EPSILON-FACTORISED DIFFERENTIAL EQUATIONS BEYOND POLYLOGARITHMS

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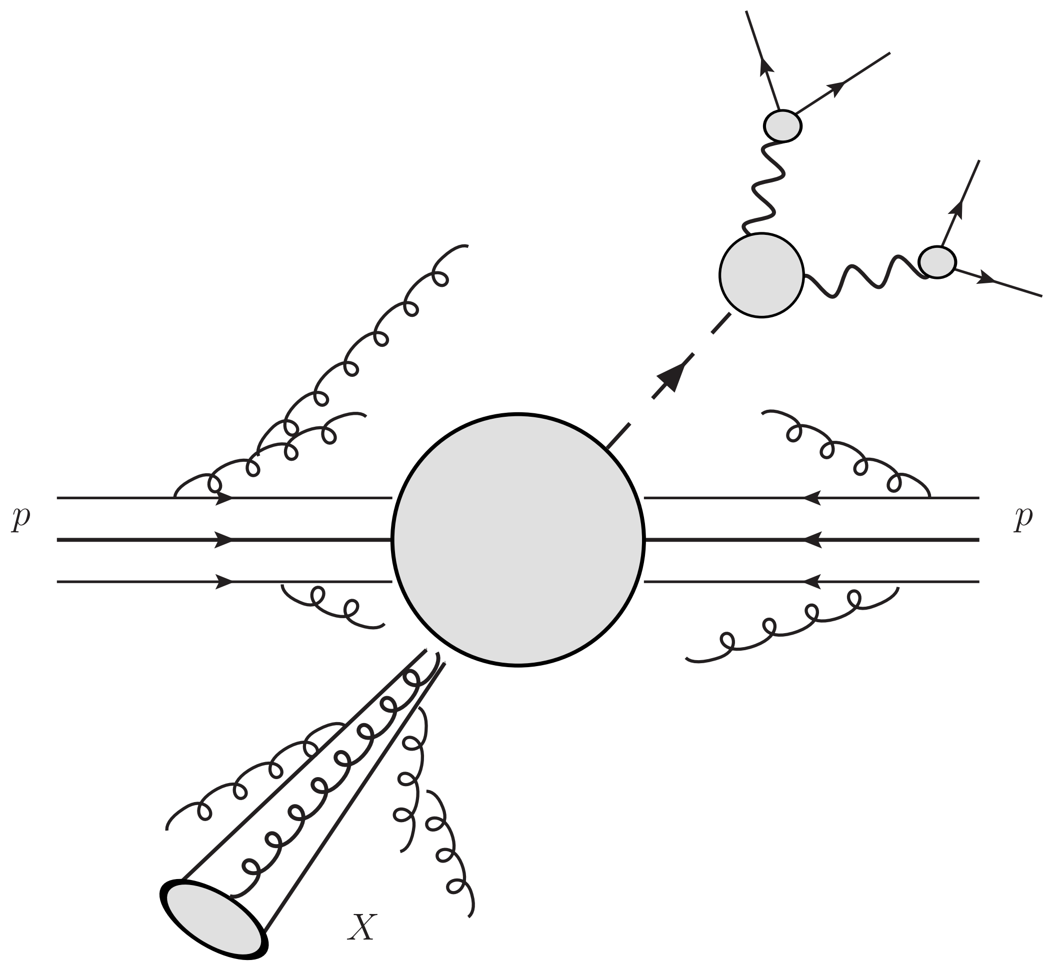
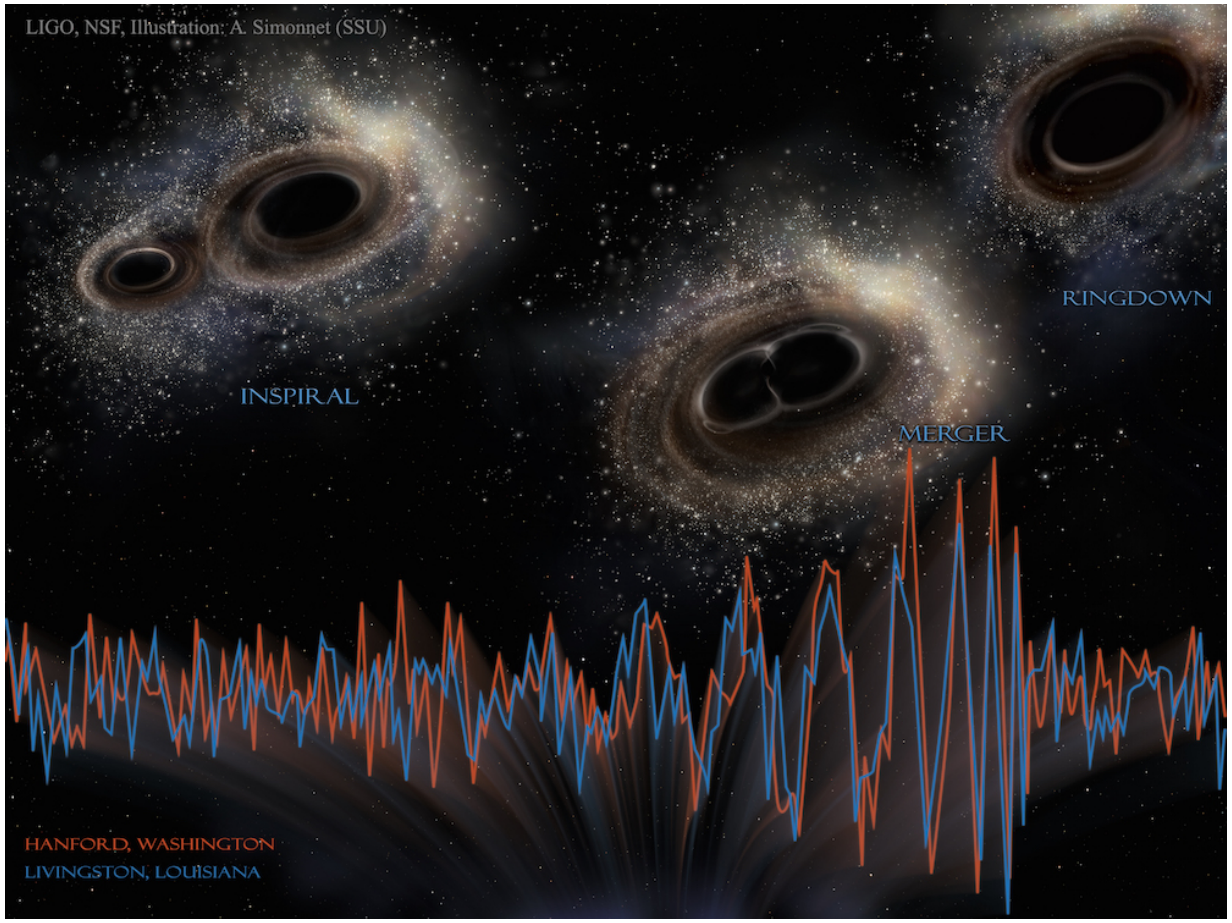
[Collaboration with L. Görges, C. Nega, F. Wagner — arXiv:2305.14090]

INTRODUCTION: AMPLITUDES AND FEYNMAN INTEGRALS

Feynman integrals are everywhere





Feynman Integrals are difficult, but also **beautiful objects**, their calculation manifests **regularities** and **structures** in ***physical observables***

ANALYTIC STRUCTURES: SPECIAL FUNCTIONS IN PARTICLE PHYSICS

The “most famous calculation” in pQFT: the **g-2 of the electron**

$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \dots$$

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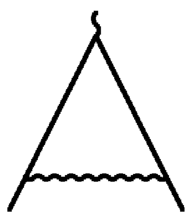
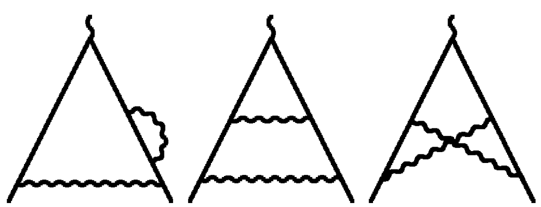
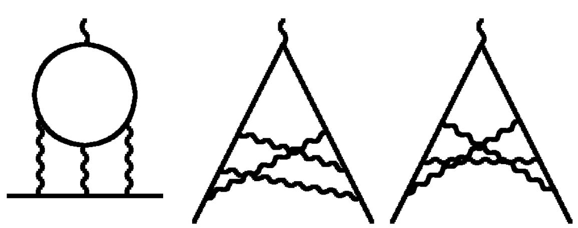
$C_1 =$  $= +0.50000000\ldots$

$C_2 =$  $= -0.328478965\ldots$

$C_3 =$  $= +1.181241456\ldots$

$C_4 =$ $= -1.912245764\ldots$

lots of Feynman diagrams

$C_5 =$ $= +6.737(159)$

} [impressive analytic 4loop by **S. Laporta**]
[impressive numeric 5 loop by **Kinoshita et al**]

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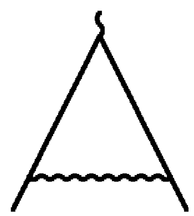
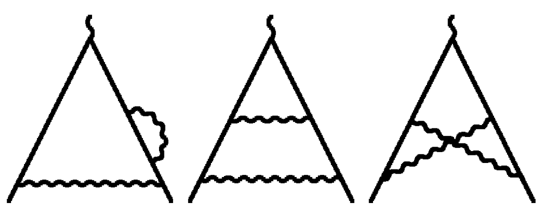
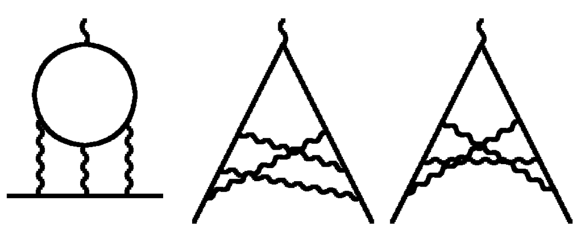
$C_1 =$  $= \frac{1}{2}$ [Schwinger ’48]

$C_2 =$  $= \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3)$ [Petermann, Sommerfield ’57]

$C_3 =$ 

$$= \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{\pi^2 \ln^2 2}{24}\right] - \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2 \ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}$$

[Laporta, Remiddi ’97]

from analytic results order by order in ϵ, iterated patterns emerges:

multiple polylogarithms evaluated at special (rational) points

DIFFERENTIAL EQUATION METHOD

We compute Feynman integrals as series in $\epsilon = (4-d)/2$

Iterated integral structure in ϵ made manifest by differentiation — powerful technique:

(Scalar) Feynman Integrals

$$\mathcal{I} = \int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \frac{S_1^{b_1} \dots S_m^{b_m}}{D_1^{a_1} \dots D_n^{a_n}}$$

with $S_i \in \{k_i \cdot k_j, \dots, k_i \cdot p_j\}$

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Integration by Parts etc

$$\int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \frac{\partial}{\partial k_l^\mu} \left[v_\mu \frac{S_1^{b_1} \dots S_m^{b_m}}{D_1^{a_1} \dots D_n^{a_n}} \right] = 0$$

Basis of Master Integrals (MIs)

$$\underline{I} = \{I_1(\underline{z}, \epsilon), \dots, I_N(\underline{z}, \epsilon)\}$$

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By differentiating MIs and re-expressing derivatives through MIs, we get system of diff. equations

$$d\underline{I} = GM(\underline{z}, \epsilon)\, \underline{I}$$

Gauss-Manin connection

Matrix of differential forms, all rational functions (from IBPs)

Basis of master integrals

[Kotikov '93; Remiddi '97; Gehrmann Remiddi '99, ...]

EPSILON-FACTORISED BASES

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Imagine to be able to perform a series of **rotations R_i on the original basis**

$$\underline{J} = \mathbf{R}(\underline{z}, \epsilon)\underline{I} \quad \text{with} \quad \mathbf{R}(\underline{z}, \epsilon) = \mathbf{R}_r(\underline{z}, \epsilon) \cdots \mathbf{R}_2(\underline{z}, \epsilon)\mathbf{R}_1(\underline{z}, \epsilon)$$

Such that

$$\mathrm{d}\underline{J} = \epsilon\,\mathbf{GM}\;(\underline{z})\underline{J}\,, \quad \text{where} \quad \epsilon\,\mathbf{GM}\;(\underline{z}) = [\mathbf{R}(\underline{z}, \epsilon)\mathbf{GM}(\underline{z}, \epsilon) + \mathrm{d}\mathbf{R}(\underline{z}, \epsilon)]\,\mathbf{R}(\underline{z}, \epsilon)^{-1}$$

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Since GM(z) does not depend on ϵ, the iterated structure in ϵ becomes manifest

We refer to such a basis as in ***epsilon-factorised form*** [Kotikov '10; J. Henn '13; Lee '13, ...]

EPSILON-FACTORISED OR CANONICAL?

What can we say about GM(z) ?

$$d\underline{J} = \epsilon\, GM(\underline{z})\, \underline{J}$$

- Is **GM(z)** **unique** ?
- Are there ϵ-factorized bases that are **better than others**?

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→ Notice: $d\underline{I} = \left[A_0(\underline{z}) + \epsilon\ B(\underline{z})\right] \underline{I} \ \rightarrow \quad \underline{I} = W \cdot \underline{J} \rightarrow \quad d\underline{J} = \epsilon\ \left[W^{-1} \cdot B(\underline{z}) \cdot W\right]\ \underline{J}$

The basis $\underline{J}$ often won't have unique properties, depends on $\underline{I}$ …

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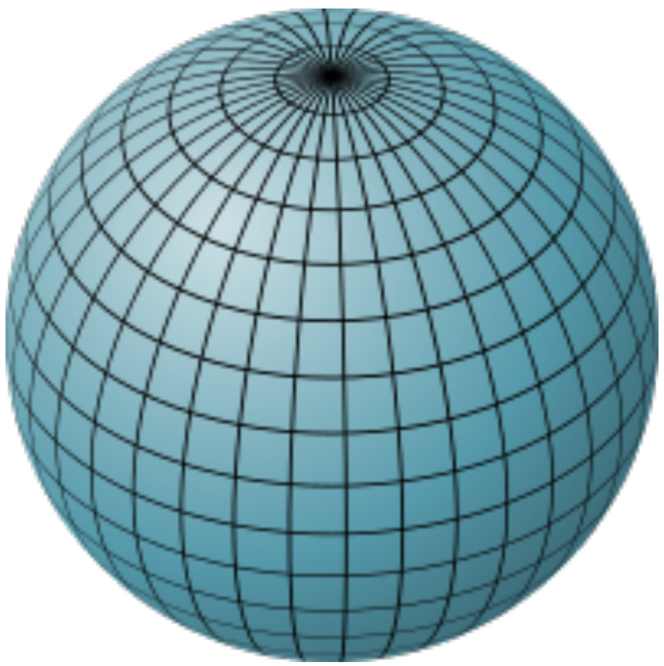
We understand a lot if solutions can be expressed in terms of **Multiple Polylogarithms (MPLs)**

MPLs span the space of **iterated integrals of rational functions** on complex plane $\mathbb{C}$

Simplest example: $\int^x \frac{dt}{(t-c)^n} \propto \frac{1}{n-1}\frac{1}{(x-c)^{n-1}}$; $\int^x \frac{dt}{t-c} \propto \log(x-c)$

[Every simple pole → non trivial residue → "new" multivalued function]

MULTIPLE POLYLOGARITHMS



MPLs can be defined as iterated integrals of rational functions with **single poles** on Riemann Sphere $\mathbb{CP}^1 \sim \mathbb{C} \cup \{\infty\}$

→ They have at most logarithmic singularities

$$G(c_1, c_2, ..., c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, ..., c_n, t_1)$$
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n = length or <u>transcendental weight</u>

[...,Remiddi, Vermaseren '99, Goncharov '00,...]

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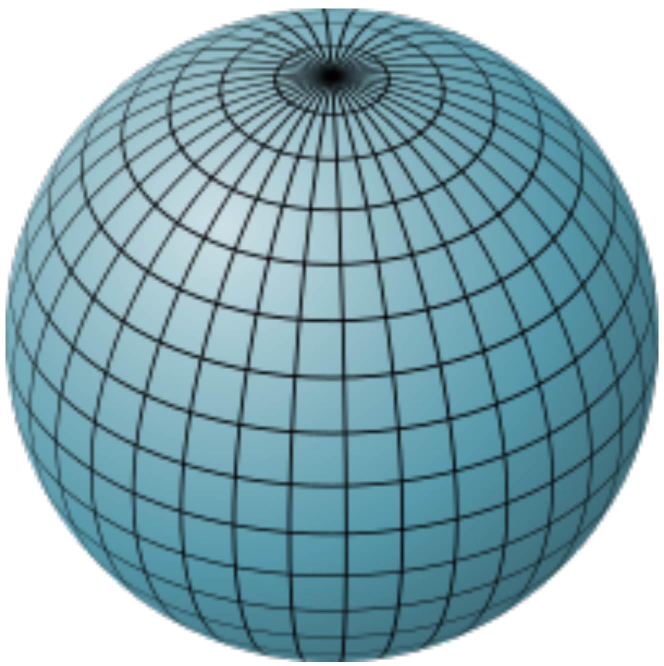

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They fulfil extremely simple (inhomogeneous!) differential equations: **unipotent**

$$\frac{d}{dx} G(c_1, \ldots, c_n; x) = \frac{1}{x - c_1} G(c_2, \ldots, c_n; x)$$

by diff. we lower the weight & length

CANONICAL BASES: THE POLYLOGARITHMIC CASE

If master integrals $\underline{J}$ can be expressed through MPLs, it makes sense to ask for more

$$d\underline{J} = \epsilon\, GM(\underline{z})\, \underline{J}$$

Look for basis that gives only **pure MPLs of uniform weight**

Require that $GM(\underline{z})$ **is in d-log form** (only simple poles)

If this is the case, we say GM(z) is in ***Canonical Form***

[Arkani Hamed et al ’10; Kotikov ‘10; J. Henn ’13]

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Riemann sphere: conjecturally we can find it by analysing **leading singularities in** $d = 2n,\ n \in \mathbb{N}$

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A Feynman Integral can be parametrized in $d = 2n - 2\epsilon$ as **(Feynman-Schwinger, Lee-Pomeransky, Baikov...)**

$$I \sim \int \prod_{i=1}^{n} \mathrm{d}x_i\, \mathcal{F}(x_i, \underline{z})\, \left(\mathcal{G}(x_i, \underline{z})\right)^{\epsilon}$$

→ can be neglected

expanding in ϵ gives on logs !

CANONICAL BASES: THE POLYLOGARITHMIC CASE

Focusing on parametrization in $d = 2n\,,\;\; n \in \mathbb{N}$

$$I \sim \int \prod_{i=1}^{n} \mathrm{d}x_i\, \mathcal{F}(x_i, \underline{z}) \quad \xrightarrow[\text{can be written as}]{\text{if}} \quad \sim \sum_i c_i \int d\log f_1{}^i \int d\log f_2{}^i \ldots \int d\log f_n{}^i\,; \qquad c_i \in \mathbb{Q}$$

Leading Singularities ~ ***iterative residues*** of the integrand in all integration variables

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Leading Singularities ~ ***iterative residues***
of the integrand in all integration variables

Conjecturally, these integrals fulfil ***canonical differential equations*** [Arkani Hamed et al '10; Kotikov '10; J. Henn '13]

Recipe (in a nutshell):

1. choose integrals whose ***integrands*** have only **simple poles and are in d-log form**
2. choose integrals whose ***iterated residues*** at all simple poles can be **normalized to numbers**

[Arkani-Hamed et al'10; Henn, Mistlberger, Smirnov, Wasser '20]

BEYOND POLYLOGARITHMS: CONCEPTUAL DIFFERENCES

Even with MPLs, insisting on ***simple poles*** in the integrand *(neglecting integration contour)* is too strong of a requirement, as it forces us to **exclude any squared propagator**!

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Double poles often imply power-like singularities in the IR which should be excluded in gauge theories

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Mathematics:

Differential forms with ***simple poles*** are intrinsically ***not enough*** to span full space for more general problems ***(elliptic curves or Tori, K3, Calabi-Yaus etc)***

→ Think about *independent integrands* in the elliptic case:

$K(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}}$ has **no poles** while $E(x) = \int_0^1 dt \frac{\sqrt{1-xt^2}}{\sqrt{1-t^2}}$ has **double pole at infinity**

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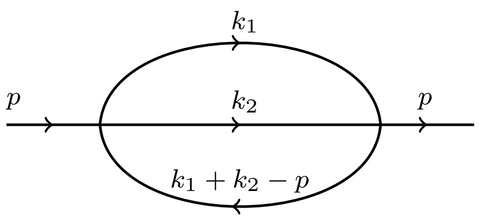
In fact, trying to apply MPL recipe to Elliptic Feynman Integrals (or beyond) fails…



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Tadpole + 2 Master Integrals $\quad I_1 = I_{0,1,1,0,0}\,, \quad I_2 = I_{1,1,1,0,0} \quad \text{and} \quad I_3 = I_{1,1,2,0,0}$

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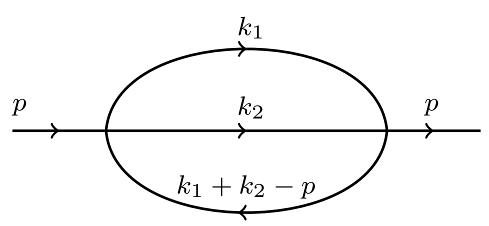

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If we try to analyse iterated residues we get to *(define* $P_4(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4)$ *)*

$$I_{1,1,1,0,0} \sim \int \frac{\mathrm{d}z_4}{\sqrt{P_4(z_4)}} \wedge \mathrm{d}\log f_3(z_3, z_4) \wedge \mathrm{d}\log f_2(z_2, z_3, z_4) \wedge \mathrm{d}\log f_1(z_1, z_2, z_3, z_4)$$

differential of 1st kind —>
$K(x)$ It has no poles at all!

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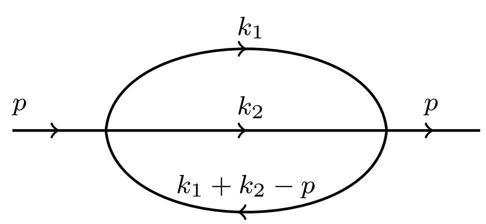

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Looking for a second candidate in this way we will ***always need candidates with double poles!*** (this is a theorem!)

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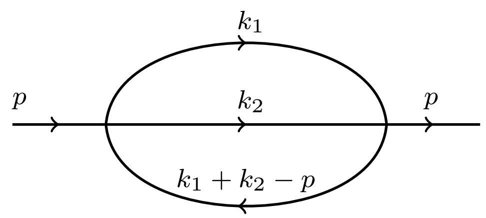


$$I_{n_1,n_2,n_3,-n_4,-n_5} = I_{n_1,n_2,n_3,-n_4,-n_5}(s,m^2;\epsilon) = \int \frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{(k_1-p)^{2n_4}(k_2-p)^{2n_5}}{(k_1^2-m^2)^{n_1}(k_2^2-m^2)^{n_2}((k_1+k_2-p)^2-m^2)^{n_3}}$$

Tadpole + 2 Master Integrals $\quad I_1 = I_{0,1,1,0,0}\,, \quad I_2 = I_{1,1,1,0,0} \quad \text{and} \quad I_3 = I_{1,1,2,0,0}$

If we try to analyse iterated residues we get to *(define* $P_4(x) = (x-a_1)(x-a_2)(x-a_3)(x-a_4)$ *)*

$$I_{1,1,1,0,0} \sim \int \frac{\mathrm{d}z_4}{\sqrt{P_4(z_4)}} \wedge \mathrm{d}\log f_3(z_3,z_4) \wedge \mathrm{d}\log f_2(z_2,z_3,z_4) \wedge \mathrm{d}\log f_1(z_1,z_2,z_3,z_4)$$

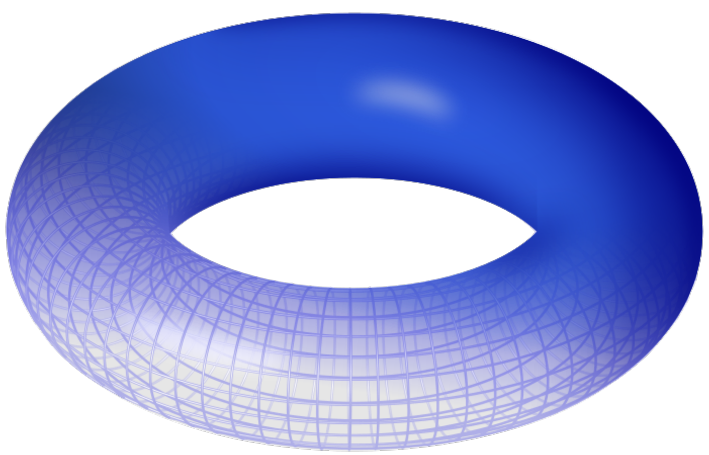
Notice that in this and other ***(single-scale)*** cases, eps-factorised basis found by an **Ansatz**

[Weinzierl, Adams '16] [Frellesvig, Weinzierl '23]
[Pögel, Wang, Weinzierl '22, '23]
[Jian, Wang, Yang, Zhao '23]

BEYOND POLYLOGARITHMS: CONCEPTUAL DIFFERENCES

Get inspiration from construction of Elliptic polylogarithms (eMPLs)

[Brown Levin '11; Brödel, Mafra, Matthes, Schlotterer '14]
[Brödel, Dulat, Duhr, Penante, Tancredi '17, '18]



We can insist on **single poles** ↔ logarithmic singularities (Gauge Theory)

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

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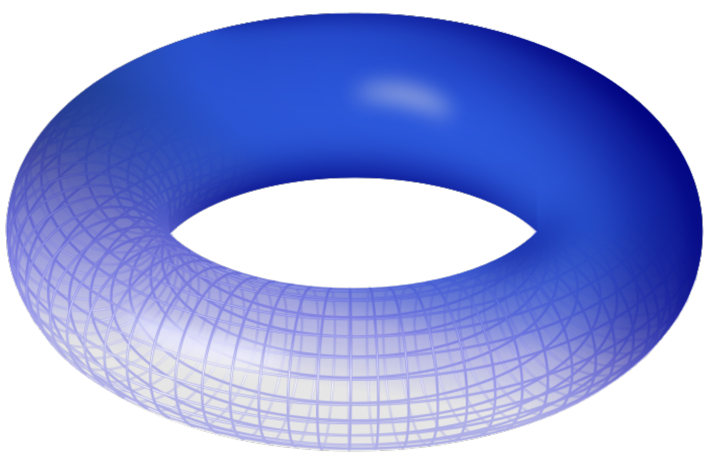

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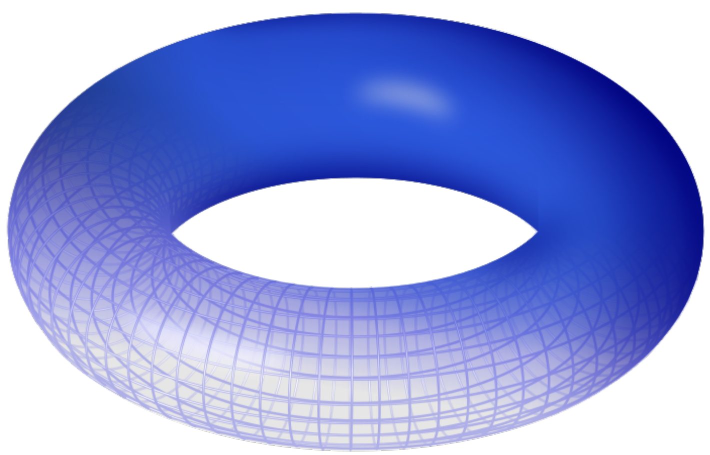

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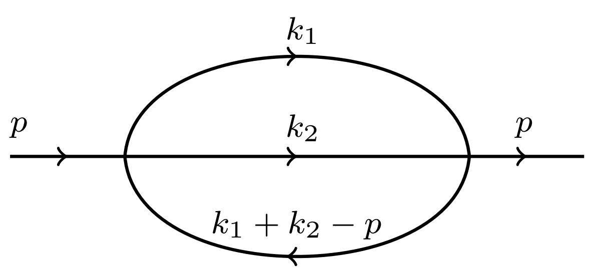
Important property of eMPLs is that they still satisfy **generalized unipotent differential equations**

$$\mathrm{d}\mathbf{W}^{\mathrm{u}} = \left(\sum_i \mathbf{U}_i(\underline{z})\ \mathrm{d}z_i\right)\mathbf{W}^{\mathrm{u}},$$

Where $U_i(\underline{z})$ are **Nilpotent matrices**: $\underbrace{U_i \cdot U_i \cdot \dots \cdot U_i}_{n} = 0$

GENERAL STRATEGY: THE IMPORTANCE OF BEING UNIPOTENT

Let's go back to sunrise case: diff equations for standard basis are coupled in $d = 2n$



$$\frac{\partial}{\partial m^2}\begin{pmatrix}\mathfrak{I}_2\\ \mathfrak{I}_3\end{pmatrix} = \begin{pmatrix} 0 & 3 \\ \frac{s-3m^2}{m^2(s-m^2)(s-9m^2)} & -\frac{s^2-20m^2s+27m^4}{m^2(s-m^2)(s-9m^2)} \end{pmatrix}\begin{pmatrix}\mathfrak{I}_2\\ \mathfrak{I}_3\end{pmatrix}$$

Matrix of homog. sol. is **not unipotent** $W = \begin{pmatrix}\omega_1 & \omega_2\\ \eta_1 & \eta_2\end{pmatrix}$

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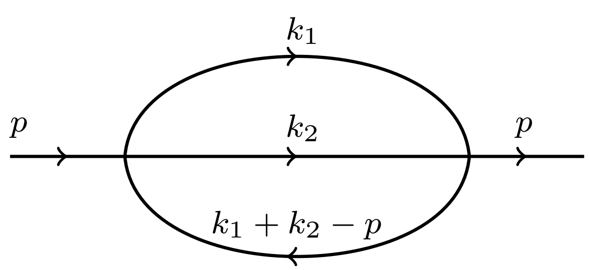

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A **non-unipotent** matrix can be split *(in a non-unique way)* into **Unipotent** and **Semi-Simple** part

$$W = S \cdot U \longrightarrow S = \begin{pmatrix}\omega_1 & 0\\ \eta_1 & -\frac{i\pi}{\omega_1}\end{pmatrix} \text{ and } U = \begin{pmatrix}1 & \frac{\omega_2}{\omega_1}\\ 0 & 1\end{pmatrix}$$

(Using Legendre Relation)

Define a new basis rotating only by semi-simple part $\underline{I} = S \cdot \underline{J}$

What remains is unipotent — indeed rotated $\underline{J}$ can be expressed in terms of ***pure eMPLs!***

[Brödel, Duhr, Dulat, Penante, LT '18]

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Same finding **confirmed** by evaluating various two- and three-point functions by **direct integration** to first few $\mathcal{O}(\epsilon)$ [Brödel, Duhr, Dulat, Penante, LT '18]

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STRATEGY: [Görges, Nega, LT, Wagner '23]

1. Start with a basis of MIs which has **no double poles in UV and IR**
2. For every **non-polylogarithmic sector**, choose **a minimally coupled basis** [achieved by analysing residues and choosing as many master integrals as possible to decouples minimally coupled block]
3. For coupled sectors, choose first integral whose residues involve **differential of first kind**
4. Perform a rotation of this minimally coupled basis to **remove semi-simple part**
5. **Conjecture 1:** after this rotation, matrix can be put in ϵ-factorized form by integrating out remaining terms that are not in the right form
6. **Conjecture 2:** The form so achieved will be a ***generalization of a canonical form beyond MPLs***

EXAMPLE 1: POLYLOGARITHMIC CASE

It works in the (simple) polylogarithmic case: Sunrise with **2 massive and 1 massless propagator**

$$I_1 = I_{0,1,1,0,0}\,, \quad I_2 = I_{1,1,1,0,0} \quad \text{and} \quad I_3 = I_{1,1,2,0,0}$$

Differential equations read: $d\underline{I} = \left[A_0 + \epsilon A_1 + \mathcal{O}(\epsilon^2)\right] \underline{I}$

Homogeneous equation in d=2
$$\frac{\partial}{\partial m^2} \begin{pmatrix} \mathfrak{I}_2 \\ \mathfrak{I}_3 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ \frac{1}{m^2(s-4m^2)} & \frac{-s+10m^2}{m^2(s-4m^2)} \end{pmatrix} \begin{pmatrix} \mathfrak{I}_2 \\ \mathfrak{I}_3 \end{pmatrix}$$

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Matrix of **homogeneous solutions** contains **algebraic functions and logs**

Split it in **semi-simple** and **unipotent** $\quad W = W^{ss}\cdot W^{u}$

$$\mathbf{W}^{\mathrm{ss}} = \begin{pmatrix} \frac{1}{r(s,m^2)} & 0 \\ \frac{s}{r(s,m^2)^3} & \frac{1}{2m^2(s-4m^2)}\end{pmatrix} \quad \text{and} \quad \mathbf{W}^{\mathrm{u}} = \begin{pmatrix} 1 & \log\left(\frac{s-r(s,m^2)}{s+r(s,m^2)}\right) \\ 0 & 1\end{pmatrix} \qquad r(s,m^2) = \sqrt{s(s-4m^2)}$$

Rotate away semi-simple part

$$\underline{I}' = \begin{pmatrix} 1 & 0 \quad 0 \\ 0 & \\ 0 & (\mathbf{W}^{\mathrm{ss}})^{-1}\end{pmatrix}\underline{I}$$

EXAMPLE 1: POLYLOGARITHMIC CASE

Clean up remaining non-factorised dependence with a rotation $\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2(s+2m^2)}{r(s,m^2)} & 1 \end{pmatrix} \begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}$

$\mathrm{d}\underline{J} = \epsilon \, \mathbf{GM}^{\epsilon} \underline{J}$ with $\underline{J} = (J_1, J_2, J_3)^T = \mathbf{T}\, \underline{I}'\,,$

$$\mathbf{GM}^{\epsilon} = \begin{pmatrix} -2\alpha_1 & 0 & 0 \\ 0 & 2\alpha_1 - \alpha_2 - 3\alpha_3 & \alpha_4 \\ 2\alpha_1 - 2\alpha_2 & -6\alpha_4 & -3\alpha_1 + \alpha_2 \end{pmatrix}$$

$$\alpha_1 = \mathrm{d}\log(m^2)\,, \quad \alpha_2 = \mathrm{d}\log(s)\,, \quad \alpha_3 = \mathrm{d}\log\left(s - 4m^2\right)\,, \quad \alpha_4 = \mathrm{d}\log\left(\frac{s - r(s,m^2)}{s + r(s,m^2)}\right)$$

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NB: by analysing leading singularities with **DLogBasis** find the same basis up to constant rotation!

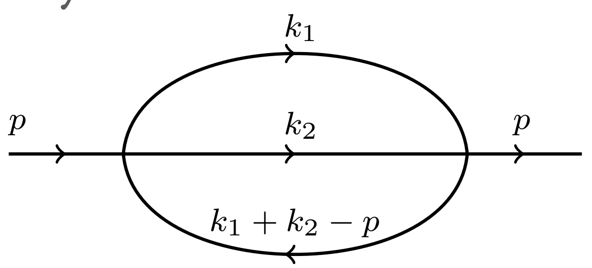
[P. Wasser '19,'20]

$$J_1 = M_1\,, \quad J_2 = M_2\,, \quad J_3 = -M_1 + 3M_3$$

NON TRIVIAL EXAMPLES: ELLIPTICS AND BEYOND

Strategy is general and **does not have to do with details of the geometry***

Applied it successfully to elliptic sunrise (equal or different masses)



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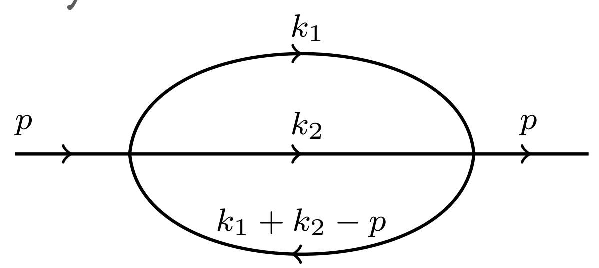
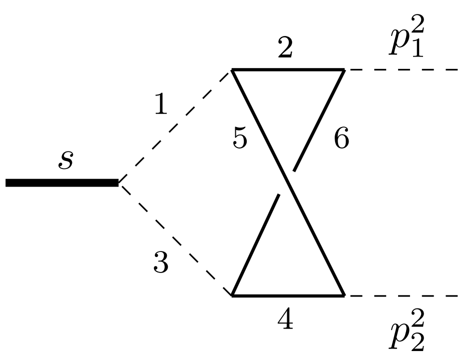
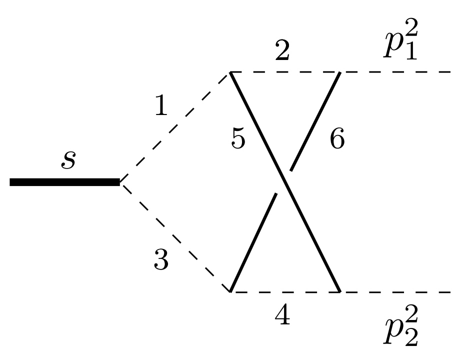
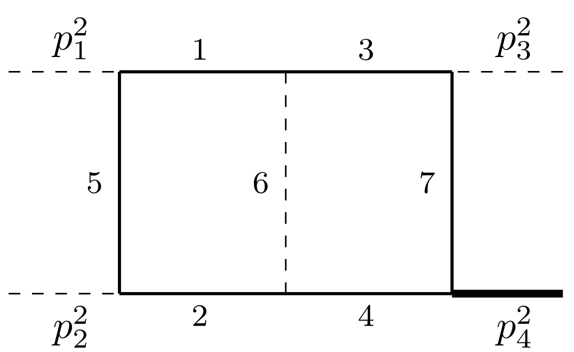


Many other multi-scale elliptic problems







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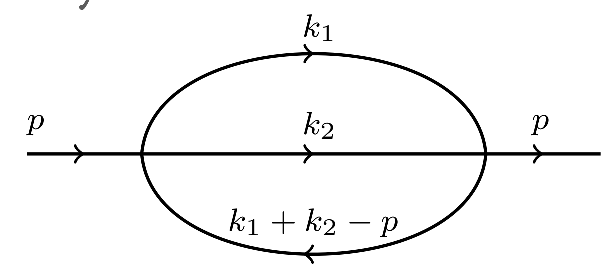

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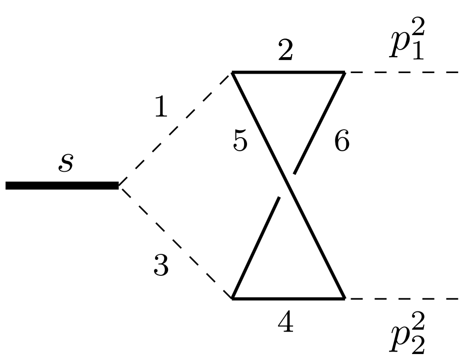
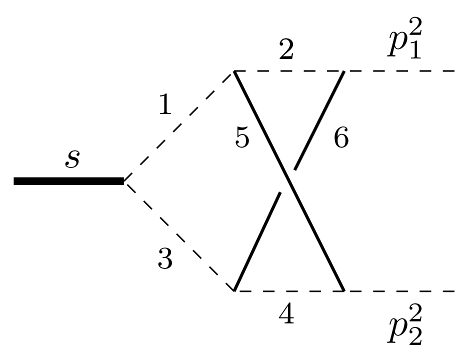
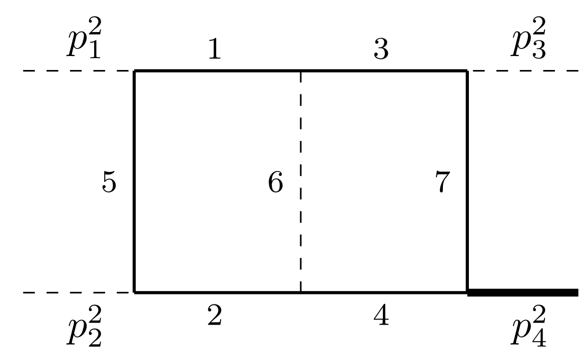
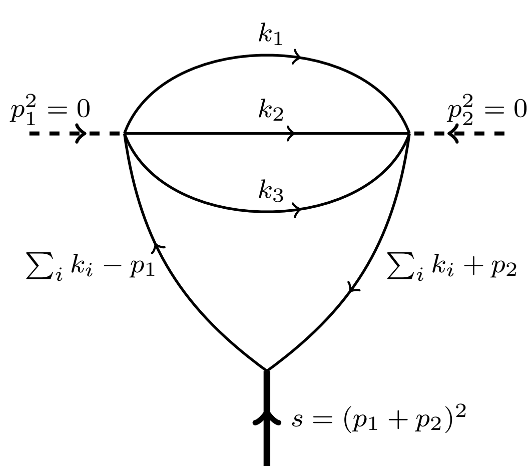
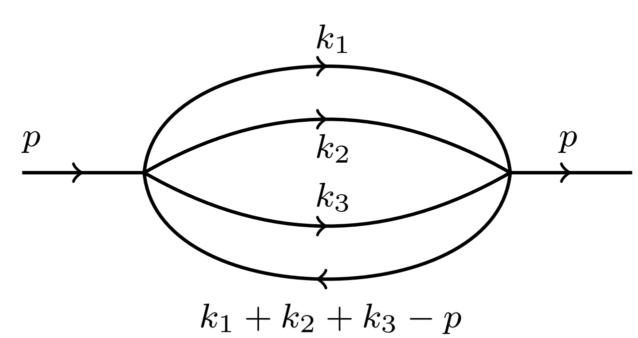






Even cases beyond 1 elliptic curve





CONCLUSIONS

Feynman integrals are difficult to compute, but they hide structures and simplicity

Epsilon-factorised bases are an important tool to make some of their structure manifest

Beyond polylogarithms, searching for integrals with simple poles (and unit leading singularities) in the traditional sense is not enough

Instead, the property of **unipotence** can be used to **build differential equations in epsilon-factorised form almost algorithmically**

In the polylogarithmic case, this construction **reproduces results obtained from analysis of leading singularities**

Beyond polylog case, we have showed that it is enough to obtain epsilon-factorised equations in multi-scale elliptic cases and beyond!

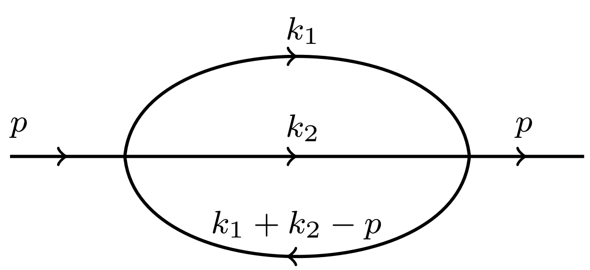
OUTLOOK: more complicated geometries (CYs, higher genus), application to physics problems…

THANK YOU FOR YOUR ATTENTION!

BACK UP

FORCING AN EPS-FACTORIZED BASIS: THE SUNRISE GRAPH

Diff equations for two sunrise MIs are coupled in d=2n



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Homogeneous solutions given by periods and their derivatives $W = \begin{pmatrix}\omega_1 & \omega_2\\ \eta_1 & \eta_2\end{pmatrix}$ with $\eta_i \propto \partial\omega_i$

→ provided by independent set of maximal cuts

[Primo, Tancredi ’16, ‘17; Frellesvig, Papadopoulos ’17; Bosma, Sogaard, Zhang ’17]
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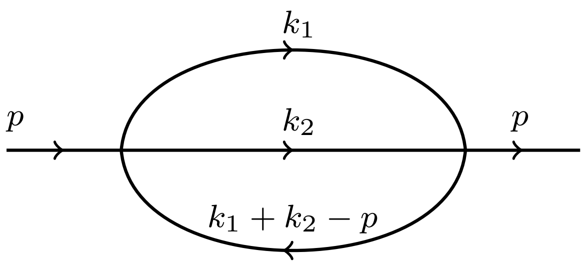

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One could attempt to rotate away the homogeneous solution and get something in eps-factorized form

$$d\underline{I} = \left[A_0(\underline{z}) + \epsilon\, B(\underline{z})\right]\underline{I} \quad \longrightarrow \quad \underline{I} = W\cdot\underline{J} \qquad d\underline{J} = \epsilon\,\left[W^{-1}\cdot B(\underline{z})\cdot W\right]\underline{J}$$

Basis $\underline{J}$ does not have right properties even in MPL case!

[Frellesvig, Weinzierl '22]
[Görges, Nega, LT, Wagner '23]