Fermilab

A new approach to QCD evolution in processes with massive partons

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Based on: 2306.xxxx with Stefan Höche (Fermilab)

Introduction

Heavy flavour production studied in multiple contexts:

Calculations at FO, NLL, FONLL: Cacciari, Frixione, Houdeau, Mangano, Nason, Ridolfi,… [arXiv:1205.6344, hep-ph/0312132, hep-ph/9801375, NPB373(1992)295, NPB373(1992)295, …]

Particle-level Monte Carlo: Norrbin, Sjöstrand, Gieseke, Stephens, Webber, Schumann, Krauss, Gehrmann-deRidder, Ritzmann, Skands,… [hep-ph/0010012, hep-ph/0010012, hep-ph/0310083, arXiv:0709.1027, arXiv:1108.617]

Matching & merging of (N)LO PS: Frixione, Nason, Webber, Mangano, Moretti, Pittau,… [hep-ph/0305252, arXiv:0707.3088, hep-ph/0108069]

This talk: generalise the Alaric shower to account for massive evolution

Required for proper description of bottom and charm jet production and quark fragmentation functions

Subtleties of heavy flavour evolution

Both **high-energy** and **threshold** regime require accurate descriptions

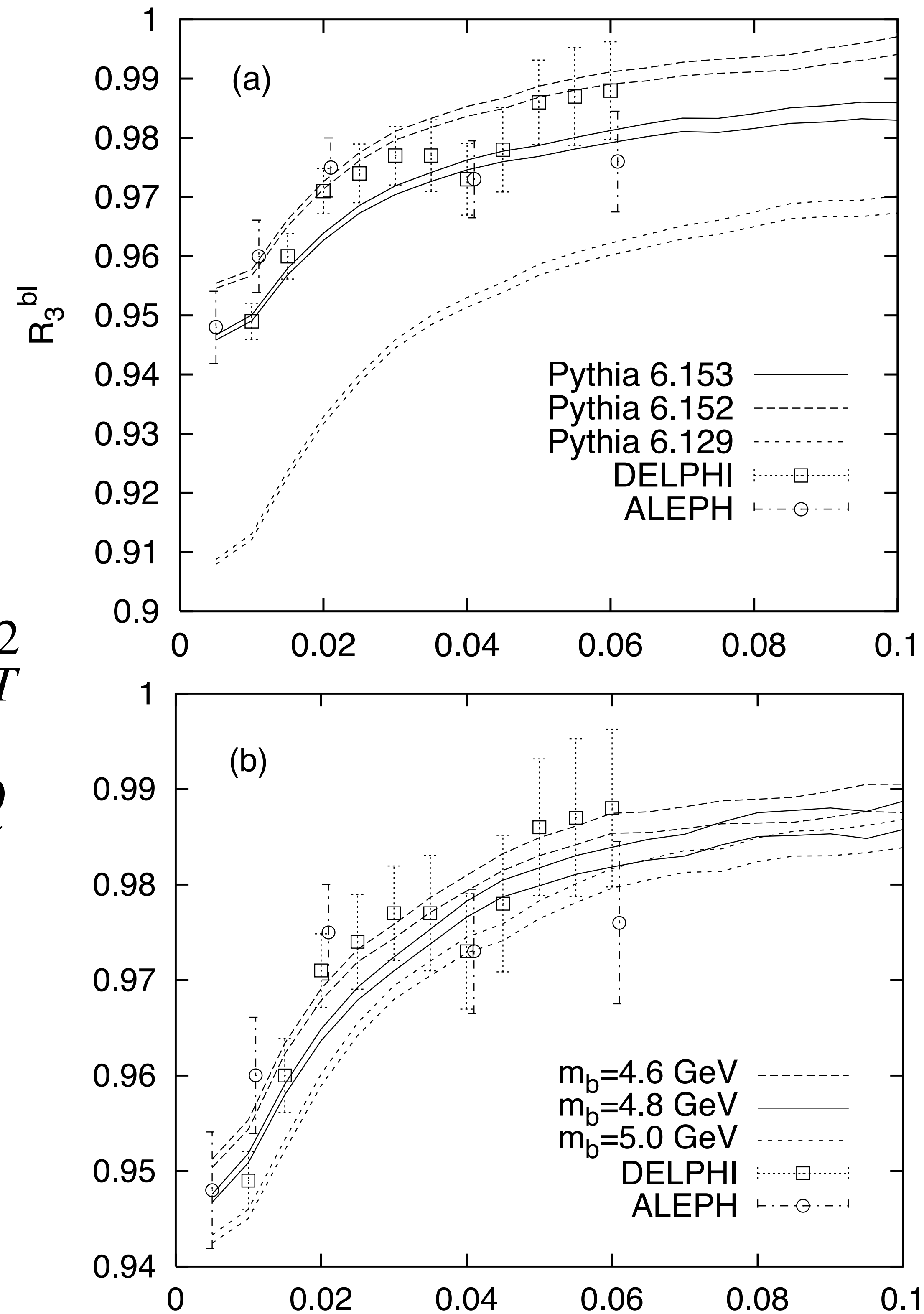
Various challenges:

IR finite prediction for $g \to Q\bar{Q}$ choice of splitting function semi-arbitrary

Soft gluon emission off light/heavy quarks associated with $\alpha_s(k_T^2) \Rightarrow$ 'correct' scale is k_T^2

No such scale-setting argument for $g \to Q\bar{Q}$ $\Rightarrow$ HQ production rate not very stable parton shower variations [Amati et al.]

Multiple prescriptions varying success in describing experiments [Norrbin, Sjöstrand], [Gieseke, Stephens], [Schumann, Krauss], [Gehrmann-deRidder, Ritzmann, Skands]



[Norrbin, Sjöstrand]

Soft-collinear matching

Squared amplitude factorizes in **quasi-collinear** limit:

$$\sum_{\lambda,\lambda'=\pm} \left\langle 1,\ldots,\not{i}(ij),\ldots,\not{j},\ldots,n \left| \frac{8\pi\alpha_s P^{\lambda\lambda'}_{(ij)i}(z)}{(p_i+p_j)^2 - m_{ij}^2} \right| 1,\ldots,\not{i}(ij),\ldots,\not{j},\ldots,\not{j},\ldots,n \right\rangle$$

Splitting function: $P_{ab}(z,\epsilon) = \delta_{ab} C_a \left(\frac{2z}{1-z} - \frac{m_i^2}{p_i p_j} \right) + C_{ab}(z,\epsilon)$

Soft gluon limit:

$$-8\pi\alpha_s \sum_{i,k\neq j} \left\langle 1,\ldots,\not{j},\ldots,n \left| \mathbf{T}_i \mathbf{T}_k \, w_{ik,j} \right| 1,\ldots,\not{j},\ldots,n \right\rangle$$

Soft radiator in terms of energies and angles [Marchesini,Webber]

$$w_{ik,j} \to \frac{2p_i p_k}{(p_i p_j)(p_j p_k)} - \frac{p_i^2}{(p_i p_j)^2} - \frac{p_k^2}{(p_k p_j)^2} = \frac{2W_{ik,j}}{E_j^2}$$

Angular radiator function:

$$W_{ik,j} = \frac{1 - v_i v_k \cos\theta_{ik}}{(1 - v_i\cos\theta_{ij})(1 - v_k\cos\theta_{jk})} - \frac{(1-v_i^2)/2}{(1-v_i\cos\theta_{ij})^2} - \frac{(1-v_k^2)/2}{(1-v_k\cos\theta_{jk})^2}$$

Soft collinear matching

Exposing individual collinear singularities by partial fractioning

$$W_{ik,j} = \bar{W}^i_{ik,j} + \bar{W}^k_{ki,j} \quad \text{with} \quad \bar{W}^i_{ik,j} = \frac{1 - \cos\theta_{ik}}{(1 - \cos\theta_{ij})(2 - \cos\theta_{ij} - \cos\theta_{kj})} W_{ik,j}$$

Bounded by $(1 - \cos\theta_{ij})\bar{W}^i_{ik,j} < 2$ and strictly positive

Maintains PD interpretation ↔ efficient MC implementation

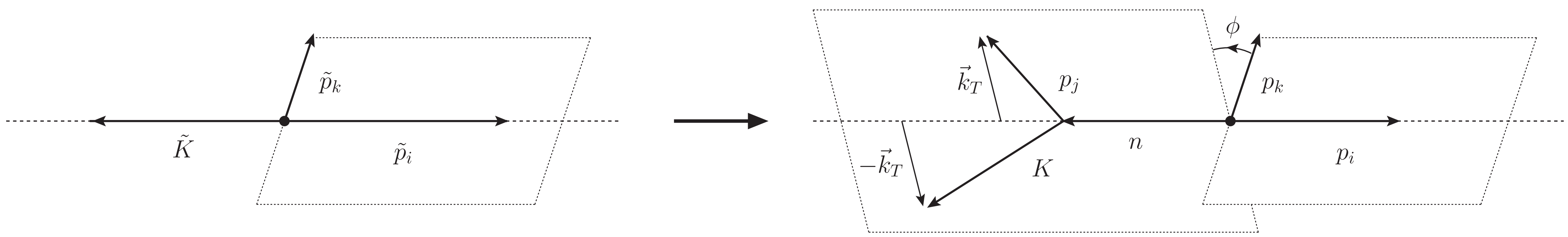
Taking quasi-collinear limit

$$w_{ik,j} \xrightarrow[m_i \propto p_i p_j]{i||j} w^{(\mathrm{coll})}_{ik,j}(z) = \frac{1}{2p_i p_j}\left(\frac{2z}{1-z} - \frac{m_i^2}{p_i p_j}\right) = \frac{P_{aa}(z,\epsilon) - C_{aa}(z,\epsilon)}{C_a\, 2p_i p_j}$$

ID'd with the **leading term** in $(1 - z)$ of splitting functions $P_{aa}(z, \epsilon)$

Kinematics mapping — Soft emissions

Collinear NLL safe inspired by ID particle CS algorithm [Catani,Seymour: hep-ph/9605323] (Florian's talk - arXiv:2208.06057)



Rescale emitter momentum with z (n aux vector) maintain collinear safety

$$p_i \xrightarrow{i||j} z\,\tilde{p}_i\,, \qquad p_j \xrightarrow{i||j} (1-z)\,\tilde{p}_i \qquad \text{where} \qquad z = \frac{p_i n}{(p_i + p_j)n}$$

Compensating remaining recoil by hard momentum $\tilde{K}$

$$p_j = (1-z)\,\tilde{p}_i + v\big(\tilde{K} - (1-z+2\kappa)\,\tilde{p}_i\big) + k_\perp, \quad K = \tilde{K} - v\big(\tilde{K} - (1-z+2\kappa)\,\tilde{p}_i\big) - k_\perp$$

Recoil effects on $\tilde{K}$ scale with $k_\perp^2/\tilde{K}^2$

Kinematics mapping — Soft emissions

Massive case: using collinear safety and NLL constraints

$$p_i^\mu = \bar{z}\,\tilde{p}_{ij}^\mu + \frac{\mu_i^2 - \bar{z}^2\mu_{ij}^2}{\bar{z}\,v_{\tilde{p}_{ij}\tilde{K}}}\left(\tilde{K}^\mu - \bar{\kappa}\,\tilde{p}_{ij}^\mu\right)$$

$$p_j^\mu = \frac{1 - z + \mu_{ij}^2 - \mu_i^2 + \mu_j^2}{v_{\tilde{p}_{ij}\tilde{K}}\zeta}\left(\tilde{p}_{ij}^\mu - \bar{\mu}_{ij}^2\tilde{K}^\mu\right)$$

$$+\bar{v}\,\frac{1 + v_{\tilde{p}_{ij}\tilde{K}}}{2v_{\tilde{p}_{ij}\tilde{K}}}\left[\left(\tilde{K}^\mu - \bar{\kappa}\tilde{p}_{ij}^\mu\right) - \frac{1 - \bar{z} + \bar{\kappa}}{\zeta}\left(\tilde{p}_{ij}^\mu - \bar{\mu}_{ij}^2\tilde{K}^\mu\right)\right] + k_\perp^\mu$$

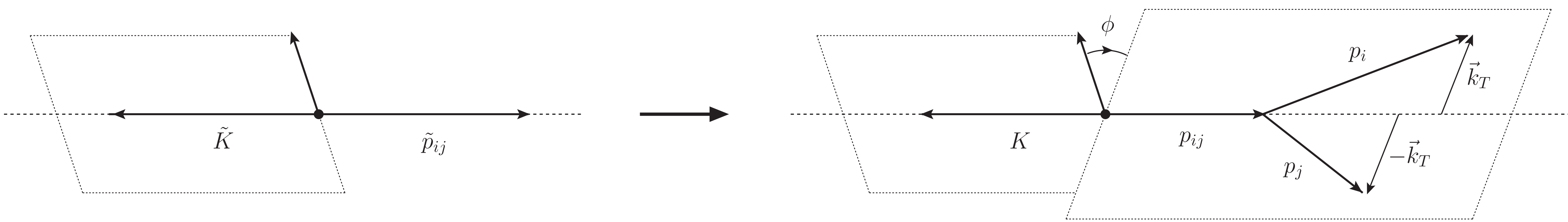
with massive analogues to z, v and $k_\perp$

$$\bar{z} = \frac{z + 2\mu_i^2}{1 + v_{\tilde{p}_i\tilde{K}} + 2\mu_i^2} + \sqrt{\left(\frac{z + 2\mu_i^2}{1 + v_{\tilde{p}_i\tilde{K}} + 2\mu_i^2}\right)^2 - \frac{2\mu_i^2(1 + \bar{\kappa})}{1 + v_{\tilde{p}_i\tilde{K}} + 2\mu_i^2}}, \quad \bar{v} = \frac{\frac{2\,v\,z}{1 + v_{\tilde{p}_i\tilde{K}}} - \frac{\bar{\mu}_i^2}{\bar{z}}\left(1 - \bar{z} + \bar{\kappa} - \frac{\bar{\kappa}}{\zeta}\right)}{\bar{z} - \frac{\bar{\mu}_i^2}{\bar{z}}\frac{1 - \bar{z} + \bar{\kappa}}{\zeta}},$$

$$\mathrm{k}_\perp^2 = \tilde{p}_{ij}\tilde{K}(1 + v_{\tilde{p}_{ij}\tilde{K}})\,\bar{v}\left[\left(1 - \frac{\bar{v}}{\zeta}\right)(1 - \bar{z}) - \frac{1 - \zeta + \bar{v}}{\zeta}\,\bar{\kappa} + \frac{\bar{\mu}_j^2}{\zeta}\right] - m_j^2$$

Kinematics mapping — Collinear splittings

Kinematics of **purely collinear** components of splitting functions



Compensating longitudinal recoil by hard momentum $\tilde{K}$ and transverse recoil by splitter

$$p_i^\mu = z\,\tilde{p}_{ij}^\mu + y(1-\bar{z})\,\tilde{K}^\mu + k_\perp^\mu, \quad p_j^\mu = (1-\bar{z})\,\tilde{p}_{ij}^\mu + y\,\bar{z}\,\tilde{K}^\mu - k_\perp^\mu$$

Not NLL safe for massless $\tilde{K}$ [Dagupta et al.,arXiv:1805.09327]

Kinematics mapping — Collinear splittings

Massive case: Using CDST kinematics with hard momentum $\tilde{K}$
[Catani,Dittmaier,Seymour,Trocszanyi: hep-ph/0201036]

$$p_i^\mu = \bar{z}\,\frac{\tilde{p}_{ij}^\mu - \bar{\mu}_{ij}^2 \tilde{K}^\mu}{\bar{z}_{ij}\, v_{\tilde{p}_{ij},\tilde{K}}} + \left(y(1-\bar{z})(1+\mu_{ij}^2-\mu_i^2-\mu_j^2) - \bar{z}(\mu_i^2+\mu_j^2) + 2\mu_i^2 \right) \frac{\tilde{K}^\mu - \bar{\kappa}\,\tilde{p}_{ij}^\mu}{v_{\tilde{p}_{ij},\tilde{K}}/z_{ij}} + k_\perp^\mu$$

$$p_j^\mu = (1-\bar{z})\,\frac{\tilde{p}_{ij}^\mu - \bar{\mu}_{ij}^2 \tilde{K}^\mu}{\bar{z}_{ij}\, v_{\tilde{p}_{ij},\tilde{K}}} + \left(y\,\bar{z}\,(1+\mu_{ij}^2-\mu_i^2-\mu_j^2) - (1-\bar{z})(\mu_i^2+\mu_j^2) + 2\mu_j^2 \right) \frac{\tilde{K}^\mu - \bar{\kappa}\,\tilde{p}_{ij}^\mu}{v_{\tilde{p}_{ij},\tilde{K}}/z_{ij}} - k_\perp^\mu$$

Can show that longitudinal recoil effects scale as $k_\perp^2/\tilde{K}^2$

Transverse recoil irrelevant at NLL if applied to collinear branching only

NLL accuracy: analysis based final-state resummation technique
[Banfi, Salam, Zanderighi, hep-ph/0407286] (Florian's talk - massless case)

Kinematical effect on NLL corrections

Massive extension: given observable to be re-summed $V(\{p\}, \{k\})$

Parametrizing hard $p_{i,j}$ and soft momenta k

$$k = (z_{i,j} - \bar{\mu}_j^2 z_{j,i})\, p_i + (z_{j,i} - \bar{\mu}_i^2 z_{i,j})\, p_j + k_{T,ij}$$

$$k_{T,ij}^2 = 2 p_i p_j \frac{2 v_{p_i,p_j}}{1 + v_{p_i,p_j}} z_{i,j}\, z_{j,i}$$

with rapidity $\eta_{ij} = 1/2 \ln(z_{i,j}/z_{j,i})$

Observable can be expressed as

$$V(k) = d_l \left(\frac{k_{T,l}}{Q} \right)^a e^{-b_l \eta_l} g_l(\phi_l)$$

with $k_{T,l} = k_{T,lj}$, $\eta_l = \eta_{lj}$ for $j \nparallel l$ in collinear limit

Kinematical effect on NLL corrections

Quasi-collinear limit: constant $z_{i,j}, z_{j,i}$ and small k_T gluon momentum

$$k \xrightarrow[m_{i/j}^2 \propto k_T^2]{k_T^2 \ll p_i p_j} z_{i,j} p_i + z_{j,i} p_j + k_{T,ij} + \mathcal{O}(k_T^2)$$

equal to massless case ⇒ value of observable unchanged

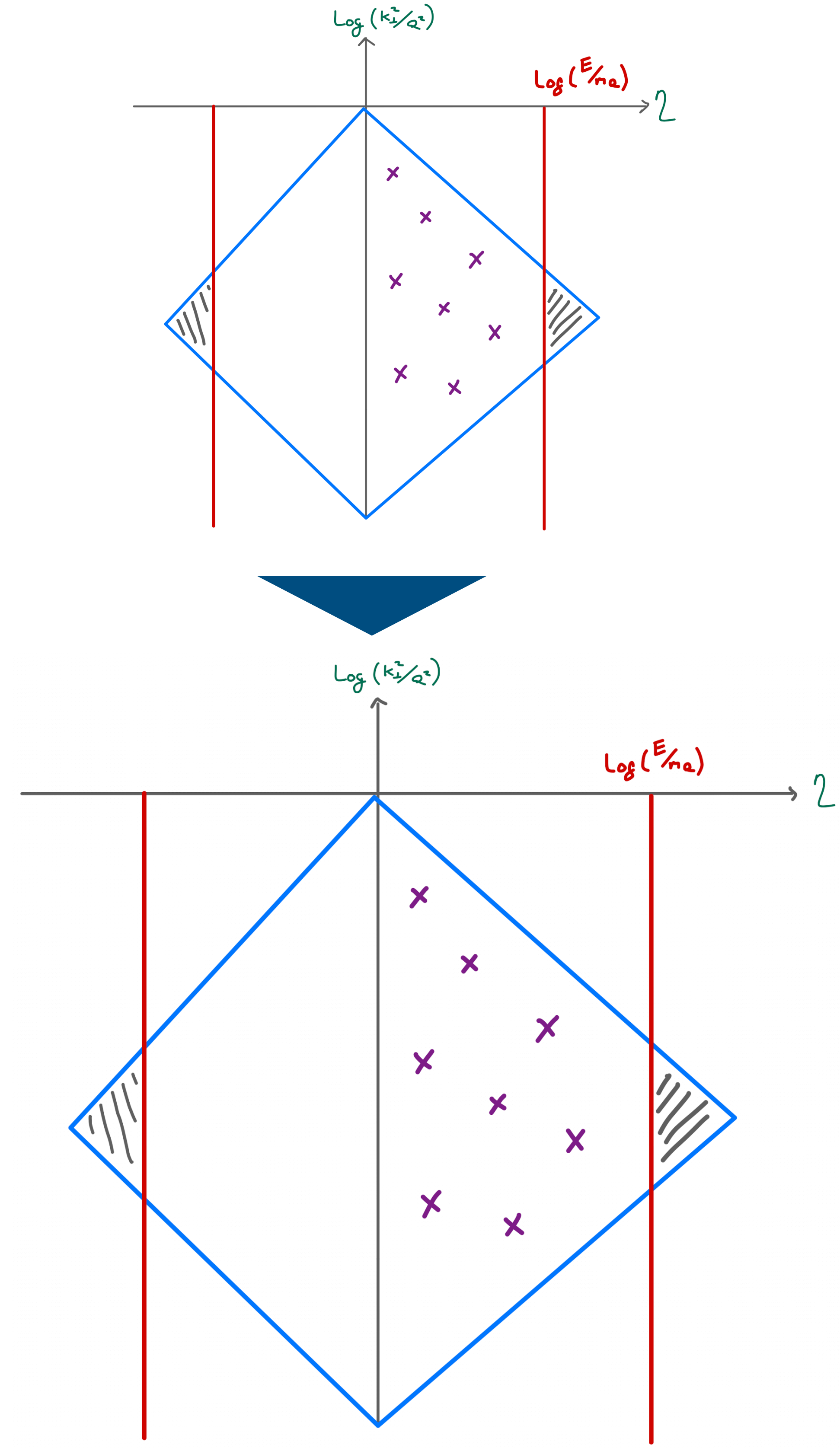
Both Sudakov and $\mathcal{F}$ change due to massive splittings and integration boundary effects

Gluon emission off a dipole containing massive quark with (m, E) has rapidity bound (**dead-cone**) at $\eta = \ln(E/m)$

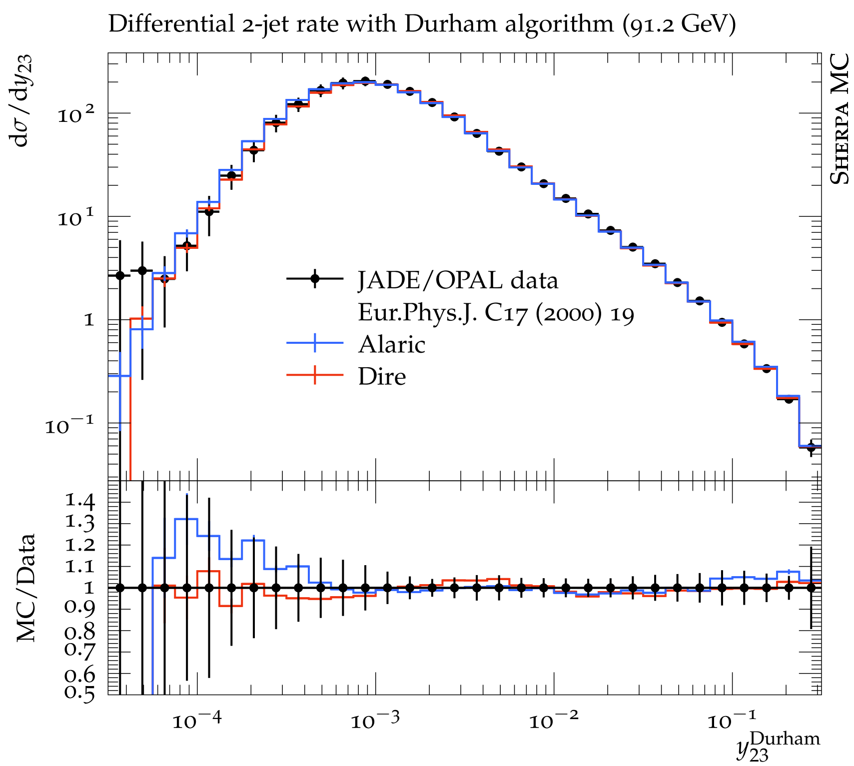
Also showed hard (quasi-)collinear emissions do **not** generate LTFs that change momenta by more than

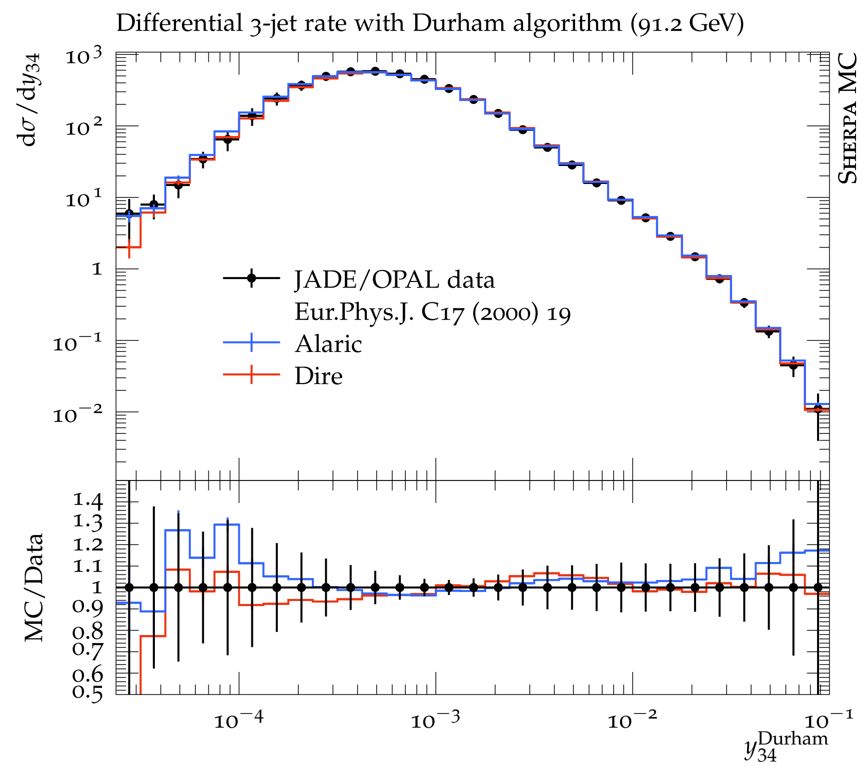
$$\mathcal{O}((k_T^2/K^2)^{\tilde{\rho}})$$

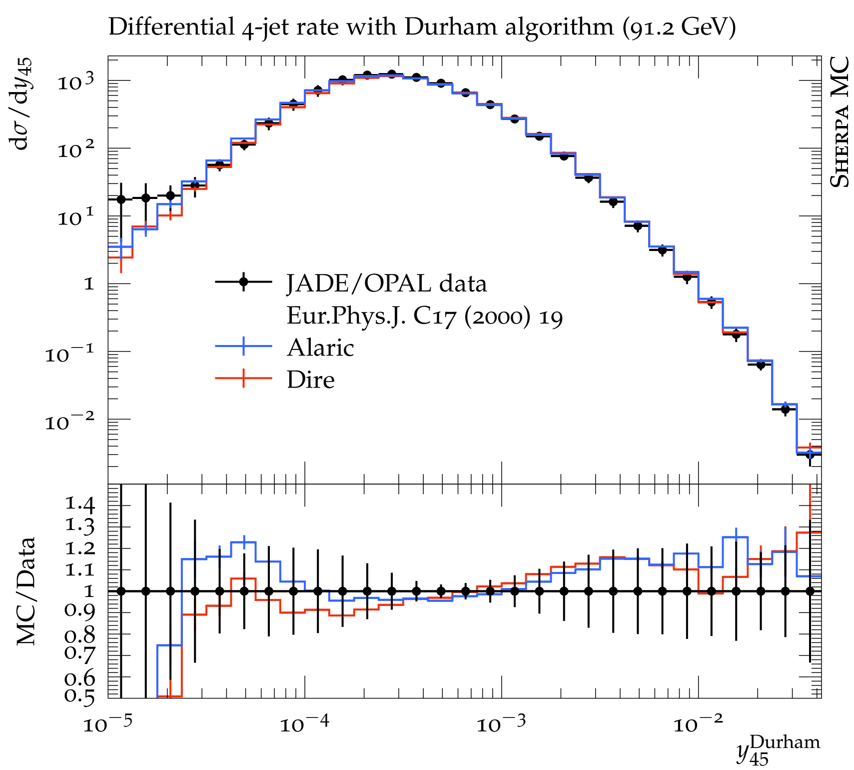
Effect of LTF vanish even for **hard quasi-collinear** splittings

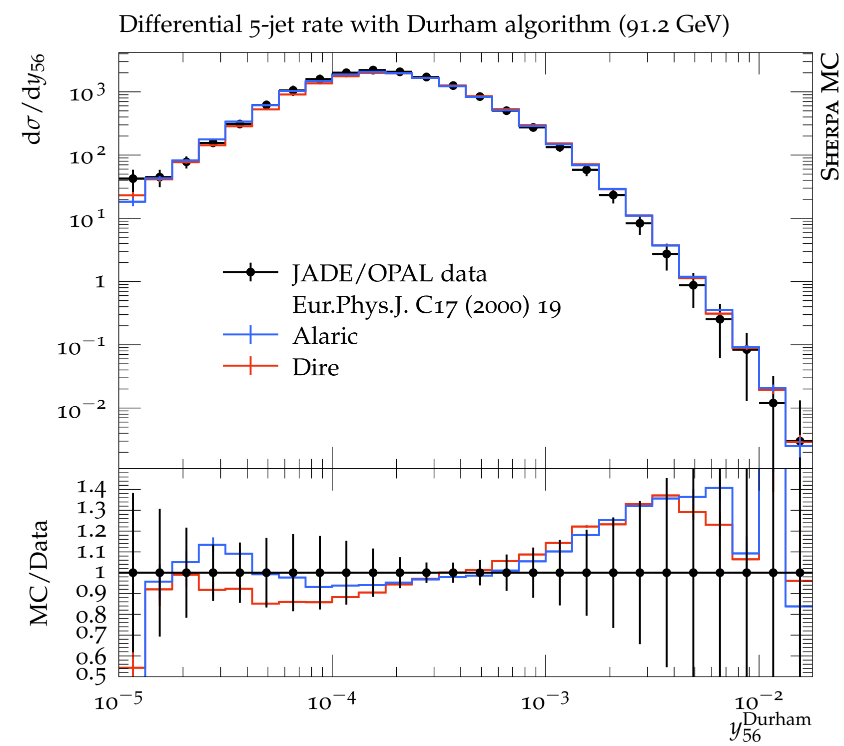


Experimental comparisons



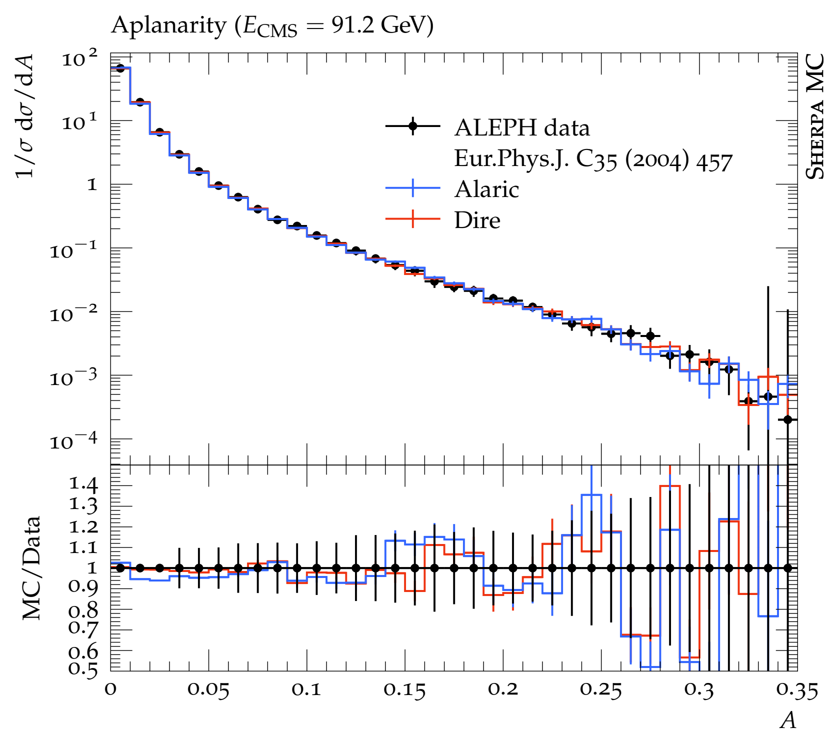


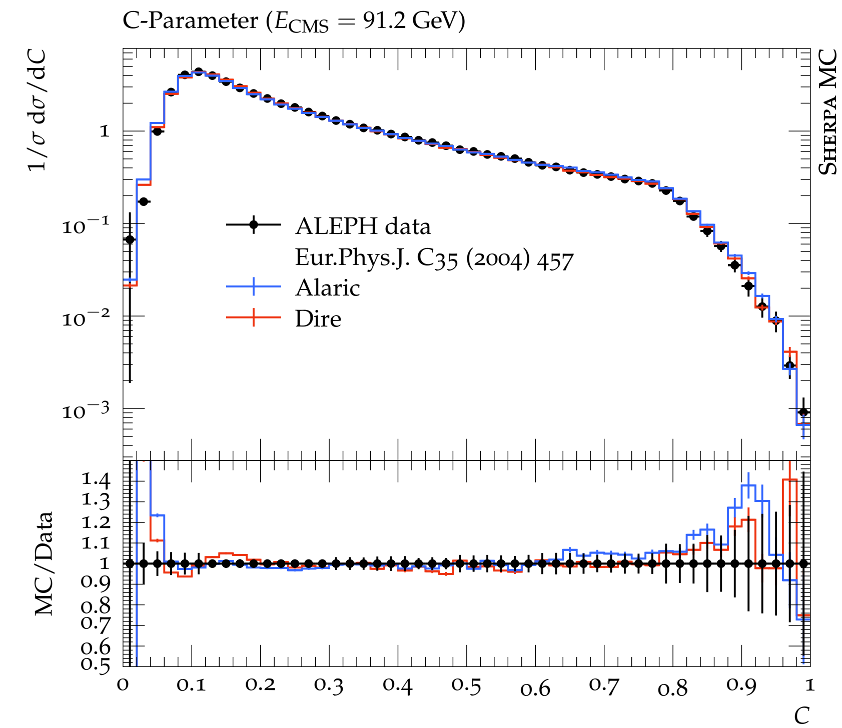


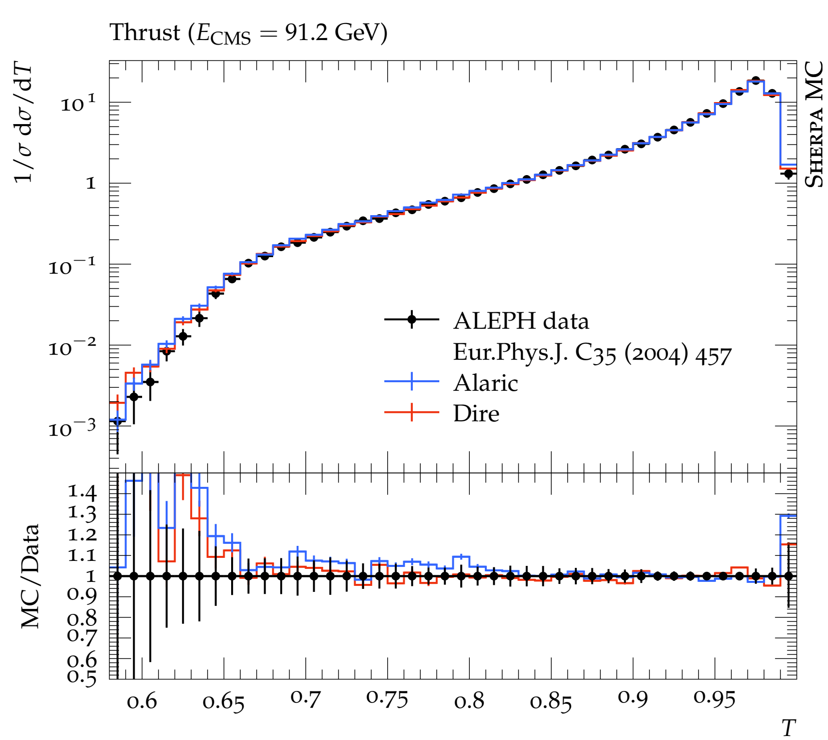


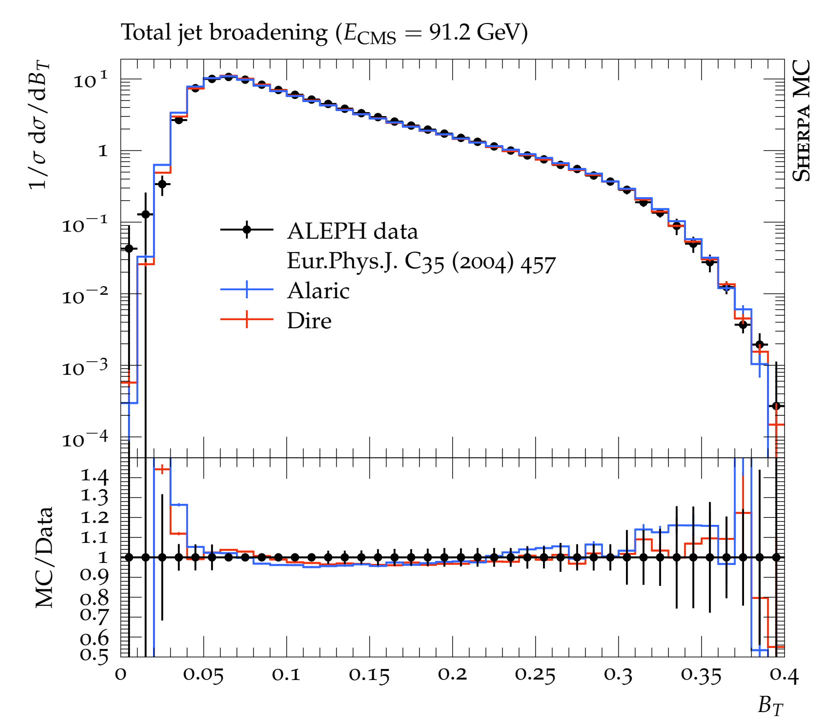
Strong agreement with data hadronization effects dominate predictions below $\sim 10^{-4}$ and quality of prediction similar to [Herren et al.] where HQ effects where modeled by threshold effects

Experimental comparisons



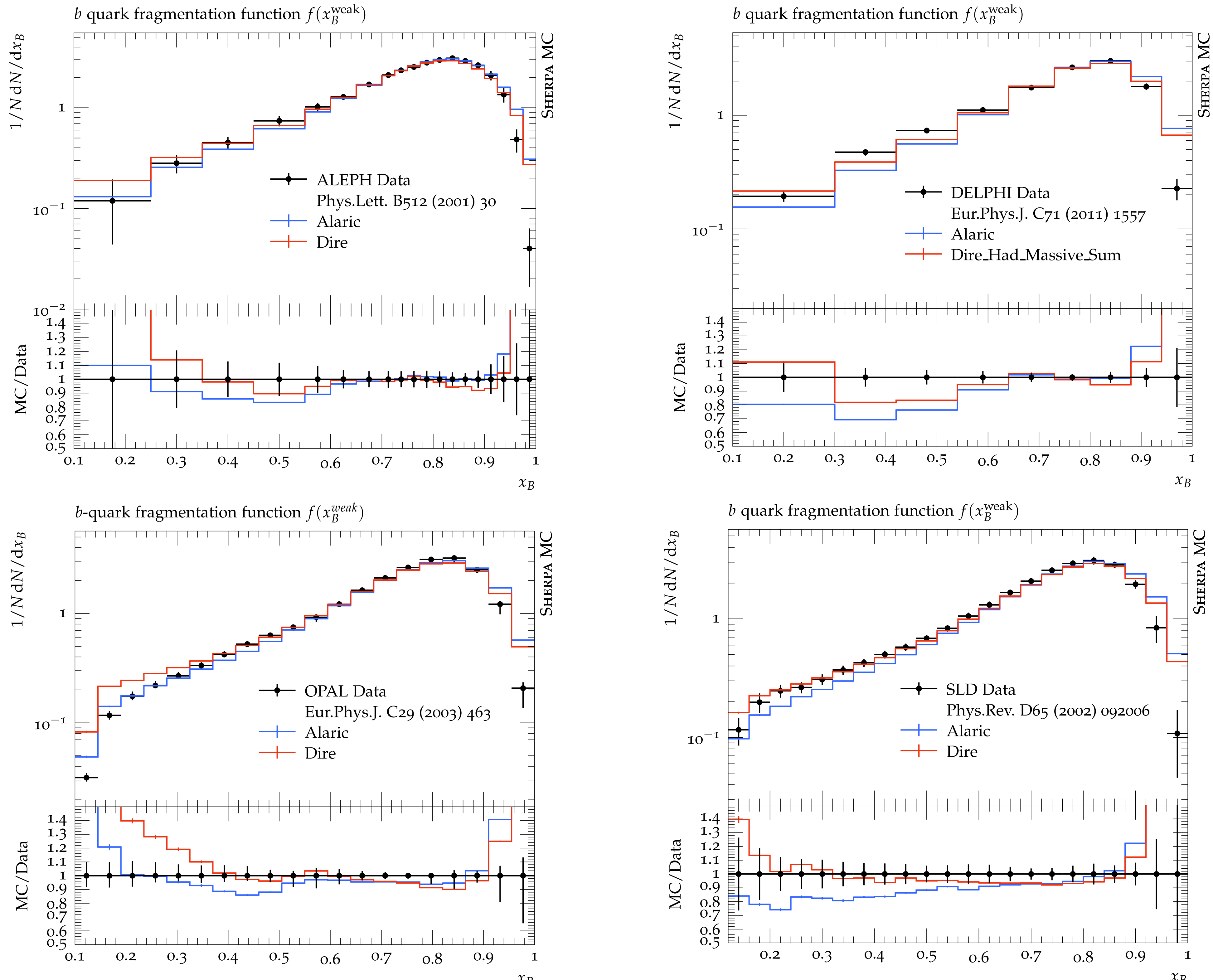






Again **agreement** and comparable to [Herren et al.] **expected** that numerical predictions will improve upon including matrix-element corrections or merging the parton shower with higher-multiplicity FO calculations

Experimental comparisons



Agreement for both Alaric and Dire predictions with experimental data (note that both parton shower implementations use same hadronization tune)

NLO matching

MC@NLO: choose subtraction terms to be evolution kernels $\Rightarrow$ compute integrated terms with our momentum mapping [Frixione, Webber, hep-ph/0204244]

$$\sigma^{\text{NLO}} = \int d\Phi_n \left[B + V + \int d\Phi_{+1} S \right] + \int d\Phi_{n+1} [R - S]$$

Combined integrated subtraction term for identified parton production with a partonic frag. function [Höche, Liebschner, Siegert, 1807.04348]

$$\int_{m+1} \mathrm{d}\sigma^S + \int_m \mathrm{d}\sigma^C = \frac{1}{2} \sum_{i=g,q,\bar{q}} \sum_{\imath=1}^{m} \int_0^1 \frac{dz}{z^{2-2\epsilon}} \int_m d\sigma^B(p_1, \ldots, \frac{p_i}{z}, \ldots, p_m) \otimes \hat{\mathbf{I}}_{\imath i}^{(\text{FS})}$$

with FS insertion operator $\hat{\mathbf{I}}_{\imath i}^{(\text{FS})} = \delta(1-z)\mathbf{I}_{\imath i} + \mathbf{P}_{\imath i} + \mathbf{H}_{\imath i}$

Require **factorization** of the differential $n+1 \to n$ particle PS element

$$\mathrm{d}\Phi_n(p_a, p_b; p_1, \ldots, p_i, \ldots, p_j, \ldots, p_n) = \left[\prod_{i=1}^{n} \frac{1}{(2\pi)^3} \frac{\mathrm{d}^3 p_i}{2p_i^0} \right] (2\pi)^4 \delta^{(4)}\Big(p_a + p_b - \sum p_i \Big)$$

The ratio of differential PS elements **after** and **before** mapping is **single emission PS element**

$$\mathrm{d}\Phi_{+1}(\tilde{p}_a, \tilde{p}_b; \tilde{p}_1, \ldots, \tilde{p}_{ij}, \ldots, \tilde{p}_n; p_i, p_j) = \frac{\mathrm{d}\Phi_n(p_a, p_b; p_1, \ldots, p_i, \ldots, p_j, \ldots, p_n)}{\mathrm{d}\Phi_{n-1}(\tilde{p}_a, \tilde{p}_b; \tilde{p}_1, \ldots, \tilde{p}_{ij}, \ldots \tilde{p}_j, \ldots, \tilde{p}_n)}$$

PS factorization — Soft kinematics

Combining $i, j \to \widetilde{ij}$ the PS-element for underlying Born PS

$$\mathrm{d}\Phi_{n-1} = \left[\prod_{\substack{k=1\\k\neq i,j}}^{n} \frac{1}{(2\pi)^3}\frac{\mathrm{d}^3\tilde{p}_i}{2\tilde{p}_i^0}\right]\frac{1}{(2\pi)^3}\frac{\mathrm{d}^3\tilde{p}_{ij}}{2\tilde{p}_{ij}^0}(2\pi)^4\delta^{(4)}\Big(\tilde{p}_a+\tilde{p}_b-\sum_{k\neq i,j}\tilde{p}_k-\tilde{p}_{ij}\Big)$$

Soft/collinear **kinematics** $\Rightarrow$ angular/energy integrals separately and combine

To derive FS emission PS in **radiation kinematics** use factorization formula

$$\mathrm{d}\Phi_3(-K;p_i,p_j,q) = \mathrm{d}\Phi_2(-K;p_j,-n)\,\frac{\mathrm{d}n^2}{2\pi}\,\mathrm{d}\Phi_2(-n;p_i,q)$$

Can **relate** PS factor $\mathrm{d}\Phi_2(-n;p_i,q)$ to underlying Born PS

$$\frac{\mathrm{d}\Phi_2(-n;p_i,q)}{\mathrm{d}\Phi_2(-\tilde{K};\tilde{p}_{ij},\tilde{q})} = \frac{\Big(1-2\mu_{ij}^2(2(\kappa-\mu_i^2)-z+\sigma_{i,ij}(1+2\mu_{ij}^2))\Big)\left[zv_{p_i,n}\right]^{1-2\epsilon}}{\left[\Big(1+2\mu_{ij}^2(1-\sigma_{i,ij})\Big)^2-4\mu_{ij}^2(1-z+\kappa+\mu_{ij}^2-\mu_i^2)\right]^{3/2-\epsilon}}$$

$\Rightarrow$ the **2-body** PS element for production of n^μ and p_j^μ

$$\mathrm{d}\Phi_2(-K;n,p_j) = \frac{(2\tilde{p}_{ij}\tilde{K})^{-\epsilon}}{(16\pi^2)^{1-\epsilon}}\,\frac{\left[(1-z+\mu_{ij}^2-\mu_i^2+\mu_j^2)v_{p_j,n}\right]^{1-2\epsilon}}{2(1-z+\kappa+\mu_{ij}^2-\mu_i^2)^{1-\epsilon}}\,\mathrm{d}\Omega_{j,n}^{2-2\epsilon}$$

PS factorization — Soft kinematics

Combining gives **single emission PS element**

$$\mathrm{d}\Phi_{+1}(\tilde{p}_a, \tilde{p}_b; \ldots, \tilde{p}_{ij}, \ldots; p_i, p_j)$$

$$= \left(\frac{2\tilde{p}_{ij}\tilde{K}}{16\pi^2}\right)^{1-\epsilon} \frac{\left(1 - 2\mu_{ij}^2(2(\kappa - \mu_i^2) - z + \sigma_{i,ij}(1 + 2\mu_{ij}^2))\right)\left[v_{p_i,n} v_{p_j,n}\right]^{1-2\epsilon}}{\left[\left(1 + 2\mu_{ij}^2(1 - \sigma_{i,ij})\right)^2 - 4\mu_{ij}^2(1 - z + \kappa + \mu_{ij}^2 - \mu_i^2)\right]^{3/2-\epsilon}}$$

$$\times \frac{\left(z(1 - z + \mu_{ij}^2 - \mu_i^2 + \mu_j^2)\right)^{1-2\epsilon}}{(1 - z + \kappa + \mu_{ij}^2 - \mu_i^2)^{1-\epsilon}} \, \mathrm{d}z \, \frac{\mathrm{d}\Omega_{j,n}^{2-2\epsilon}}{4\pi}$$

Simplifies in **massless case** to

$$\mathrm{d}\Phi_{+1}(\tilde{p}_a, \tilde{p}_b; \ldots, \tilde{p}_{ij}, \ldots; p_i, p_j) = \left(\frac{2\tilde{p}_{ij}\tilde{K}}{16\pi^2}\right)^{1-\epsilon} \frac{\left(z(1 - z)\right)^{1-2\epsilon}}{(1 - z + \kappa)^{1-\epsilon}} \, \mathrm{d}z \, \frac{\mathrm{d}\Omega_{j,n}^{1-2\epsilon}}{4\pi}$$

which agrees with previous result [Catani, Seymour, hep-ph/9605323]

PS factorization — Collinear kinematics

Splitting kinematics: factorize with recoiler as remaining FS parton

$$\mathrm{d}\Phi_3(q;p_i,p_j,K) = \mathrm{d}\Phi_2(q;K,p_{ij})\,\frac{\mathrm{d}p_{ij}^2}{2\pi}\,\mathrm{d}\Phi_2(p_{ij};p_i,p_j)$$

Relating PS factor to underlying Born PS element $\mathrm{d}\Phi_2(q;\tilde{K},\tilde{p}_{ij})$

$$\frac{\mathrm{d}\Phi_2(q;K,p_{ij})}{\mathrm{d}\Phi_2(q;\tilde{K},\tilde{p}_{ij})} = (1-y)^{1-2\epsilon}(1+\mu_{ij}^2-\mu_i^2-\mu_j^2)^{1-2\epsilon}\left(\frac{v_{p_{ij},K}}{v_{\tilde{p}_{ij},\tilde{K}}}\right)^{1-2\epsilon}$$

Decay of intermediate off-shell parton associated with

$$\mathrm{d}\Phi_2(p_{ij};p_i,p_j) = \frac{(2\tilde{p}_{ij}\tilde{K})^{-\epsilon}}{(16\pi^2)^{1-\epsilon}}\,\frac{\left(y^2(1+\mu_{ij}^2-\mu_i^2-\mu_j^2)^2-4\mu_i^2\mu_j^2\right)^{1/2-\epsilon}}{2\left(y(1+\mu_{ij}^2-\mu_i^2-\mu_j^2)+\mu_i^2+\mu_j^2\right)^{1-\epsilon}}\,\mathrm{d}\Omega_{i,ij}^{2-2\epsilon}$$

PS factorization — Collinear kinematics

Again combining to obtain the **single-emission PS element**

$$\mathrm{d}\Phi_{+1}(\tilde{p}_a, \tilde{p}_b; \tilde{p}_1, \ldots, \tilde{p}_{ij}, \ldots, \tilde{p}_n; p_i, p_j)$$

$$= \left(\frac{2\tilde{p}_{ij}\tilde{K}}{16\pi^2}\right)^{1-\epsilon} (1-y)^{1-2\epsilon}(1+\mu_{ij}^2-\mu_i^2-\mu_j^2)^{2-2\epsilon}\left(\frac{v_{p_{ij},K}}{v_{\tilde{p}_{ij},\tilde{K}}}\right)^{1-2\epsilon}$$

$$\times \frac{\left(y^2(1+\mu_{ij}^2-\mu_i^2-\mu_j^2)^2-4\mu_i^2\mu_j^2\right)^{1/2-\epsilon}}{\left(y(1+\mu_{ij}^2-\mu_i^2-\mu_j^2)+\mu_i^2+\mu_j^2\right)^{1-\epsilon}}\,\mathrm{d}y\,\frac{\mathrm{d}\Omega_{i,ij}^{2-2\epsilon}}{4\pi}$$

We show **equivalence** to fully massive element of CDST [hep-ph/0201036]

Again this simplifies dramatically in **massless limit**

$$\mathrm{d}\Phi_{+1}(\tilde{p}_a, \tilde{p}_b; \tilde{p}_1, \ldots, \tilde{p}_{ij}, \ldots, \tilde{p}_n; p_i, p_j) = \left(\frac{2\tilde{p}_{ij}\tilde{K}}{16\pi^2}\right)^{1-\epsilon} (1-y)^{1-2\epsilon}\, y^{-\epsilon}\,\mathrm{d}y\,\frac{\mathrm{d}\Omega_{i,ij}^{2-2\epsilon}}{4\pi}$$

Subtraction at NLO — Soft integrals

Upon factorizing we can compute the **angular** and **energy** integrals

The partial fractioned **soft Eikonal** can be written

$$\bar{W}^i_{ik,j} = \frac{1}{2l_i l_j}\left(\frac{l_{ik}^2}{l_{ik}l_j} - \frac{l_i^2}{l_i l_j} - \frac{l_k^2}{l_k l_j}\right) \quad \text{with} \quad l^\mu_{pq} = \sqrt{q^2}\,\frac{p^\mu}{pq}$$

In terms of **angular master integrals** (required to $\mathcal{O}(\epsilon)$ to due to **pole in Energy integral**)

$$\frac{1}{\Omega(2-2\epsilon)}\int \mathrm{d}\Omega_j\, \bar{W}^i_{ik,j} = I^{(2)}_{1,1}\left(\frac{l_i l_{ik}}{2}, l_i^2, \frac{l_{ik}^2}{4}\right)\frac{1 - v_i v_k \cos\theta_{ik}}{2}$$
$$+\frac{1-v_i^2}{2}\left[\frac{1}{2}I^{(2)}_{1,1}\left(\frac{l_i l_{ik}}{2}, l_i^2, \frac{l_{ik}^2}{4}\right) - I^{(1)}_2(l_i^2)\right]$$
$$+\frac{1-v_k^2}{2}\left[\frac{1}{2}I^{(2)}_{1,1}\left(\frac{l_i l_{ik}}{2}, l_i^2, \frac{l_{ik}^2}{4}\right) - I^{(2)}_{1,1}\left(l_i l_k, l_i^2, l_k^2\right)\right]$$

Recursion relations from simpler integrals known [Lyubovitskij, Wunder, Zhevlakov, arXiv:2102.08943]

Subtraction at NLO — Soft integrals

E.g. Integral with **two massive denominators** given by

$$I^{(2)}_{1,1}(v_{11}, v_{12}, v_{22}) = \frac{\pi}{\sqrt{v_{12}^2 - v_{11}v_{22}}} \left\{ \log \frac{v_{12} + \sqrt{v_{12}^2 - v_{11}v_{22}}}{v_{12} - \sqrt{v_{12}^2 - v_{11}v_{22}}} + \epsilon \left(\frac{1}{2} \log^2 \frac{v_{11}}{v_{13}^2} - \frac{1}{2} \log^2 \frac{v_{22}}{v_{23}^2} \right.\right.$$

$$+2\mathrm{Li}_2\left(1 - \frac{v_{13}}{1 - \sqrt{1 - v_{11}}}\right) + 2\mathrm{Li}_2\left(1 - \frac{v_{13}}{1 + \sqrt{1 - v_{11}}}\right)$$

$$\left.\left. -2\mathrm{Li}_2\left(1 - \frac{v_{23}}{1 - \sqrt{1 - v_{22}}}\right) - 2\mathrm{Li}_2\left(1 - \frac{v_{23}}{1 + \sqrt{1 - v_{22}}}\right)\right) + \mathcal{O}(\epsilon^2) \right\}$$

Massless limit only one simpler integral [arXiv:2208.06057]

$$I^{(1)}_{1,1}(v_{12}, v_{12}) = -\frac{\pi}{v_{12}^{1+\epsilon}} \left\{ \frac{1}{\epsilon} + \epsilon\, \mathrm{Li}_2\left(1 - v_{12}\right) + \mathcal{O}(\epsilon^2) \right\}$$

Soft-subtraction counter term

Remainder: z-dependence from energy denominator ⇒ expand integrand in Laurent series to obtain **soft subtraction counterterms**

$$\mathrm{d}\sigma_S^S = -8\pi\alpha_s\mu^{2\epsilon}\sum_{\widetilde{ij},k}\mathrm{d}\Phi_{+1}(\tilde{p}_a,\tilde{p}_b;\ldots,\tilde{p}_{ij},\ldots;p_i,p_j)\,\mathbf{T}_{\widetilde{ij}}\mathbf{T}_k\frac{n^2}{(p_jn)^2}\,\bar{W}^i_{ik,j}$$

$$= -\frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\epsilon)}\sum_{\widetilde{ij},k}\frac{\mathbf{T}_{\widetilde{ij}}\mathbf{T}_k}{\mathbf{T}^2_{\widetilde{ij}}}\left(\frac{4\pi\mu^2\,(p_kn)}{2(p_kp_i)(p_in)}\right)^{\epsilon}\left\{-\frac{\delta(1-z)}{\epsilon}+\frac{2z}{\left[1-z\right]_+}-4\epsilon\,z\left[\frac{\log(1-z)}{1-z}\right]_+\right\}$$

$$\times\,\mathcal{J}_{ik}(z,\mu_i^2,\kappa)\,(l_{ik}^2)^{\epsilon}\,\frac{C_{\widetilde{ij}}}{\pi}\frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\left[\frac{\mathrm{d}\Omega^{2-2\epsilon}_{j,n}}{\Omega(1-2\epsilon)}\,\bar{W}^i_{ik,j}\right]$$

Massless limit is given by $\mathcal{J}_{ik}(z{,}0{,}\kappa) = 1$

All $1/\epsilon$ **poles extracted** ⇒ can integrate over delta functions and plus distributions in z

Complete counterterm

To compute purely **collinear counterterm** use splitting kinematics

Combining soft + collinear ⇒ complete counterterm upon Laurent **expansion** (using CDST definitions [hep-ph/0201036])

$$\int \mathrm{d}\Phi_{+1}(q;\tilde{p}_{ij},\tilde{K};p_i,p_j)\frac{8\pi\alpha_s\mu^{2\epsilon}C_{ab}}{(p_i+p_j)^2-m_{ij}^2}$$

$$= \frac{\alpha_s}{2\pi}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\frac{\Omega(3-2\epsilon)}{(4\pi)^{1-2\epsilon}}(1-\hat{\mu}_i^2-\hat{\mu}_j^2-\hat{\kappa})^{2-2\epsilon}\lambda(1,\hat{\mu}_{ij}^2,\hat{\kappa})^{-1/2+\epsilon}$$

$$\times\int \mathrm{d}y\,(1-y)^{1-2\epsilon}\frac{\left[y(1-\hat{\mu}_i^2-\hat{\mu}_j^2-\hat{\kappa})+\hat{\mu}_i^2+\hat{\mu}_j^2\right]^{-\epsilon}}{y(1-\hat{\mu}_i^2-\hat{\mu}_j^2-\hat{\kappa})+\hat{\mu}_i^2+\hat{\mu}_j^2-\hat{\mu}_{ij}^2}\left(z_+-z_-\right)^{1-2\epsilon}\bar{\gamma}_{ab}(\epsilon)$$

with **purely collinear** anomalous dimension

$$\bar{\gamma}_{ab}(\epsilon) = \frac{\Omega(2-2\epsilon)}{\Omega(3-2\epsilon)}\left(\frac{z_+-z_-}{2}\right)^{-1+2\epsilon}\int_{z_-}^{z_+}\mathrm{d}z\,\Big((z-z_-)(z_+-z)\Big)^{-\epsilon}C_{ab}(\epsilon)$$

Complete counterterm

First of **three variants** collinear splitting massive quark to quark & gluon ($\hat{\mu}_i = \hat{\mu}_{ij} > 0$, $\hat{\mu}_j = 0$)

$$\int \mathrm{d}\Phi_{+1}(q; \tilde{p}_{ij}, \tilde{K}; p_i, p_j) \frac{8\pi\alpha_s \mu^{2\epsilon} C_{qq}}{(p_i + p_j)^2 - m_{ij}^2} = \frac{\alpha_s}{2\pi} \frac{(1 - \hat{\mu}_i^2 - \hat{\kappa})}{\sqrt{\lambda(1, \hat{\mu}_i^2, \hat{\kappa})}}$$

$$\times \int \frac{\mathrm{d}y}{y(1 - \hat{\mu}_i^2 - \hat{\kappa}) + \hat{\mu}_i^2} \sqrt{\left[(1-y)(1 - \hat{\mu}_i^2 - \hat{\kappa}) + 2\hat{\kappa}\right]^2 - 4\hat{\kappa}}\; \bar{\gamma}_{qq}(0)$$

Result **IR finite** since poles arise in soft gluon emission accounted for by eikonal integral

Subtraction scheme has **manifest hierarchy** between soft and collinear enhancements properly classifying leading and sub-leading logs (matches NSCS scheme [Caola, Melnikov, Röntsch, 1702.01352])

Remaining cases: gluon split to 2 massive quarks ($\hat{\mu}_i = \hat{\mu}_j > 0,\ \hat{\mu}_{ij} = 0$) and collinear decay of massless parton to 2 massless partons ($\hat{\mu}_i = \hat{\mu}_j = \hat{\mu}_{ij} = 0$)

Summary

Implemented NLL accurate massive parton extension to Alaric

Momentum mapping preserves directions of hard partons

Maintained strict positivity of evolution kernels

Determined of counterterms and integrals for NLO matching

Outlook

Implement complete NLO matching and merging

Extension of algorithm to NNLL and NNLO matching feasible [arXiv:2108.07133]

Include spin correlations and dominant subleading color effects

Non-trivial color dependence at Born level e.g. $t\bar{t}$-production and inclusive jet/di-jet production at LHC