Two-Loop Electroweak Corrections with Fermion Loops to $e^+e^- \to ZH$

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based on: arXiv:2209.07612, arXiv:2101.00308

1. Motivation
2. Evaluation Method
3. ~~Phenomenology~~
4. Conclusion

Motivation

Eur. Phys. J. Spec. Top. ***228***, *1306.6352[hep-ph]*
1811.10545 [hep-ex]

- Anticipated high precision measurement for $\sigma(e^+e^- \to ZH)$: (0.4%-1.1%)
- Theoretical uncertainty up to NNLO(EW+QCD): 0.8%, greater than expected experimental one

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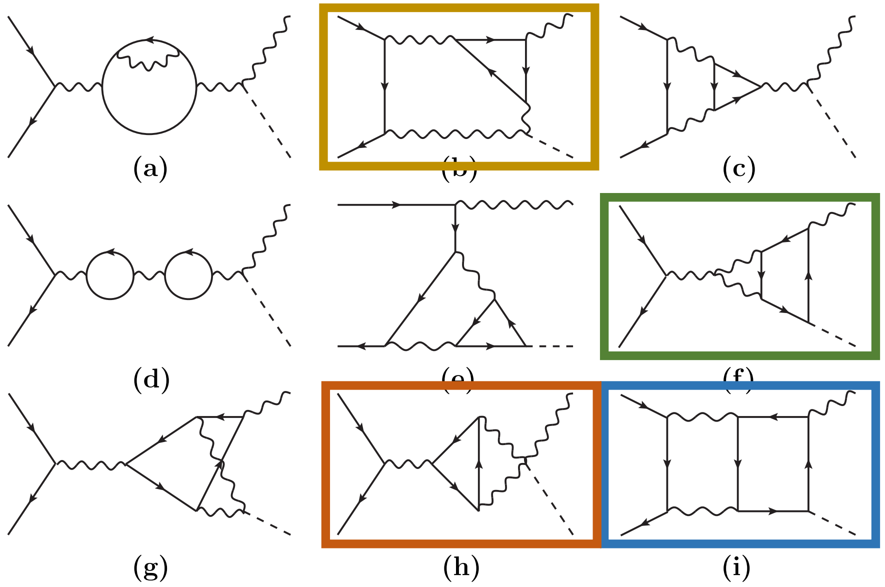
- NNLO(EW+EW) corrections must be included

$$\sigma^{\mathrm{SM}}(e^+e^- \to ZH) = \boxed{\sigma^{\mathrm{LO}} + \Delta\sigma^{\mathrm{NLO(EW)}} + \Delta\sigma^{\mathrm{NNLO(EW+QCD)}}} + \boxed{\Delta\sigma^{\mathrm{NNLO(EW+EW)}}} + \cdots$$

- NNLO(EW+EW) corrections can involve 6 mass scales: $\{m_t, m_Z, m_H, m_W, s, t\}$
- Special technique must be developed for complex multi-scale amplitude

Motivation

- Our method is a semi-numerical method based on Feynman parameterization and dispersion relation.
- A few subtraction terms are introduced to deal with UV divergence.
- The Feynman integrals are reduced to ≤3-fold numerical integral, which can be evaluated with ***typically 3-4 digits precision within minutes(6 digit, ~seconds) on a single CPU core.***
- Our method has been cross-checked by calculating some simple diagrams
- Choose two-loop double box diagram(UV finite) and two-loop VZH vertex diagram(UV div) to demonstrate our method



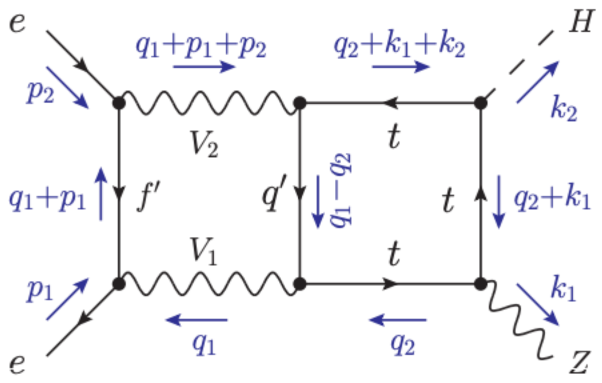
Examples of two loop Feynman diagrams with fermion loops

UV finite diagram: Planar double-box diagram

- For simplicity, numerator = 1

$$I_{plan} = \int d^D_{q_1} d^D_{q_2} \frac{1}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)(q_1-q_2)^2 - m_{q'}^2}$$

$$\frac{1}{(q_2^2 - m_t^2)((q_2+k_1)^2 - m_t^2)((q_2+k_1+k_2)^2 - m_t^2)}$$

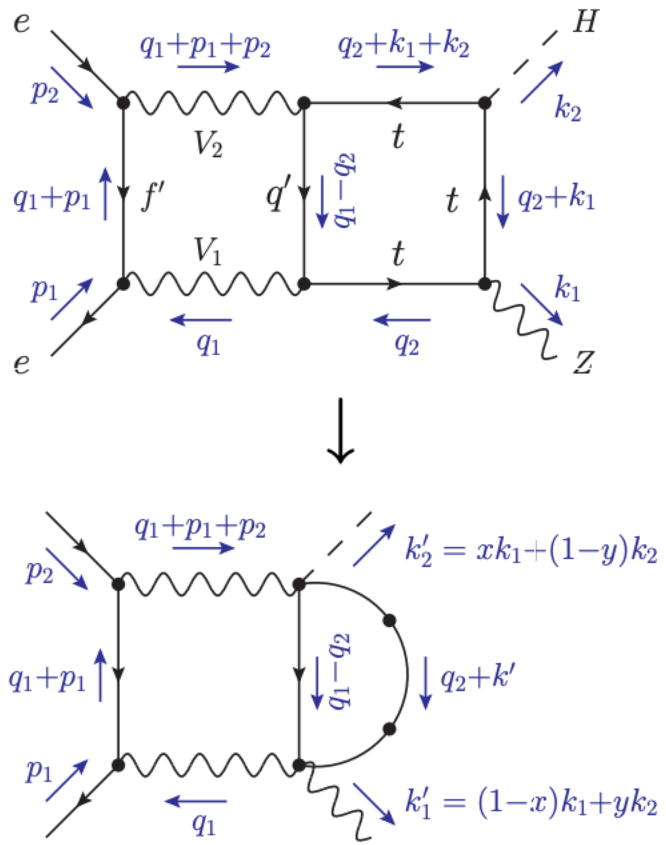


UV finite diagram: Planar double-box diagram

- use Feynman parametrization to simplify the denominators only involve q2

$$I_{plan} = \int d^D_{q_1} d^D_{q_2} \frac{1}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)(q_1-q_2)^2 - m_{q'}^2}$$

$$\underbrace{\frac{1}{(q_2^2 - m_t^2)((q_2+k_1)^2 - m_t^2)((q_2+k_1+k_2)^2 - m_t^2)}}_{\int_0^1 dx \int_0^{1-x} dy \frac{1}{((q_2+k')^2 - m'^2)^3} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \frac{1}{(q_2+k')^2 - m'^2}}$$



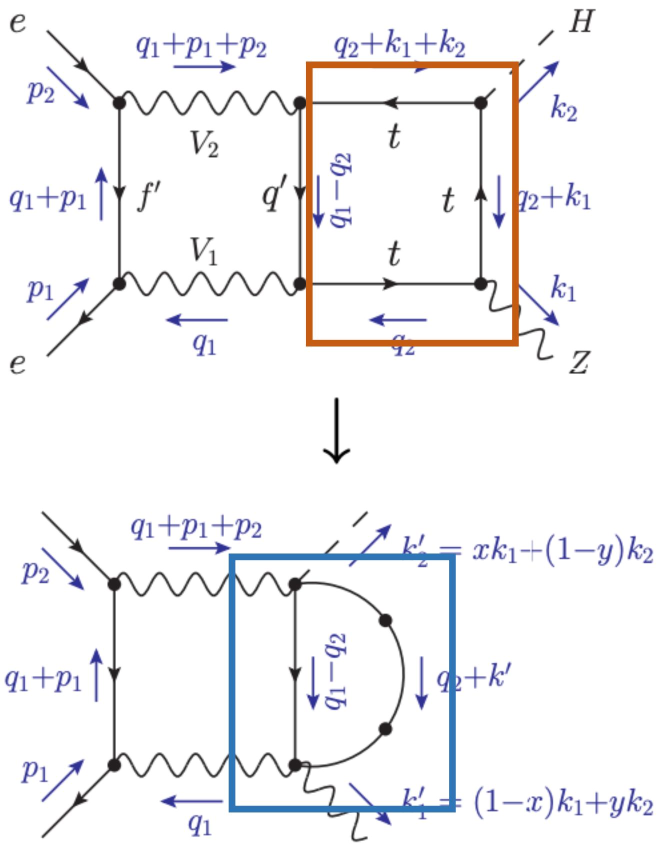
UV finite diagram: Planar double-box diagram

- use Feynman parametrization to simplify the denominators only involve q2

$$I_{plan} = \int d^D_{q_1} d^D_{q_2} \frac{1}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)(q_1-q_2)^2 - m_{q'}^2}$$

$$\frac{1}{(q_2^2 - m_t^2)((q_2+k_1)^2 - m_t^2)((q_2+k_1+k_2)^2 - m_t^2)}$$

$$\int_0^1 dx \int_0^{1-x} dy \frac{1}{((q_2+k')^2 - m'^2)^3} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \frac{1}{(q_2+k')^2 - m'^2}$$



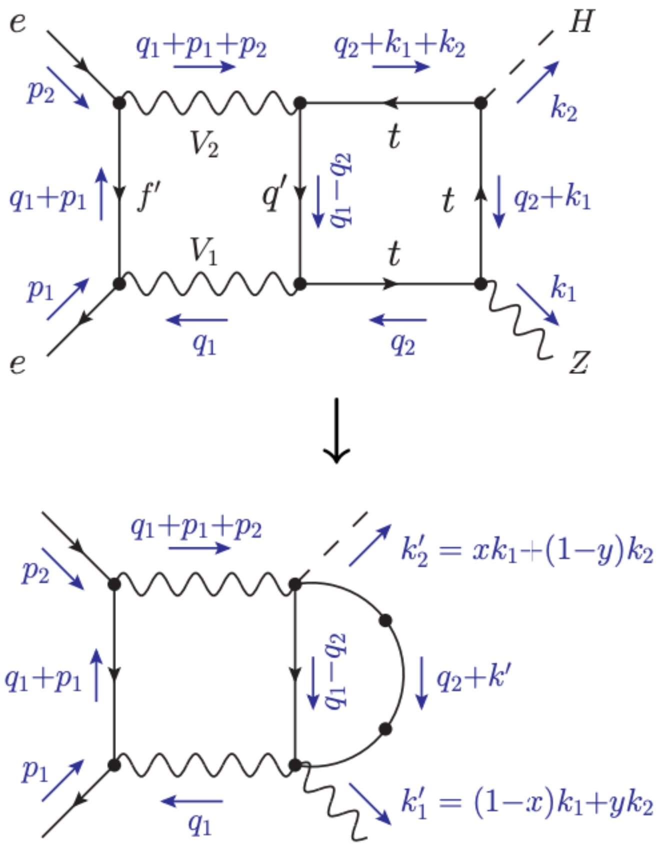
UV finite diagram: Planar double-box diagram

- use Feynman parametrization to simplify the denominators only involve q2

$$I_{plan} = \int d^D_{q_1} d^D_{q_2} \frac{1}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)(q_1-q_2)^2 - m_{q'}^2} \underbrace{\frac{1}{(q_2^2 - m_t^2)((q_2+k_1)^2 - m_t^2)((q_2+k_1+k_2)^2 - m_t^2)}}$$

$$\int_0^1 dx \int_0^{1-x} dy \frac{1}{((q_2+k')^2 - m'^2)^3} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \frac{1}{(q_2+k')^2 - m'^2}$$

$$= \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int d^D_{q_1} \frac{B_0((q_1+k')^2, m_{q'}^2, m'^2)}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)}$$



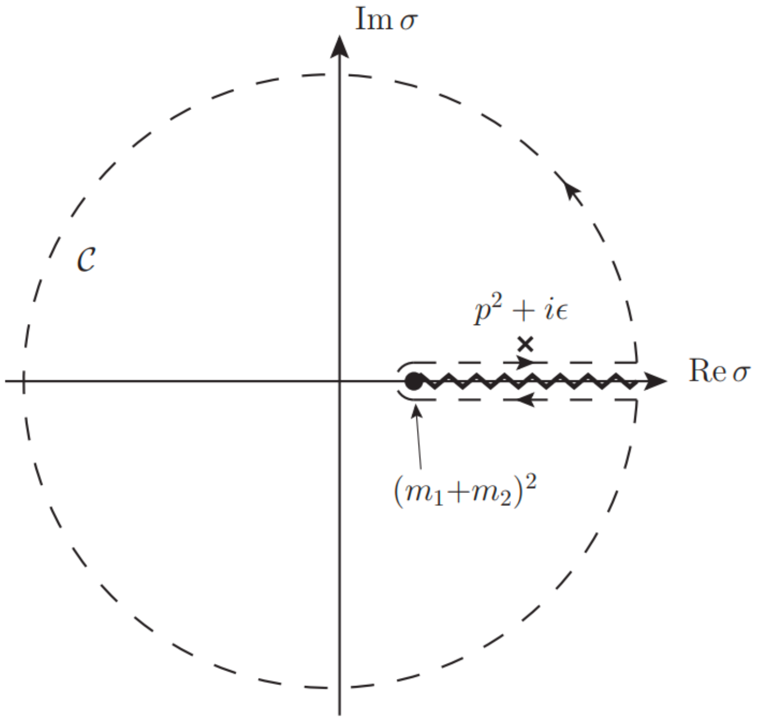
- integrate over q2: B0 function
- loop momentum q1 appears in B0 function, how to integrate over q1? dispersion relation

UV finite diagram: dispersion relation

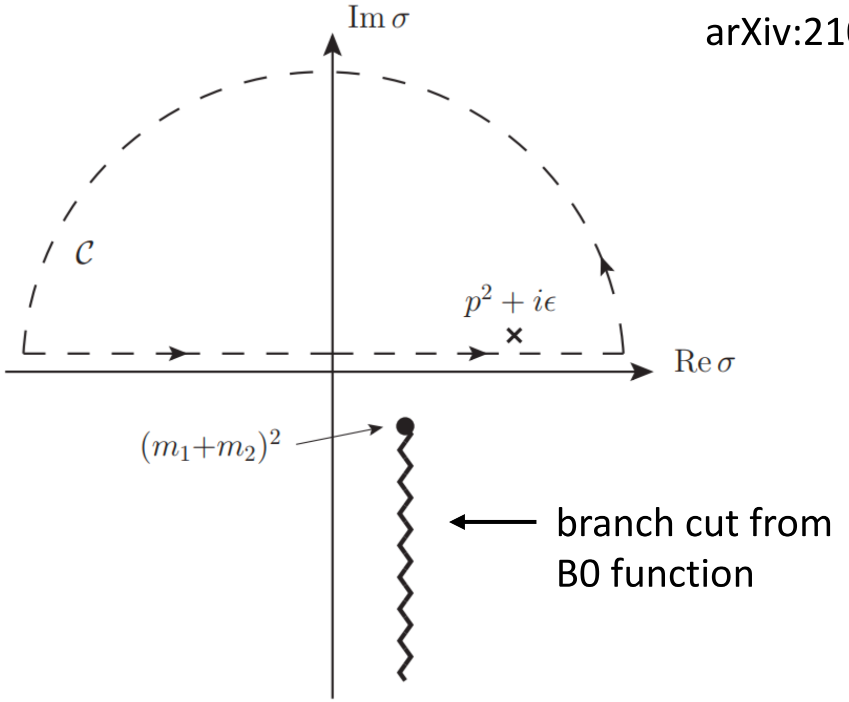
$m_1^2 \geq 0\ ,\ m_2^2 \geq 0$

$m_1^2 > 0\ ,\ m_2^2 < 0$

arXiv:2101.00308



integration contours

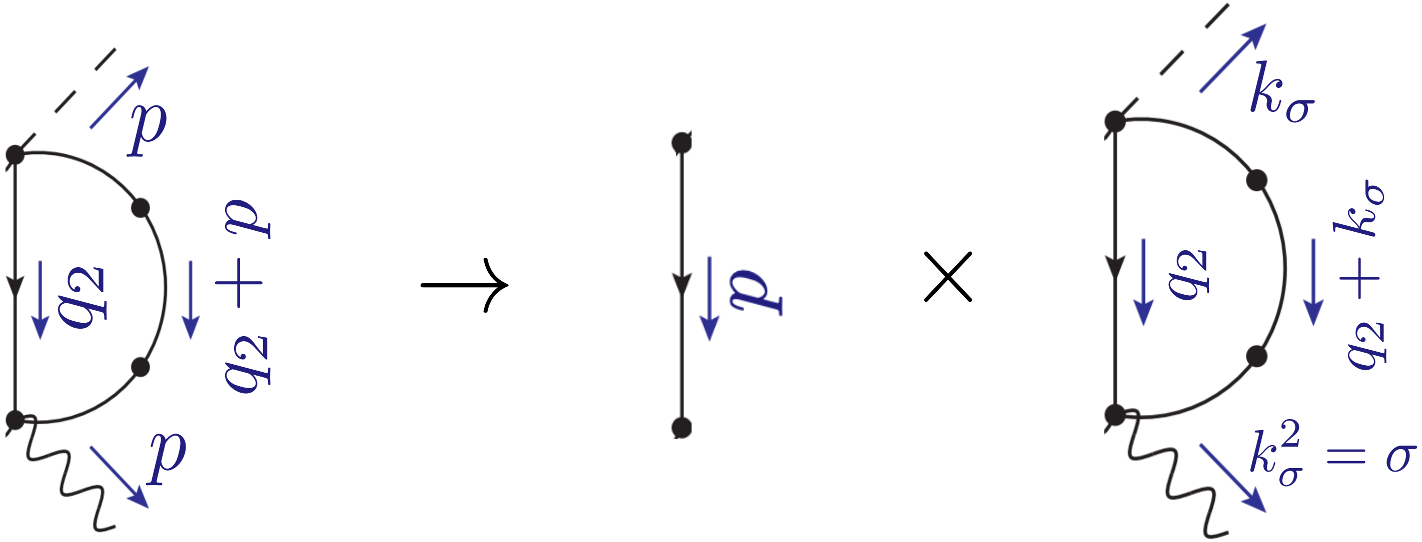


formulas

$$B_0(p^2, m_1^2, m_2^2) = \frac{1}{2\pi i}\oint d\sigma \frac{B_0(\sigma, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon} = \int_{(m_1+m_2)^2}^{\infty} d\sigma \frac{\Delta B_0(\sigma, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon}$$

$$B_0(p^2, m_1^2, m_2^2) = \frac{1}{2\pi i}\oint_C d\sigma \frac{B_0(\sigma, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon} = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\sigma \frac{B_0(\sigma, m_1^2, m_2^2)}{\sigma - p^2 - i\epsilon}$$

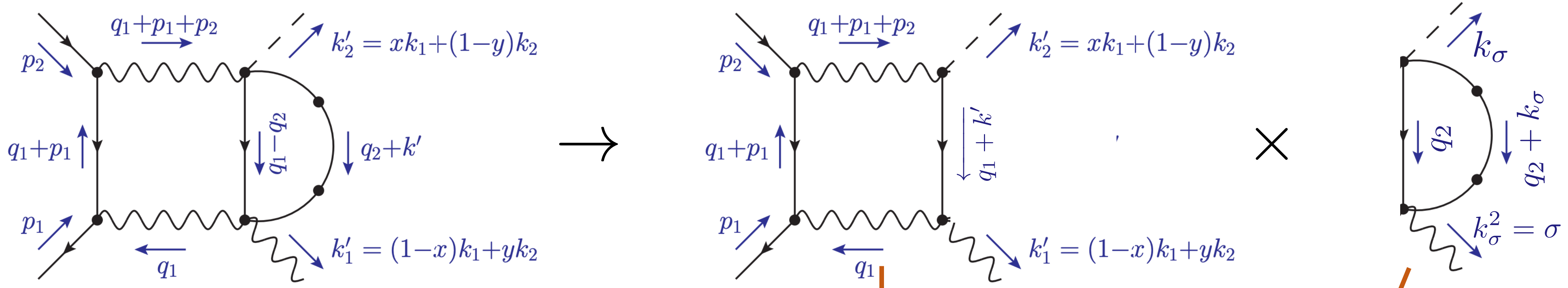
UV finite diagram: dispersion relation



$$B_0(p^2, m_1^2, m_2^2) \rightarrow \frac{1}{p^2 - \sigma} \times \begin{cases} \Delta B_0(\sigma, m_1^2, m_2^2), m_1^2 \geq 0 \wedge m_2^2 \geq 0 \\ B_0(\sigma, m_1^2, m_2^2), m_1^2 \geq 0 \vee m_2^2 \geq 0 \end{cases}$$

independent on p!

UV finite diagram: Planar double-box diagram



$$\int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int d^D_{q_1} \frac{B_0((q_1+k')^2, m^2_{q'}, m'^2)}{(q_1^2 - m^2_{V_1})((q_1+p_1)^2 - m^2_{f'})((q_1+p_1+p_2)^2 - m^2_{V_2})}$$

$$\rightarrow \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int_{(m_{q'}+m')^2}^{\infty} d\sigma \; D_0(k_i^2; m^2_{V_1}, m^2_{f'}, m^2_{V_2}, \sigma) \times \Delta B_0(\sigma, m^2_{q'}, m'^2)$$

UV finite diagram: Planar double-box diagram

$$I_{plan} = \int d^D_{q_1} d^D_{q_2} \frac{1}{(q_1^2 - m_{V_1}^2)((q_1+p_1)^2 - m_{f'}^2)((q_1+p_1+p_2)^2 - m_{V_2}^2)(q_1-q_2)^2 - m_{q'}^2}$$
$$\frac{1}{(q_2^2 - m_t^2)((q_2+k_1)^2 - m_t^2)((q_2+k_1+k_2)^2 - m_t^2)}$$

$$\text{numerator} = \{q_2^2, q_2 \cdot q_1, q_2 \cdot p_1 ...\} \neq 1$$

$$I_{\text{plan}} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(s, m'^2, m_{q'}^2) D_0(..., \sigma)$$

integrand is much more complicated

UV finite diagram: Planar double-box diagram

$$I_{\text{plan}} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(s, m'^2, m^2_{q'}) D_0(..., \sigma)$$

$$\frac{q_2^\mu \cdots q_2^\nu}{[q_2^2 - m'^2][(q_2+p)^2 - m^2_{q'}]} \to \{B_0, p^\mu B_1, g^{\mu\nu} B_{00}, p^\mu p^\nu B_{11}, \cdots\}$$

analytical formulas are known

$$I_{\text{plan}} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int_{(m'+m_{q'})^2}^{\infty} d\sigma [b_0 \Delta B_0 + b_1 \Delta B_1 + b_{00} \Delta B_{00} + \cdots]$$

$$\times [c_0 A_0 + c_1 B_0 + c_2 C_0 + \cdots]$$

UV finite diagram: Planar double-box diagram

$$I_{\text{plan}} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(s, m'^2, m^2_{q'}) D_0(..., \sigma)$$

$$\frac{q_1^{\mu} \cdots q_1^{\nu}}{[q_1^2 - m^2_{V_1}] \cdots [(q_1 + p)^2 - \sigma]} \to \{A_0, B_0, C_0 \cdots\}$$

$$I_{\text{plan}} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int_{(m'+m_{q'})^2}^{\infty} d\sigma [b_0 \Delta B_0 + b_1 \Delta B_1 + b_{00} \Delta B_{00} + \cdots]$$

$$\times [c_0 A_0 + c_1 B_0 + c_2 C_0 + \cdots]$$

implement LoopTools

UV finite diagram: Planar double-box diagram

$$I_{\text{plan}} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(s, m'^2, m^2_{q'}) D_0(\dots, \sigma)$$

integrand are obtained with private code

$$I_{\text{plan}} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int_{(m'+m_{q'})^2}^{\infty} d\sigma [b_0 \Delta B_0 + b_1 \Delta B_1 + b_{00} \Delta B_{00} + \cdots] \times [c_0 A_0 + c_1 B_0 + c_2 C_0 + \cdots]$$

numerically integrated in C++

integrand must be UV finite!

UV divergent diagram

- Subtraction terms to deal with UV divergence:
  - subtract few simple terms($I_{\rm subtra}$) to make it UV finite
  - $I_{\rm subtra}$ must be simple enough to be integrated analytically
  - add $I_{\rm subtra}$ back analytically

$$|M_0 M_2^*| \sim \int dx \int dy \int d\sigma \times \underbrace{[\text{integrand}]}_{\text{UV div}}$$

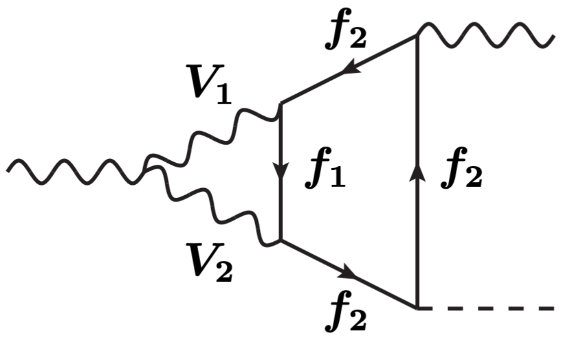
$$= \int dx \int dy \int d\sigma \times \underbrace{[\text{integrand} - I_{\rm subtra}]}_{\text{UV finite, integrate numerically}}$$

$$+ \int dx \int dy \int d\sigma \times \underbrace{[I_{\rm subtra}]}_{\text{UV div, integrate analytically}}$$

UV divergent diagram

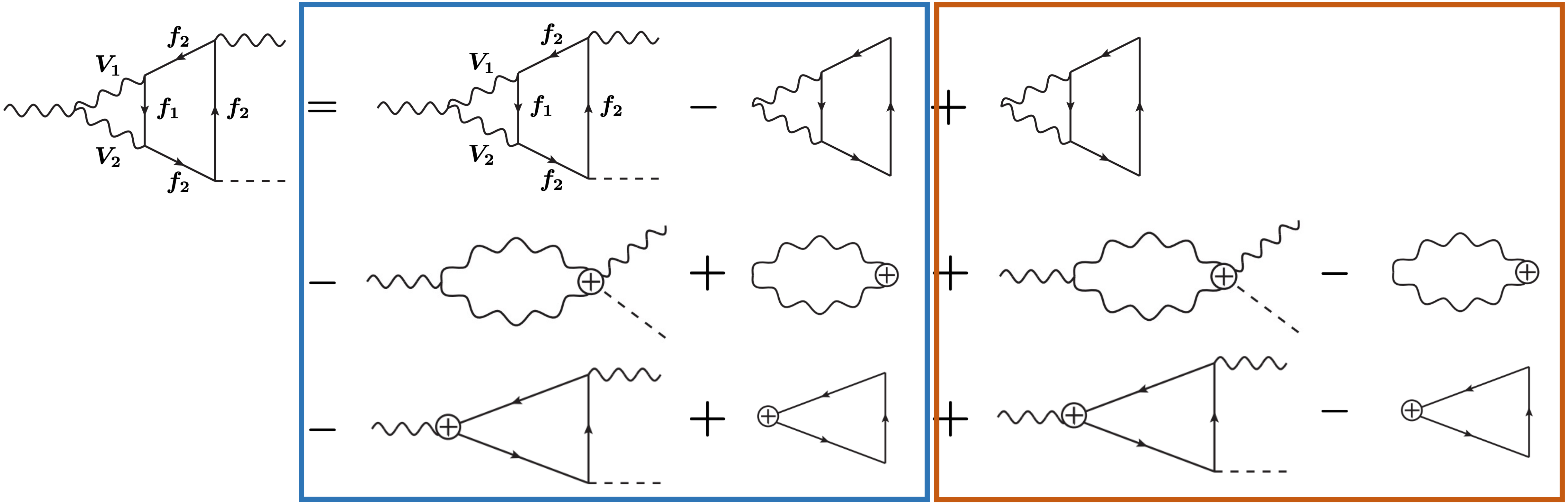
- Subtraction terms to deal with UV divergence:
  - subtract few simple terms(I_{subtra}) to make it UV finite
  - I_{subtra} must be simple enough to be integrated analytically
  - add I_{subtra} back analytically
- 3 types subtraction terms $\leftrightarrow$ 1 global divergence(highest order divergence)
  + 2 local divergences(divergence from subloops)
- Use two-loop VZH vertex diagram as an example
  - contains 1 global and 2 local divergences
  - 3 types subtraction terms are needed

most complicated UV divergence structure at two-loop



$$\sim \frac{A}{\epsilon^2} + \frac{B}{\epsilon}$$

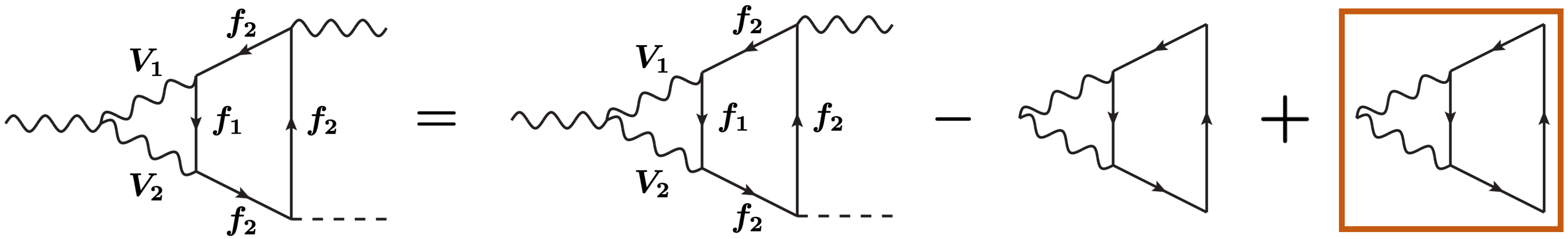
UV divergent diagram: VZH vertex



UV finite,
numerically integrate in C++

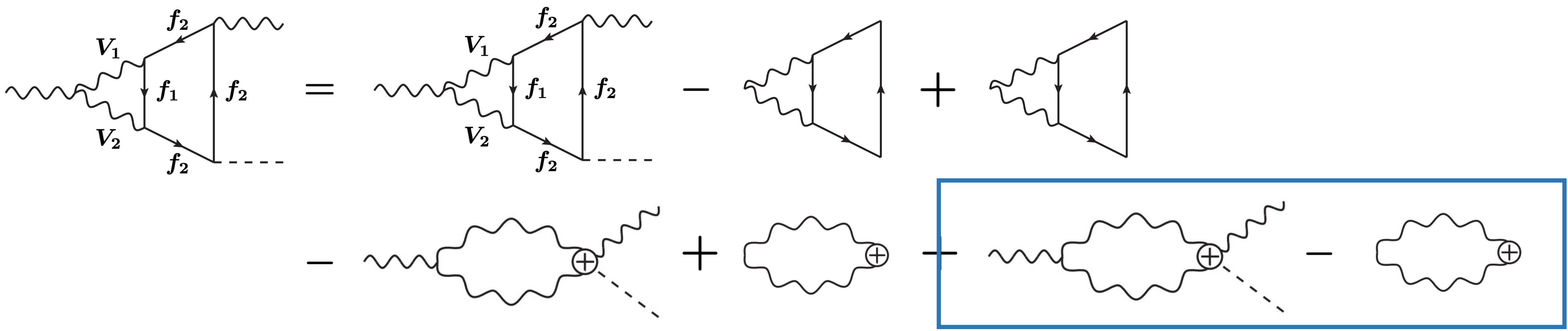
UV divergent,
analytically integrate
→ all divergences cancel after combine counter-term diagrams

UV divergent diagram: VZH vertex



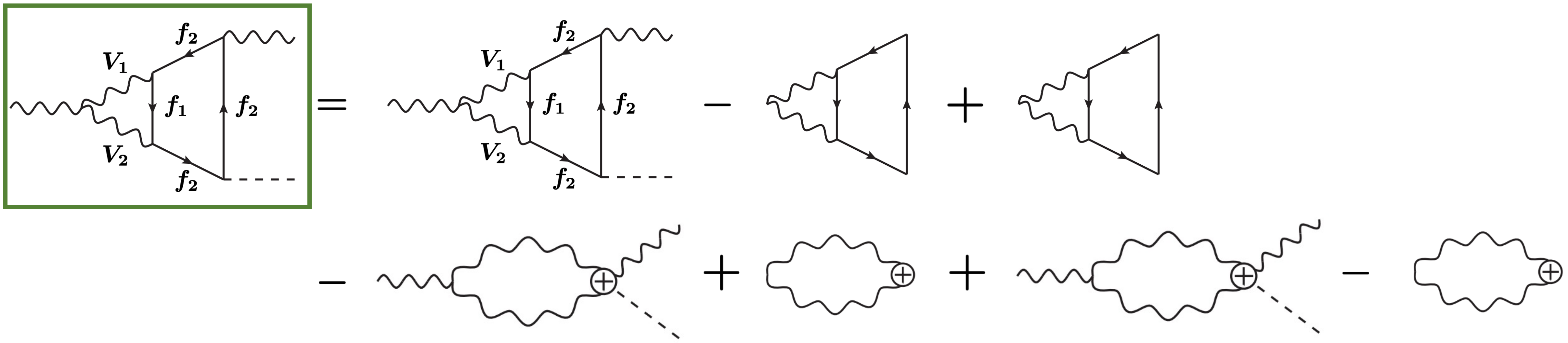
- vacuum diagram is used to cancel the global divergence
- vacuum diagram has same intermediate particles, external momenta set to 0
- Add vacuum diagram back analytically
  - tensor decomposition: reduce tensor integral to scalar integral
  - reduce scalar integral to master one using FIRE *Comput.Phys.Commun. 247 (2020) 106877*
  - master integral can be evaluated analytically with TVID *JHEP 01 (2020) 024*

UV divergent diagram: VZH vertex



- Subtract two "diagrams" to cancel divergence from fermionic loop(loop momenta q1)
- "diagrams" = mathematical formulas at UV divergence limit, set all momenta in q1 propagators to 0

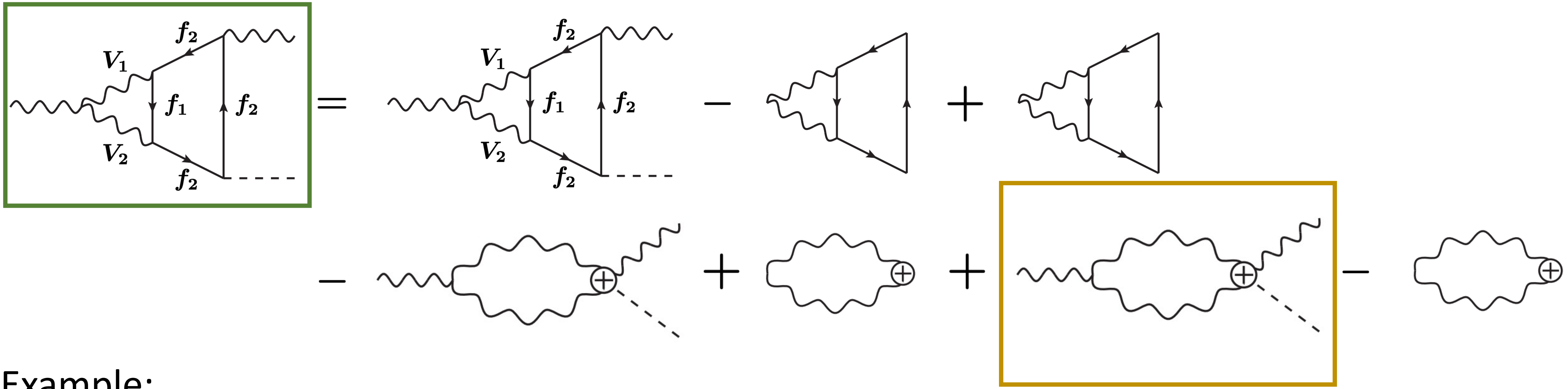
UV divergent diagram: VZH vertex



Example:

$$\mathcal{I} = \int\int \frac{q_1^4}{(q_2^2 - m_{V_2}^2)((q_2+p)^2 - m_{V_1}^2)((q_2+q_1)^2 - m_{f_1}^2)(q_1^2 - m_{f_2}^2)((q_1-p_h)^2 - m_{f_2}^2)((q_1-p)^2 - m_{f_2}^2)}$$

UV divergent diagram: VZH vertex



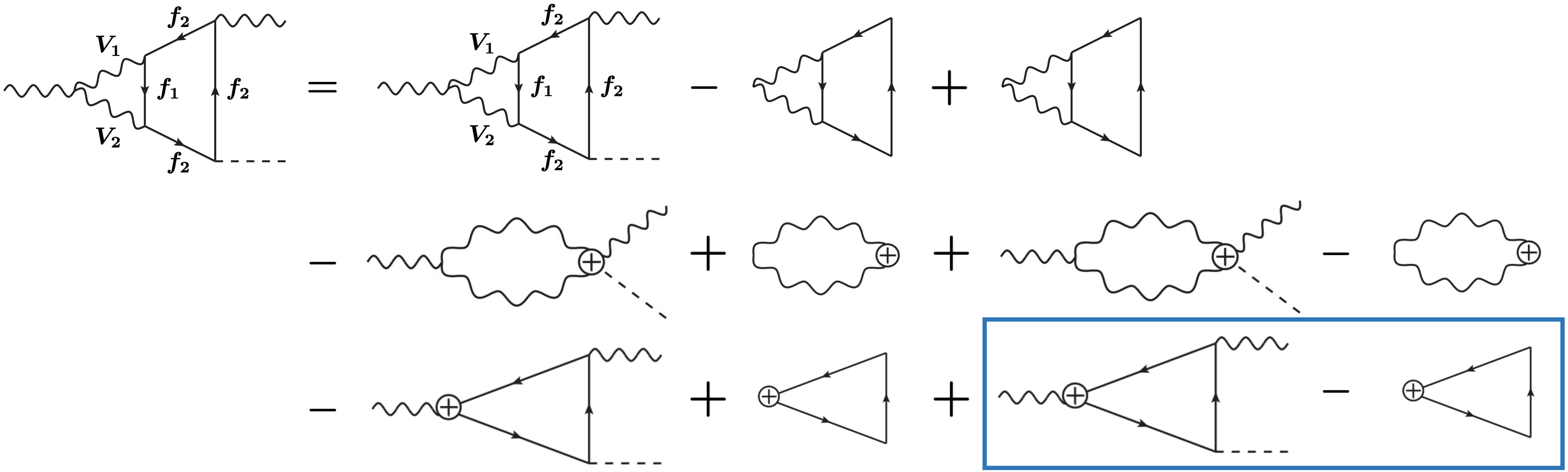
Example:

$$\mathcal{I} = \int\int \frac{q_1^4}{(q_2^2 - m_{V_2}^2)((q_2+p)^2 - m_{V_1}^2)((q_2+q_1)^2 - m_{f_1}^2)(q_1^2 - m_{f_2}^2)((q_1-p_h)^2 - m_{f_2}^2)((q_1-p)^2 - m_{f_2}^2)}$$

$$\mathcal{I}^{q_1}_{\text{subtra}} = \int\int \frac{1}{(q_2^2 - m_{V_2}^2)((q_2-p)^2 - m_{V_1}^2)} \frac{q_1^4}{(q_1^2 - m_{f_2}^2)(q_1^2 - m_{f_2}^2)(q_1^2 - m_{f_2}^2)(q_1^2 - m_{f_1}^2)}$$

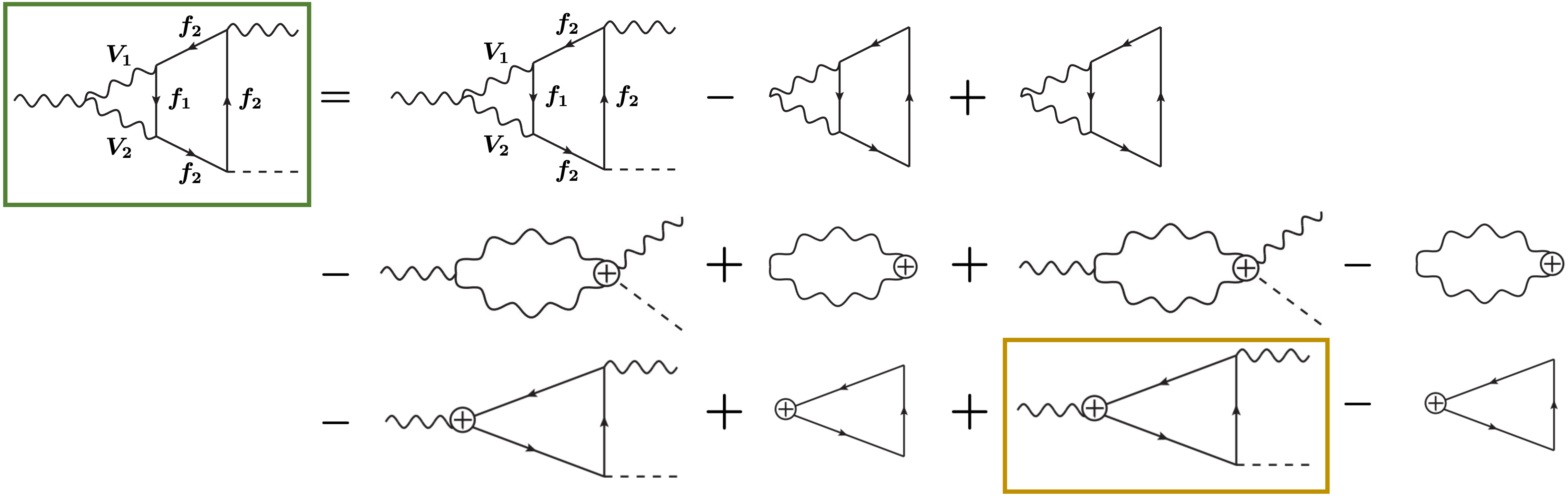
$$= [B_0(p^2, m_{V_2}^2, m_{V_1}^2) - B_0(0, m_{V_2}^2, m_{V_1}^2)] \times [c_1 A_0(m_{f_1}^2) + c_2 A_0(m_{f_2}^2)]$$

UV divergent diagram: VZH vertex



- ➢ Subtract two “diagrams” to cancel divergence from bosonic loop(loop momenta q2)
- ➢ “diagrams” = mathematical formulas at UV divergence limit, set all momenta in q2 propagators to 0

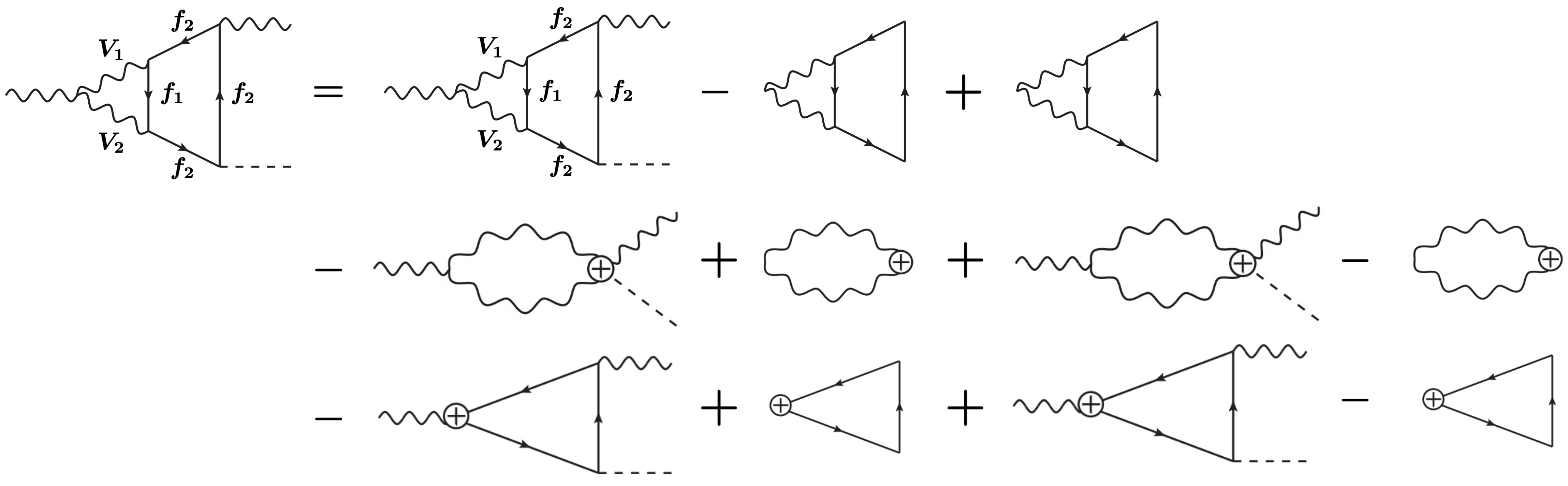
UV divergent diagram: VZH vertex



$$\mathcal{I} = \int\int \frac{q_2^2}{(q_2^2 - m_{V_2}^2)((q_2+p)^2 - m_{V_1}^2)((q_2+q_1)^2 - m_{f_1}^2)(q_1^2 - m_{f_2}^2)((q_1-p_h)^2 - m_{f_2}^2)((q_1-p)^2 - m_{f_2}^2)}$$

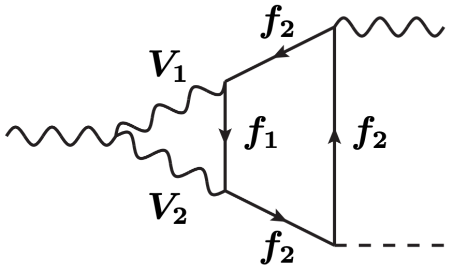
$$\mathcal{I}^{q_2}_{\text{subtra}} = \int\int \frac{q_2^2}{(q_2^2 - m_{V_2}^2)(q_2^2 - m_{V_1}^2)(q_2^2 - m_{f_1}^2)} \frac{1}{(q_1^2 - m_{f_2}^2)((q_1-p_h)^2 - m_{f_2}^2)((q_1-p)^2 - m_{f_2}^2)}$$

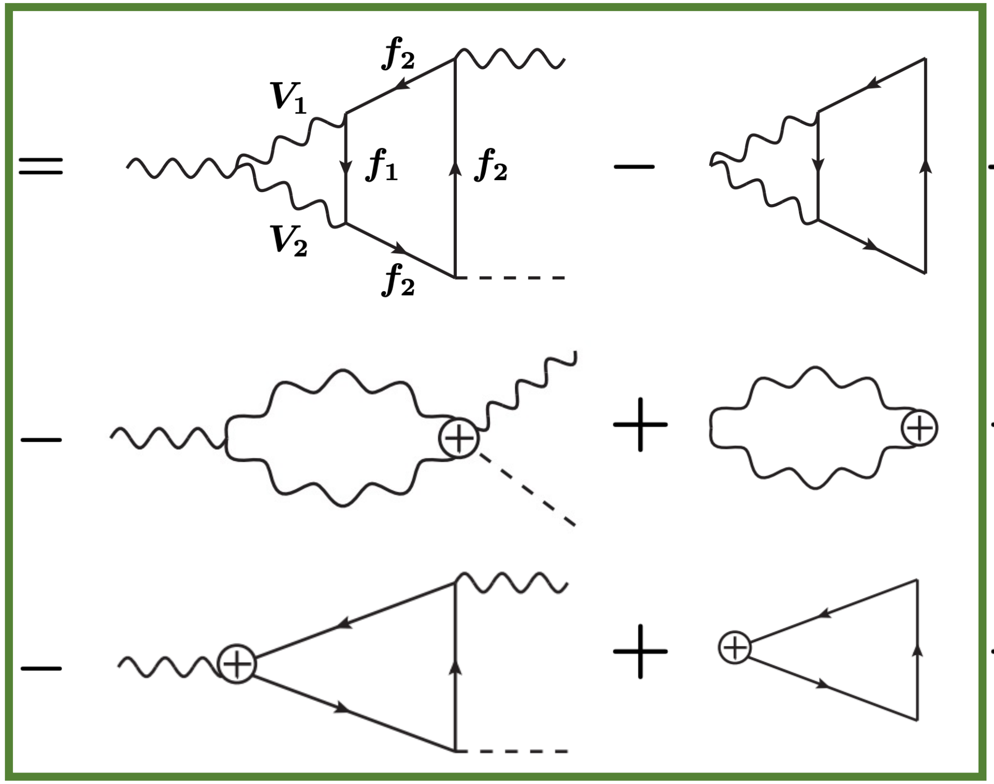
UV divergent diagram: VZH vertex

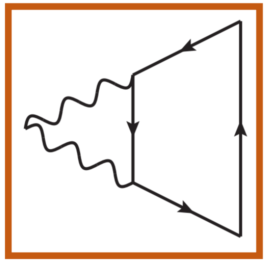


$$\mathcal{I}^{q_2}_{\text{subtra}} = \int_0^1 dx \frac{\partial}{\partial m_x^2} B_{00}(m^2, m_x^2, m_{f_1}^2) C_0 - \int_0^1 dy \frac{\partial}{\partial m_y^2} B_{00}(m^2, m_y^2, m_{f_1}^2) C_0$$

UV divergent diagram: VZH vertex

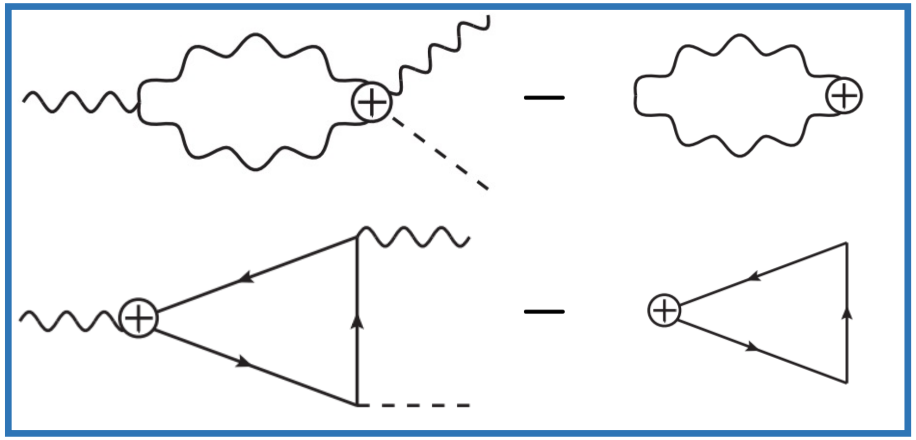






cancel global divergence

$\sim \frac{A}{\epsilon^2}$



UV finite,
numerically integrate in C++

cancel local divergence $\sim \frac{B}{\epsilon}$

Summary

- Two-loop Electroweak corrections must be included because of expected high precision at future e^+e^- colliders
- Our evaluation method is based on Feynman parametrization and dispersion relation, a few subtraction terms are introduced to deal with UV divergence
- With our method, amplitude is simplified to ≤3-fold numerical integral

$$\mathcal{I} = \underbrace{\int_0^1 dx \int_0^1 dy}_{\text{Feynman Parametrization}} \overbrace{\int_{\sigma_0}^{\infty} d\sigma}^{\text{dispersion relation}} \overbrace{F(x,y,\sigma) - F_{\text{subtra}}}^{\text{UV finite, numerical integral}} + \underbrace{F_{\text{subtra}}}_{\text{UV div,analytical}}$$

- Numerical integration can be evaluated with typically 3-4 digits precision within minutes on a single CPU core.

Thank you!

Unphysical divergence from lower bound

Integrating over q1 gets the D0 function.
Use Leibiniz's rule to put the derivative inside the integral: ΔB_0 is divergent at the lower bound, it can be fixed by subtracting one term to make the integrand become 0 at the lower bound.

$$I_{plan} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2} \int_{(m'+m_{q'})^2}^{\infty} d\sigma \int d^D_{q_1} \Delta B_0(s, m'^2, m^2_{q'}) \frac{1}{(q_1^2 - m^2_{V_1})((q_1+p_1)^2 - m^2_{f'})((q_1+p_1+p_2)^2 - m^2_{V_2})(s-(q_1+k')^2)}$$

$$= -\int_0^1 dx \int_0^{1-x} \partial^2_{m'^2} \int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(\sigma, m'^2, m^2_{q'}) D_0(p_1^2, p_2^2, k'^2_2, k'^2_1, s, t, m^2_{V_1}, m^2_{f'}, m^2_{V_2}, \sigma)$$

Leibiniz's rule:

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt + f(x,b(x))\frac{db(x)}{dx} - f(x,a(x))\frac{da(x)}{dx}$$

Unphysical divergence from lower bound

$$\partial^2_{m'^2}\int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(\sigma, m'^2, m^2_{q'})(D_0(...,\sigma) - \frac{\sigma_0}{\sigma}D_0(...,\sigma_0) + \frac{\sigma_0}{\sigma}D_0(...,\sigma_0))$$

$$\frac{\Delta B_0(s, m_1^2, m_2^2)}{s - s_0}\Big|_{s=s_0} = \frac{1}{0}$$

$$= \partial^2_{m'^2}\int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(\sigma, m'^2, m^2_{q'})(D_0(...,\sigma) - \frac{\sigma_0}{\sigma}D_0(...,\sigma_0))$$

→ 0 at the lower bound, so derivative van be put inside the integral

$$+ \partial^2_{m'^2}\int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(\sigma, m'^2, m^2_{q'})\frac{\sigma_0}{\sigma}D_0(...,\sigma_0)$$

→ integrate over σ gives $B_0(0, m'^2, m^2_{q'})$ (dispersion relation)

$$I_{\text{plan}} = \int_0^1 dx \int_0^{1-x} dy \partial^2_{m'^2}\int_{(m'+m_{q'})^2}^{\infty} d\sigma \Delta B_0(s, m'^2, m^2_{q'}) D_0(p_1^2, p_2^2, k'^2_2, k'^2_1, s, t, m^2_{V_1}, m^2_{f'}, m^2_{V_2}, \sigma)$$

$$= \int_0^1 dx \int_0^{1-x} dy \int_{(m'+m_{q'})^2}^{\infty} d\sigma \partial^2_{m'^2}\Delta B_0(s, m'^2, m^2_{q'})(D_0(...,\sigma) - \frac{\sigma_0}{\sigma}D_0(...,\sigma_0))$$

$$+ \int_0^1 dx \int_0^{1-x} dy \sigma_0 D_0(...,\sigma_0)\partial^2_{m'^2}B_0(0, m'^2, m^2_{q'})$$

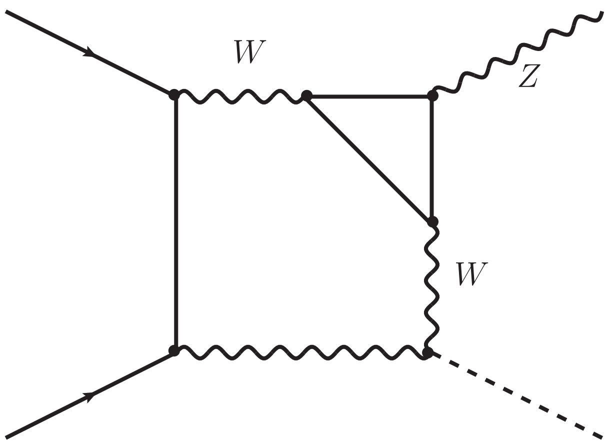
Tensor two-point functions

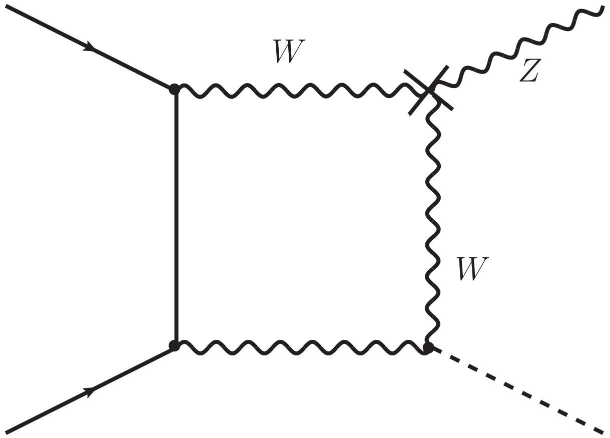
$$\int d^4q_2' \, \frac{q_2'^\mu}{[q_2'^2 - m'^2]^3[(q_2' - q_1 - k')^2 - m_{q'}^2]} = -\tilde{q}_1^\mu \, \partial_{m'}^2 B_1(\tilde{q}_1^2, m'^2, m_{q'}^2) \, ,$$

$$\int d^4q_2' \, \frac{q_2'^\mu q_2'^\nu}{[q_2'^2 - m'^2]^3[(q_2' - q_1 - k')^2 - m_{q'}^2]}$$

$$= g^{\mu\nu} \, \partial_{m'}^2 B_{00}(\tilde{q}_1^2, m'^2, m_{q'}^2) + \tilde{q}_1^\mu \tilde{q}_1^\nu \, \partial_{m'}^2 B_{11}(\tilde{q}_1^2, m'^2, m_{q'}^2) \, ,$$

Subtraction term: local divergence





div: (-2.808566*10^-6 + 1.176922*10^-6*I)/ϵ
finite: 2.871395*10^-5 - 1.562527*10^-5*I

div: (2.808566*10^-6 - 1.176922*10^-6*I)/ϵ
finite: (-2.317469*10^-5 + 1.049388*10^-5*I)
-(1.288853*10^-4 - 5.102569*10^-5*)*deltaAlpha

Sum of loop and CT is UV finite