdot Q)(p_k \cdot Q)(2x_i) - p_j \cdot p_k\right]\left[\delta(\mathcal{M}_{jk}(x_i))\right]^k = \left[\delta(\mathcal{M}_{jk}(x_i))\right]^{k-1}$$

[Dixon, Luo, Shtabovenko, Yang, Zhu, 1801.03219]

- It turns out IBP reduction is NOT efficient (e.g. FIRE6) with three nonlinear propagators
- Instead, we expand the phase space in $d = 4 - 2\epsilon$ and do the integration directly
- Introduce a nice parameterization for all Mandelstam variables:

$$z_1 = \frac{2p_1 \cdot q}{Q^2}, \quad z_2 = \frac{2p_2 \cdot q}{Q^2}, \quad z_3 = \frac{2p_3 \cdot q}{Q^2},$$

$$\Rightarrow \begin{cases} z_1 = s_{12} + s_{13} + s_{14}, \\ z_2 = s_{12} + s_{23} + s_{24}, \\ z_3 = s_{13} + s_{23} + s_{34} \end{cases}$$

$$\Rightarrow \begin{cases} s_{12} = z_1 z_2 x_3, \\ s_{13} = z_1 z_3 x_2, \\ s_{23} = z_2 z_3 x_1, \\ s_{14} = z_1(1 - z_2 x_3 - z_3 x_2), \\ s_{24} = z_2(1 - z_1 x_3 - z_3 x_1), \\ s_{34} = z_3(1 - z_1 x_2 - z_2 x_1) \end{cases}$$

Phase space integration

- The main difficulty of integrating the four-particle phase space comes from the Gram determinant $\Theta(-\Delta_4)$. However, with our parameterization:

$$\begin{aligned}
&\int ds_{12}ds_{13}ds_{14}ds_{23}ds_{24}ds_{34}\,\Theta(-\Delta_4)\delta(\sum_{i<j} s_{ij}-1)\prod_{l=1,2,3}\delta\left(x_l-\frac{1-\cos\theta_{mn}}{2}\right)\\
=&\int dz_1dz_2dz_3(z_1^2z_2^2z_3^2)\Theta(-\Delta_4)\frac{1}{1-x_2z_1-x_1z_2}\delta\left(z_3-\frac{z_1+z_2-x_3z_1z_2-1}{z_1x_2+z_2x_1-1}\right)\\
=&\Theta(-\widetilde{\Delta}_4)\int dz_1dz_2dz_3(z_1^2z_2^2z_3^2)\frac{1}{1-x_2z_1-x_1z_2}\delta\left(z_3-\frac{z_1+z_2-x_3z_1z_2-1}{z_1x_2+z_2x_1-1}\right)\,,
\end{aligned}$$

with factorized Θ function:

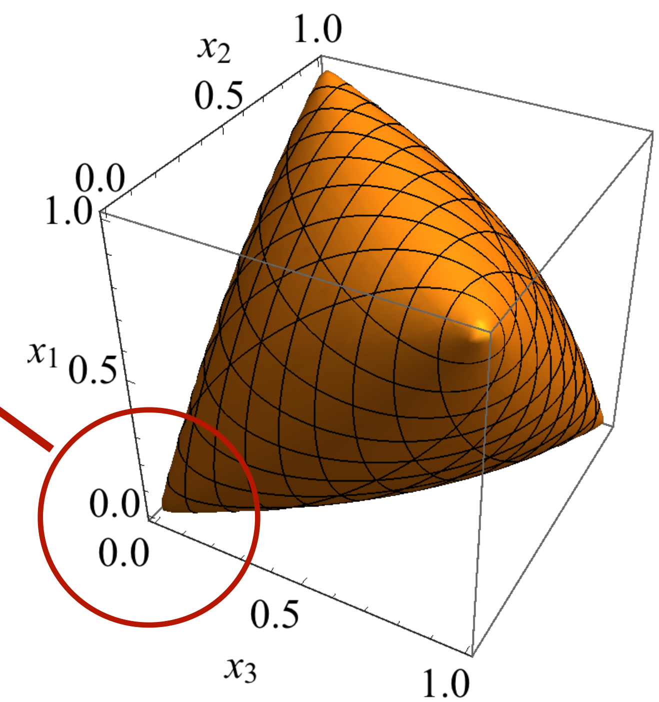
$$\frac{\Delta_4}{z_1^2z_2^2z_3^2}=x_1^2+x_2^2+x_3^2-2x_1x_2-2x_1x_3-2x_2x_3+4x_1x_2x_3\equiv\widetilde{\Delta}_4\,.$$

- In the triple collinear limit, $x_1, x_2, x_3\sim\lambda^2$, the kinematic space is reduced to a triangle

$$\widetilde{\Delta}_4\approx\widetilde{\Delta}_4^{\text{coll}}=x_1^2+x_2^2+x_3^2-2x_1x_2-2x_1x_3-2x_2x_3$$

where $\sqrt{x_1},\sqrt{x_2},\sqrt{x_3}$ form a triangle by Helen's area formula

('Zongzi'-shaped) kinematic space:



Phase space integration

- Eventually the EEEC integration takes the form

$$\int dz_1 dz_2 dz_3 \delta\left(z_3 - \frac{z_1 + z_2 - x_3 z_1 z_2 - 1}{z_1 x_2 + z_2 x_1 - 1}\right) f(x_1, x_2, x_3, z_1, z_2, z_3)$$

$$= \int_0^1 dz_1 \int_0^{\frac{1-z_1}{1-x_3 z_1}} dz_2 f\left(x_1, x_2, x_3, z_1, z_2, \frac{z_1 + z_2 - x_3 z_1 z_2 - 1}{z_1 x_2 + z_2 x_1 - 1}\right)$$

with two square roots

$$\sqrt{x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3} = \sqrt{\widetilde{\Delta}_4^{\mathrm{coll}}},$$

$$\sqrt{x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3 + 4x_1x_2x_3} = \sqrt{\widetilde{\Delta}_4}$$

- There are two different parameterizations to rationalize both of them

cross-ratio variables: $\frac{x_1}{x_3} = z\bar{z}, \quad \frac{x_2}{x_3} = (1-z)(1-\bar{z}), \quad x_3 = \frac{t^2 - (z-\bar{z})^2}{4z\bar{z}(1-z)(1-\bar{z})}$

$$\Rightarrow \quad \sqrt{x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3} = x_3(z-\bar{z}) = \frac{t^2 - (z-\bar{z})^2}{4z\bar{z}(1-z)(1-\bar{z})}(z-\bar{z}),$$

$$\sqrt{x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3 + 4x_1x_2x_3} = x_3 t = \frac{t^2 - (z-\bar{z})^2}{4z\bar{z}(1-z)(1-\bar{z})} t$$

The $\{t, z, \bar{z}\}$ variable set manifests the symmetry and geometry in the triple-collinear limit

Phase space integration

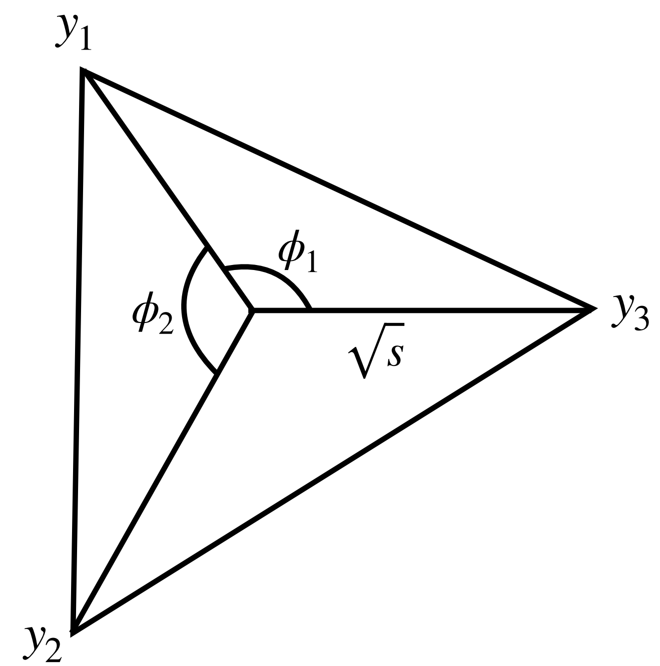
- The angles are mapped onto the the distances between three points on celestial sphere

celestial variables:

$$x_1 = \frac{|y_2 - y_3|^2}{(1+|y_2|^2)(1+|y_3|^2)},\ x_2 = \frac{|y_1 - y_3|^2}{(1+|y_1|^2)(1+|y_3|^2)},\ x_3 = \frac{|y_1 - y_2|^2}{(1+|y_1|^2)(1+|y_2|^2)}$$

further map to

$$y_1 = \sqrt{s}\,\underbrace{e^{i\phi_1}}_{\equiv \tau_1},\ y_2 = \sqrt{s}\,\underbrace{e^{i(\phi_1+\phi_2)}}_{\equiv \tau_1\tau_2},\ y_3 = \sqrt{s}$$



$$\begin{cases} \sqrt{\widetilde{\Delta}_4^{\text{coll}}} = \dfrac{s(1-\tau_1)(1-\tau_2)(1-\tau_1\tau_2)}{(1+s)^2\tau_1\tau_2} \\ \sqrt{\widetilde{\Delta}_4} = \dfrac{s(1-s)(1-\tau_1)(1-\tau_2)(1-\tau_1\tau_2)}{(1+s)^3\tau_1\tau_2} \end{cases}$$

The $\{s, \tau_1, \tau_2\}$ variable set is easier for understanding the alphabet and simplifying the result

- With either variable set, we can directly perform the integration over z_1, z_2 in HyperInt, and express the result in terms of GPLs (up to weight-2)

$$G(a_1, \cdots a_n; x) \equiv \int_0^x \frac{dt}{t-a_1} G(a_2, \cdots a_n; t); \quad G(;x) \equiv 1,\ G(\vec{0}_n; x) \equiv \frac{1}{n!}\ln^n(x)$$

[Panzer, 1403.3385]

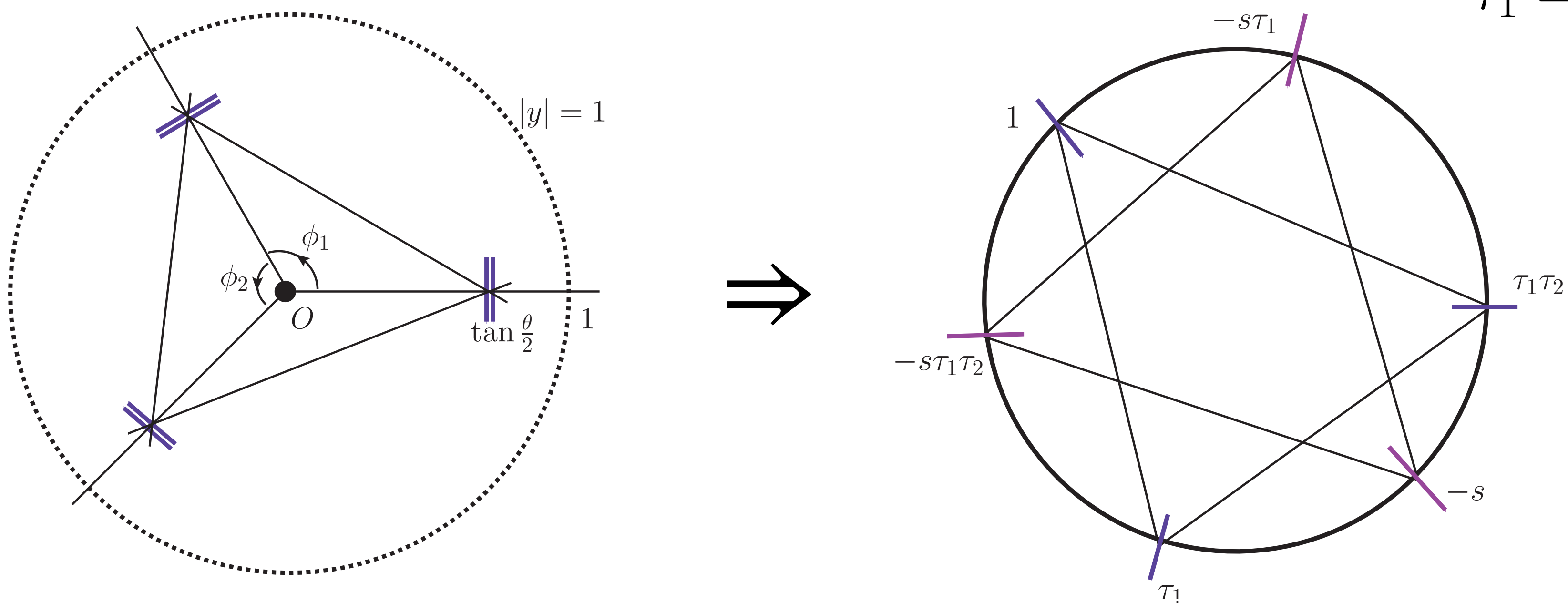
Embedding

- The EEEC kinematic space can be embedded into a hexagon on a unit circle. From the definition of three angles:

$$x_k = \frac{q^2(p_i \cdot p_j)}{2(q \cdot p_i)(q \cdot p_j)} = \frac{\langle p_i p_j \rangle \langle \xi_i \xi_j \rangle}{\langle p_i \xi_j \rangle \langle p_i \xi_j \rangle}, \qquad |\xi_j\rangle \equiv q|j]$$

We can embed: $|p_i\rangle \equiv |2i-1\rangle, |\xi_i\rangle \equiv |2i+2\rangle$

$\tau_1 = e^{i\phi_1}, \tau_2 = e^{i\phi_2}$



Symmetry: D_6 dihedral group $\qquad Z_i = \{1, -s\tau_1\tau_2, \tau_1, -s, \tau_1\tau_2, -s\tau_1\}, \; I = \infty$

- Cyclic permutation (σ) : $Z(a+2) = Z(a)$
- Parity (P) : $Z(a+3) = Z(a)$
- Flip (τ) : $Z(8-a) = Z(a)$

Symbol alphabet

- From the output of HyperInt, we found 17 symbols:

$$\{s-1,\, s,\, s+1,\, \tau_1-1,\, \tau_1,\, \tau_2-1,\, \tau_2,\, s+\tau_1,\, 1+s\tau_1,\, s+\tau_2,\, 1+s\tau_2,\, s+\tau_1\tau_2,\, 1+s\tau_1\tau_2,\, \tau_1\tau_2-1,\, \tau_1-\tau_2,\, \tau_1^2\tau_2-1,\, \tau_1\tau_2^2-1\}$$

- However it turns out that only 16 symbols are independent
- This can be written as a close set under D_6 using three conformal invariant ratios:

$$u_1 \equiv -\frac{\langle 51\rangle\langle 62\rangle\langle 43\rangle}{\langle 35\rangle\langle 16\rangle\langle 24\rangle} = -\frac{s+\tau_1}{1+s\tau_1},$$
$$u_2 \equiv -\frac{\langle 31\rangle\langle 5I\rangle}{\langle 15\rangle\langle I3\rangle} = \frac{\tau_1-1}{1-\tau_1\tau_2},$$
$$u_3 \equiv -\frac{\langle 13\rangle\langle 56\rangle}{\langle 35\rangle\langle 61\rangle} = \frac{(1-\tau_1)(s+\tau_2)}{(1-\tau_2)(1+s\tau_1)},$$

$$\{u_1\,, 1+u_1\,, u_2\,, 1+u_2\,, u_3\,, 1+u_3\,, u_1+u_3\,, 1+u_1+u_3\,, u_2+u_3+u_2u_3\,, u_1+u_2+u_3+u_2u_3\,, 1+u_1+u_2+u_3+u_2u_3\,, 1+u_1+u_2+2u_3+u_2u_3\,, u_2+u_1u_2+u_2^2+u_3+2u_2u_3+u_2^2u_3\,, 1+u_2+u_1u_2+u_2^2+u_3+2u_2u_3+u_2^2u_3\,, 1+u_1+u_2+u_1u_2+u_2^2+u_3+2u_2u_3+u_2^2u_3\,, 1+u_1+u_2+u_1u_2+u_2^2+2u_3+2u_2u_3+u_2^2u_3\}$$

- Finally we are able to reconstruct the minimal EEEC function space (up to weight-2)

Function space

- Arguments: 15 conformal invariant ratios that cover the χ- coordinate in [Golden, Paulos, Spradlin, Volovich, 1401.6446]; additional variable $r_1 \equiv -\frac{\langle 14\rangle\langle 3I\rangle}{\langle 13\rangle\langle 4I\rangle}$ and the images under D_6
- First entry conditions: (1). Single-value requirement in the physical region; (2). Free of logarithmic divergence in the triple collinear limit $s \to 0$
- Eventually, we find 3 weight-1 basis and 11 weight-2 basis

$$\frac{d^3\sigma}{dx_1 dx_2 dx_3} = \sum_{i=1}^{3} b_i^{(1)} f_i + \sum_{i=1}^{11} b_i^{(2)} g_i + \text{perm of } x_1, x_2, x_3$$

- Weight-1 is simple

$$\frac{\tilde{\Delta}_4^{\text{coll}} + 4x_3^2 x_2 - 4x_3 x_2^2}{x_1^2 x_2^2 x_3^2} \underbrace{\ln(1-x_3)}_{f_1} + \frac{2(x_3 - x_1)}{x_3 x_2^2 x_1} \underbrace{\ln \frac{x_1}{x_3}}_{f_2} - \frac{\sqrt{\tilde{\Delta}_4}}{x_3 x_2^2 x_1^2} \underbrace{\ln \frac{2 - x_1 - x_2 - x_3 - \sqrt{\tilde{\Delta}_4}}{2 - x_1 - x_2 - x_3 + \sqrt{\tilde{\Delta}_4}}}_{f_3}$$

with

$$\widetilde{\Delta}_4 = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3 + 4x_1x_2x_3$$
$$\widetilde{\Delta}_4^{\text{coll}} = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$$

Function space

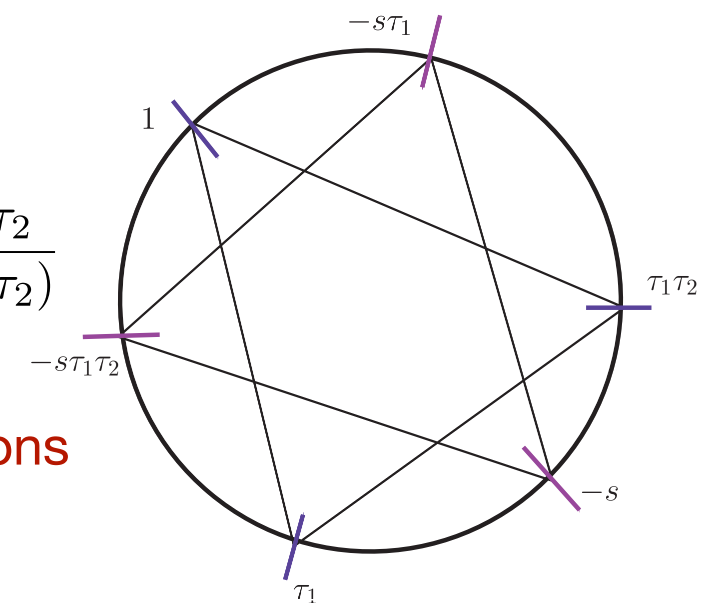
- For weight-2 functions, we need to introduce a few variables from the embedding

$$z_1 \equiv \frac{\langle 56\rangle\langle 13\rangle}{\langle 35\rangle\langle 16\rangle} = \frac{(\tau_1 - 1)(s+\tau_2)}{(1-\tau_2)(1+s\tau_1)}, \quad \bar{z}_1 \equiv P(z_1) = \frac{\langle 23\rangle\langle 46\rangle}{\langle 26\rangle\langle 34\rangle} = \frac{(1+s\tau_2)(1-\tau_1)}{(s+\tau_1)(1-\tau_2)}$$

$$w_1 \equiv \frac{\langle 16\rangle\langle 25\rangle}{\langle 56\rangle\langle 12\rangle} = -\frac{(1+s)(1+s\tau_1)\tau_2}{(s+\tau_2)(1+s\tau_1\tau_2)}, \quad \bar{w}_1 \equiv P(w_1) = \frac{\langle 34\rangle\langle 25\rangle}{\langle 23\rangle\langle 45\rangle} = -\frac{(1+s)(s+\tau_1)\tau_2}{(1+s\tau_2)(s+\tau_1\tau_2)}$$

$$v_1 \equiv \frac{\langle 26\rangle\langle 35\rangle}{\langle 56\rangle\langle 23\rangle} = -\frac{s(1-\tau_2)^2}{(s+\tau_2)(1+s\tau_2)}$$

$\{z_2, z_3, \cdots\}$ is obtained by cyclic permutations



$$g_1 = \mathrm{Li}_2(-v_2)$$
$$g_2 = \mathrm{Li}_2(1+w_3) + \mathrm{Li}_2(1+\bar{w}_3) + 2\,\mathrm{Li}_2(-v_3) - \mathrm{Li}_2(1+w_1) - \mathrm{Li}_2(1+\bar{w}_1) - 2\,\mathrm{Li}_2(-v_1)$$
$$g_3 = \mathrm{Li}_2(-z_2) - \mathrm{Li}_2(-\bar{z}_2) + \frac{1}{2}\ln|z_2|^2 \ln\frac{1+z_2}{1+\bar{z}_2}$$
$$g_4 = \mathrm{Li}_2(1+w_1) - \mathrm{Li}_2(1+\bar{w}_1) + \mathrm{Li}_2(1+w_2) - \mathrm{Li}_2(1+\bar{w}_2) + \mathrm{Li}_2(1+w_3) - \mathrm{Li}_2(1+\bar{w}_3)$$
$$g_5 = \pi^2$$
$$g_6 = \ln^2\frac{\bar{w}_1}{w_1}$$
$$g_7 = \ln\frac{\bar{w}_1}{w_1}\ \ln|z_2|^2$$
$$g_8 = \ln(1+v_3)\ln|z_1|^2 - \ln(1+v_1)\ln|z_3|^2$$

Additional variable under D_6

$$r_a = \frac{\langle a\, a+3\rangle\langle a+2\, I\rangle}{\langle a\, a+2\rangle\langle I\, a+3\rangle}, \quad \bar{r}_a = \frac{\langle a+5\, a+2\rangle\langle a+3\, I\rangle}{\langle a+5\, a+3\rangle\langle I\, a+2\rangle}$$

$$g_{11} = \sum_{i=1}^{6} \mathrm{Li}_2(-r_i) - \mathrm{Li}_2(-\bar{r}_i) + \frac{1}{2}\ln|r_i|^2\ln\frac{1+r_i}{1+\bar{r}_i}$$
$$g_9 = \frac{1}{2}\mathrm{Li}_2\left(1-\frac{\bar{w}_1}{w_1}\right) - \frac{1}{2}\mathrm{Li}_2\left(1-\frac{w_1}{\bar{w}_1}\right)$$
$$g_{10} = \mathrm{Li}_2(1-|z_2|^2) + \frac{1}{2}\ln|z_2|^2\ln|1-z_2|^2$$

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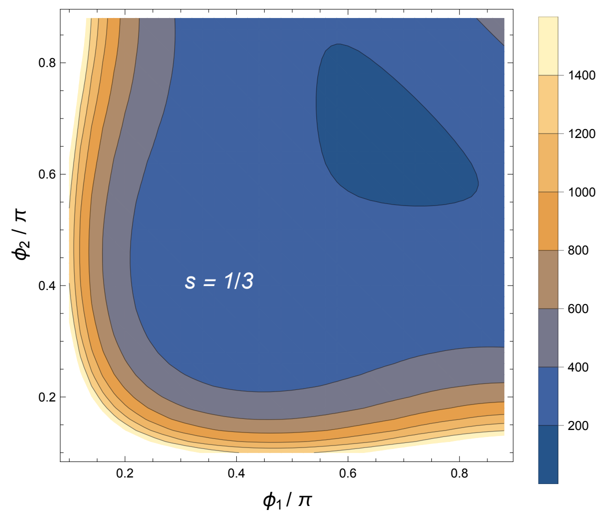
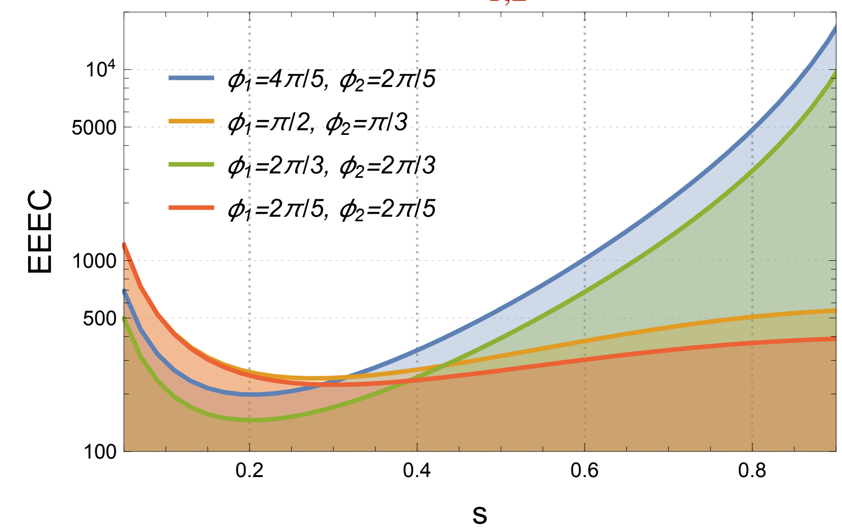
$\mathcal{N} = 4$ result

- Visualize the 3D distribution:

Fixed s



Fixed $\phi_{1,2}$



- Cross check with the triple collinear limit: $\dfrac{d^3\sigma}{dx_1 dx_2 dx_3} \sim \dfrac{1}{x_L^3} G(z)$ $\quad z \equiv \dfrac{1-\tau_1}{1-\frac{1}{\tau_2}},\ \bar{z} \equiv \dfrac{1-\frac{1}{\tau_1}}{1-\tau_2}$

$$
\begin{aligned}
G(z) = & \frac{(1+|z|^2+|1-z|^2)}{2|z|^2|1-z|^2}(1+\zeta_2) + \frac{(-1+|z|^2+|z|^4-|z|^6-|1-z|^4-|z|^2|1-z|^4+2|1-z|^6)}{2|z|^2|1-z|^2(z-\bar{z})^2}\log|1-z|^2 \\
& + \frac{(-1-|z|^4+2|z|^6+|1-z|^2-|z|^4|1-z|^2+|1-z|^4-|1-z|^6)}{2|z|^2|1-z|^2(z-\bar{z})^2}\log|z|^2 \\
& + \frac{|z|^4-1}{2|z|^2|1-z|^4} D_2^+(z) + \frac{|1-z|^4-1}{2|z|^4|1-z|^2} D_2^+(1-z) + \frac{(|z|^2-|1-z|^2)(|z|^2+|1-z|^2)}{2|z|^2|1-z|^2} D_2^+\left(\frac{z}{z-1}\right) \\
& + \frac{2iD_2^-(z)}{2|1-z|^4|z|^4(z-\bar{z})^3} p_3(|z|^2,|1-z|^2)\,.
\end{aligned}
$$

$$
\begin{aligned}
p_3(|z|^2,|1-z|^2) = & (-1+|z|^2-|1-z|^2)(1+|z|^2-|1-z|^2)(-1+|z|^2+|1-z|^2) \\
& \times \Big[(-1+|1-z|^2)^2|1-z|^2+|z|^6(1+|1-z|^2) \\
& - 2|z|^4(1+|1-z|^2)^2 + |z|^2(1+|1-z|^2)(1+(-5+|1-z|^2)|1-z|^2)\Big]
\end{aligned}
$$

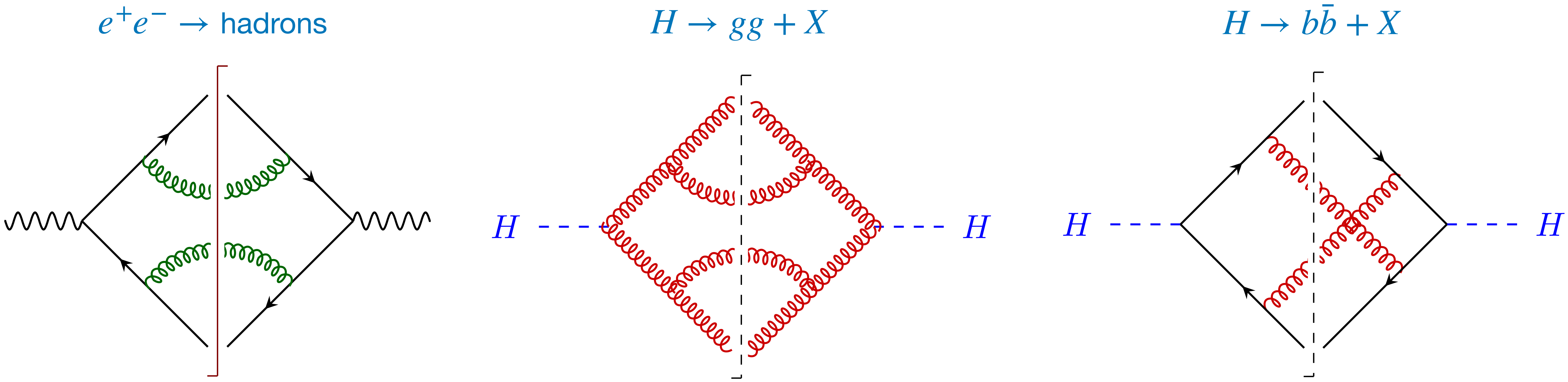
$$
\begin{aligned}
2iD_2^-(z) &\equiv \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\ln\frac{1-z}{1-\bar{z}}\ln(z\bar{z}), \\
D_2^+(z) &\equiv \mathrm{Li}_2\left(1-|z|^2\right) + \frac{1}{2}\ln\left(|1-z|^2\right)\ln\left(|z|^2\right)
\end{aligned}
$$

[Chen, Luo, Moult, Yang, **XYZ**, Zhu, 1912.11050]

EEEC in QCD

[Yang, **XYZ**, 2208.01051 and in progress work]

- For LO EEEC in QCD, we consider both two processes:



$\gamma^\star \to q\bar{q}q'\bar{q}',\ q\bar{q}q\bar{q},\ q\bar{q}gg$ $\qquad$ $H \to gggg,\ q\bar{q}gg,\ q\bar{q}q'\bar{q}',\ q\bar{q}q\bar{q}$ $\qquad$ $H \to q\bar{q}gg,\ q\bar{q}q'\bar{q}',\ q\bar{q}q\bar{q}$

- For Higgs-EEEC, we use the HEFT and work in the massless quark limit

$$\mathcal{L}_{\rm int} = -\frac{1}{4}\lambda H \mathrm{Tr}(G^{\mu\nu}G_{\mu\nu}) + \sum_q \frac{y_q}{\sqrt{2}} H\bar{\psi}_q\psi_q$$

- All matrix elements are calculated in QFRAF and FORM, with axial gauge

$$\sum_{\lambda=1}^{2} \epsilon^\mu(p_i,\lambda)\epsilon^{*\nu}(p_i,\lambda) = -g^{\mu\nu} + \frac{\bar{n}^\mu p_i^\nu + \bar{n}^\nu p_i^\mu}{\bar{n}\cdot p_i} - \frac{\bar{n}^2 p_i^\mu p_i^\nu}{(p_i\cdot\bar{n})^2}$$

EEEC in QCD

- For QCD calculation, we need the topology identification

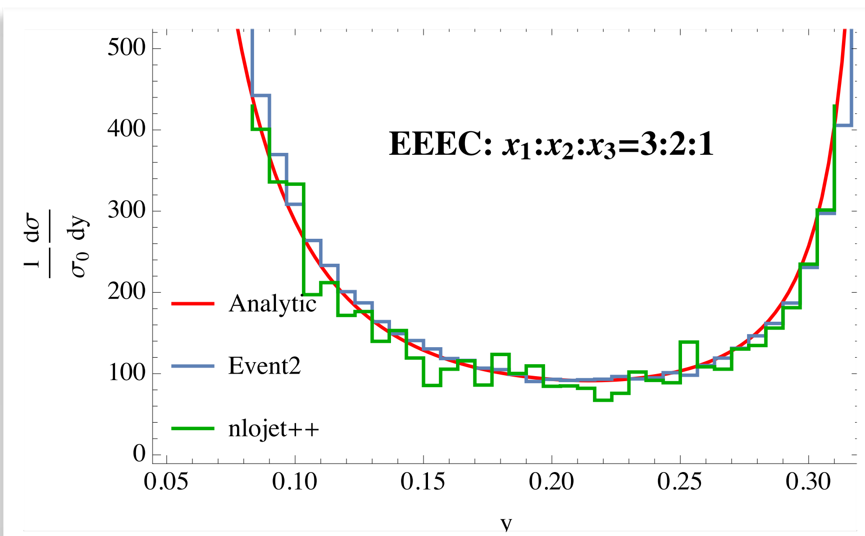
$$
\begin{aligned}
&\sum_{i\neq j\neq k\in\{1,2,3,4\}}\int \frac{E_iE_jE_k}{Q^3}dPS_4\Pi_{ijk}|\mathcal{M}(p_1,p_2,p_3,p_4)|^2\\
=&\sum_{a\neq b\neq c\in\{1,2,3\}}\int \frac{E_aE_bE_c}{Q^3}dPS_4\Pi_{abc}\left(|\mathcal{M}(p_a,p_b,p_c,p_4)|^2+|\mathcal{M}(p_a,p_b,p_4,p_c)|^2\right.\\
&\qquad\left.+|\mathcal{M}(p_a,p_4,p_c,p_b)|^2+|\mathcal{M}(p_4,p_a,p_b,p_c)|^2\right)\\
=&\Bigg[\int \frac{E_1E_2E_3}{Q^3}dPS_4\Pi_{123}\left(|\mathcal{M}(p_1,p_2,p_3,p_4)|^2+|\mathcal{M}(p_1,p_2,p_4,p_3)|^2+|\mathcal{M}(p_1,p_4,p_3,p_2)|^2\right.\\
&\qquad\left.+|\mathcal{M}(p_4,p_1,p_2,p_3)|^2\right)\Bigg]+\text{permutations of } x_1\,,x_2\,,x_3\,, \qquad (2.3)
\end{aligned}
$$

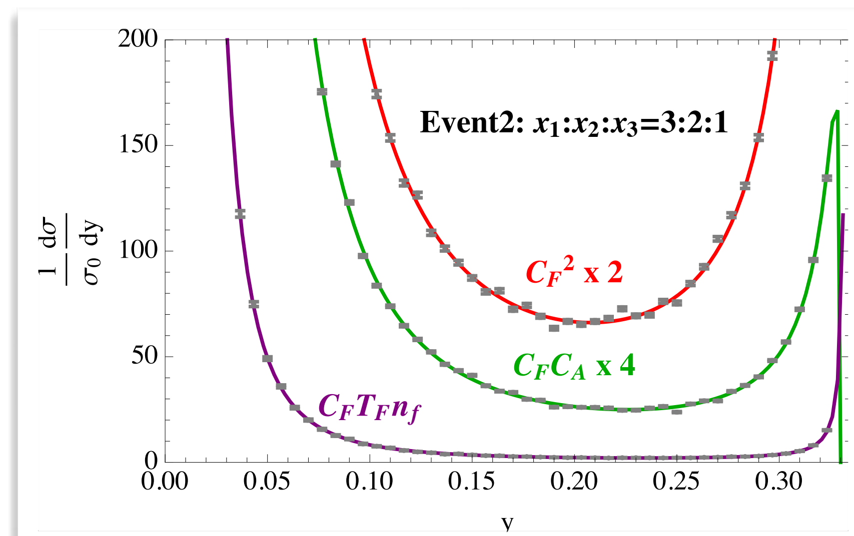
where $$\Pi_{ijk}=\delta\left(x_1-\frac{1-\cos\theta_{jk}}{2}\right)\delta\left(x_2-\frac{1-\cos\theta_{ik}}{2}\right)\delta\left(x_3-\frac{1-\cos\theta_{ij}}{2}\right)$$

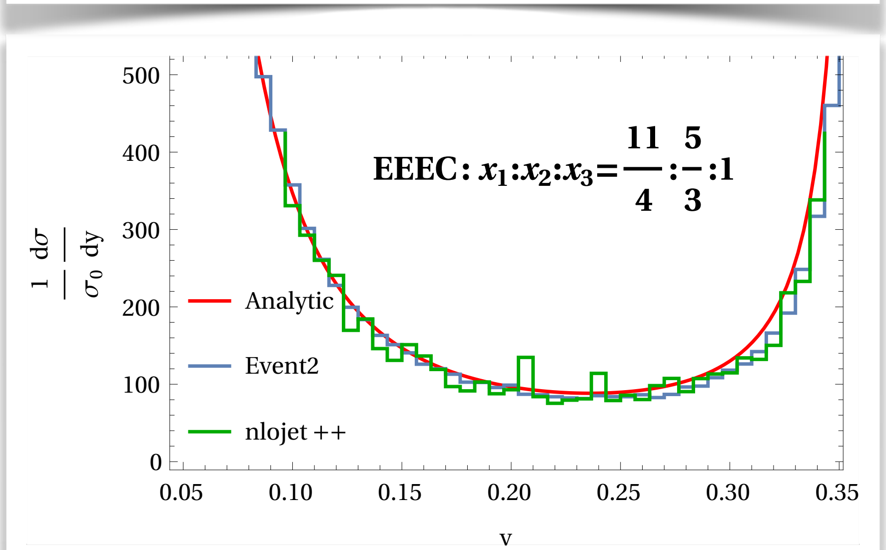
- The phase space parameterization and integration are the same as $N=4$, and we still express the result in terms of GPLs.
- It has been observed that $N=4$ and QCD shares a lot of common properties, in both amplitude level and cross-section level. For LO EEEC, $N=4$ and QCD turns out to have exactly the same function space.

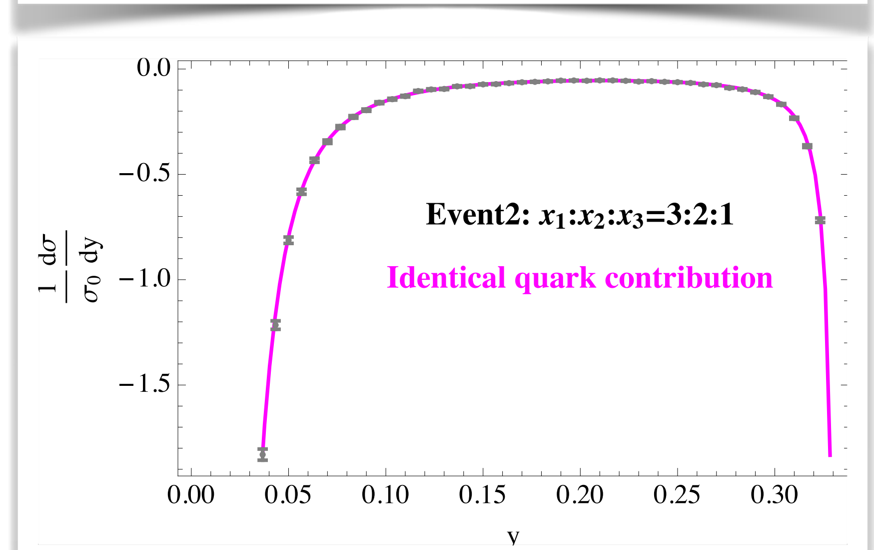
QCD result

- To simplify the QCD result, we build the linear relation between the GPL function space and the $N = 4$ classical polylogarithmic space via PSLQ, and project the QCD result onto it.
- Example: $e^+e^- \to$ hadrons   6.3 MB→1.2 MB   200 digits precision within 4 sec for a regular point
- Comparison with Monte Carlo programs: Event2: fifty billion points are sampling; NLOJet++: ten billion points



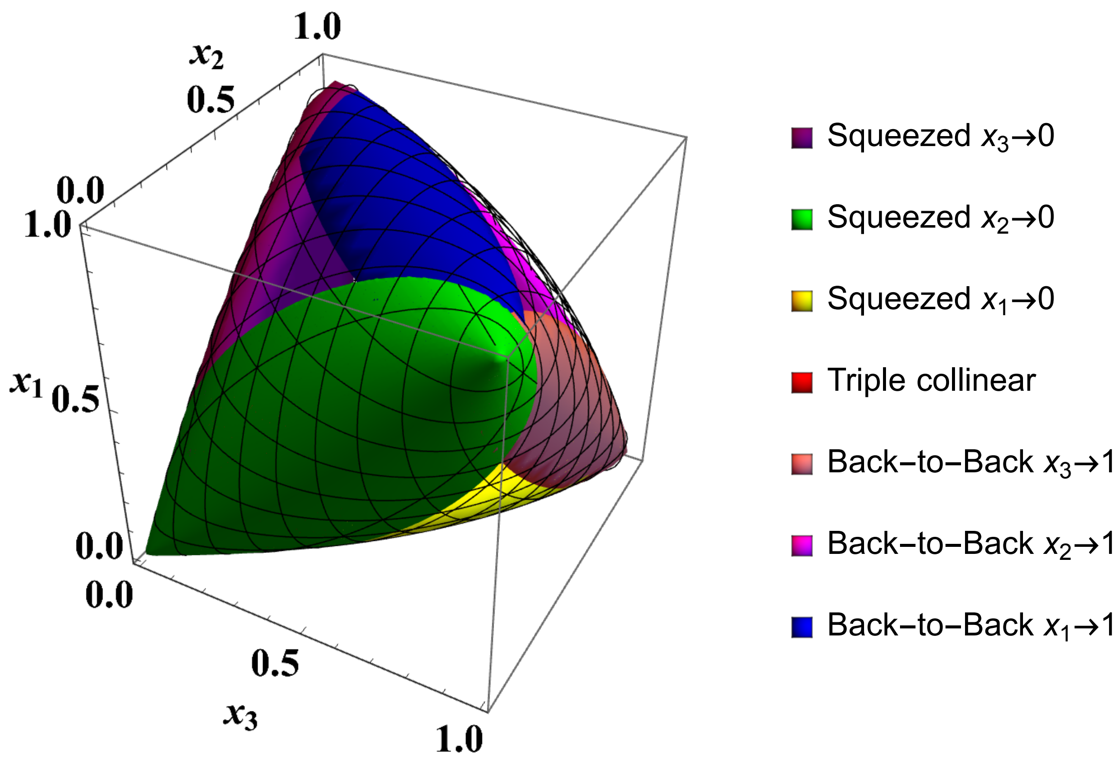


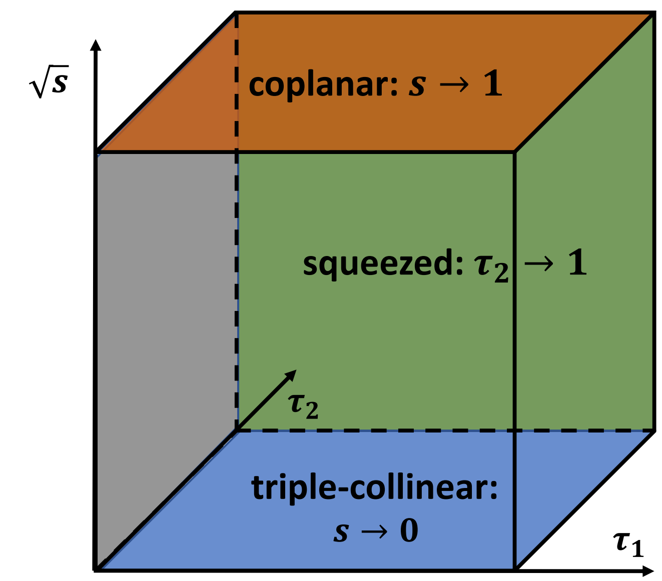




Kinematic limits

- Different ways to visualize the kinematic limits:





Its boundary corresponds to the coplanar limit

$$x_1 = -\frac{s}{(s+1)^2}\frac{(1-\tau_1)^2}{\tau_1},\ x_2 = -\frac{s}{(s+1)^2}\frac{(1-\tau_2)^2}{\tau_2},$$
$$x_3 = -\frac{s}{(s+1)^2}\frac{(1-\tau_1\tau_2)^2}{\tau_1\tau_2}$$

- For triple-collinear limit:
  - The LP is understood.
  - Our result also provides perturbative data for NLP and beyond
  - The NLP triple-collinear function space is the same as LP:

$$\left\{\ln x_1;\ \pi^2,\ 2iD_2^-(z),\ \mathrm{Li}_2\left(1-\frac{x_2}{x_1}\right)+\frac{1}{2}\ln\frac{x_1}{x_3}\ln\frac{x_1}{x_2}\right\} + \text{permutations}$$

Kinematic limits

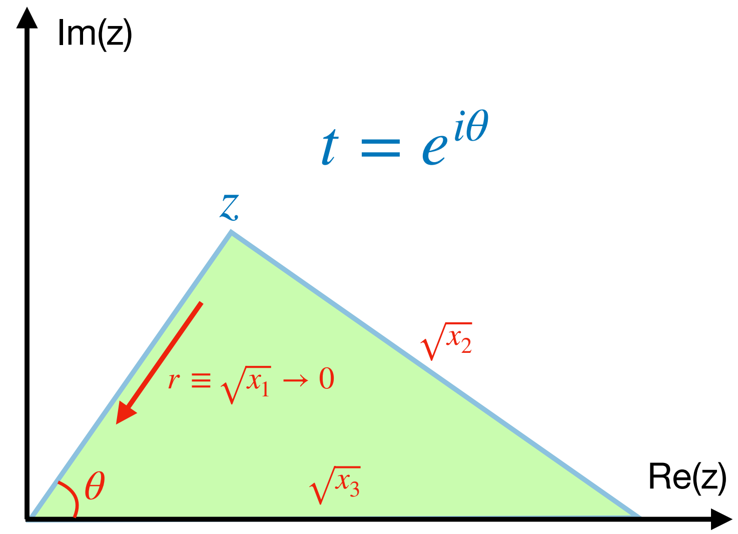
- Squeezed limit: $$\frac{1}{\sigma_{\rm tot}}\frac{d^3\sigma}{dx_1dx_2dx_3} \overset{x_1\to 0, x_{2,3}\sim\eta}{\approx} \left(\frac{\alpha_s}{4\pi}\right)^2 \frac{1}{4\pi\sqrt{-s_2^2}}\left(\frac{B(\eta)}{x_1}+\mathcal{O}(x_1^0)\right)$$

$x_1 \to 0, x_2 \sim x_3 \to \eta$

$$\begin{aligned} B(\eta) = {} & C_F n_f T_F\left(\frac{4\left(28\eta^2-82\eta+63\right)\log(1-\eta)}{15\eta^6}-\frac{67\eta^3-702\eta^2+1362\eta-756}{45(1-\eta)\eta^5}\right) \\ & + C_F^2\left(-\frac{6\left(12\eta^3-41\eta^2+40\eta-9\right)\log(1-\eta)}{(1-\eta)\eta^6}-\frac{31\eta^3-288\eta^2+426\eta-108}{2(1-\eta)\eta^5}\right) \\ & + C_A C_F\left(\frac{4\left(166\eta^2-544\eta+441\right)\log(1-\eta)}{15\eta^6}-\frac{349\eta^3-4374\eta^2+9174\eta-5292}{45(1-\eta)\eta^5}\right) \end{aligned}$$

- However, the squeeze limit itself is ambiguous since it is path-dependent
- If we take the double limit: triple-collinear +squeezed $x_1 \sim x_2 \sim x_3 \to 0, z \to 0$

$$\begin{aligned} \frac{1}{\sigma_{\rm tot}}\frac{d^3\sigma}{dx_1dx_2dx_3} \approx {} & \left(\frac{\alpha_s}{4\pi}\right)^2 \frac{1}{4\pi\sqrt{-s_2^2}}\left[\frac{C_F^2}{x_3}\left(\frac{16}{5r^2}+\frac{8(1+t^2)}{5rt}\right)\right. \\ & + \frac{C_F C_A}{x_3}\left(\frac{5+273t^2+5t^4}{225r^2t^2}+\frac{10+273t^2+273t^4+10t^6}{450rt^3}\right) \\ & + \frac{C_F T_F n_f}{x_3}\left(-\frac{10-39t^2+10t^4}{225r^2t^2}+\frac{-20+39t^2+39t^4-20t^6}{450rt^3}\right) \\ & + C_F n_f T_F\left(-\frac{24t^4-31t^2+24}{225r^2t^2}-\frac{4(t-1)^2\left(t^2+1\right)(t+1)^2}{75rt^3}\right) \\ & \left. + C_A C_F\left(\frac{12t^4+367t^2+12}{225r^2t^2}+\frac{2(t-1)^2\left(t^2+1\right)(t+1)^2}{75rt^3}\right)+\frac{47C_F^2}{10r^2}\right] \end{aligned}$$

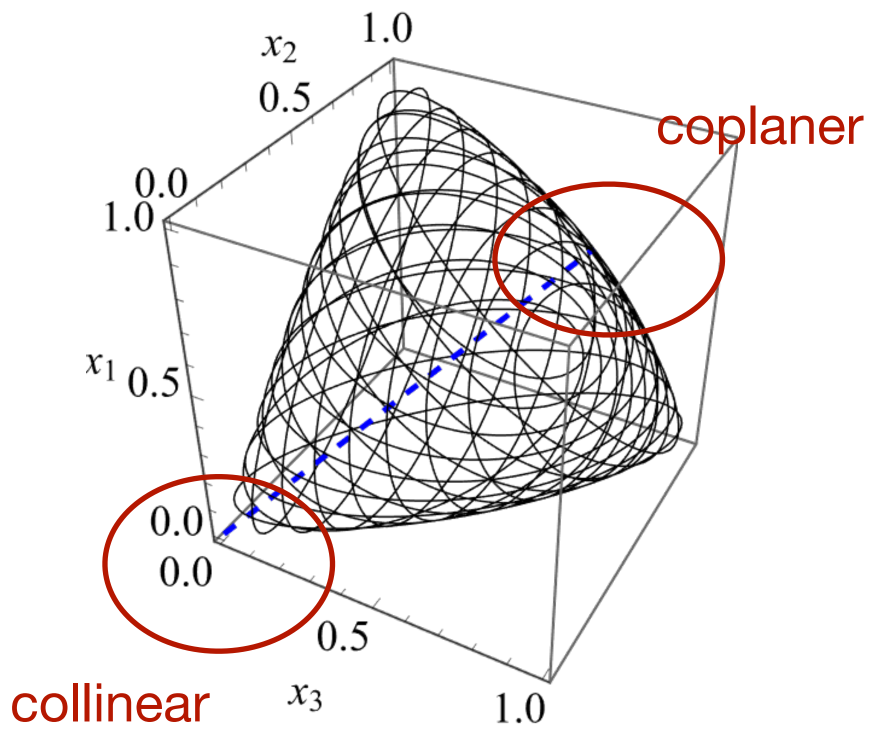


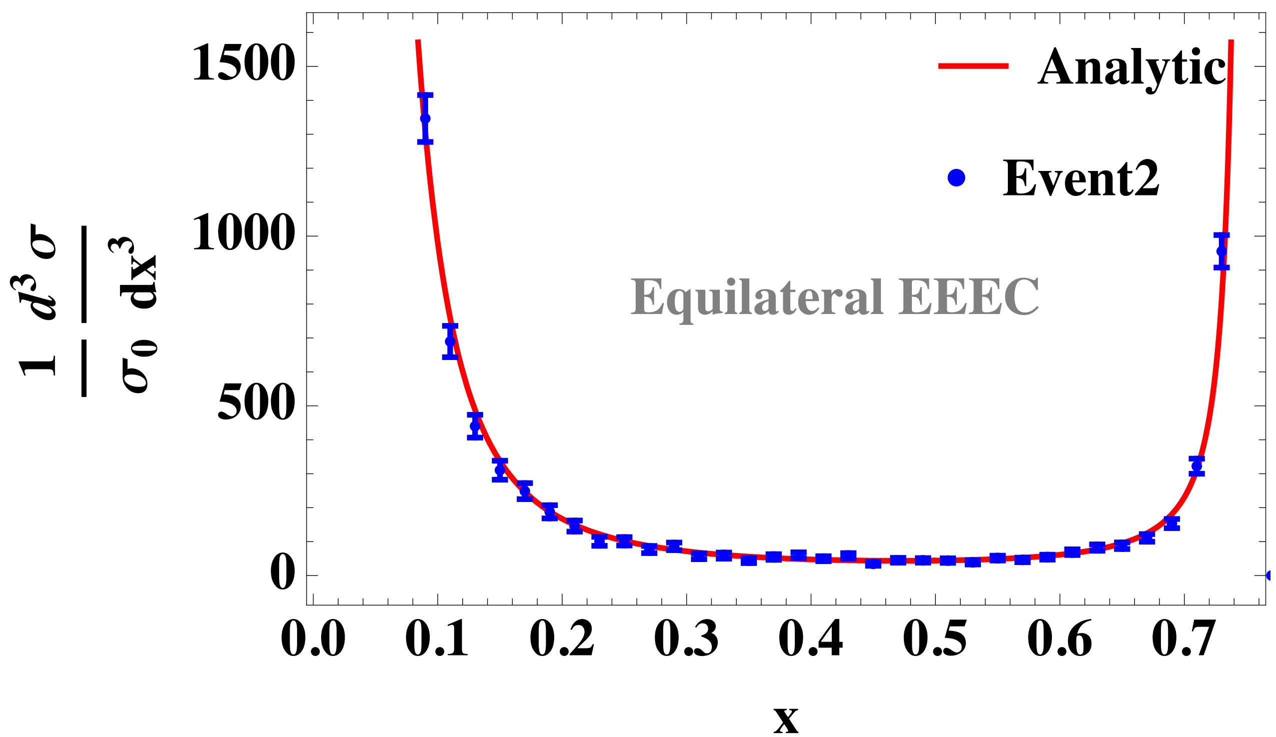
t-dependence disappears for $N=1$ SYM

[Chen, Moult, Zhu, 2104.00009]

Kinematic limits

- There are also various possibilities on slicing the EEEC kinematic space. The simplest one is the equilateral limit $x_1 = x_2 = x_3 \equiv x$





- It will be interesting to study factorization and the joint resummation of different kinematic limits/slicing
- It might be also possible to develop a ENC bootstrap program using all these kinematic limits

Simplification game: necessary but challenging

- The most difficult part of EEEC calculation is how to simplify the result. It is important because:

  - Fast and accurate numerical evaluation ⇒ essential for collider physics
  - Mathematical structure understanding ⇒ physical singularities

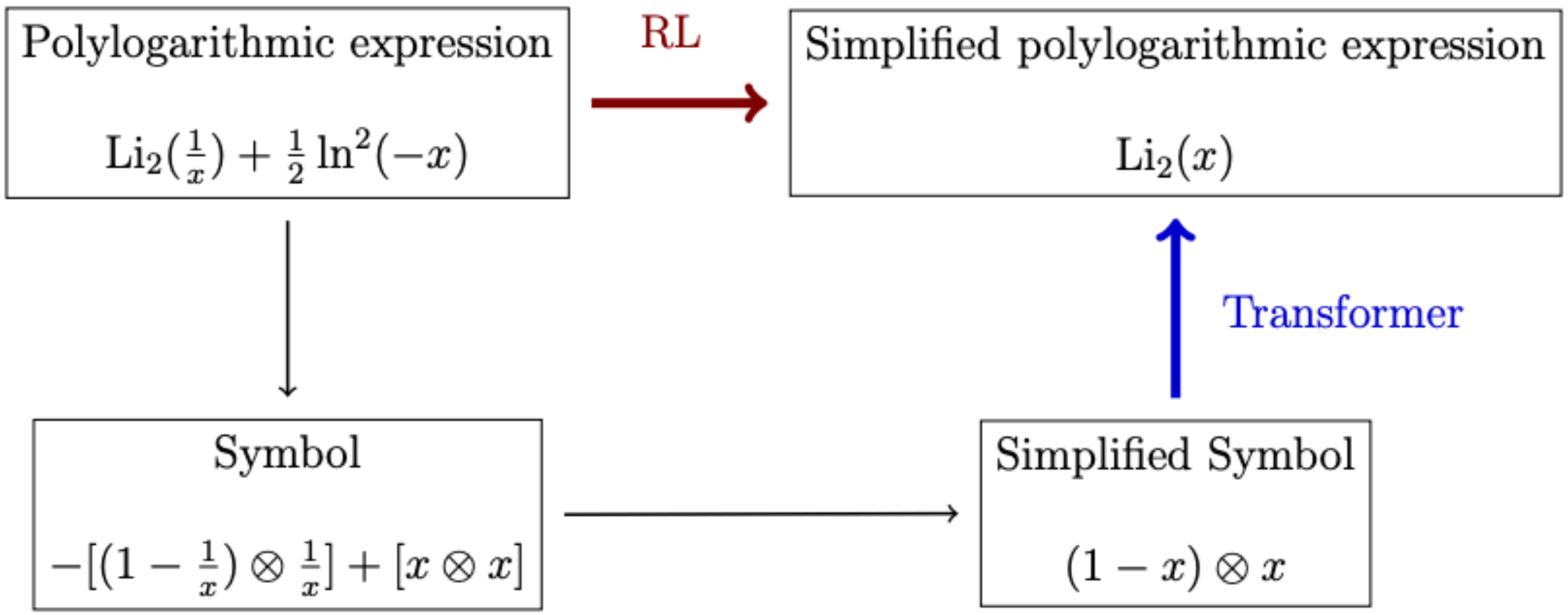
e.g. [Hannesdottir, McLeod, Schwartz, Vergu, 2109.09744]

- Observables involving various cuts (θ/δ functions) usually lead to complicate symbol and alphabet. It is hard to lift the symbol into simplified polylogarithms.
- Even for standard loop integrals and phase space integrals, the raw expressions could involve spurious singularities.

Can Symbolic ML help with any step?

Polylogarithm meets symbolic ML

[Dersy, Schwartz and **XYZ**, 2206.04115]



- Approach 1: Reinforcement learning
- Approach 2: transformer networks

[Vaswani et al, 2017]

[Facebook: Deep learning for integration and solving ODEs, Lample, Charton, 1912.01412]

Simplifying polylogarithms with transformer

- A baby step: consider single-variable expressions without any square root
- Procedure:
  - Compute the symbol expressions with PolyLogTools [Duhr, Dulat, 1904.07279]
  - Translation task in ML: find the simplest result corresponding a given symbol
  - Transformer could give a wrong result, but it is easy to verify

$$-\frac{23}{4}(1-x)\otimes x - 6\left(x^2+x+1\right)\otimes x$$

"French"

Transformer →

$$2\,\mathrm{Li}_2\left(x^3\right) - \frac{1}{4}\,\mathrm{Li}_2(x)$$

"English"

- We do weight-2,3,4, and only calculate the leading transcendental term.
- Training: Generate simple expressions ($\sim$ 1-2M) and compute their symbol

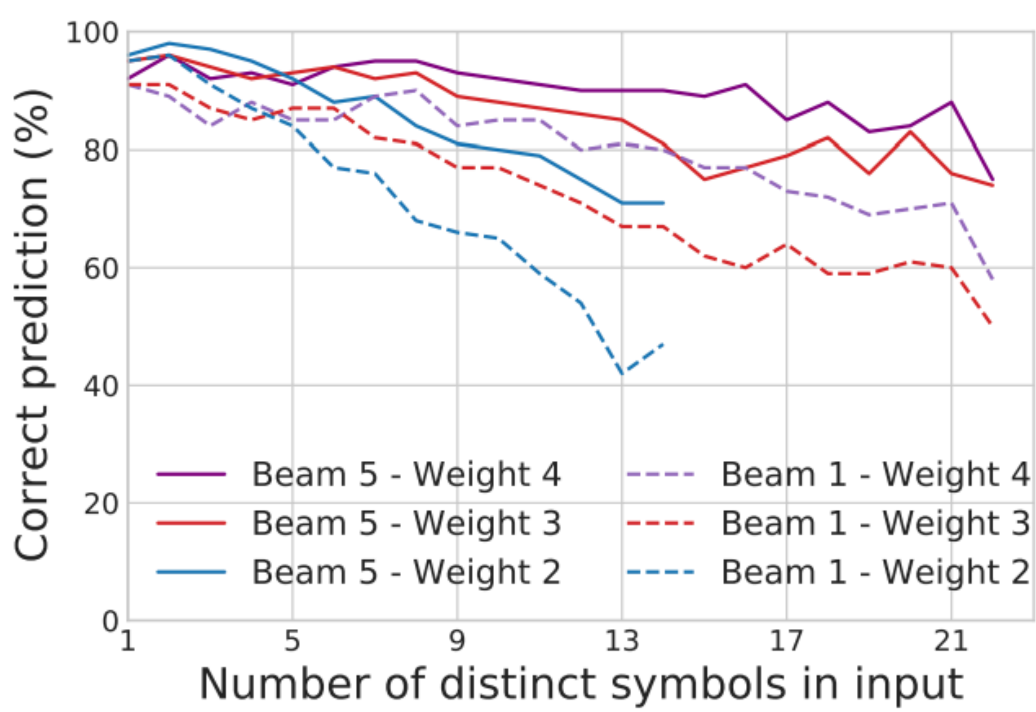
Weight 2	Weight 3	Weight 4
$\mathrm{Li}_2(x)$	$\mathrm{Li}_3(x)$	$\mathrm{Li}_4(x)$
$\ln(x)\ln(y)$	$\mathrm{Li}_2(x)\ln(y)$	$\mathrm{Li}_3(x)\ln(y)$
	$\ln(x)\ln(y)\ln(z)$	$\mathrm{Li}_2(x)\mathrm{Li}_2(y)$
		$\mathrm{Li}_2(x)\ln(y)\ln(z)$
		$\ln(w)\ln(x)\ln(y)\ln(z)$

Simplifying polylogarithms with transformer

- Evaluation: We ask the transformer for N guess (Beam evaluation)

Result: we can integrate ~90% symbols

		Beam Size 1	Beam Size 5
Weight 2	Transformer	82%	**91%**
	Classical Algorithm	59%	59%
Weight 3	Transformer	78%	**88%**
Weight 4	Transformer	80%	**89%**



Sample problems

Input symbol $\mathcal{S}_i$	Simple expression F_i
$-(-x^2-x+1)\otimes(1-x)+(-x^2-x+1)\otimes x$ $-(-x^2-x+1)\otimes(x+1)+x\otimes(1-x)-x\otimes x+x\otimes(x+1)$	$\mathrm{Li}_2\left(\frac{(1-x)(x+1)}{x}\right)$
$-\frac{23}{4}(1-x)\otimes x-6\,(x^2+x+1)\otimes x$	$2\,\mathrm{Li}_2\left(x^3\right)-\frac{1}{4}\,\mathrm{Li}_2(x)$
$-40\,(6-x^2)\otimes(6-x^2)-3(1-x)\otimes(-x^6-x^2+3)$ $-3(x+1)\otimes(-x^6-x^2+3)-3\,(x^4+x^2+2)\otimes(-x^6-x^2+3)$ $+\frac{1}{4}(5-2x)\otimes(2-x)$	$3\mathrm{Li}_2\left(-x^6-x^2+3\right)$ $-\frac{1}{4}\,\mathrm{Li}_2(2x-4)$ $-20\ln^2\left(x^2-6\right)$
$8\,\frac{x^2-x-1}{x-1}\otimes x-8\left((x+1)\left(x^2-x-1\right)\right)\otimes x$ $+8(1-x)\otimes(-x^3+x^2-x-1)-8\frac{1}{x-1}\otimes x$ $-8(1-x)\otimes\left(x\left(x^3-x^2+x+1\right)\right)$	$4\,\mathrm{Li}_2\left(x^2\right)$

- Unlike humans, transformer does NOT care about transcendentally weights!

Conclusions

- Multi-point energy correlator lives in the intersection of studies involving jet substructure, fixed-order calculation and effective field theories.
- We present the analytic calculation of leading order EEEC in both $N = 4$ super Yang-Mills theory and QCD (e^+e^- annihilation and Higgs decays).
  - Instead of IBP+DE, we preform direct phase space integration with proper parameterization.
  - The result is converted to polylogarithms (up to t.w.2) and simplified using symbol and symmetries.
- Our analytic result provides perturbative data for various kinematic limits at leading power and beyond: triple-collinear, squeezed, back-to-back, coplanar etc. It will be useful once we initialize the study on factorization and joint resummation.
- We also initialize the symbolic ML project (mainly transformer) on simplifying polylogarithms in Feynman integral calculations.

Thank you for your attention!