Feynman integrals from positivity constraints

Mao Zeng, University of Edinburgh

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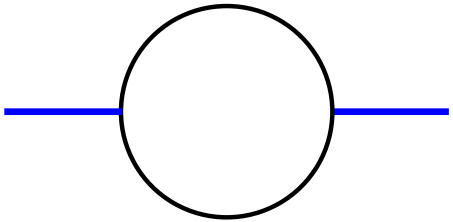
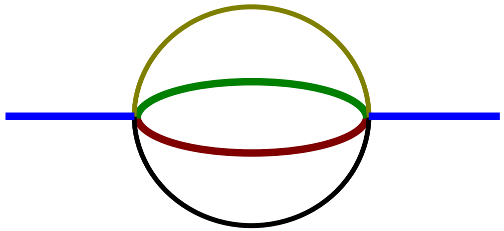
arXiv:2303.15624, MZ

Outline

- Background
  - Positivity in theoretical physics
- Toy example: 1-loop massive bubble integral
  - ad hoc positivity constraints in momentum space
  - Precise numerics from semidefinite programming
  - Feynman parameter space, ε-expansion, etc.
- 3-loop banana integrals with unequal masses





Background

Evaluating Feynman integrals

- A classic problem, but also an active and expanding frontier.
  Applications in collider physics, gravitational waves, cosmology...

- Many methods available: Feynman parameter integration, Mellin Barnes representation, differential equations, difference equations, sector decomposition, Loop-tree duality and related representations, tropical geometry integration, bootstrap from symbols & boundary conditions, Yangian symmetry…

- Automated, widely applied ***numerical methods:*** sector decomposition [Binoth, Heinrich, ’00], numerical series solution of differential equations, e.g. recent work of [Moriello, ’19, Hidding, ’20. Liu, Ma, ’21. Hidding, Usovitsch, ’22]

- Still challenging in practise. ***New explorations warranted***

New idea: *inequality constraints*

- Inequality constraints: fruitful idea in recent theoretical physics.

  Conformal bootstrap: positivity of squared 3-point structure function.
  S-matrix bootstrap: positivity of Im(M) in forward limit / partial wave basis.
  EFT positivity bounds: no superluminal propagation. “Not anything goes”.

- Direct inspiration: bootstrapping quantum mechanics.
  - Schrodinger’s equation with potential $V(x) = x^d$ and eigenstate $|\psi\rangle$.
  - Unitarity ⇒ $\langle\psi|\mathcal{O}^\dagger\mathcal{O}|\psi\rangle \geq 0$. Consider e.g. $\mathcal{O} =$ any polynomial in $\hat{x}$.
  - **Algebraic recusion relations** reduce all moments sums of $\langle x^i\rangle,\ i < d$
    - analogous to integration-by-parts relations for Feynman integrals!

[Lin, ’20. Han, Hartnoll, Kruthoff, ’20. Berenstein, Hulsey, ’21. Anderson, Kruczenski, ’16. Kazakov, Zheng, ’22. Cho, Lin, Rodriguez, Sandor, Yin, ’22...]

New method for Feynman integrals

[arXiv:2303.15624, MZ]

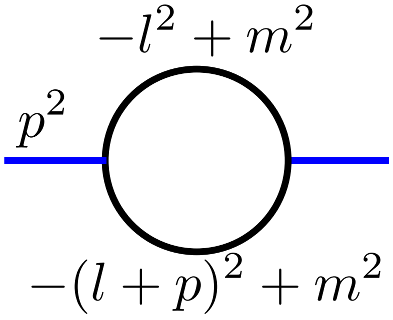
- First method based on *inequality constraints* & numerical technique of *semidefinite programming* [Vandenberghe, Boyd, '96. Poland, Simmons-Duffin, Vichi, '12]
- Disclaimer: practicality yet to be assessed. But hopeful signs:
  - ✔ Can reach (moderately) high precision, e.g. 10 digits in a nontrivial 3-loop example, a few CPU hours
  - ✔ Can handle dimensional regularization, ε expansion.
  - ✔ Applicable to wide class of integrals.

One-loop toy example

Positivity for 1-loop integrals: first attempt

- Bubble integral with massive external and internal lines.
  Simplest case: $p^2 < 0$. *Will later extend to* $p^2 < 4m^2$.

$$I_{a_1,a_2} \equiv \int \frac{d^d l\, e^{\gamma_E \epsilon}}{i\pi^{d/2}} \frac{1}{(-l^2+m^2)^{a_1}[-(p+l)^2+m^2]^{a_2}}$$



Wick rotation

$$I_{a_1,a_2} \equiv \int \frac{d^d \boldsymbol{l}\, e^{\gamma_E \epsilon}}{\pi^{d/2}} \frac{1}{(\boldsymbol{l}^2+m^2)^{a_1}[(\boldsymbol{p}+\boldsymbol{l})^2+m^2]^{a_2}}, \quad \boldsymbol{p}^2 = -p^2$$

- This is, obviously, *positive*. Can we say something more precise? **Yes.**

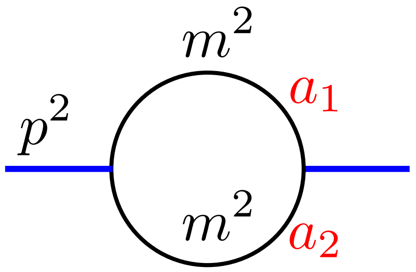
Integration-by-parts reduction

[Chetyrkin, Tkachov, '81]

- Undergraduate calculus: gamma function integrals

$$\Gamma(n) = \int_0^\infty dx\, x^{n-1} e^{-x}$$

- Integration by parts: $\Gamma(n) = (n-1)\Gamma(n-1)$. "Master integral" $\Gamma(1)$
- Feynman integrals in dim. reg.: total derivatives integrate to zero.



I_{a_1,a_2} expressed as linear combinations of $I_{1,1},\ I_{1,0}$

Or alternatively *finite master integral basis* $I_{2,1},\ I_{3,0}$

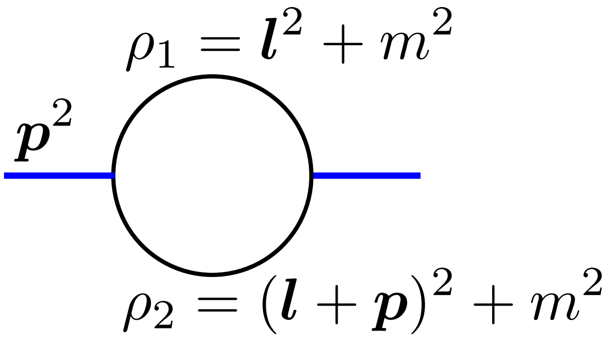
Coefficients are rational in $p^2,\ m^2,\ d$

bubble tadpole

ad hoc positivity constraints

- Try a few (Euclidean) positive integrals

(1) $$\int \frac{d^d \boldsymbol{l}\, e^{\gamma_E \epsilon}}{\pi^{d/2}} \frac{1}{\rho_1^2 \rho_2^2} = I_{2,2} > 0$$



IBP reduction

$$\frac{2}{\boldsymbol{p}^2(\boldsymbol{p}^2 + 4m^2)} \left[(\boldsymbol{p}^2 + 2m^2) I_{2,1} - 2m^2 I_{3,0} \right] > 0$$

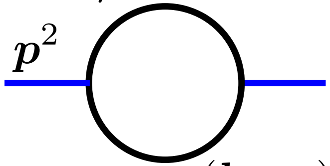
tadpole integral

$$I_{3,0}\Big|_{d=4} = \frac{1}{2m^2}$$

$$\Longrightarrow I_{2,1}/I_{3,0} \geq \frac{2m^2}{\boldsymbol{p}^2 + 2m^2} \quad \text{for } d = 4$$

Two-sided bounds

Try two (Euclidean) positive integrals

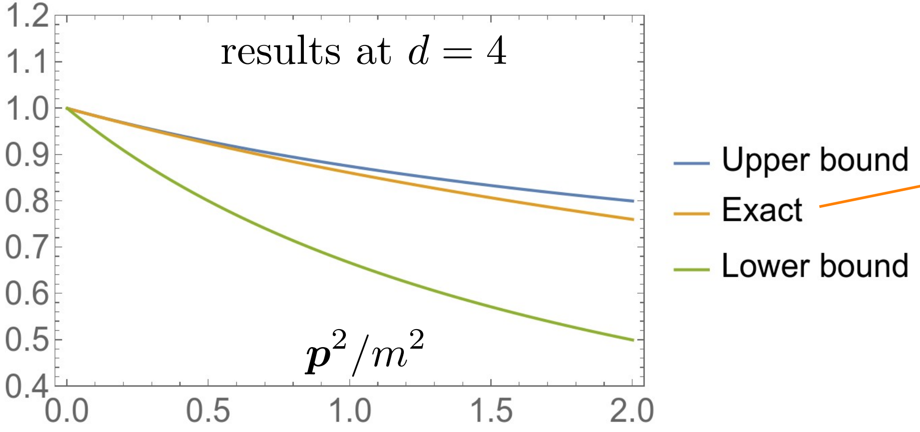


Lower bound

(1) $$\int \frac{d^d \boldsymbol{l}\, e^{\gamma_E \epsilon}}{\pi^{d/2}} \frac{1}{\rho_1^2 \rho_2^2} > 0 \implies \frac{2m^2}{\boldsymbol{p}^2 + 2m^2} \le I_{2,1}/I_{3,0}$$

(2) $$\int \frac{d^d \boldsymbol{l}\, e^{\gamma_E \epsilon}}{\pi^{d/2}} \frac{1}{\rho_1 \rho_2} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right)^2 > 0 \implies I_{2,1}/I_{3,0} \le \frac{\boldsymbol{p}^2 + 6m^2}{2\boldsymbol{p}^2 + 6m^2}$$

upper bound



$$I_{2,1}/I_{3,0} = \frac{2m^2}{\beta \boldsymbol{p}^2} \log \frac{\beta + 1}{\beta - 1},$$

$$\beta \equiv \sqrt{1 + \frac{4m^2}{\boldsymbol{p}^2}}$$

Precision numerics: semidefinite programming

- A class of positive integrals parameterized by vector $\vec{\alpha}$.

$$\int \frac{d^d \boldsymbol{l}\, e^{\gamma_E \epsilon}}{\pi^{d/2}} \frac{1}{\rho_1^2 \rho_2} (\alpha_1 + \alpha_2/\rho_1 + \alpha_3/\rho_2 + \alpha_4/\rho_1^2 + \dots)^2 = \vec{\alpha}^T \mathbb{M}\, \vec{\alpha} \geq 0$$

cutoff at e.g. degree 3

- Symmetric matrix $\mathbb{M}$ is *positive semidefinite. All eigenvalues must be non-negative.*

$$\mathbb{M} = \begin{pmatrix} I_{2,1} & I_{3,1} & I_{2,2} & \dots \\ I_{3,1} & I_{4,1} & I_{3,2} & \\ I_{2,2} & I_{3,2} & I_{2,3} & \\ \vdots & & & \ddots \end{pmatrix} \overset{\text{IBP}}{=} I_{3,0} \cdot \mathbb{M}_1 + I_{2,1} \cdot \mathbb{M}_2 \succcurlyeq 0$$

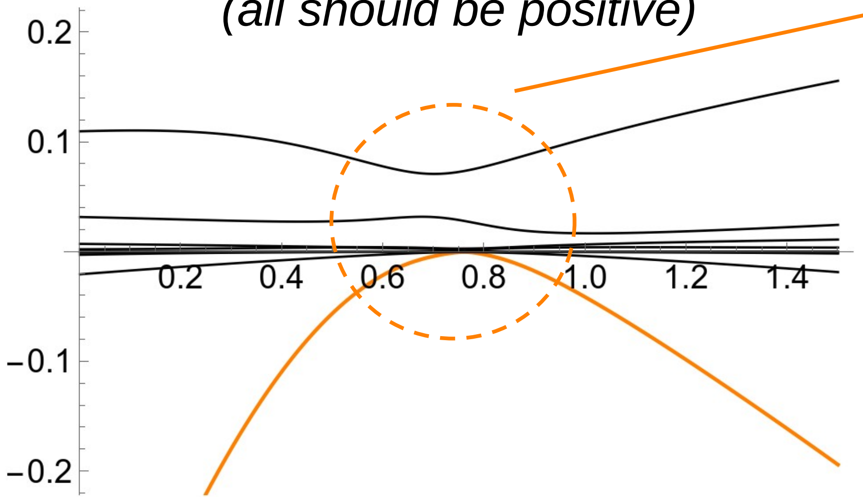
known tadpole integral

bubble integral is constrained

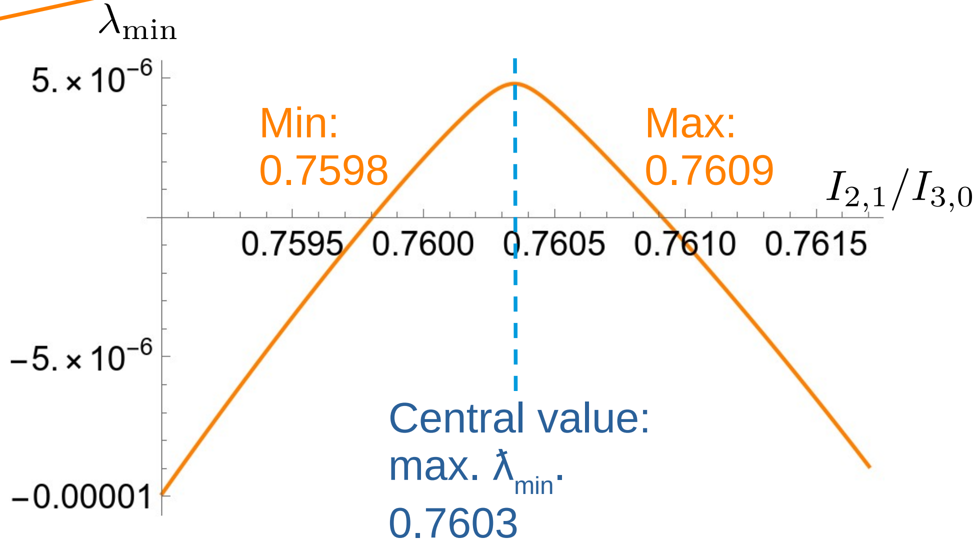
Semidefinite programming example

- With cutoff degree 3, 10×10 symmetric matrix.

Eigenvalues vs. $I_{2,1}/I_{3,0}$
(all should be positive)



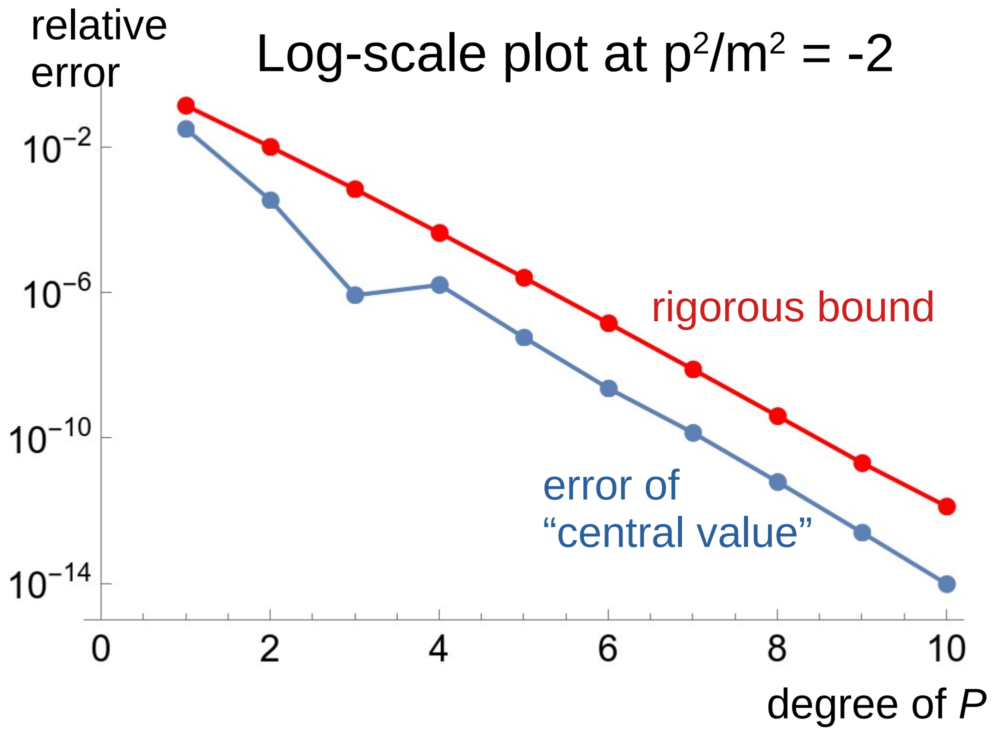
Zooming in: ***tiny positive region***



Error vs. cutoff degree

$$\int \frac{d^d \boldsymbol{l}\, e^{\gamma_E \epsilon}}{\pi^{d/2}} \frac{1}{\rho_1^2 \rho_2} P(1/\rho_1, 1/\rho_2)^2 > 0$$

- Polynomial *P* with arbitrary monomial terms up to some cutoff degree.
- **Exponential convergence** as cutoff degree raised.
- *The more IBP computation you do, the more accruate the result.*

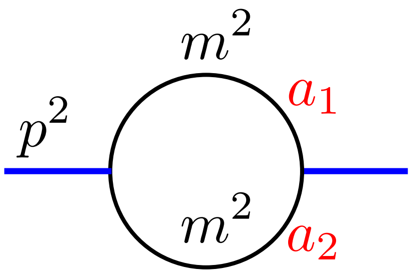


Positivity in Feynman parameter space

- More general "Euclidean region": below Cutkosky cut threshold, not necessarily in Euclidean spacetime.

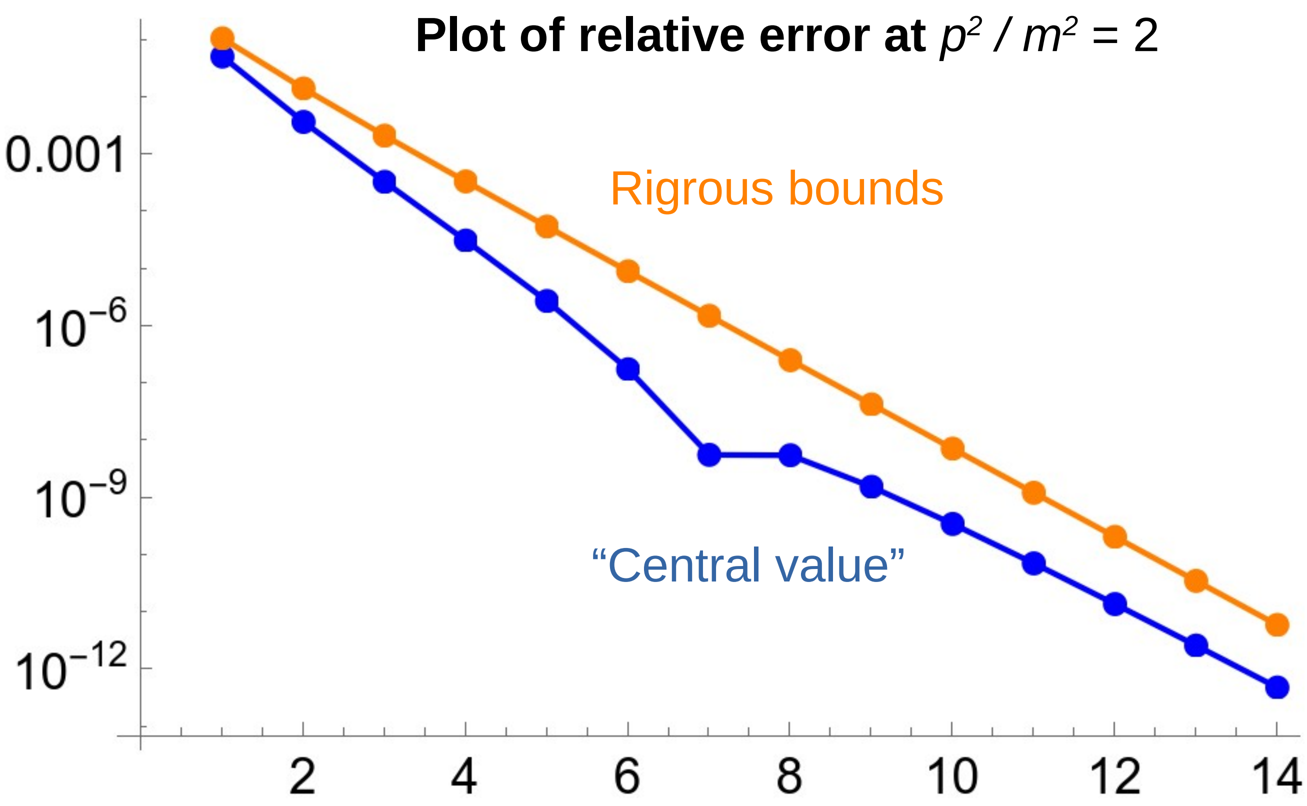
$$I^d_{a_1,a_2} \propto \int_0^1 dx\, x^{a_1-1}(1-x)^{a_2-1}[m^2 - p^2 x(1-x) - i0^+]^{d/2-a}, \quad a \equiv a_1 + a_2$$

F polynomial, **always positive** if $p^2 < 4m^2$, so $i0^+$ negligible. Real integrand.



If replaced by arbitrary positive polynomial, get positive integral. Monomials correspond to integrals in shifted dimensions.

Error vs. cutoff degree – Feyn. param. space



Exponential convergence again

Dimensional regularization – ε expansion

- ***Any*** convergent *l*-loop Euclidean integral, no numerator:

$$I\Big|_{\epsilon^n} \propto \int_0^1 dx_1 \cdots \int_0^1 dx_n\, \delta(x_1 + \cdots + x_n - 1) \frac{U^{a-(l+1)d_0/2}}{F^{a-ld_0/2}} \frac{1}{n!} \log^n \frac{U^{l+1}}{F^l}$$

$$(d = d_0 - 2\epsilon)$$

- Replace last term by square of any polynomial *P*,

$$0 < \int \ldots\ldots P^2 \left(\log \frac{U^{l+1}}{F^l} \right)$$

- ε expansion terms must form positive-semidefinite matrix.
Consistency condition hidden in plain sight (and decades of data).

Consistency condition for ε expansion

$$0 \le \int_0^1 dx_1 \cdots \int_0^1 dx_n \, \delta(x_1 + \cdots + x_n - 1) \frac{U^{a-(l+1)d_0/2}}{F^{a-ld_0/2}} P^2(\log G)$$

for any polynomial *P* of $G \equiv U^{l+1}/F^l$,

- Hankel matrix with $H_n = n! \cdot I\Big|_{\epsilon^n}$ is positive semidefinite.

$$\mathbb{H} = \begin{pmatrix} H_0 & H_1 & H_2 & \cdots \\ H_1 & H_2 & H_3 & \\ H_2 & H_3 & H_4 & \\ \vdots & & & \ddots \end{pmatrix}$$

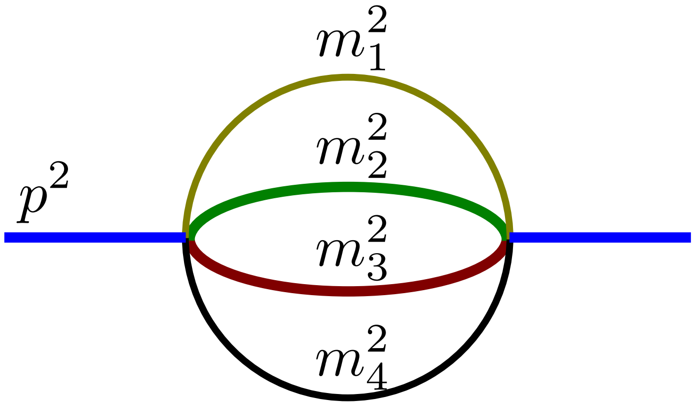
- e.g. any *n*×*n* top-left block has non-negative determinant.

"not anything goes."

3-loop example

3-loop example: banana integrals

$$I_{a_1 a_2 a_3 a_4} \equiv \left(\prod_i \int \frac{d^d l_i \, e^{\gamma_E \epsilon}}{i \pi^{d/2}} \right) \frac{1}{(-l_1^2 + m_1^2)^{a_1}} \frac{1}{(-l_2^2 + m_2^2)^{a_2}} \frac{1}{(-l_3^2 + m_3^2)^{a_3}} \frac{1}{[-(p + l_1 + l_2 + l_3)^2 + m_4^2]^{a_4}}$$

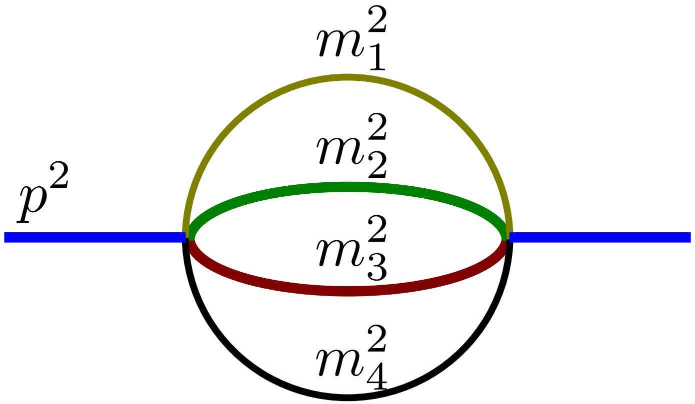


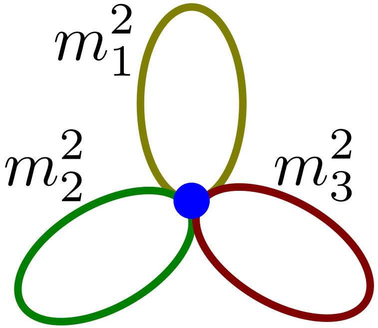
Attracted much attention, e.g. in connction with Calabi-Yau manifolds.
Groote, Korner, Pivovarov, '05. Klemm, Nega, Safari, '19. Bonisch, Fischbach, Klemm, Nega, Safari, '20. Bonisch, Durh, Fischback, Klemm, Nega, '21. Kreimer, '22. Forum, von Hippel, '22. Pogel, Wang, Weinzierl, '22...

Nontrivial integral, but high-precision numerics available for comparision in the literature, e.g. from `DiffExp` package [Hidding, '20] which includes ε expansion terms

3-loop example: banana integrals

$$I_{a_1 a_2 a_3 a_4} \equiv \left(\prod_i \int \frac{d^d l_i \, e^{\gamma_E \epsilon}}{i\pi^{d/2}} \right) \frac{1}{(-l_1^2+m_1^2)^{a_1}} \frac{1}{(-l_2^2+m_2^2)^{a_2}} \frac{1}{(-l_3^2+m_3^2)^{a_3}} \frac{1}{[-(p+l_1+l_2+l_3)^2+m_4^2]^{a_4}}$$





11 top-level master integrals *to calculate from positivity constraints, in d = 2 - 2ε:*

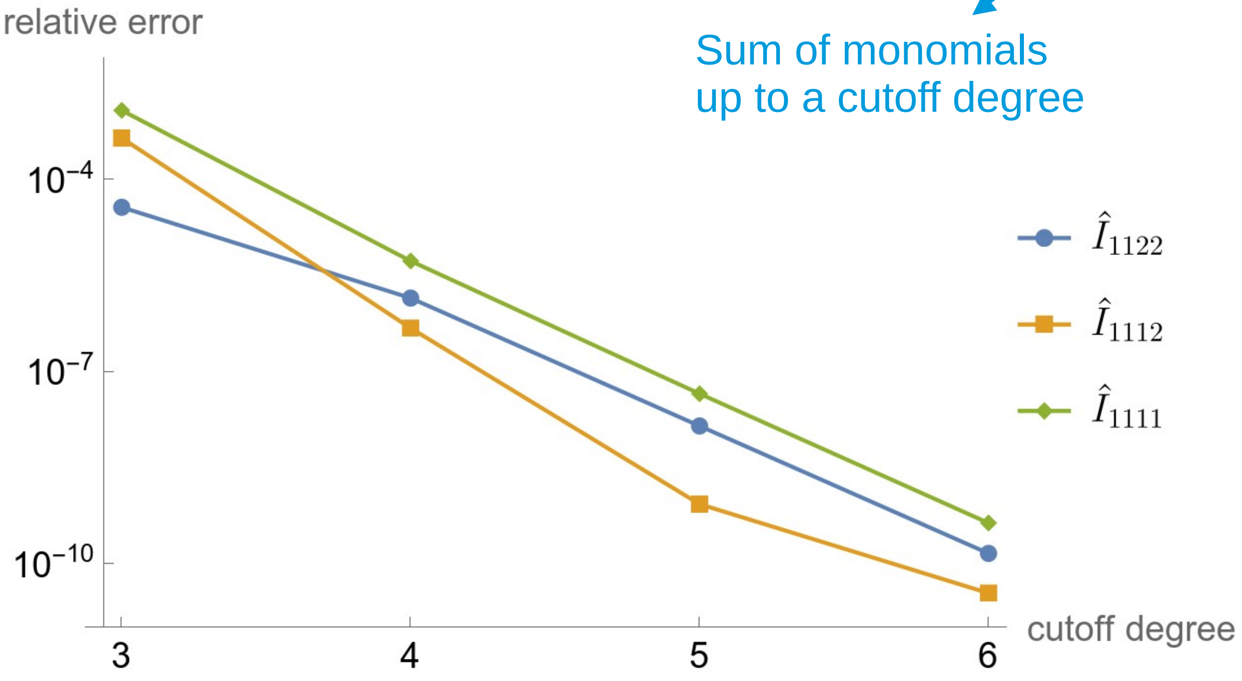
$I_{1,1,2,2}, I_{1,1,1,2}, I_{1,1,1,1}$ + index perms.

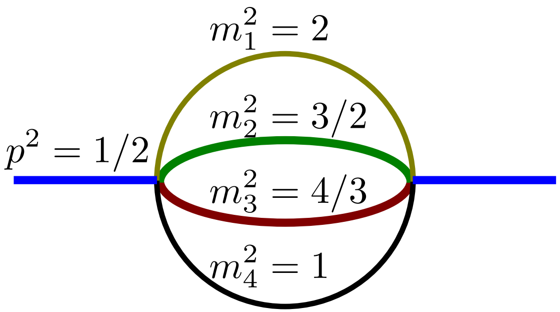
4 trivial tadpole^3 sub-topologies, if any a_i = 0. **Values used as input** (get consistency check for free)

Error vs. cutoff degree for d = 2

$$0 < \int_0^1 dx_1 \cdots \int_0^1 dx_4\, \delta(x_1 + \cdots + x_4 - 1)\, \tilde{x}_i P^2(\tilde{x}_i) \frac{U^{4-(l+1)d/2}}{F^{4-ld/2}}\,, \qquad \tilde{x}_i \equiv x_i U/F$$

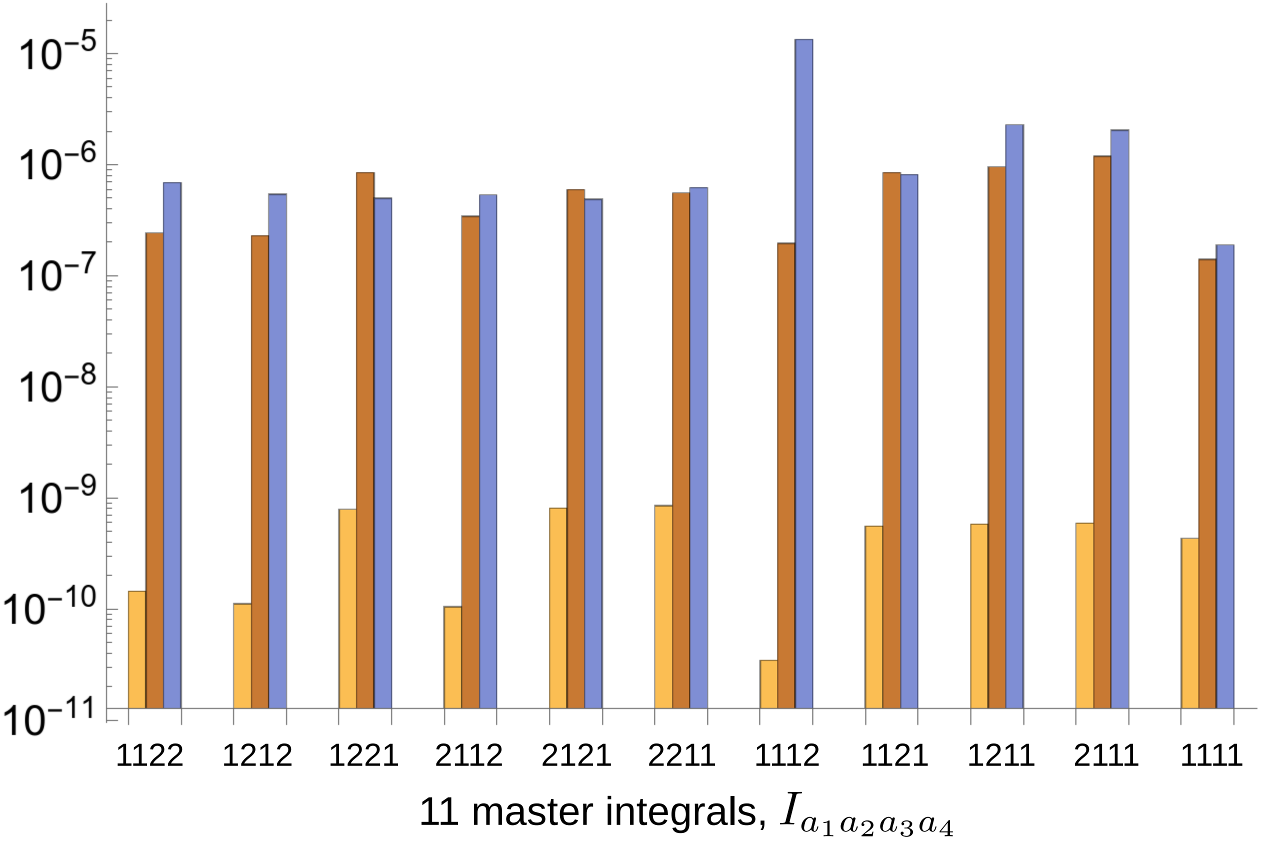
Sum of monomials
up to a cutoff degree





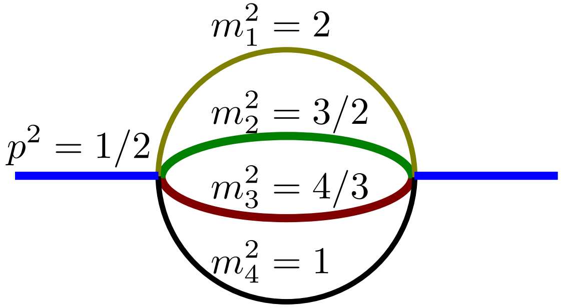
Exponential convergence, as in 1-loop example

Error from ε expansion constraints

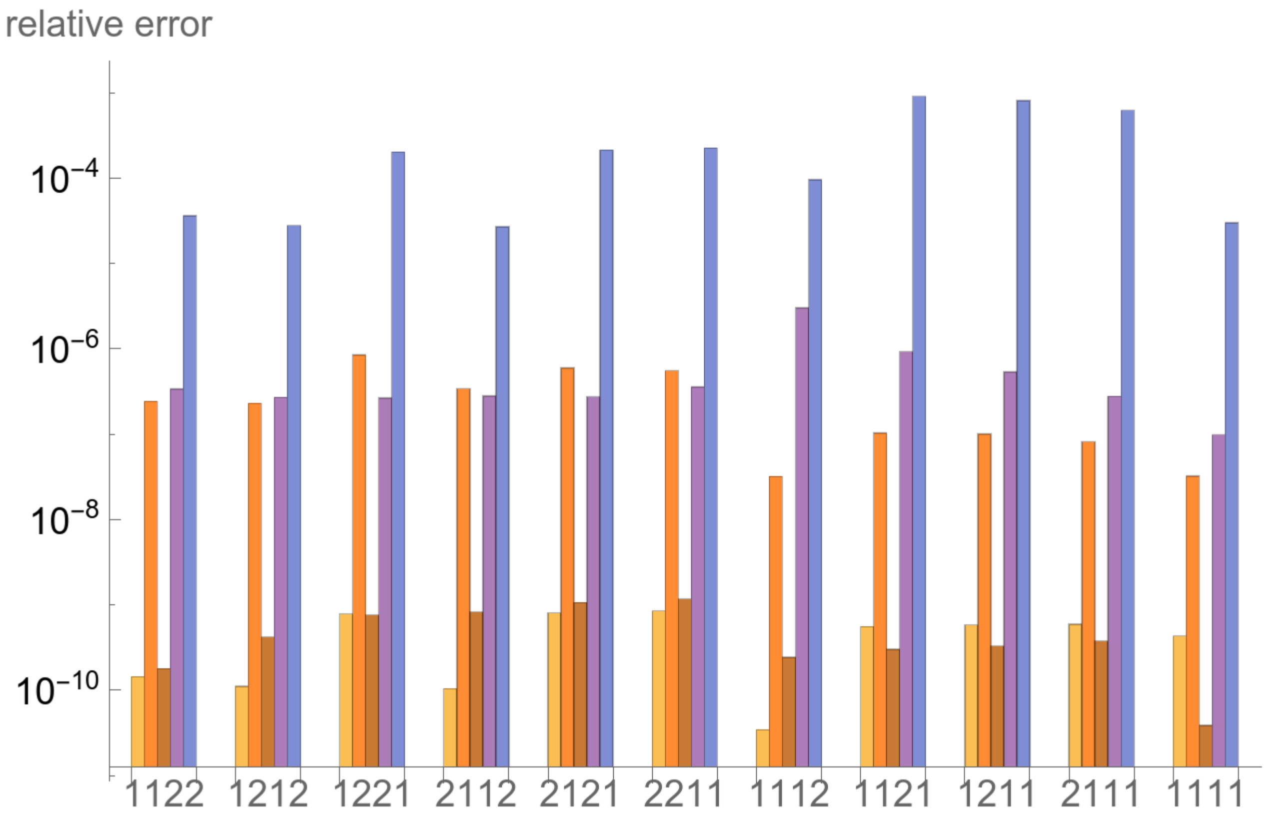


(Cutoff degree 6)

Reference results: `DiffExp` package [Hidding, '20]



Error from numerical differentiation



4th order finite difference

$$\epsilon = 0, \pm\delta, \pm 2\delta, \delta = 10^{-3}$$

- ϵ^0
- ϵ^1, cons
- ϵ^1, diff ✓
- ϵ^2, cons ✓
- ϵ^2, diff

Conclusions

- New method for evaluating Feynman integrals, using ***inequality constraints.*** *Evidence: **constraints strong enough** to uniquely fix Euclidean integrals.*

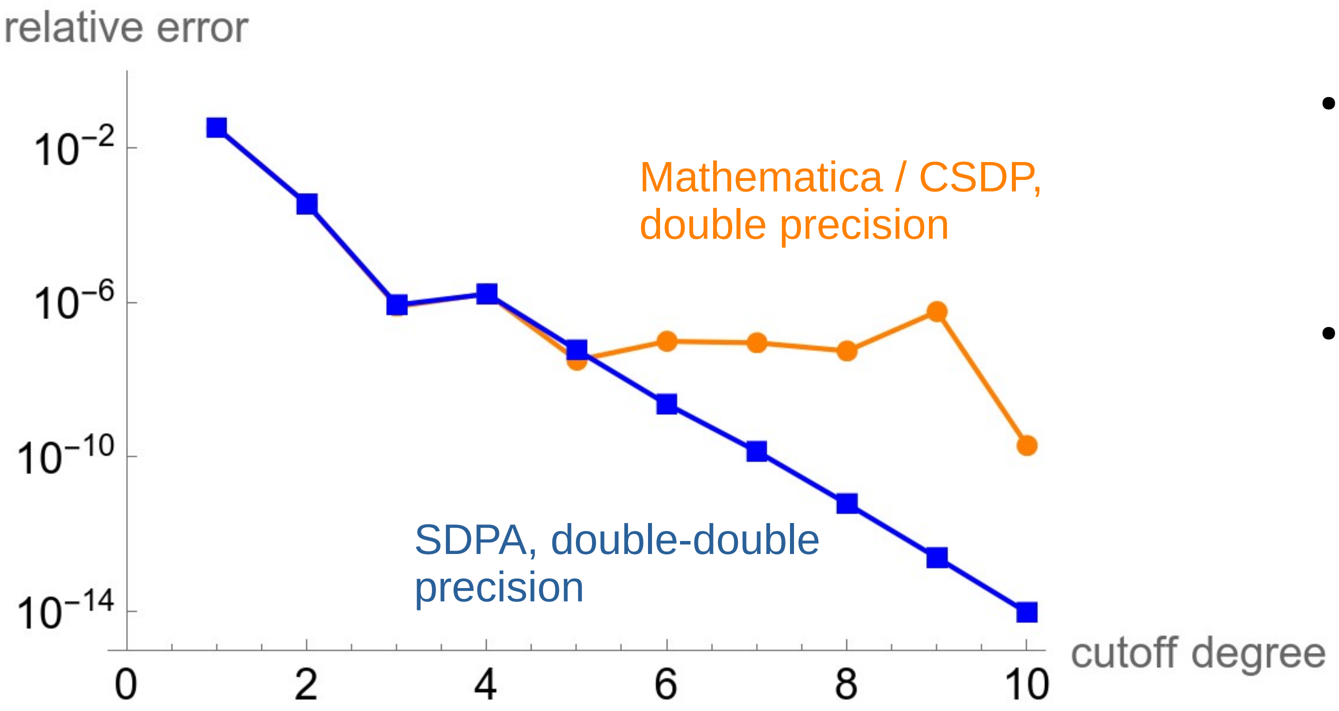
- ***New consistency relations*** for ε expansion terms of any convergent Euclidean integral.

- ***Linear identities*** (IBP, dimension shifting) crucial; facilitates exchanging techniques from perturbative and nonperturbative contexts.
  - QM: moment recursion. Lattice YM: loop equation. Ising: spin flip identities...
  - In reverse direction, perturbative technique of ***IBP, intersection theory, differential equations*** applied to lattice correlation functions. [Weinzierl, '20. Gasparotto, Rapakoulias, Weinzierl, '22. Cacciatori, Mastrolia, '22.]

- Explore positivity constraints for entire amplitudes order by order perturbatively? E.g. numerical evidence in [Dixon, von Hippel, McLeod, Trnka, '16]

Backup Slides

Need for higher-precision arithmetic



- Later, will also need quad-double precision or higher
- Similar to observations in applying semidefinte programming to conformal bootstrap etc. [e.g. Simmons-Duffin, 1502.02033]

Details: Feynman parameter space positivity

- Positive integral: multiplying $I_{2,1}$ integrand by square of polynomial

$$\int_0^1 dx\, x\, P^2(x)[m^2 - p^2 x(1-x)]^{-1-\epsilon} \geq 0\,,$$

$$\text{for any } P(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n\,,$$

- As in the momentum-space case, get semidefinite constraint

$$I_{3,0} \cdot \mathbb{M}_1 + I_{2,1} \cdot \mathbb{M}_2 \succcurlyeq 0$$

known tadpole integral

bubble integral is constrained

More positivity for ε expansion

$$0 \le \int_0^1 dx_1 \cdots \int_0^1 dx_n \, \delta(x_1 + \cdots + x_n - 1) \frac{U^{a-(l+1)d_0/2}}{F^{a-ld_0/2}}$$
$$\times \, P^2(\log G) \cdot (\log G - \log G_{\min})$$

if log G is bounded below. E.g. one-loop bubble

$$\log[m^2 - p^2 x(1-x)] \ge \log(m^2 - p^2/4), \quad 0 \le x \le 1$$

Similarly, if log G is bounded below, use $P^2(\log G) \cdot (\log G_{\max} - \log G)$

More generally, use square of polynomial in both x_i and log G, to determine the ε expansion of all master integrals together.

Bubble integral ε expansion result

At $p^2/m^2 = 2$, cutoff degree 14. relative deviation from exact result

$$I_{2,1}/I_{3,0} = 1.5707963267941302820 \quad \text{------} \quad 5 \times 10^{-13}$$

$$+ \epsilon \cdot 0.74313814320585636075 \quad \text{------} \quad 4 \times 10^{-12}$$

$$+ \epsilon^2 \cdot 0.20810874445217139583 \quad \text{------} \quad 2 \times 10^{-11}$$

More calculation details on backup slides

- *Alternatively,* perform numerical differentiation of *d*-dimensional SDP result. Evaluations at $d = 4 - 2\epsilon, \ \epsilon = 0, \pm 10^{-3}, \pm 2 \times 10^{-3}$
  - Use 4th order finite-difference approximation
  - Relative errors at *O(ε) and O(ε²)*: $3 \times 10^{-12}, \ 3 \times 10^{-10}$

Positive integrands beyond 1 loop

- Feynman parameterization

$$I_{a_1,a_2,a_3,a_4} \propto \int_0^1 dx_1 \cdots \int_0^1 dx_4\, \delta(x_1+\cdots+x_4-1)\, x_1^{a_1-1}\ldots x_4^{a_4-1} \frac{U^{a-(l+1)d/2}}{F^{a-ld/2}},$$

$$a \equiv a_1+\cdots+a_4$$

- Beyond 1 loop, inserting powers of x_i alone is trickly... instead consider

$$0 < \int_0^1 dx_1 \cdots \int_0^1 dx_4\, \delta(x_1+\cdots+x_4-1) Q(x_i U/F) \frac{U^{4-(l+1)d/2}}{F^{4-ld/2}},$$

where Q is any positive polynomial.

$\tilde{x}_i \equiv x_i U/F$,
maintains "projective invariance"

Positive integrands for 3-loop banana integrals

- With polynomials *P* with max. degree 6, we consider +ve integrands:

$$Q(\tilde{x}_i) = \tilde{x}_1 P^2(\tilde{x}_i),\ x_2 P^2(\tilde{x}_i),\ \tilde{x}_3 P^2(\tilde{x}_i),\ \tilde{x}_4 P^2(\tilde{x}_i)$$

- Matrix with four 210×210 blocks, depending linearly on 11 unknown master integrals, fixed by positive semidefiniteness.

- Beyond 1 loop, inserting powers of x_i alone is trickly... instead consider

$$0 < \int_0^1 dx_1 \cdots \int_0^1 dx_4\, \delta(x_1 + \cdots + x_4 - 1) Q(x_i U/F) \frac{U^{4-(l+1)d/2}}{F^{4-ld/2}}\,,$$

where *Q* is any positive polynomial.

$\tilde{x}_i \equiv x_i U/F\,,$
keeps U and F with "correct" power