EPSILON-FACTORISED DIFFERENTIAL EQUATIONS BEYOND POLYLOGARITHMS

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Lorenzo Tancredi - Technical University Munich

[Collaboration with L. Görges, C. Nega, F. Wagner — arXiv:2305.14090]

Technische Universität München

INTRODUCTION: AMPLITUDES AND FEYNMAN INTEGRALS

Feynman integrals are everywhere

Feynman Integrals are difficult, but also **beautiful objects**, their calculation manifests **regularities** and **structures** in *physical observables*

ANALYTIC STRUCTURES: SPECIAL FUNCTIONS IN PARTICLE PHYSICS

The "most famous calculation" in pQFT: the g-2 of the electron

$$
a_e^{QED} = C_1 \left(\frac{\alpha}{\pi}\right) + C_2 \left(\frac{\alpha}{\pi}\right)^2 + C_3 \left(\frac{\alpha}{\pi}\right)^3 + C_4 \left(\frac{\alpha}{\pi}\right)^4 + C_5 \left(\frac{\alpha}{\pi}\right)^5 + \dots
$$

QED Mass-independent term: 1-loop contribution **ANALYTIC STRUCTURES: SPECIAL FUNCTIONS IN PARTICLE PHYSICS**

QED Mass-independent term: 2-loop contribution calculation in p Q F i: the g - z of the electron Γ be "most femous coloulation" in Γ The "most famous calculation" in pQFT: the **g-2 of the electron**

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+ C³

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:

"3

 $+ + + +$

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$$

\n
$$
C_1 = \bigcup_{\substack{e \text{ odd}}} \bigcup_{\sub
$$

QED Mass-independent term: 1-loop contribution QED Mass-independent term: 1-loop contribution **ANALYTIC STRUCTURES: SPECIAL FUNCTIONS IN PARTICLE PHYSICS**

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• The final analytical expression was obtained by S.L. and Ettore Remiddi in 1996.

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$$

\n
$$
C_1 = \bigotimes_{\alpha=2}^{1} \text{[Schwinger '48]}
$$

\n
$$
C_2 = \bigotimes_{\alpha=3}^{1} \bigotimes_{\alpha=4}^{197} + \frac{1}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3) \text{ [Peterminan, Sommerfield '57]}
$$

\n
$$
= \frac{83}{72}\pi^2 \zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3} \left[\left(\text{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24} \right) - \frac{\pi^2 \ln^2 2}{24} \right] - \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2 \ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}
$$

[Laporta, Remiddi '97]

QED Mass-independent term: 3-loop contribution

from analytic results order by order in ϵ , iterated patterns emerges: unfortunately the multiple polyl esults order by order in ϵ , iterated pay atterns emei • The final analytical expression was obtained by S.L. and Ettore Remiddi in 1996. from analytic results order by order in ϵ , iterated patterns emerges: *multiple polylogarithms* evaluated at special (rational) points

DIFFERENTIAL EQUATION METHOD

We compute Feynman integrals as series in $\epsilon = (4 - d)/2$

Iterated integral structure in ϵ made manifest by differentiation — powerful technique:

(Scalar) Feynman Integrals

$$
\mathcal{F} = \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \frac{S_1^{b_1} \dots S_m^{b_m}}{D_1^{a_1} \dots D_n^{a_n}}
$$

with $S_i \in \{k_i \cdot k_j, \ldots, k_i \cdot p_j\}$

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Basis of Master Integrals (MIs)

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$$

Integration by Parts etc

$$
\underline{I} = \{I_1(\underline{z}, \epsilon), \ldots, I_N(\underline{z}, \epsilon)\}
$$

$$
\iiint_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \frac{\partial}{\partial k_l^{\mu}} \left[v_{\mu} \frac{S_1^{b_1} \dots S_m^{b_m}}{D_1^{a_1} \dots D_n^{a_n}} \right] = 0
$$

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$$

(Scalar) Feynman Integrals Basis of Master Integrals (MIs)

 $I = \{I_1(z, \epsilon), \ldots, I_N(z, \epsilon)\}\$

By differentiating MIs and re-expressing derivatives through MIs, we get system of diff. equations

$Gauss-Manin$ connection

Matrix of differential forms, all rational functions (from IBPs)

$$
dI = GM(z, \epsilon)I
$$

Basis of master integrals

[Kotikov '93; Remiddi '97; Gehrmann Remiddi '99, …]

EPSILON-FACTORISED BASES

 $\begin{array}{cccccccccccccc} \bullet & \bullet \end{array}$

 $dI = GM(z, \epsilon)I$ In this form, iterated structure *hidden* in arbitrary dependence on ϵ

satisfies a system of linear first-order differential equations with respect to the kinematic variables and the masses the problem depends on. We indicate the set of all variables with We indicate them in vector form as *I*. Moreover, any choice of basis of master integrals *I* **EPSILON-FACTORISED BASES**satisfies a system of linear first-order differential equations with respect to the kinematic res

 $d\underline{I} = GM(\underline{z}, \epsilon)\underline{I}$ d*I* = GM(*z,* ✏)*I .* (2.2) $\overline{\mathbf{u}}$ and it takes the general formulation $\overline{\mathbf{u}}$ In this form, iterated structure *hidden* in arbitrary dependence on ϵ

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$$
\underline{J} = \mathbf{R}(\underline{z}, \epsilon) \underline{I} \quad \text{with} \quad \mathbf{R}(\underline{z}, \epsilon) = \mathbf{R}_r(\underline{z}, \epsilon) \cdots \mathbf{R}_2(\underline{z}, \epsilon) \mathbf{R}_1(\underline{z}, \epsilon)
$$

 σ cast that case σ system of differential equations in σ Such that

 $dJ = G\mathbf{M}$ (\cdot) $I = -\frac{1}{2}$ and (\cdot) $(\mathbf{R}(\cdot), \cdot)$ $G\mathbf{M}$ (\cdot, \cdot) (\cdot, \cdot) $\mathbf{R}(\cdot, \cdot)$ $\mathrm{d}\underline{J}=\epsilon\,\mathbf{GM}\,\left(\underline{z}\right)\underline{J}\,,\quad\text{where}\quad\epsilon\,\mathbf{GM}\,\left(\underline{z}\right)=[\mathbf{R}(\underline{z},\epsilon)\mathbf{GM}(\underline{z},\epsilon)+\mathrm{d}\mathbf{R}(\underline{z},\epsilon)]\,\mathbf{R}(\underline{z},\epsilon)^{-1}\,.$

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GM✏ $(m(z)$ does not depend on c, the <u>netated structure in a becomes</u> (*z*)]*ij* are given in terms Since GM(z) does not depend on *ε*, the <u>iterated structure in *ε* becomes manifest</u>

 N/a rafer to such a basis as in *ensilon-factorised form* [Kotikov^{210.}] Henn^{213.} Lee²¹³ An ✏-factorised form implies that the iterative structure of the solution is manifest in We refer to such a basis as in *epsilon-factorised form*. I [Kotikov '10; J. Henn '13; Lee '13, ...] **[Kotikov '10; J. Henn '13; Lee '13, …]**

What can we say about GM(z) ?

.

 $d\underline{J} = \epsilon$ $GM(\underline{z})\underline{J}$

- Is **GM(z) unique** ?

- Are there *ϵ*-factorized bases that are **better than others***?*

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Notice:
$$
dI = [A_0(z) + \epsilon B(z)] I \rightarrow I = W \cdot I \rightarrow dI = \epsilon [W^{-1} \cdot B(z) \cdot W] I
$$

The basis *J* often won't have unique properties, depends on *I* …

What can we say about GM(z) ?

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Can we define an **optimal basis** of master integrals for a given problem?

What can we say about $GM(z)$?

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Can we define an **optimal basis** of master integrals for a given problem?

We understand a lot if solutions can be expressed in terms of **Multiple Polylogarithms (MPLs)**

MPLs span the space of iterated integrals of rational functions on complex plane $\mathbb C$

Simplest example:
$$
\int_{0}^{x} \frac{dt}{(t-c)^n} \propto \frac{1}{n-1} \frac{1}{(x-c)^{n-1}};
$$

$$
\int_{0}^{x} \frac{dt}{t-c} \propto \log(x-c)
$$

 Γ Every simple pole \rightarrow non trivial residue → "new" multivalued function **]**

MULTIPLE POLYLOGARITHMS

MPLs can be defined as iterated integrals of rational functions with $\operatorname{\textbf{single poles}}$ on Riemann Sphere $\mathbb{CP}^1\sim \mathbb{C}\cup\{\infty\}$

 \rightarrow They have at most **logarithmic singularities**

$$
G(c_1, c_2, ..., c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, ..., c_n, t_1)
$$

=
$$
\int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} ... \int_0^{t_{n-1}} \frac{dt_n}{t_n - c_n}
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[…,Remiddi, Vermaseren '99, Goncharov '00,…] n = length or transcendental weight

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$$

[…,Remiddi, Vermaseren '99, Goncharov '00,…]

They fulfil extremely simple (inhomogeneous!) differential equations: **unipotent**

n = length or transcendental weight

$$
\frac{d}{dx}G(c_1, ..., c_n; x) = \frac{1}{x - c_1}G(c_2, ..., c_n; x)
$$

by diff. we lower the weight & length

CANONICAL BASES: THE POLYLOGARITHMIC CASE

If master integrals *J* can be expressed through MPLs, it makes sense to ask for more

 $d\underline{J} = \epsilon$ $GM(z)\underline{J}$

Look for basis that gives only **pure MPLs of uniform weight Require** that *GM*(*z*) **is in d-log form** (only simple poles)

.

If this is the case, we say GM(z) is in *Canonical Form*

[Arkani Hamed et al '10; Kotikov '10; J. Henn '13]

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How do we get a **canonical form** instead of any ϵ -factorized form?

Riemann sphere: conjecturally we can find it by analysing **leading singularities in** $d = 2n, \,\, n \in \mathbb{N}$

CANONICAL BASES: THE POLYLOGARITHMIC CASE

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 $dJ = \epsilon GM(z)J$

For a cannot constructed in terms in terms of all algebraic functions and in terms of analyzing terms of uniform weight

 $\frac{dy}{dx}$ C $\frac{dy}{dx}$ **Require** that $GM(z)$ is in d-log form (only simple poles) information.5 In fact, as electrons, as electrons, as electrons, as electrons, as electrons, as electrons, as

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Riemann sphere: conjecturally we can find it by analysing leading s The analysis is usually performed in *d* = *d*02✏ dimensions, with typically *d*⁰ = 2*,* 4*,* 6. As a Riemann sphere: conjecturally we can find it by analysing **leading singularities in** $d = 2n, \,\, n \in \mathbb{N}$

 Λ Formmon Integral can be nerametrized in $d-2n$, 2000 (Figures realising that for *d* = *d*⁰ 2✏ the integrand can be schematically parameterised as A Feynman Integral can be parametrized in *d* = 2*n* − 2*ϵ* as **(Feynman-Schwinger, Lee-Pomeransky, Baikov…)**

$$
I \sim \int \prod_{i=1}^{n} dx_i \mathcal{F}(x_i, \underline{z}) \frac{(\mathcal{G}(x_i, \underline{z}))^{\epsilon}}{(\mathcal{G}(x_i, \underline{z}))^{\epsilon}}
$$
 can be neglected
expanding in ϵ gives on logs !

CANONICAL BASES: THE POLYLOGARITHMIC CASE **achieved in practice by an analysis of the iterated residual integrands, of the corresponding integrands, and c**

Focusing on parametrization in $d = 2n$, $n \in \mathbb{N}$

in any suitable representation (Feynman-Schwinger parameters, Baikov representation, etc).

x i are the Baikov variables the Baikov variables of the integrand in all integration variables if the integrand is in definition is in definition order corrections in \mathcal{C} **Leading Singularities** ~ *iterative residues*

CANONICAL BASES: THE POLYLOGARITHMIC CASE **achieved in practice by an analysis of the iterated residual integrands, of the corresponding integrands, and c**

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[Arkani Hamed et al '10; Kotikov '10; J. Henn '13] σ and σ and μ they will not conclude \mathcal{C} This analysis is the complete one minimum and one might wonder the complete derivative for *Conjecturally,* these integrals fulfil *canonical differential equations*

be achieved in a nutshell): Recipe (in a nutshell):

- 1. choose integrals whose *integrands* have only simple poles and are in d-log form
- α ding singularities has at least two drawbacks. First of all, in a general multi-parameter case and for increasing the increasing of the local methods, it becomes computed for local differences computed to 2. choose integrals whose *iterated residues* at all simple poles can be **normalized to numbers**

analyse all iterated residues. In these cases, a simpler analysis restricted to some specific restricted to so
In the simpler analysis restricted to some specific restricted to some specific restricted to some specific re **[Arkani-Hamed et al'10; Henn, Mistlberger, Smirnov, Wasser '20]**

Even with MPLs, insisting on *simple poles* in the integrand *(neglecting integration contour)* is too strong of a requirement, as it forces us to **exclude any squared propagator**!

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Physics:

Double poles often imply power-like singularities in the IR which should be excluded in gauge theories

Typically true when dealing with massless propagators

Massive propagators can be squared at will, *without changing IR behaviour* and (actually) *improving UV behaviour*

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Mathematics:

Differential forms with *simple poles* are intrinsically *not enough* to span full space for more general problems *(elliptic curves or Tori, K3, Calabi-Yaus etc)*

$$
\rightarrow
$$

Think about *independent integrands* in the elliptic case:

 $K(x) = \int_0^{\frac{u}{\sqrt{(1 - t^2)(1 - xt^2)}}}$ has **no poles** while $E(x) = \int_0^{\frac{v}{\sqrt{1 - t^2}}} dt \frac{\sqrt{1 - x^2}}{\sqrt{1 - t^2}}$ has **double pole at infinity** 1 θ *dt* $(1 - t^2)(1 - xt^2)$ $E(x) = \int$ 1 $\overline{0}$ *dt* $1 - xt^2$ $1 - t^2$

BEYOND POLYLOGARITHMS: CONCEPTUAL DIFFERENCES integrals of a given sector, by integrals in lower sectors. This is in principle a straightforward procedure. In practice, however, it can become tricky as also the functional dependence OFIAND I AFIFAANIIIII. *^r*(*s, m*2) = ^p*s*(*^s* ⁴*m*2)*.* (3.8) associated to the threshold for the production of two massive and one massless particle.

sector considered, the introduction of new functions might be required. These are likewise

 $3 - 200$ by $\frac{1}{2}$ or $\frac{1}{2}$ e enply MDI recipe to Elliptic Formon Integrals (or beyond) feils checked that these master integrals satisfy indeed differential equations in canonical form. In canonical form In fact, trying to apply MPL recipe to Elliptic Feynman Integrals (or beyond) fails…

$$
\sum_{k_1+k_2=p}^{k_1} \sum_{k_2+k_3=p}^{k_2} I_{n_1,n_2,n_3,-n_4,-n_5} = I_{n_1,n_2,n_3,-n_4,-n_5}(s,m^2;\epsilon) = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{(k_1-p)^{2n_4}(k_2-p)^{2n_5}}{(k_1^2-m^2)^{n_1}(k_2^2-m^2)^{n_2}((k_1+k_2-p)^2-m^2)^{n_3}}
$$

we chose it such that it is manning that it is mand it is the factor threshold *s 4m2. The factor secondation serie*
The factor of the factor series in the factor of the factor series in the factor of the factor series in

 Γ_1 dpole + 2 Master Integrals $I_1 = I_0$ $\frac{1}{2}$, $\frac{1}{2}$ shown in $\frac{1}{2}$ shown in $\frac{1}{2}$ space-time dimensions in $\frac{1}{2}$ $I_1 = I_{0,1,1,0,0} \,, \quad I_2 = I_{1,1,1,0,0} \, \quad \text{and} \quad I_3 = I_{1,1,2,0,0} \,.$

BEYOND POLYLOGARITHMS: CONCEPTUAL DIFFERENCES **Der number of master integrals, the steps in complexity.** However, the steps in complexity. However, the steps in integrals of a given sector, by integrals in lower sectors. This is in principle a straightforward procedure. In practice, however, it can become tricky as also the functional dependence OFIAND I AFIFAANIIIII. *^r*(*s, m*2) = ^p*s*(*^s* ⁴*m*2)*.* (3.8) associated to the threshold for the production of two massive and one massless particle.

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if analysing the leading singularities does not provide enough candidates, our method sector considered, the introduction of new functions might be required. These are likewise We chose it such that it is manifestly real above the threshold *^s* ⁴*m*2. The factor ✏² is

$$
\sum_{k_1+k_2=p}^{k_1} I_{n_1,n_2,n_3,-n_4,-n_5} = I_{n_1,n_2,n_3,-n_4,-n_5}(s,m^2;\epsilon) = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{(k_1-p)^{2n_4}(k_2-p)^{2n_5}}{(k_1^2-m^2)^{n_1}(k_2^2-m^2)^{n_2}((k_1+k_2-p)^2-m^2)^{n_3}}
$$

Tadpole + 2 Master Integrals
$$
I_1 = I_{0,1,1,0,0}
$$
, $I_2 = I_{1,1,1,0,0}$ and $I_3 = I_{1,1,2,0,0}$

analy se iterated residues we get to $\det^{n}(P(x)) = (x - a_1)(x - a_2)(x - a_3)(x - a_1)$ If we try to analyse iterated residues we get to $(define P_4(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4))$ ² *.* (3.1)

$$
I_{1,1,1,0,0} \sim \int \frac{\mathrm{d}z_4}{\sqrt{P_4(z_4)}} \wedge \mathrm{d}\log f_3(z_3, z_4) \wedge \mathrm{d}\log f_2(z_2, z_3, z_4) \wedge \mathrm{d}\log f_1(z_1, z_2, z_3, z_4)
$$

 $K(x)$ It has *K*(*x*) It has no poles at all!

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could be used completely a set of the complete complete complete complete complete complete complete complete
Complete complete co $3 - 200$ by $\frac{1}{2}$ or $\frac{1}{2}$ e enply MDI recipe to Elliptic Formon Integrals (or beyond) feils checked that these master integrals satisfy indeed differential equations in canonical form. In canonical form In fact, trying to apply MPL recipe to Elliptic Feynman Integrals (or beyond) fails…

$$
\sum_{k_1+k_2=p}^{k_1} I_{n_1,n_2,n_3,-n_4,-n_5} = I_{n_1,n_2,n_3,-n_4,-n_5}(s,m^2;\epsilon) = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{(k_1-p)^{2n_4}(k_2-p)^{2n_5}}{(k_1^2-m^2)^{n_1}(k_2^2-m^2)^{n_2}((k_1+k_2-p)^2-m^2)^{n_3}}
$$

if and an algorities the leading singularities and allow the singularities of the singularities of the singula
The leading singularities of the singularities of the singularities of the singularities of the singularities o

we chose it such that it is manning that it is mand it is the factor threshold *s 4m2. The factor secondation serie*
The factor of the factor series in the factor of the factor series in the factor of the factor series in

Tadpole + 2 Master Integrals $I_1 = I_{0,1,1,0,0}$, $I_2 = I_{1,1,1,0,0}$ and $I_3 = I_{1,1,2,0,0}$ $\frac{1}{2}$, $\frac{1}{2}$ shown in $\frac{1}{2}$ shown in $\frac{1}{2}$ space-time dimensions in $\frac{1}{2}$

> analy se iterated residues we get to $\det^{n}(P(x)) = (x - a_1)(x - a_2)(x - a_3)(x - a_1)$ If we try to analyse iterated residues we get to $(define P_4(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4))$ ² *.* (3.1)

BEYOND POLYLOGARITHMS: CONCEPTUAL DIFFERENCES Der number of master integrals, the steps in complexity. However, the steps in complexity. However, the steps in integrals of a given sector, by integrals in lower sectors. This is in principle a straightforward procedure. In practice, however, it can become tricky as also the functional dependence OFIAND I AFIFAANIIIII. *^r*(*s, m*2) = ^p*s*(*^s* ⁴*m*2)*.* (3.8) associated to the threshold for the production of two massive and one massless particle.

sector considered, the introduction of new functions might be required. These are likewise

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Complete complete co $3 - 200$ by $\frac{1}{2}$ or $\frac{1}{2}$ e enply MDI recipe to Elliptic Formon Integrals (or beyond) feils checked that these master integrals satisfy indeed differential equations in canonical form. In canonical form In fact, trying to apply MPL recipe to Elliptic Feynman Integrals (or beyond) fails…

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$$
I_{1,1,1,0,0} \sim \int \frac{\mathrm{d}z_4}{\sqrt{P_4(z_4)}} \wedge \mathrm{d}\log f_3(z_3,z_4) \wedge \mathrm{d}\log f_2(z_2,z_3,z_4) \wedge \mathrm{d}\log f_1(z_1,z_2,z_3,z_4)
$$

Notice that in this and other *(single-scale)* cases, eps-factorised basis found by an **Ansatz** *z*1*z*2*z*³ Notice that in this and other *(single-scale)* cases, eps-factorised basis found by an Ansatz

> **[Weinzierl, Adams '16] [Frellesvig, Weinzierl '23] [Pögel, Wang, Weinzierl '22, '23] [Jian, Wang, Yang, Zhao '23]**

if analysing the leading singularities does not provide enough candidates, our method

Get inspiration from **construction of Elliptic polylogarithms (eMPLs)** Γ in the calculation of F^{11} integrals that evaluate to functions of Λ F^{11} . the representation of Empire polylogarithms (CMI E3)
Fown Levin '11: Brödel, Mafra, Matthes, Schlotterer '141

[Brown Levin '11; Brödel, Mafra, Matthes, Schlotterer '14] [Brödel, Dulat, Duhr, Penante, Tancredi '17, '18]

We can insist on **single poles** \leftrightarrow <u>logarithmic singularities</u> (Gauge Theory)

pure functions and one-forms with at most logarithmic singularities.

$$
\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \ldots & n_k \\ c_1 & \ldots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt \, \Psi_{n_1}(c_1, t, \vec{a}) \, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \ldots & n_k \\ c_2 & \ldots & c_k \end{smallmatrix}; t, \vec{a}\right)
$$

BEYOND POLYLOGARITHMS: CONCEPTUAL DIFFERENCES *pure functions and one-forms with at most logarithmic singularities*.

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$$

with the requirement of the representing the requirement to the set of the side of integrals. Price to pay: infinite tower of **transcendental kernels [can't be obtained from "residue of integrand"]**

BEYOND POLYLOGARITHMS: CONCEPTUAL DIFFERENCES polis in the Gauss-Manin connection matrices, we start by connection matrices, we start by connection matrices **BEYUNU PULYLUGARII HMS: CONCEPTUAL DIFFERENCES** *pure functions and one-forms with at most logarithmic singularities*.

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[Brown Levin '11; Brödel, Mafra, Matthes, Schlotterer '14] [Brödel, Dulat, Duhr, Penante, Tancredi '17, '18] as the *Wronskian* matrix or *period* matrix. Beyond the polylogarithmic case, W contains \Box and \Box functions for which we can always derive a representations \Box and \Box

discarding any contributions from integrals whose own differential equations do not couple

We can insist on single poles \leftrightarrow logarithmic singularities (Gauge Theory)

$$
\mathcal{E}_4(\begin{smallmatrix} n_1 & \ldots & n_k \\ c_1 & \ldots & c_k \end{smallmatrix}; x, \vec{a}) = \int_0^x dt \Psi_{n_1}(c_1, t, \vec{a}) \mathcal{E}_4(\begin{smallmatrix} n_2 & \ldots & n_k \\ c_2 & \ldots & c_k \end{smallmatrix}; t, \vec{a})
$$

we convert the **anseemental Kerners** plan the with the requirement of the representing the requirement to the set of the side of integrals. Price to pay: infinite tower of **transcendental kernels [can't be obtained from "residue of integrand"]**

Important property of eMPLs is that they still satisfy generalized unipotent differential equations $\mathcal{L} = \mathcal{L} \times \mathcal{L}$ and the type and type and type \mathcal{L}

$$
d\mathbf{W}^{u} = \left(\sum_{i} \mathbf{U}_{i}(\underline{z}) d z_{i}\right) \mathbf{W}^{u}, \qquad \text{Where } U_{i}(\underline{z}) \text{ are } \mathbf{Nilpotent} \text{ matrices: } U_{i} \cdot U_{i} \cdot \dots \cdot U_{i} = 0
$$

\overline{O} of the coefficient functions in the complications in the complications might be complicated. Moreover, if a given \overline{O} **SUBSTER SINGLED I:** THE IMPORTANCE OF **E** IMPORTANCE OF (1+2✏)(1+3✏)(*s*3*m*²) *^m*2(*sm*2)(*s*9*m*2) *s*220*m*2*s*+27*m*4+✏(*s*230*m*2*s*+45*m*⁴) **GENERAL STRATEGY: THE IMPORTANCE OF BEING UNIPOTENT**

(iterated) integrals of functions already present in the differential equations.

Let's go back to sunrise case: diff equations for standard basis are coupled in $d=2n$ relation. The tadpole integral *I*¹ is the same as before and remains a suitable candidate for case, uni equations for standard basis are coupled in $a = 2n$

$$
\begin{array}{c}\n\stackrel{k_1}{\longrightarrow} \\
\hline\n\stackrel{k_2}{\longrightarrow} \\
\hline\n\stackrel{k_1}{\longrightarrow} \\
\hline\n\stackrel{k_2}{\longrightarrow}\n\end{array}\n\qquad\n\frac{\partial}{\partial m^2}\n\begin{pmatrix}\n\mathfrak{I}_2 \\
\mathfrak{I}_3\n\end{pmatrix}\n=\n\begin{pmatrix}\n0 & 3 \\
\frac{s-3m^2}{m^2(s-m^2)(s-9m^2)} - \frac{s^2-20m^2s+27m^4}{m^2(s-m^2)(s-9m^2)}\n\end{pmatrix}\n\begin{pmatrix}\n\mathfrak{I}_2 \\
\mathfrak{I}_3\n\end{pmatrix}
$$

where, as before, the Gauss-Manin connection matrix with respect to *s* follows from a scaling

INALET US INDITING SOFTS INTO THE CALIFORNIA WORKS η_1 31, 72–79] shown in figure 1. We work as it is customary in *d* = 22✏ space-time dimensions Matrix of homog. sol. is **not unipotent** $W = \begin{pmatrix} 1 & 1 & 1 \\ \eta_1 & \eta_2 \end{pmatrix}$ will not pro *ω*¹ *ω*² *η*¹ *η*2) **[Solving by variation of constants will not produce unipotent results]**

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$$
\begin{array}{c}\n\begin{array}{c}\n\hline\n\end{array}\n\\
\hline\n\end{array}\n\\
\begin{array}{c}\n\hline\n\end{array}\n\\
\hline\n\end{array}\n\\
\begin{array}{c}\n\hline\n\end{array}\n\\
\hline\n\end{array}\n\\
\begin{array}{c}\n\frac{\partial}{\partial m^2} \left(\frac{\Im_2}{\Im_3} \right) = \left(\frac{0}{m^2 (s - m^2)(s - 9m^2)} - \frac{s^2 - 20m^2 s + 27m^4}{m^2 (s - m^2)(s - 9m^2)} \right) \left(\frac{\Im_2}{\Im_3} \right)\n\end{array}
$$

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Matrix of homoge soft is not umpotent $w - \begin{pmatrix} \eta_1 & \eta_2 \end{pmatrix}$ will not pro 31, 72–79] shown in figure 1. We work as it is customary in *d* = 22✏ space-time dimensions **[Solving by variation of constants** Matrix of homog. sol. is **not unipotent** $W = \begin{pmatrix} 1 & r^2 \\ r_1 & r_2 \end{pmatrix}$ will not produce unipotent results *ω*¹ *ω*² *η*¹ *η*2)

by the following three propagators and two irreducible numerators: A **non-unipotent** matrix can be split *(in a non-unique way)* into **Unipotent** and **Semi-Simple** part

$$
W = S \cdot U \qquad S = \begin{pmatrix} \omega_1 & 0 \\ \eta_1 & -\frac{i\pi}{\omega_1} \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & \frac{\omega_2}{\omega_1} \\ 0 & 1 \end{pmatrix} \quad \text{(Using Legendre Relation)}
$$

 D_{α} C_{α} is two space-time dimensions. We find out the change of variables D_{α} Define a new basis rotating only by semi-simple part $I = S \cdot J$

> *II*, *II*, *I*, *a*₂ d*z*₄ $2t \sim1$ What remains is unipotent — indeed rotated *J* can be expressed in terms of *pure eMPLs!*

> > **[Brödel, Duhr, Dulat, Penante, LT '18]**

GENERAL STRATEGY: THE IMPORTANCE OF BEING UNIPOTENT

Same finding **confirmed** by evaluating various two- and three-point functions by **direct integration** to first few (*ϵ*) **[Brödel, Duhr, Dulat, Penante, LT '18]**

Can we generalize it to all orders at the diff eq level?

GENERAL STRATEGY: THE IMPORTANCE OF BEING UNIPOTENT

Same finding **confirmed** by evaluating various two- and three-point functions by **direct integration** to first few (*ϵ*) **[Brödel, Duhr, Dulat, Penante, LT '18]**

Can we generalize it to all orders at the diff eq level?

STRATEGY: [Görges, Nega, LT, Wagner '23]

- 1. Start with a basis of MIs which has **no double poles in UV and IR**
- 2. For every **non-polylogarithmic sector**, choose **a minimally coupled basis [achieved by analysing residues and choosing as many master integrals as possible to decouples minimally coupled block]**
- 3. For coupled sectors, choose first integral whose residues involve **differential of first kind**
- 4. Perform a rotation of this minimally coupled basis to **remove semi-simple part**
- 5. Conjecture 1: after this rotation, matrix can be put in ϵ -factorized form by integrating out remaining terms that are not in the right form
- 6. **Conjecture 2:** The form so achieved will be a *generalization of a canonical form beyond MPLs*

EXAMPLE 1: POLYLOGARITHMIC CASE Instead of following this standard approach, let us pretend that we were unable to **EXAMPLE 1:** POLYLOGARITHMIC CASE

It works in the (simple) polylogarithmic case: Sunrise with **2 massive and 1 massless propagator** *I*¹ *WOTKS* and Euler's theorem on homogeneous functions functions functions functions functions \mathcal{A} 1.000 million and the contract of the second product of the second p

 $I_1 = I_{0,1,1,0,0}$, $I_2 = I_{1,1,1,0,0}$ and $I_3 = I_{1,1,2,0,0}$

use this example this example this example to illustrate the steps in our procedure. As we will see, they will see, t

Differential equations read: $dI = \left[A_0 + \epsilon A_1 + \mathcal{O}(\epsilon^2)\right]I$

Homogeneous equation in
$$
d=2
$$

$$
\frac{\partial}{\partial m^2} \begin{pmatrix} \mathfrak{I}_2 \\ \mathfrak{I}_3 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ \frac{1}{m^2(s-4m^2)} & \frac{-s+10m^2}{m^2(s-4m^2)} \end{pmatrix} \begin{pmatrix} \mathfrak{I}_2 \\ \mathfrak{I}_3 \end{pmatrix}
$$

one with respect to *s* follows at each step from a scaling relation implied by dimensional

 $\binom{1}{2}$ Matrix of **homogeneous solutions** contains **algebraic functions and logs**

EXAMPLE 1: POLYLOGARITHMIC CASE Instead of following this standard approach, let us pretend that we were unable to **EXAMPLE 1:** POLYLOGARITHMIC CASE use this example this example this example to illustrate the steps in our procedure. As we will see, they will see, t WHE **T**₂ mally again tunned over to sector with the inverse of W to solve the system from eq. (3.12) at \sim 0.12 at \sim 0. However, this system from eq. (3.12) at \sim 0. However, the system from eq. (3.12) at \sim 0.12 at \sim 0.12 at \sim 0.12 at \sim

It works in the (simple) polylogarithmic case: Sunrise with 2 massive and 1 massless propagator and Euler's theorem on homogeneous functions functions functions functions functions \mathcal{A} 1.000 million and the contract of the second product of the second p This case with 2 massive and 1 massless propagator $\mathbf r$ $\frac{1}{2}$ in the (simple) polyloge ithmic sees. Suprise with 2 meeting and 1 meetings $\frac{1}{2}$ we follow $\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{2}$ into a lower-triangular splitting of $\frac{1}{2}$

step does not lead to a canonical basis, not even to a factorisation of ✏. Moreover, the coeffi-

one with respect to *s* follows at each step from a scaling relation implied by dimensional

 $I_1 = I_{0,1,1,0,0}$, $I_2 = I_{1,1,1,0,0}$ and $I_3 = I_{1,1,2,0,0}$ $I_1-I_{0,1,1,0,0}$ $I_2-I_{1,1,1,0,0}$ and $I_2-I_{1,1,0,0,0}$ choice for the first integral integral integral in the top sector \overline{I} already has uniform transcendental weight \overline{I}

Differential equations read: $dI = [A_0 + \epsilon A_1 + \mathcal{O}(\epsilon^2)] I$ case is multivariate and of similar complexity as the generic complexity as the generic case of three differen as we are we anticipated from the properties of its integrand. Explicitly, we find α

Homogeneous equation in d=2
$$
\frac{\partial}{\partial m^2} \begin{pmatrix} \mathfrak{I}_2 \\ \mathfrak{I}_3 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ \frac{1}{m^2(s-4m^2)} & \frac{-s+10m^2}{m^2(s-4m^2)} \end{pmatrix} \begin{pmatrix} \mathfrak{I}_2 \\ \mathfrak{I}_3 \end{pmatrix}
$$

Matrix of homogeneous solutions contains algebraic functions and logs Split it in semi-simple and unipotent $W - W^{SS}$ W^{μ} as we are we are we find the properties of its integrand. It since \mathcal{L}_1 m_{max} Matrix of **homogeneous solutions** contains **algebraic functions and logs** Split it in **semi-simple** and **unipotent** $W = W^{ss} \cdot W^u$ directly or using for example \mathcal{L} . The example DlogBasis \mathcal{L} for example to be quickly found to be \mathcal{L} ² *I*1*,*1*,*0*,*0*,*⁰ *, M*² = ✏ $\frac{1}{2}$ $\frac{1}{2}$ ontains algebraic: Itains algebraic fund
 $W - W^{ss} \cdot W^{u}$ μ α α β

$$
\mathbf{W}^{\text{ss}} = \begin{pmatrix} \frac{1}{r(s,m^2)} & 0\\ \frac{s}{r(s,m^2)^3} & \frac{1}{2m^2(s-4m^2)} \end{pmatrix} \text{ and } \mathbf{W}^{\text{u}} = \begin{pmatrix} 1 & \log\left(\frac{s-r(s,m^2)}{s+r(s,m^2)}\right) \\ 0 & 1 \end{pmatrix} \qquad r(s,m^2) = \sqrt{s(s-4m^2)}
$$

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2011年

 \overline{I}

 $\sqrt{1}$ Rotate away semi-simple partg $\underline{I}' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ The $\underline{I'} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (**V** W^u *.* (3.15) *r*(*s, m*2) *s* + *r*(*s, m*2) $\frac{1}{\sqrt{2}}$ $\frac{1}{1}$ l *m*² $\left.\begin{array}{c} \mathbf{V}^{\mathrm{SS}} \\ \mathbf{V}^{\mathrm{SS}} \end{array}\right| \, \frac{\tilde{I}}{2}$ *s* p_0 and λ Γ $\sum_{i=1}^{n}$ identify all these candidates and the above analysis and the above analysis and t $\underline{I}^{\prime}=% {\textstyle\sum\nolimits_{\alpha}} e_{\alpha}/2\pi\varepsilon_{0}\left\vert \varphi_{\alpha}\right\rangle$ $\sqrt{2}$ $\overline{}$ 1 0 0 $\overline{0}$ $\begin{array}{c} \rm{0} \ \rm{(W^{ss})^{-1}} \end{array}$ $\sum_{i=1}^{n}$ $\overline{}$

EXAMPLE 1: POLYLOGARITHMIC CASE $\mathsf{A}\mathsf{S}\mathsf{E}$ is, however, and algebraic function algebraic func

 \sim \sim

0
0
0
0
0

Clean up remaining non-factorised dependence with a rotation

$$
\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2(s+2m^2)}{r(s,m^2)} & 1 \end{pmatrix} \begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}
$$

and it is therefore easily integrated out. Therefore easily integrated out. These manipulations can be summari

$$
d\underline{J} = \epsilon \text{ GM}^{\epsilon} \underline{J}
$$
 with $\underline{J} = (J_1, J_2, J_3)^T = \mathbf{T} \underline{I}'$,

$$
GM^{\epsilon} = \begin{pmatrix}\n-2\alpha_1 & 0 & 0 \\
0 & 2\alpha_1 - \alpha_2 - 3\alpha_3 & \alpha_4 \\
2\alpha_1 - 2\alpha_2 & -6\alpha_4 & -3\alpha_1 + \alpha_2\n\end{pmatrix}
$$

$$
\alpha_1 = d \log(m^2) \,, \ \ \alpha_2 = d \log(s) \,, \ \ \alpha_3 = d \log\left(s - 4m^2\right) \,, \ \ \alpha_4 = d \log\left(\frac{s - r(s, m^2)}{s + r(s, m^2)}\right)
$$

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$$

$$
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$$
GM^{\epsilon}=\left(\begin{array}{ccc}-2\alpha_1&0&0\\0&2\alpha_1-\alpha_2-3\alpha_3&\alpha_4\\2\alpha_1-2\alpha_2&-6\alpha_4&-3\alpha_1+\alpha_2\end{array}\right)
$$

$$
\alpha_1 = d \log(m^2), \ \alpha_2 = d \log(s), \ \alpha_3 = d \log(s - 4m^2), \ \alpha_4 = d \log\left(\frac{s - r(s, m^2)}{s + r(s, m^2)}\right)
$$

and it is therefore easily integrated out. These manipulations can be summarised in the

alysing leading singularities with **DLogBasis** find the same basis up to cons *J*¹ = *M*¹ *, J*² = *M*² *, J*³ = *M*¹ + 3*M*³ *.* (3.20) simple constant rotation that is given by We would like to stress the following points concerning the application of our procedure in **[P. Wasser '19,'20]***J*₂ *M*₂ *J*₃ *J*₄ *J*₄ *J*₄ *M*₂ *M*₂ *M*₂ *M*₄ *314 .* (3.20) NB: by analysing leading singularities with **DLogBasis** find the same basis up to constant rotation!

$$
J_1 = M_1 \,, \quad J_2 = M_2 \,, \quad J_3 = -M_1 + 3M_3
$$

NON TRIVIAL EXAMPLES: ELLIPTICS AND BEYOND \mathcal{O} and \mathcal{O} is the differential equations already present in the differential equations. The differential equations are differential equations.

Strategy is general and **does not have to do with details of the geometry***

Applied it successfully to elliptic sunrise (equal or different masses)

NON TRIVIAL EXAMPLES: ELLIPTICS AND BEYOND LED: ELLIPTICS AND BEYOND on does not increase the number of master integrals in the top sector. This can be integrals in the top sector. preted as the fact that, contrary to the sunrise, the integrand integrand in the integrand integrand in the integrand integrand in the integrand in the integrand integrand in the integrand integrand in the integrand integr \mathbf{S} and introduction of new functions might be required. The required might be required. These are likewise are likewise are likewise \mathbf{S} \mathcal{O} and \mathcal{O} is the differential equations already present in the differential equations. The differential equations are differential equations.

Strategy is general and **does not have to do with d** ϵ and ϵ that the basis of the period in ϵ is the top in ϵ basis from the literature \mathbf{C} . Notice that in this problem, no additional new functions \mathbf{C} differential equations. Second, there is instead a different singularity structure in the sub-Strategy is general and **does not have to do with details of the geometry***

Applied it successfully to elliptic sunrise (equal or nccessfully to elliptic sunrise (equal or different masses) $\frac{p}{2}$ and $\frac{k_2}{2}$ and $\frac{p}{2}$ Applied it successfully to elliptic sunrise (equal or different masses)

of the maximal cut, which is linearly independent under integration by parts. This implies the part of this implicit in part of the maximal cut.

Many other multi-scale elliptic problems

two-loop sunrise graph, showing how our procedure allows us to obtain ✏-factorised systems

NON TRIVIAL EXAMPLES: ELLIPTICS AND BEYOND LED: ELLIPTICS AND BEYOND sector. This can be integrals integrals in the top sector. This can be integrals might be required. The integrals might be interpreted as the fact that, contrary to the sunrise, the integrand integrand in the integrand integrand in the integrand integrand in the integrand in the integrand integrand in the integrand integrand in the integrand integr \mathcal{O} and \mathcal{O} is the differential equations already present in the differential equations. The differential equations are differential equations.

Strategy is general and **does not have to do with d** easily, that the basis for the topsector in *J* yielded by our approach reproduces the known basis from the literature \mathbf{C} . Notice that in this problem, no additional new functions \mathbf{C} differential equations. Second, there is instead a different singularity structure in the sub-Strategy is general and **does not have to do with details of the geometry***

Applied it successfully to elliptic sunrise (equal or nccessfully to elliptic sunrise (equal or different masses) $\frac{p}{2}$ and $\frac{k_2}{2}$ and $\frac{p}{2}$ Applied it successfully to elliptic sunrise (equal or different masses)

of the maximal cut, which is linearly independent under integration by parts. This implies the part of this implicit in part of the maximal cut.

Many other multi-scale elliptic problems

two-loop sunrise graph, showing how our procedure allows us to obtain ✏-factorised systems

 $Deyona$ is emplied can be Even cases beyond 1 elliptic curve

CONCLUSIONS

Feynman integrals are difficult to compute, but they hide structures and simplicity

Epsilon-factorised bases are an important tool to make some of their structure manifest

Beyond polylogarithms, searching for integrals with simple poles (and unit leading singularities) in the traditional sense is not enough

Instead, the property of **unipotence** can be used to **build differential equations in epsilonfactorised form almost algorithmically**

In the polylogarithmic case, this construction **reproduces results obtained from analysis of leading singularities**

Beyond polylog case, we have showed that it is enough to obtain epsilon-factorised equations in multi-scale elliptic cases and beyond!

OUTLOOK: more complicated geometries (CYs, higher genus), application to physics problems…

THANK YOU FOR YOUR ATTENTION!

BACK UP

FORCING AN EPS-FACTORIZED BASIS: THE SUNRISE GRAPH on ✏ of the coefficient functions in these shifts might be complicated. Moreover, if a given subsector exhibits singularities which were not present in the homogeneous equations for the $\overline{}$ 2✏2*s m*4(*sm*2)(*s*9*m*2) $SIS:$ THE SUNRISE GRAPH

Diff equations for two sunrise MIs are coupled in $d=2n$ relation. The tadpole integral *I*¹ is the same as before and remains a suitable candidate σ sums and coupled in $q-zn$

(iterated) integrals of functions already present in the differential equations.

Let us see explicitly how our procedure works in the case of the two-loop sunrise graph [28– Homogeneous solutions given by periods and their derivatives $W = \left($

$$
= \begin{pmatrix} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{pmatrix} \quad \text{with} \quad \eta_i \propto \partial \omega_i
$$

provided by independent set of maximal cuts

² *^m*² **[Primo, Tancredi '16, '17; Frellesvig, Papadopoulos '17; Bosma, Sogaard, Zhang '17] [Frellesvig '21]**

where, as before, the Gauss-Manin connection matrix with respect to *s* follows from a scaling

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$$
\sum_{k_1 + k_2 = p}^{k_1} \sum_{k_2 + k_3 = p}^{p} \frac{\partial}{\partial m^2} \left(\frac{\mathfrak{I}_2}{\mathfrak{I}_3} \right) = \begin{pmatrix} 0 & 3 \\ \frac{s - 3m^2}{m^2 (s - m^2)(s - 9m^2)} & -\frac{s^2 - 20m^2 s + 27m^4}{m^2 (s - m^2)(s - 9m^2)} \end{pmatrix} \begin{pmatrix} \mathfrak{I}_2 \\ \mathfrak{I}_3 \end{pmatrix}
$$

Let us see explicitly how our procedure works in the case of the two-loop sunrise graph [28– Homogeneous solutions given by periods and their derivatives $W = \left($ *ω*¹ *ω*² $\eta_1 \eta_2$ *n*₂ with $\eta_i \propto \partial \omega_i$

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where, as before, the Gauss-Manin connection matrix with respect to *s* follows from a scaling

We assume *m*1*, m*2*, m*³ *>* 0 and, in the remainder of this section, we set *p*² = *s*. One could attempt to rotate away the homogeneous solution and get something in eps-factorized form

$$
d\underline{I} = \begin{bmatrix} A_0(\underline{z}) + \epsilon \ B(\underline{z}) \end{bmatrix} \underline{I} \longrightarrow \underline{I} = W \cdot \underline{J} \qquad d\underline{J} = \epsilon \begin{bmatrix} W^{-1} \cdot B(\underline{z}) \cdot W \end{bmatrix} \underline{J}
$$

*I*1*,*1*,*1*,*0*,*⁰ ⇠ d*z*¹ d*z*² d*z*³ d*z*⁴ *z*₁*z*₁*z* Basis **[Frellesvig, Weinzierl '22]** *^J* does not have right properties even in MPL case! **[Görges, Nega, LT, Wagner '23]**