TO APPEAR THIS SUMMER

3 Loop Anomalous Dimension & The EFT For QED Corrections To Beta Decay

RYAN PLESTID | WALTER BURKE INSTITUTE, CALTECH

COLLABORATORS | R. HILL, (U KENTUCKY/FNAL)

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PART 1

MOTIVATION

- Who cares about beta decays?
- Why this is an interesting theory problem.
- Requirements and challenges for theory.

PART 2

EFT SETUP

- Hierarchy of scales + heavy particle EFT.
- Gauge invariant subclasses & bkg field model.

PART 3

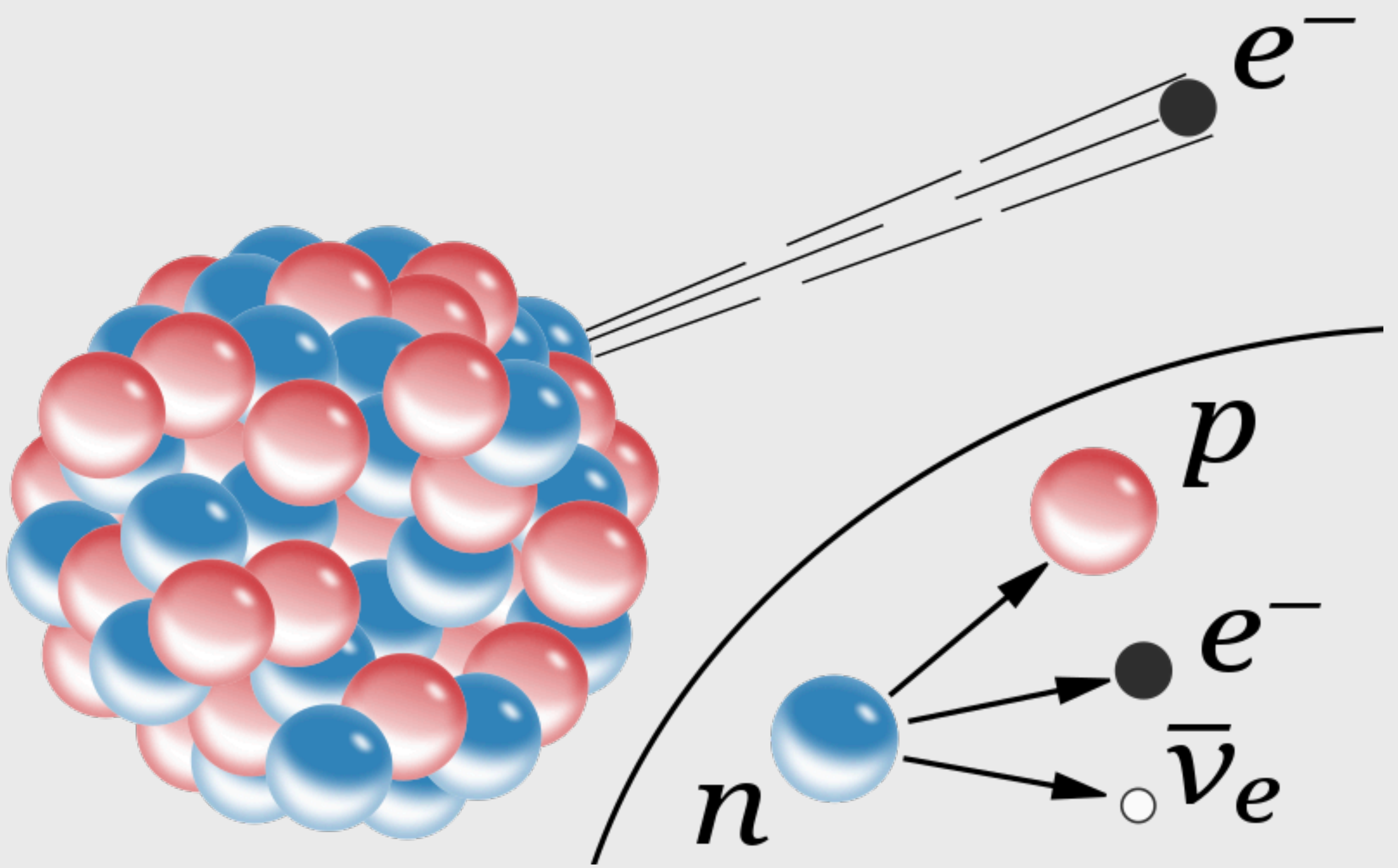
RESULTS

- All orders/exact resummation of leading $Z\alpha$ series.
- Control over "π^2-enhanced" contributions to amplitude.
- New master integrals & computation of anomalous dim.

Beta Decays & Particle Physics

- Most precise extraction of V_{ud} comes from super allowed beta decays.

$$\delta V_{\rm ud} \sim 10^{-4}$$



- Unresolved problem/anomaly with first row CKM unitarity.
- Important input to SMEFT fits.

CKM Unitarity

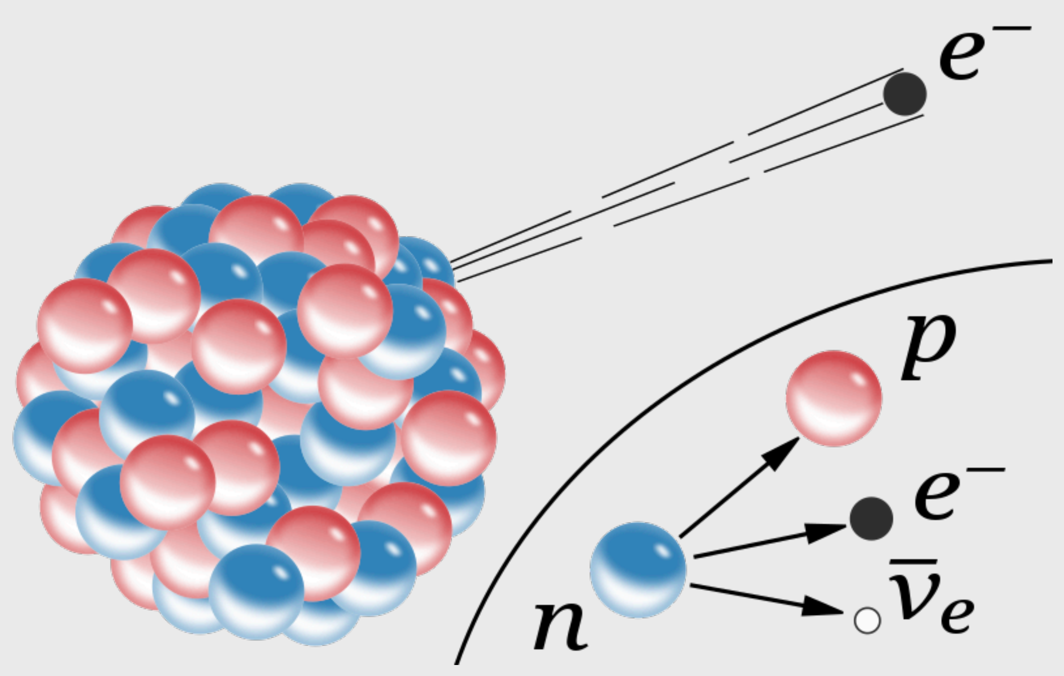
FIRST ROW UNITARITY

$$|V_{ud}|^2 + |V_{uc}|^2 + |V_{ub}|^2 = 1$$

IN WOLFENSTEIN NOTATION

$$1 - \lambda_{ud}^2 + \lambda_{uc}^2 + O(\lambda^6) = 1$$

$|V_{ud}|^2$



$|V_{uc}|^2$

$$\frac{\Gamma(K \to \mu\nu(\gamma))}{\Gamma(\pi \to \mu\nu(\gamma))}$$

- Percent-level accuracy in Kaon decay demands 100 ppm accuracy in $0^+ \to 0^+$ beta decays

CKM Unitarity

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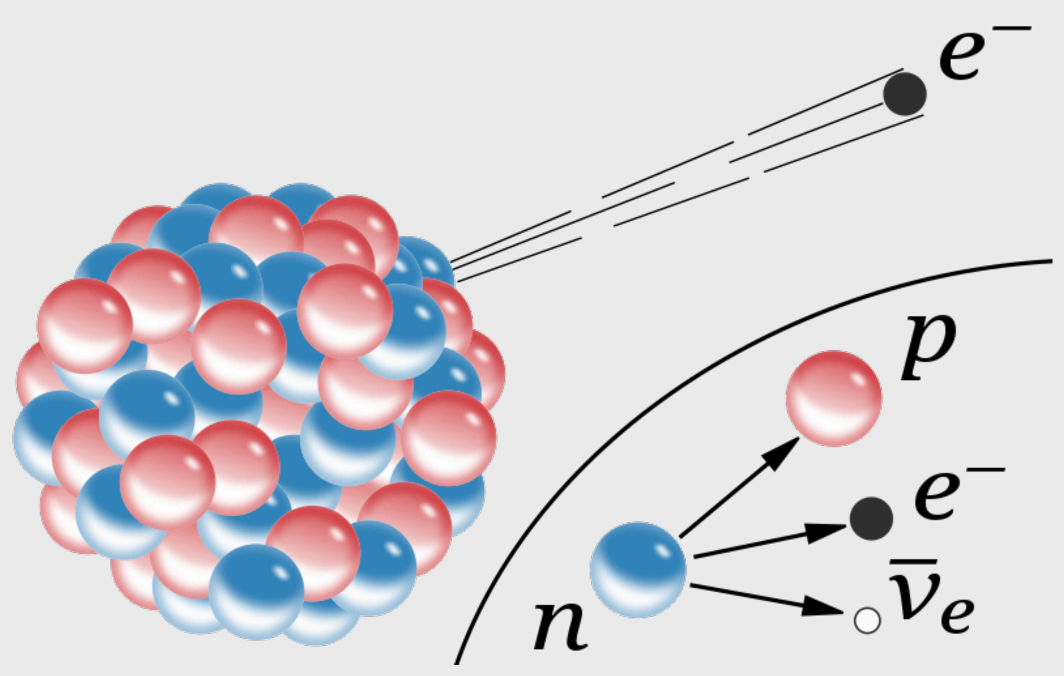
$$|V_{ud}|^2$$

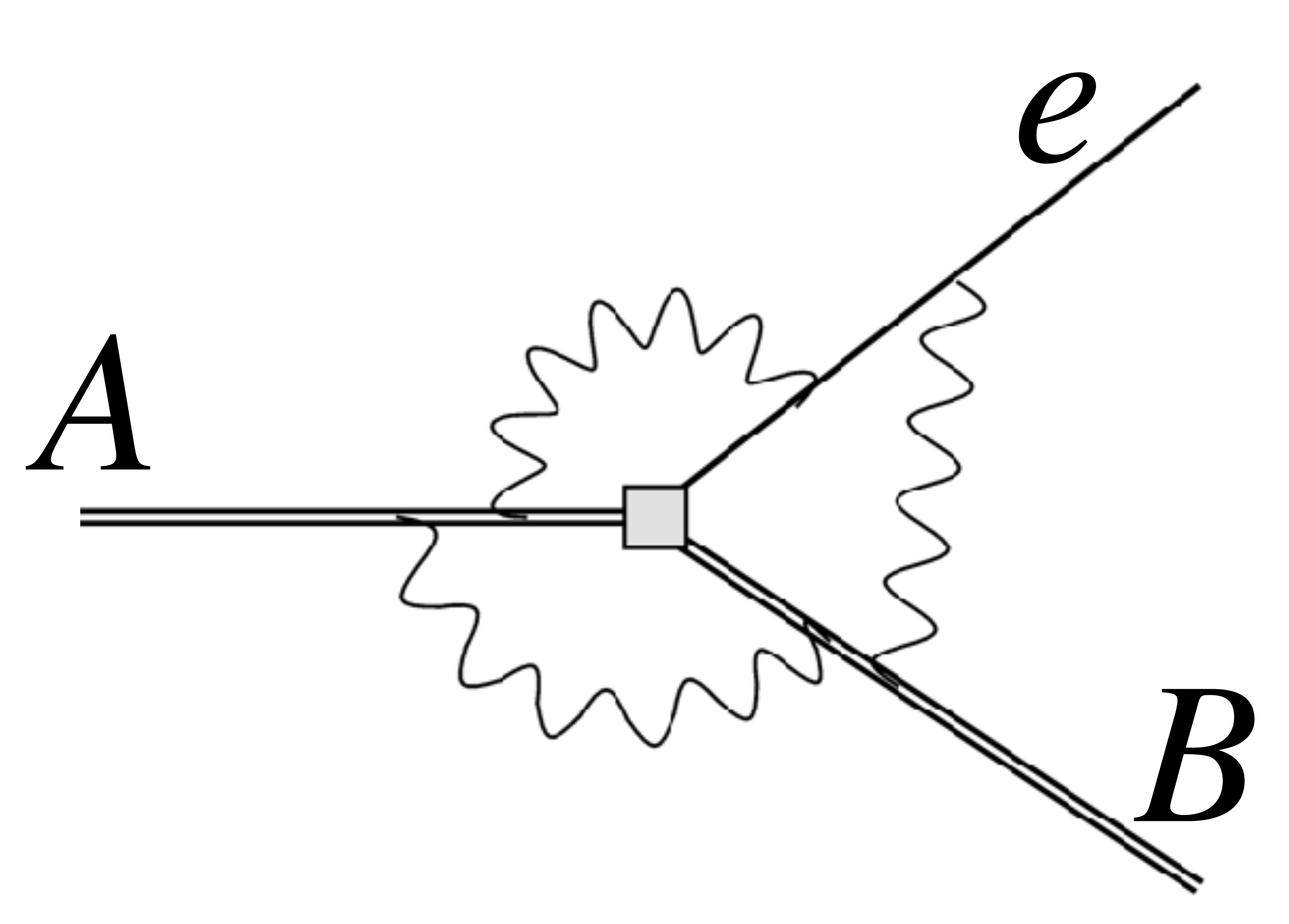


$$|V_{uc}|^2 \quad \frac{\Gamma(K \to \mu\nu(\gamma))}{\Gamma(\pi \to \mu\nu(\gamma))}$$

2-3σ Tension

- C.-Y. Seng, M. Gorchtein, H. H. Patel, and M. J. Ramsey-Musolf (2018)
- A. Czarnecki, W. J. Marciano, and A. Sirlin (2019)

A Multiscale Problem



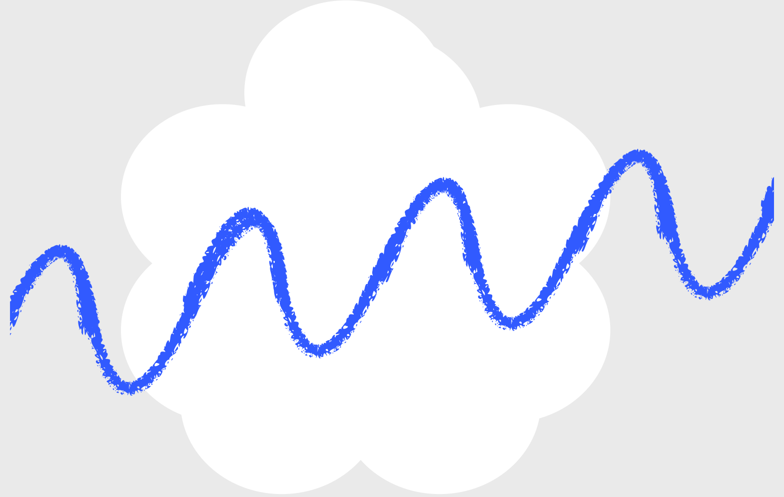
$$\int \frac{d^4L}{(2\pi)^4}$$

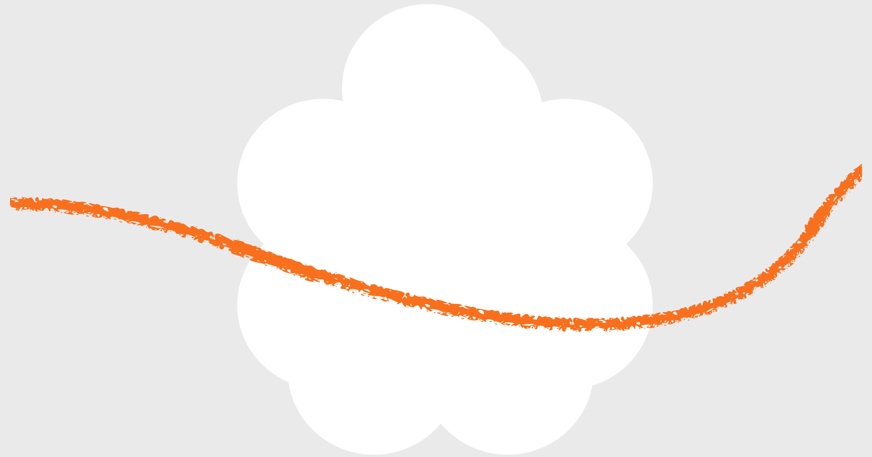
- Loops run over many scales
- Naturally suited to EFT methods.

W-mass	80 GeV	α
QCD-scale	1 GeV	
Fermi-motion	200 MeV	
Nuclear Radius	50 MeV	$Z\alpha$
Lepton Energy	5 MeV	

Effective Field Theory

- Separate scales in loops.
- EFTs ↔ Regions.





$$\int_{\text{short}} [\mathrm{d}L] + \int_{\text{long}} [\mathrm{d}L]$$

W-mass	80 GeV	$q, \bar{q}, g$
QCD-scale	1 GeV	N
Fermi-motion	200 MeV	N
Nuclear Radius	50 MeV	$^{14}\mathrm{O}$
Lepton Energy	5 MeV	$^{14}\mathrm{O}$

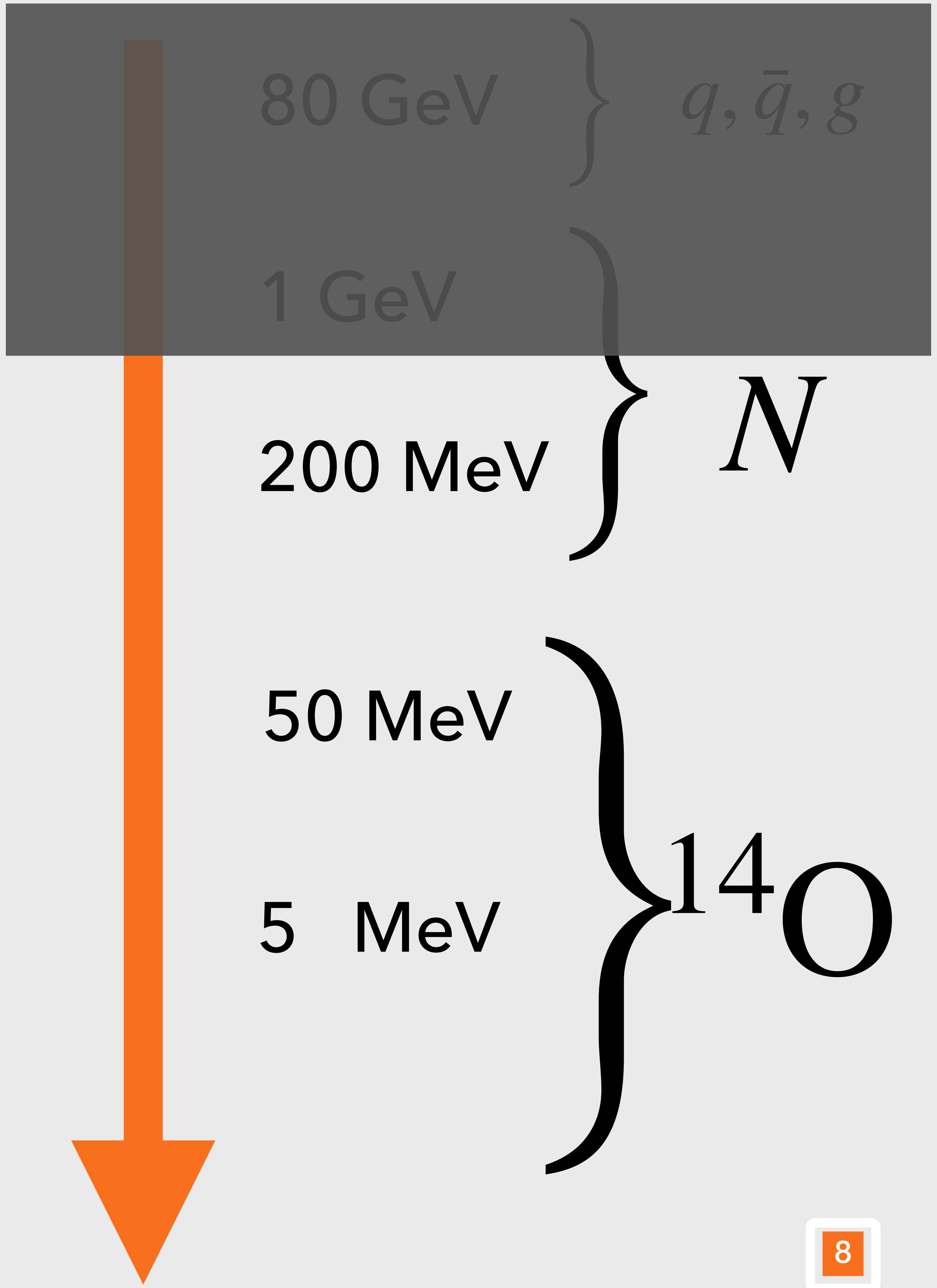
Factorization Of Amplitude

- $0^+ \rightarrow 0^+$ decays fixed by symmetry with isospin + CVC (up to QED corr.)
- Short-distances calculable with pQCD and nucleon-level EFT.

SHORT DISTANCE WILSON COEFF.

$$\mathscr{M} = C(\mu)\mathscr{M}(\mu, p_F, R_A, p, \lambda)$$

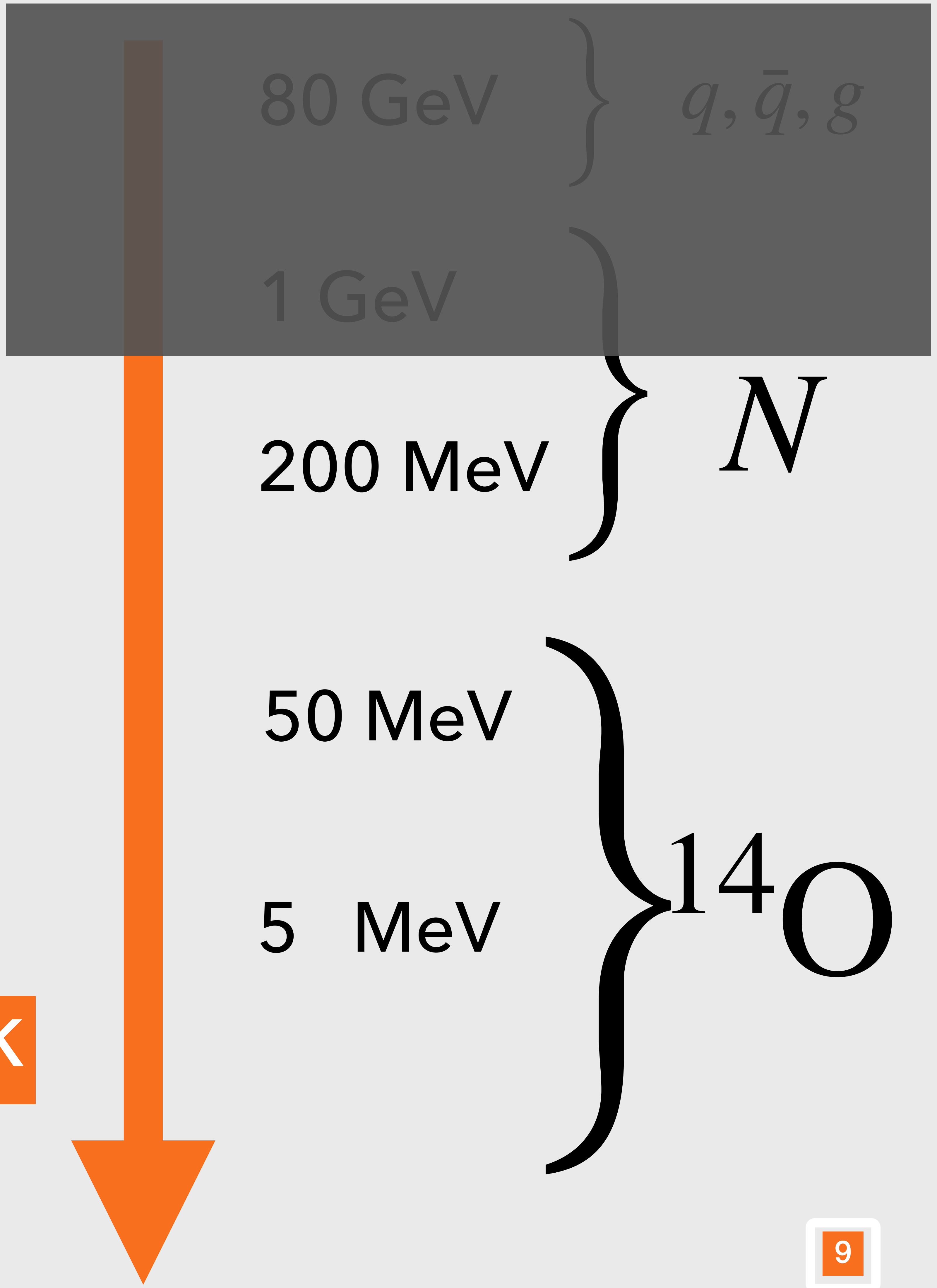
LONG DISTANCE MATRIX ELEMENT



Further Factorization At Low Energies

$$\mathscr{M}(\mu, p_F, R_A, p, \lambda)$$

- $O(\alpha)$ structure dependent $\mu \sim p_F$
- $O(Z\alpha)$ structure dependent $\mu \sim 1/R_A$
- $O(Z\alpha)$ universal/Coulomb $\mu \sim p$ THIS TALK
- $O(Z\alpha)$ IR/atomic effects $\mu \sim \lambda$



Leading Power Structure Of Amplitude

$$\mathscr{M}(\mu, p_F, R_A, p, \lambda)$$

$$C(\mu_1)\mathscr{M}_1(\mu_1, \mu_2 p_F)\mathscr{M}_2(\mu_2, \mu_3, R_A)\mathscr{M}_3(\mu_3, \mu_4, p)\mathscr{M}_4(\mu_4, \lambda)$$

- $C(\mu_1)$: CRIGLIAGNO ET. AL.
- $\mathscr{M}_1$: SENG ET. AL. TRIUMF GROUP (AB INITIO)
- $\mathscr{M}_2$: STRUCTURE DEPENDENCE
- $\mathscr{M}_3$: THIS TALK
- $\mathscr{M}_4$: UNIVERSAL SOFT FUNCTION

80 GeV } $q, \bar{q}, g$

1 GeV

200 MeV } N

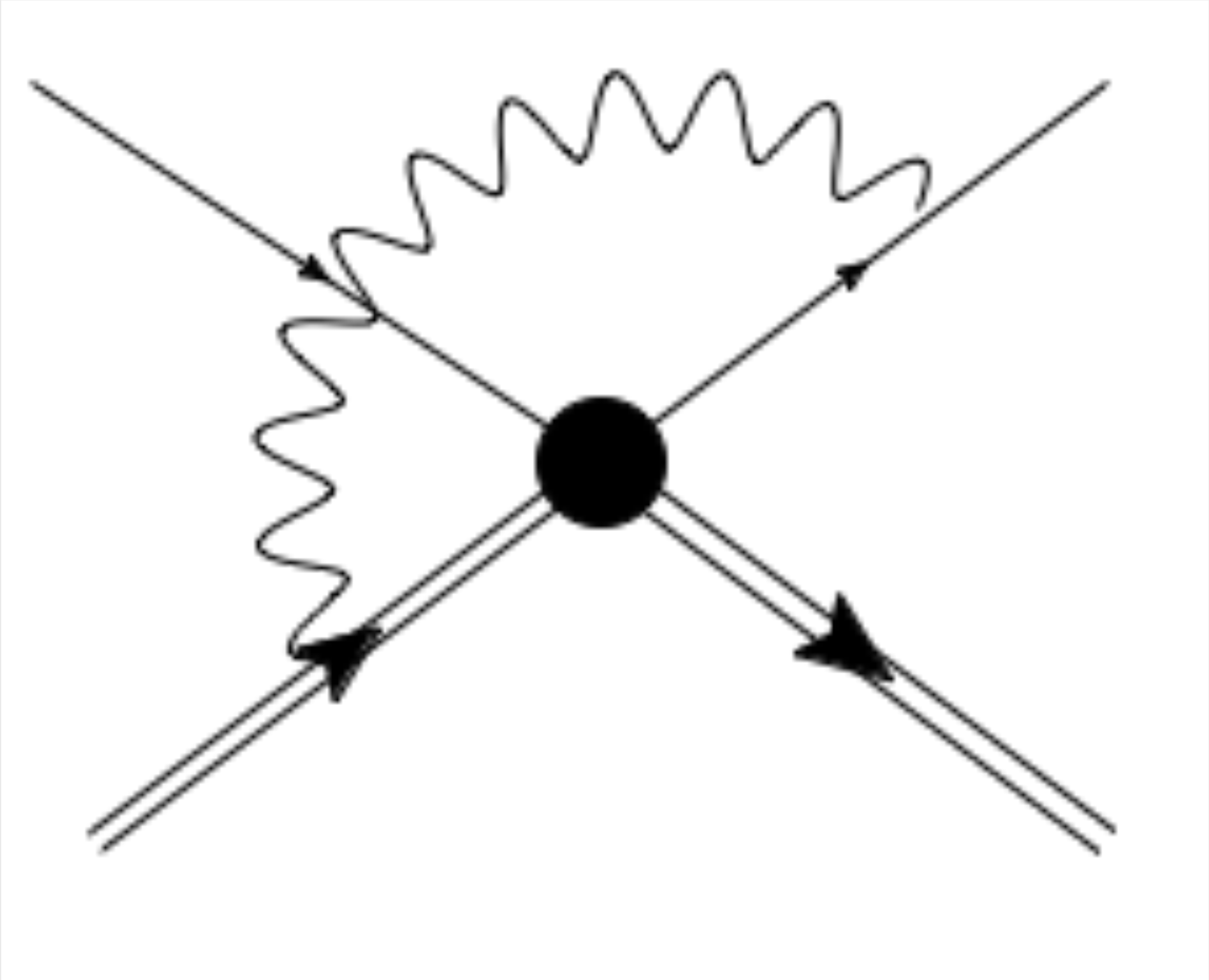
50 MeV

5 MeV } ^{14}O

- Improved theory input is active!

Coherent Enhancements

- At long wavelengths the nucleus can couple ***coherently*** to photons.
- Radiative corrections are enhanced by the the ***charge of the nucleus.***
- These corrections arise from ***long distance*** regions (low energy EFT).



$$\lambda \gtrsim R$$

$$\alpha \to Z\alpha \quad \text{for} \quad \mathbf{L} \ll 1/R$$

LOOP MOMENTUM.

π^2 Enhancements

- In non-relativistic theory Coulomb effects can be captured using Schrodinger wavefunction.

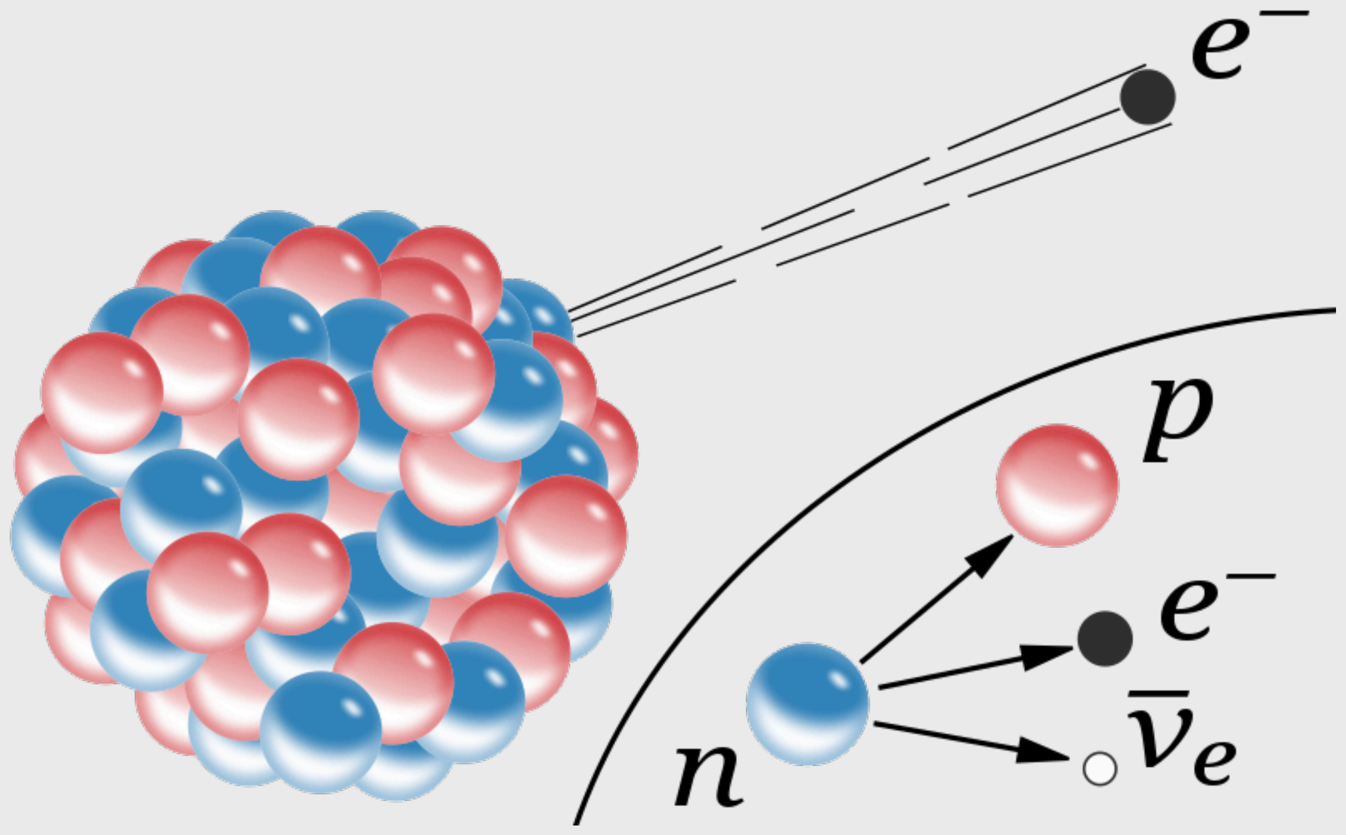
$$F_{\rm NR}(Z,E) = |\psi(0)|^2 = 1 \;+\; \pi\frac{Z\alpha}{v} \;+\; \frac{\pi^2}{3}\frac{Z^2\alpha^2}{v^2} \;+\; \ldots$$

- Violates naive counting in α/π by systematic factors of π^2. Comes from IR logs & reduced dimensionality.

- How do we systematically account for large π^2 terms?

Working To High Order

$Z \sim 10$ $\log(2p/m_e) \sim \log(2pR) \sim 5$



$\delta V_{\text{ud}} \sim 10^{-4}$

DEFINE POWER COUNTING

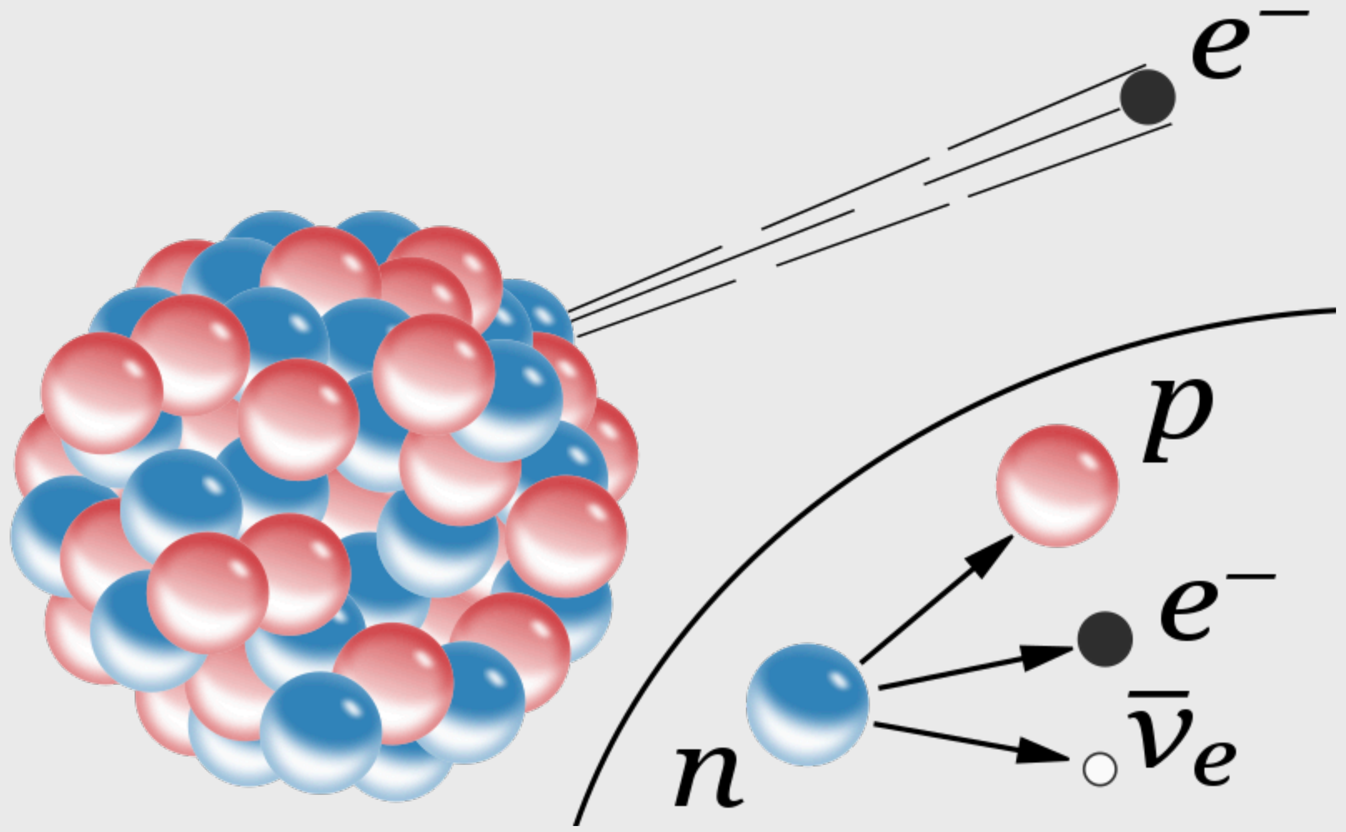
$$Z \sim \mathrm{L} \sim 1/\sqrt{\alpha}$$

- Let us aim for $O(\alpha^{3/2})$ precision
- Conservative: ignores $(1/\pi)^n$.

- 2-loops
  $\alpha^2\mathrm{L}$, $(Z\alpha)\alpha$
- 3-loops
  $\alpha^3\mathrm{L}^3$, $(Z\alpha)^2\alpha\mathrm{L}$, $(Z\alpha)\alpha^2\mathrm{L}^2$, $(Z\alpha)^3$
- 4-loops
  $(Z\alpha)^3\alpha\mathrm{L}^2$, $(Z\alpha)^4\mathrm{L}$
- 5-loops
  $(Z\alpha)^5\,\mathrm{L}^2$
- 6-loops
  $(Z\alpha)^6\,\mathrm{L}^3$

Working To High Order

$Z \sim 10$ $\log(2p/m_e) \sim \log(2pR) \sim 5$



$\delta V_{\rm ud} \sim 10^{-4}$

DEFINE POWER COUNTING

$$Z \sim \mathrm{L} \sim 1/\sqrt{\alpha}$$

QED WITH QCD LIKE DEMANDS OF PERTURBATION THEORY

- 2-loops

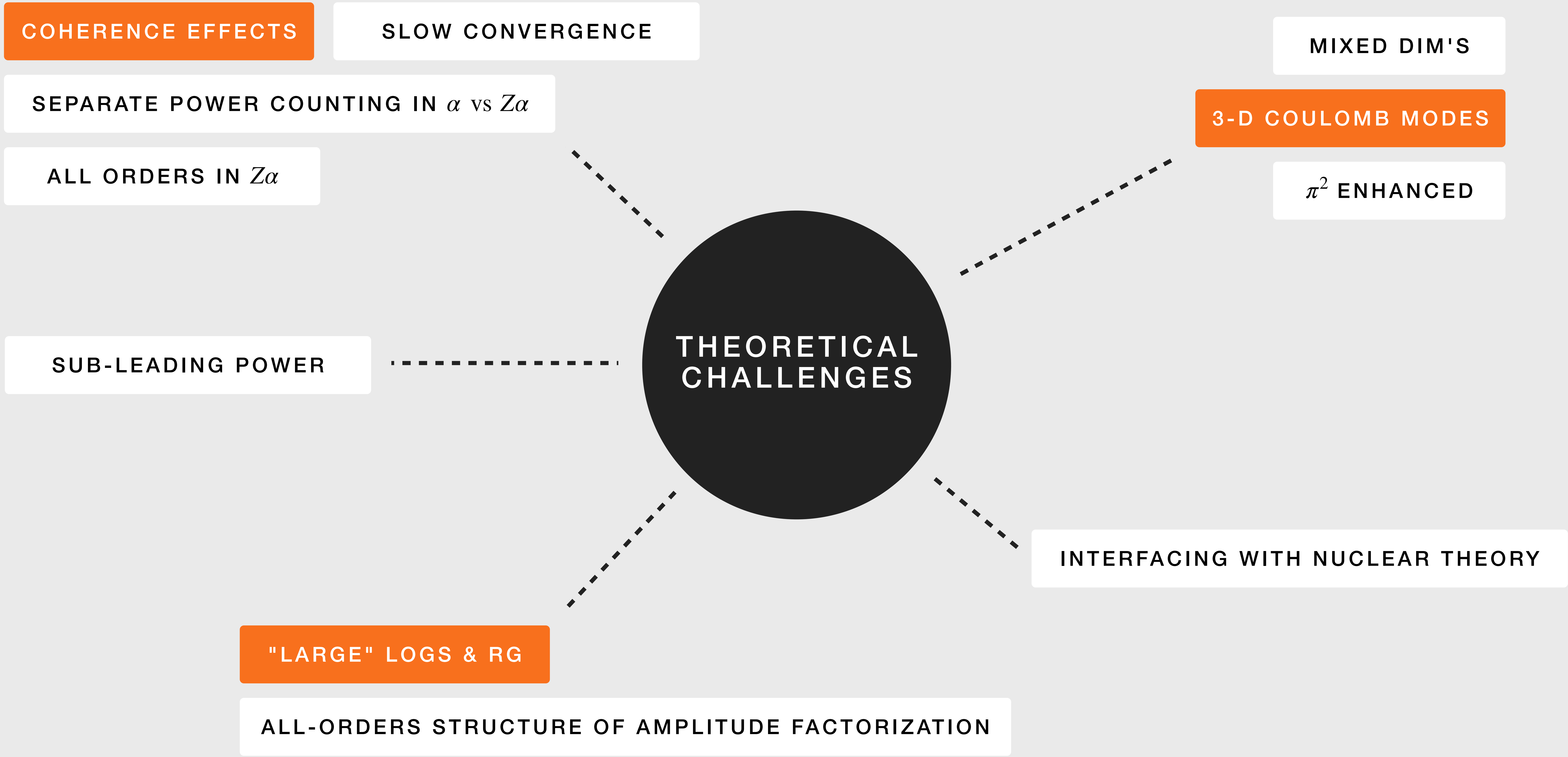
  $\alpha^2 \mathrm{L}$, $(Z\alpha)\alpha$
- 3-loops

  $\alpha^3 \mathrm{L}^3$, $(Z\alpha)^2 \alpha \mathrm{L}$, $(Z\alpha)\alpha^2 \mathrm{L}^2$, $(Z\alpha)^3$
- 4-loops

  $(Z\alpha)^3 \alpha \mathrm{L}^2$, $(Z\alpha)^4 \mathrm{L}$
- 5-loops

  $(Z\alpha)^5\, \mathrm{L}^2$
- 6-loops

  $(Z\alpha)^6\, \mathrm{L}^3$


COHERENCE EFFECTS
SLOW CONVERGENCE
SEPARATE POWER COUNTING IN α vs $Z\alpha$
ALL ORDERS IN $Z\alpha$
SUB-LEADING POWER
THEORETICAL CHALLENGES
MIXED DIM'S
3-D COULOMB MODES
π^2 ENHANCED
INTERFACING WITH NUCLEAR THEORY
"LARGE" LOGS & RG
ALL-ORDERS STRUCTURE OF AMPLITUDE FACTORIZATION

Effective Theory For Outer Corrections

Definition Of The Effective Theory

STATIC LIMIT $v_A = v_B$

$$\mathscr{L} = h_A^\dagger(iv \cdot \partial - Z_A v \cdot A)h_A \quad + \quad h_B^\dagger(iv \cdot \partial - Z_B v \cdot A)h_B$$

HQET FEYNMAN RULES

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad + \quad \bar{\psi}_e(i\gamma_\mu D^\mu + m_e)\psi_e$$

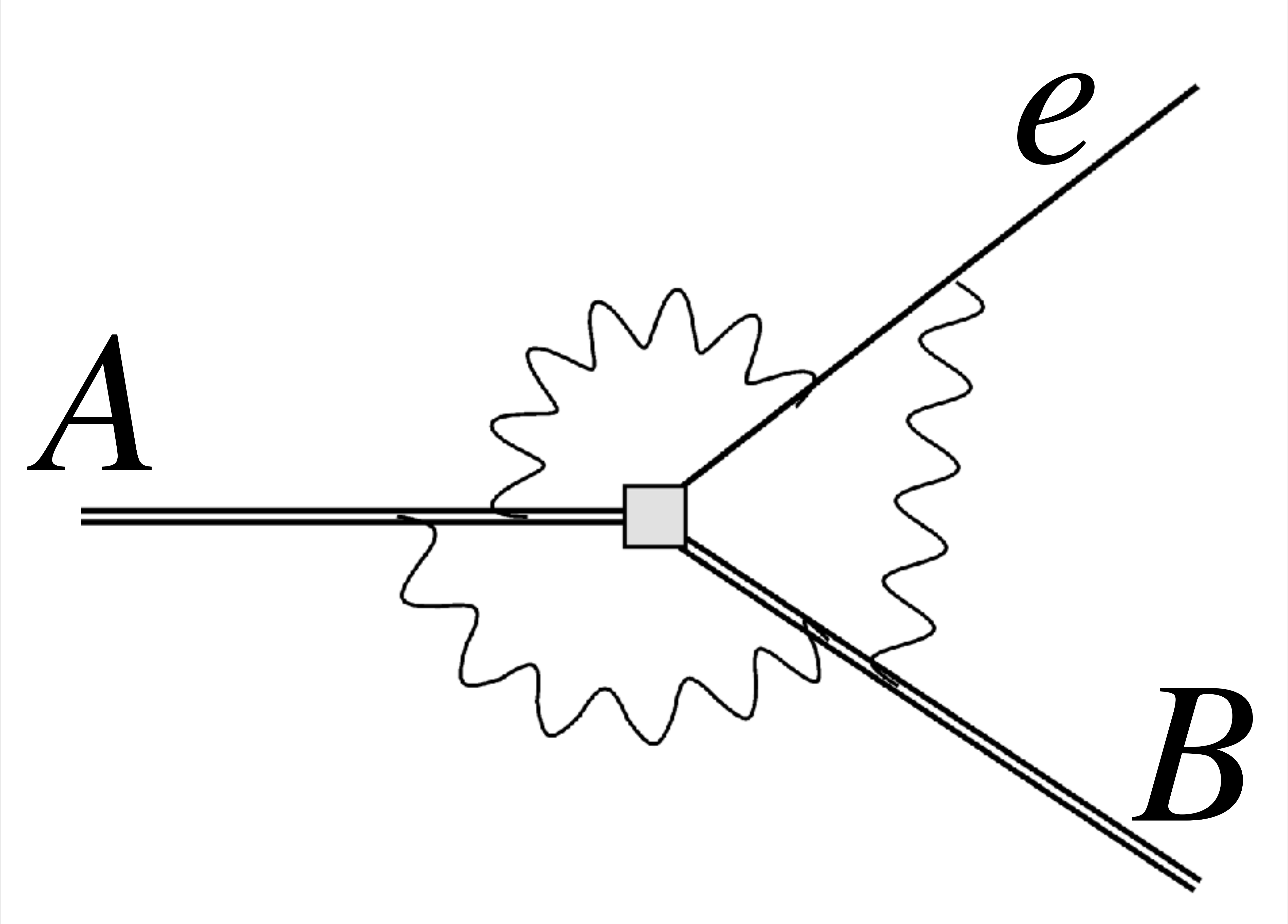
STANDARD QED

$$\mathcal{O}_{\text{ext}} = -\sqrt{2}G_F\, h_B^\dagger h_A\, \bar{\psi}_e v_\mu \gamma^\mu \psi_e$$

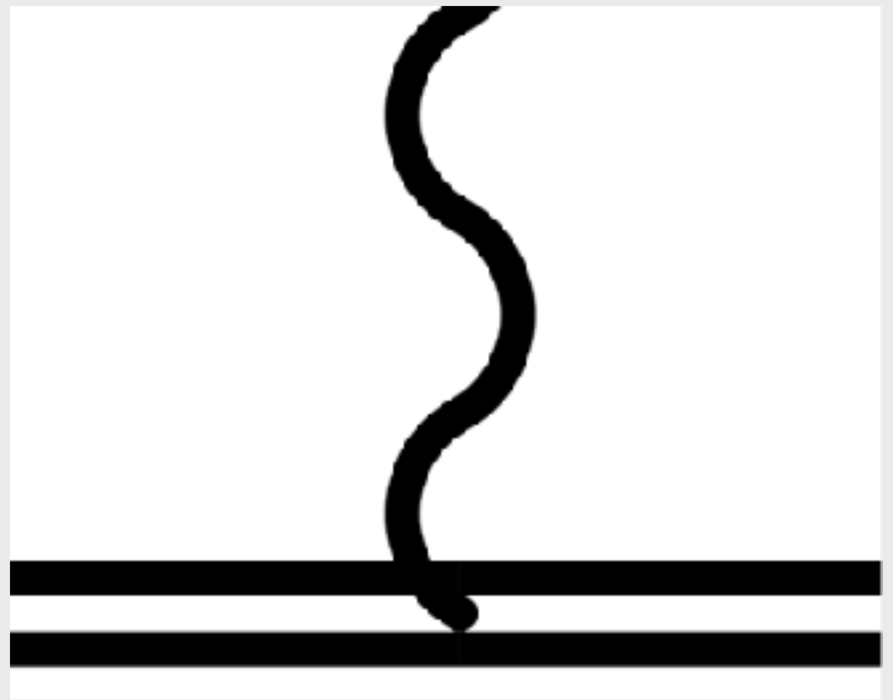
WEAK CHARGED CURRENT

- Largest QED corrections arise from the long-distance dynamics

Example Diagram At 3 Loops



$$= \frac{\mathrm{i}}{v \cdot q + \mathrm{i}\varepsilon}$$



$$= \mathrm{i}(Ze)v_{\mu}$$

Strategy For Amplitude

$$\mathscr{M} = C(\mu) \quad \mathscr{M}_{\text{EFT}}(p,\mu) \quad S(\mu,\lambda)$$

Wilson Coefficient

Single scale matrix element

Soft function

Strategy For Amplitude

$\mu \sim 2p \rightarrow$ minimize logs

$$\mathscr{M} = C(\mu_H) \times \frac{C(\mu)}{C(\mu_H)} \; \mathscr{M}_{\mathrm{EFT}}(p, \mu) \; S(\mu, \lambda)$$

Input from UV matching

Re-sum logs with RG

All orders in PT

$\gamma_{\mathcal{O}}$ Anomalous dimension

Gauge Invariant Subclasses

Mapping To Background Field

- Can use **new eikonal identities** to identify gauge invariant sub-classes of diagrams. Class-I & Class-II

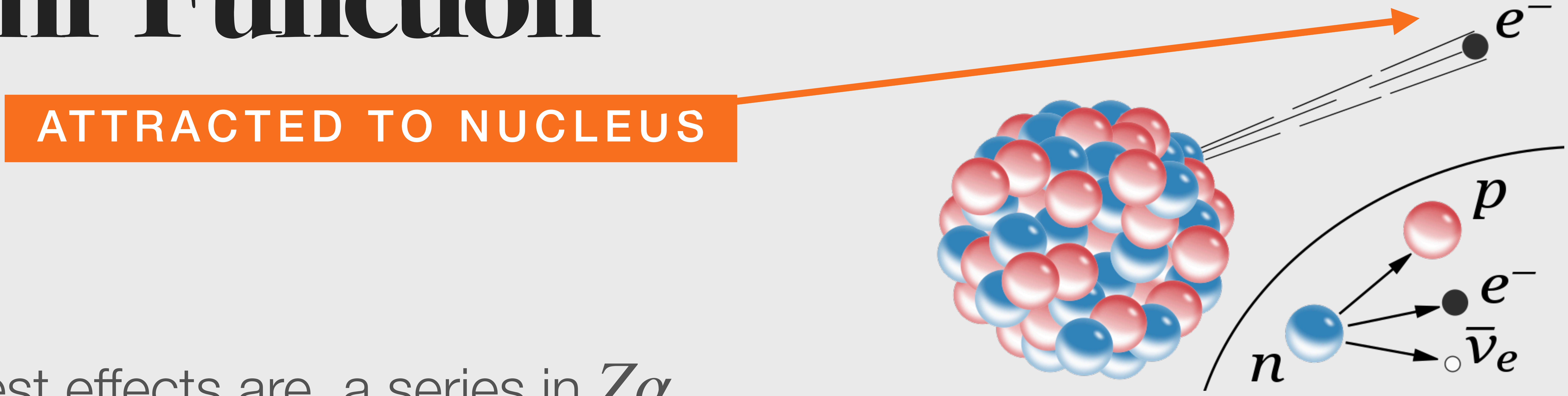
$$\mathscr{L}_{\text{bkg}} \subset h_A^\dagger \mathrm{i} v \cdot \partial h_A + h_B^\dagger(\mathrm{i} v \cdot \partial - e v \cdot A) h_B + \bar{e}\gamma_0 \mathscr{A}(x) e$$

- Class-I reproduces $\mathscr{L}_{\text{bkg}}$ order-by-order ($\therefore$ gauge inv .)

- Class-II vanishes diagram-by-diagram in Coulomb gauge. ($\therefore$ vanishes in all gauges)

Resummation Of Leading-Z

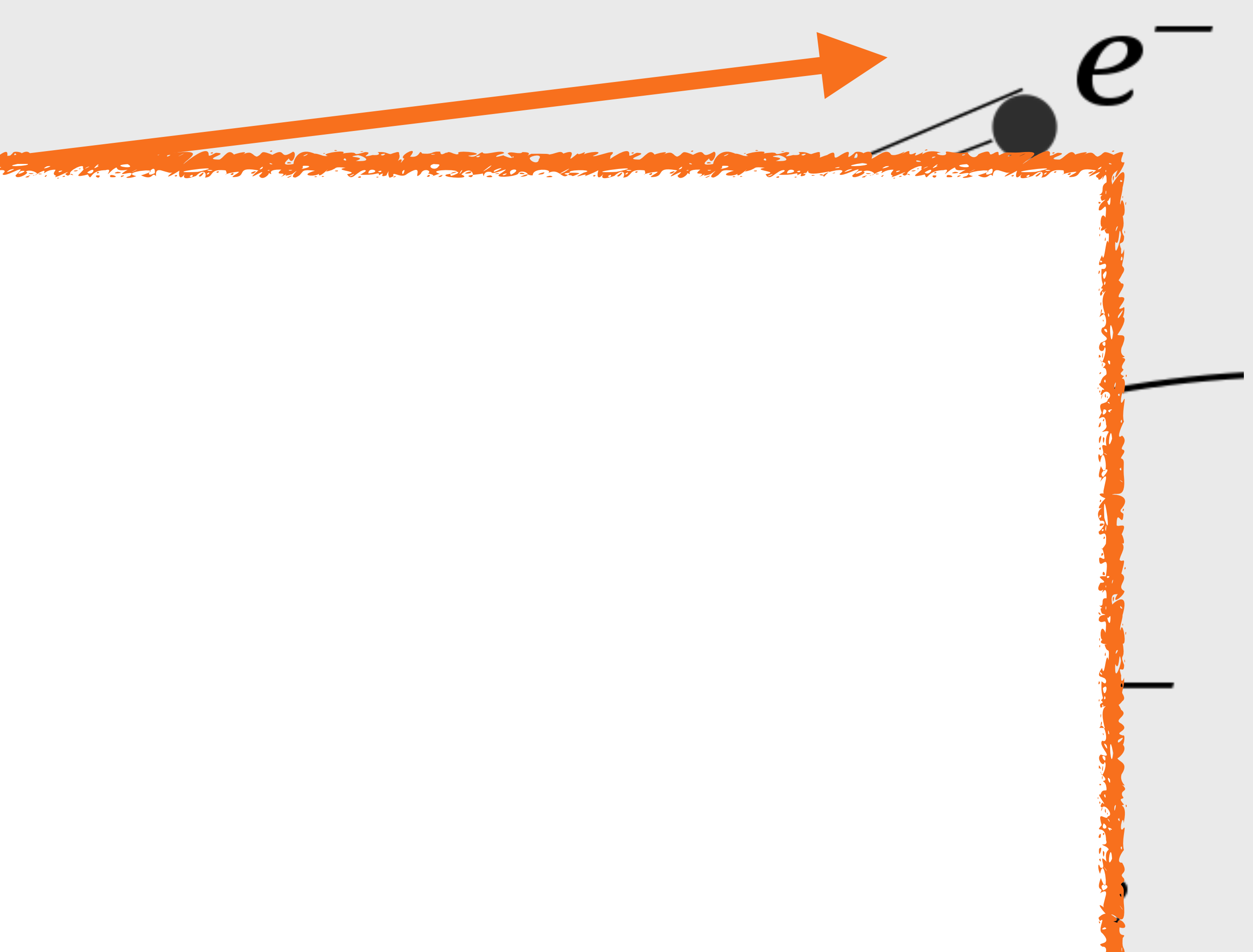
Fermi Function



- Largest effects are a series in $Z\alpha$
- Historically done with finite-distance regulator

$$\langle e^- | \bar{\psi}(\mathbf{x}) | 0 \rangle \sim \left(\frac{1}{|\mathbf{x}|} \right)^{\nu} \qquad \nu = \sqrt{1 - Z^2\alpha^2} - 1$$

Fermi Function



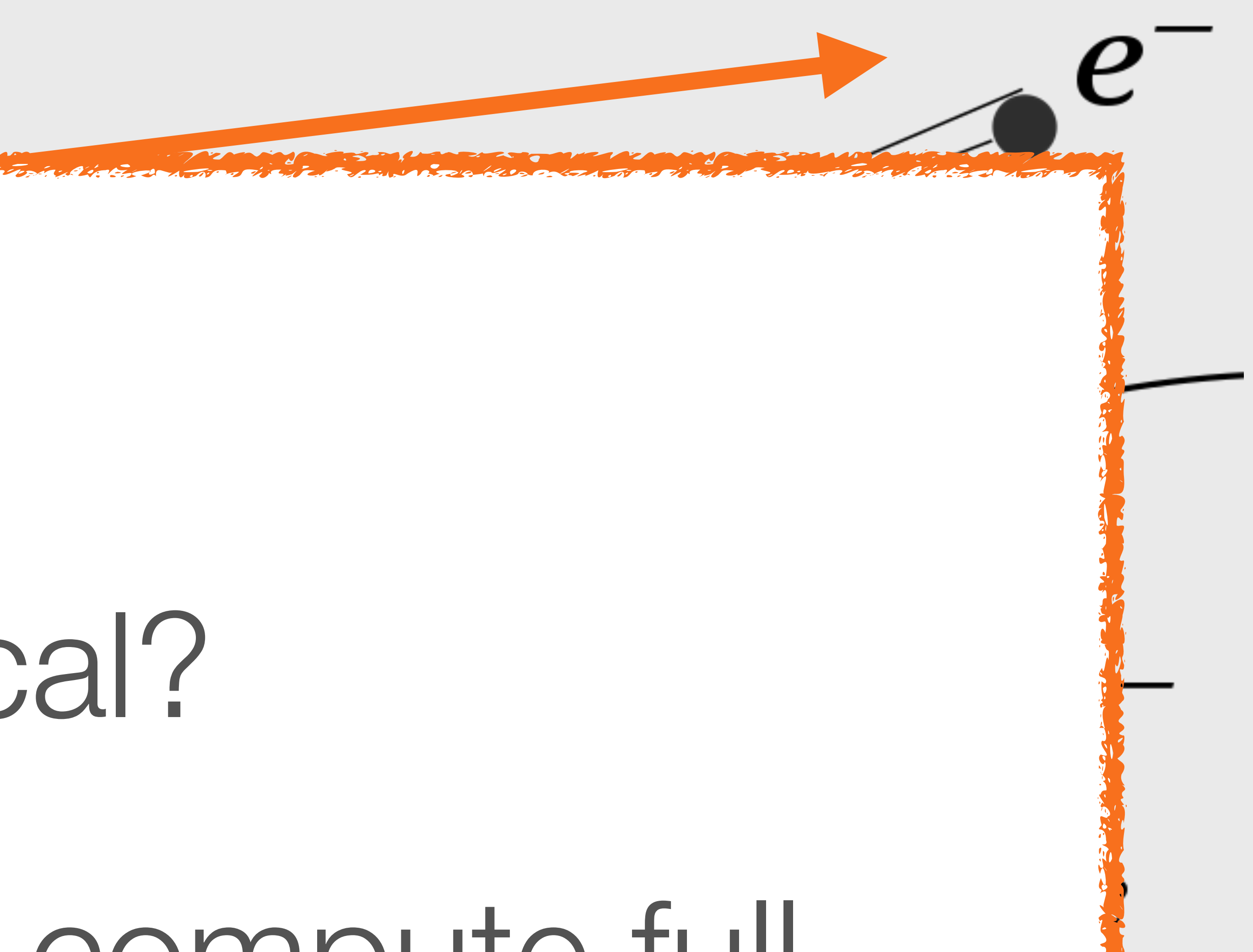
Pros

- Manifests universal point-charge limit.
- Resums series in $Z\alpha$.

- La
- Hi

$$\langle e^- | \bar{\psi}(\mathbf{x}) | 0 \rangle \sim \left(\frac{1}{|\mathbf{x}|} \right)^{\nu} \qquad \nu = \sqrt{1 - Z^2\alpha^2} - 1$$

Fermi Function



Cons

- Regulator dependent. Is this even physical?
- No clear way to interface with UV except compute full answer and then divide. Convention dependent etc.

- La
- Hi

$$\langle e^- | \bar{\psi}(\mathbf{x}) | 0 \rangle \sim \left(\frac{1}{|\mathbf{x}|} \right)^{\nu} \qquad \nu = \sqrt{1 - Z^2\alpha^2} - 1$$

Direct Computation At 2-Loops

$$\mathcal{M}_H(\mu_S,\mu_H) = 1 + \frac{Z\alpha}{\beta}\left[i\left(\log\frac{2p}{\mu_S} - \frac{i\pi}{2}\right) + \frac{i}{2}\left(\frac{m}{E}\gamma^0 - 1\right)\right] + \left(\frac{Z\alpha}{\beta}\right)^2\left\{\frac{-\pi^2}{12} - \frac{1}{2}\left(\log\frac{2p}{\mu_S} - \frac{i\pi}{2}\right)^2\right.$$
$$\left. - \frac{1}{2}\left(\log\frac{2p}{\mu_S} - \frac{i\pi}{2}\right)\left(\frac{m}{E}\gamma^0 - 1\right) + \left[\frac{5}{4} - \frac{1}{2}\left(\log\frac{2p}{\mu_H} - \frac{i\pi}{2}\right)\right]\beta^2\right\} + \mathcal{O}(\alpha^3),$$

- Complicated interplay of IR & UV logs, non-trivial Dirac structure etc.
- No obvious pattern. Resummation unlikely by brute force.

Wavefunctions And Feynman Diagrams

- Wavefunctions admit a loop-expansion (Lippmann-Schwinger Eq).

$$|\psi_p^{(\pm)}\rangle = |\phi_p\rangle + \frac{1}{H - E_p \pm i\varepsilon} V |\phi_p\rangle + \frac{1}{H - E_p \pm i\varepsilon} V \frac{1}{H - E_p \pm i\varepsilon} V |\phi_p\rangle + \ldots$$

Loop With A Phase Factor!

$$\langle x | \psi_p^{(\pm)}\rangle = e^{i\mathbf{p}\cdot\mathbf{x}} \left(1 + \int \frac{d^3Q}{(2\pi)^3} \frac{1}{2\mathbf{P}\cdot\mathbf{Q} + \mathbf{Q}^2 \pm i\varepsilon} \frac{Z\alpha}{\mathbf{Q}^2} e^{i\mathbf{Q}\cdot\mathbf{x}} + \ldots \right)$$

Factorization Of Dirac Wavefunction

- Wavefunctions satisifies same factorization theorem as amplitude

$$\mathscr{M} = \mathscr{M}_S(\mu_S)\mathscr{M}_H(\mu_S, \mu_H)\mathscr{M}_{\mathrm{UV}}(\mu_H, \Lambda)$$

SAME

$$\Psi(\mathbf{x}) = \mathscr{M}_S(\mu_S)\mathscr{M}_H(\mu_S, \mu_H)\tilde{\mathscr{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

Short Distance Matrix Element

$$\tilde{\mathscr{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

- Finite distance acts as regulator.
- Compute at threshold ($\mathbf{p} = 0$)
- Iteratively one-loop at all orders in PT. Re-sum!

Factorization Of Dirac Wavefunction

$$\tilde{\mathscr{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

CLOSED FORM INTEGRALS AT ARBITRARILY HIGH ORDER

$$\mathcal{I}_1^{(n)} = \left[\prod_{j=1}^{n-1} C(\nu_j)\right] \times \frac{\Gamma(d-\nu_n-1)}{(4\pi)^d\Gamma(\nu_n)} B(\tfrac{d}{2}-1, 1+\tfrac{d}{2}-\nu_n)\left(\frac{\mathbf{x}^2}{4}\right)^{\nu_n+1-d},$$

$$\mathcal{I}_2^{(n)} = \left[\prod_{j=1}^{n} C(\nu_j)\right]\left[\frac{2\Gamma(\frac{d}{2}-\nu_{n+1}+1)}{(4\pi)^{d/2}\Gamma(\nu_{n+1})}\right]\left[\frac{\mathbf{x}^2}{4}\right]^{\nu_{n+1}-(d+1)/2} \times \frac{i\gamma_0\boldsymbol{\gamma}\cdot\mathbf{x}}{2|\mathbf{x}|}.$$

Factorization Of Dirac Wavefunction

$$\tilde{\mathscr{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

BARE AMPLITUDE MAY BE SUMMED TO ALL ORDERS

$$F_1^{\text{bare}} = 2^{\frac{1}{4\epsilon}-\frac{1}{2}} \left(\frac{\sqrt{\tilde{g}}}{\epsilon}\right)^{1-\frac{1}{2\epsilon}} \Gamma\left(\frac{1}{2\epsilon}\right) J_{\frac{1}{2\epsilon}-1}\left(\frac{\sqrt{8}\sqrt{\tilde{g}}}{\epsilon}\right),$$

$$(Z\tilde{\alpha})^{-1} F_2^{\text{bare}} = 2^{\frac{1}{4\epsilon}} \left(\frac{\sqrt{\tilde{g}}}{\epsilon}\right)^{-\frac{1}{2\epsilon}} \Gamma\left(1+\frac{1}{2\epsilon}\right) J_{\frac{1}{2\epsilon}}\left(\frac{\sqrt{8}\sqrt{\tilde{g}}}{\epsilon}\right).$$

Factorization Of Dirac Wavefunction

$$\tilde{\mathcal{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

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$$(Z\tilde{\alpha})^{-1} F_2^{\text{bare}} = 2^{\frac{1}{4\epsilon}} \left(\frac{\sqrt{\tilde{g}}}{\epsilon}\right)^{-\frac{1}{2\epsilon}} \Gamma\left(1+\frac{1}{2\epsilon}\right) J_{\frac{1}{2\epsilon}}\left(\frac{\sqrt{8}\sqrt{\tilde{g}}}{\epsilon}\right).$$

Factorization Of Dirac Wavefunction

$$\tilde{\mathscr{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

RESULT CAN BE RENORMALIZED AT ALL-ORDERS IN $Z\alpha$

$$\mathcal{M}^{R}_{\mathrm{UV}}(\mu) = (\mu r e^{\gamma_{\mathrm{E}}})^{\eta-1} \frac{1+\eta}{2\sqrt{\eta}} \left[1 + \frac{Z\alpha}{1+\eta} \frac{i\gamma_0 \boldsymbol{\gamma} \cdot \mathbf{x}}{|\mathbf{x}|} \right] ,$$

$$\eta = \sqrt{1-(Z\alpha)^2}$$

Extraction Of Hard Matrix Element

$$\Psi(\mathbf{x}) = \mathscr{M}_S(\mu_S)\mathscr{M}_H(\mu_S, \mu_H)\tilde{\mathscr{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

KNOWN TO ALL ORDERS

All-Orders Hard Matrix Element

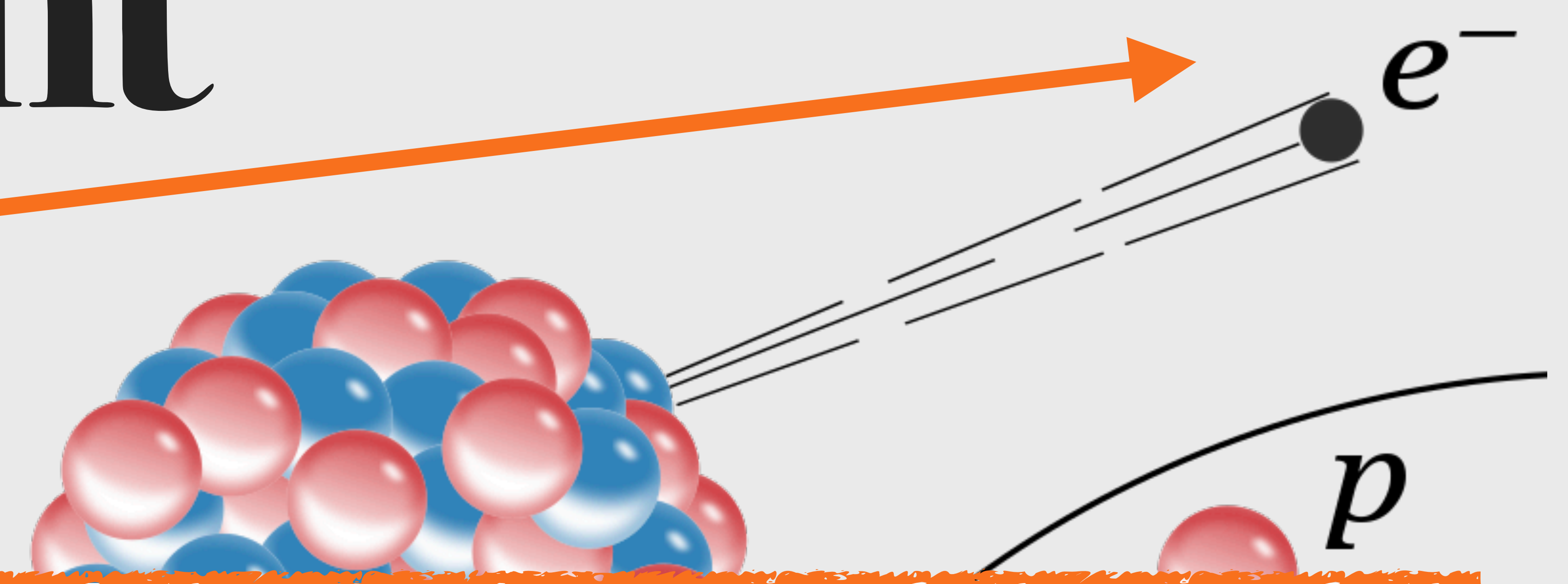
$$\mathcal{M}_H(\mu_S,\mu_H) = \mathrm{e}^{\frac{\pi\xi}{2}+i\xi\left(\log\frac{2p}{\mu_S}-\gamma_\mathrm{E}\right)-i(\eta-1)\frac{\pi}{2}}\frac{2\Gamma(\eta-i\xi)}{\Gamma(2\eta+1)}$$

$$\sqrt{\frac{\eta-i\xi}{1-i\xi\frac{m}{E}}}\sqrt{\frac{E+\eta m}{E+m}}\sqrt{\frac{2\eta}{1+\eta}}\left(\frac{2p\mathrm{e}^{-\gamma_\mathrm{E}}}{\mu_H}\right)^{\eta-1}\times\left[\frac{1+M^*}{2}+\frac{1-M^*}{2}\gamma^0\right]$$

- $\eta = \sqrt{1 - Z^2\alpha^2}$
- $\xi = Z\alpha/\beta$
- $M = (E+m)(1+\mathrm{i}\xi m/E)/(E+\eta m)$

Coulomb Enhancement

ATTRACTED TO NUCLEUS



- Well defined EFT matrix element. Can be evolved with RG to re-sum logs.

NEW RESULT IN EFT

ALL ORDERS IN $Z\alpha$

$$\mathcal{M}_H(\mu_S,\mu_H) = e^{\frac{\pi\xi}{2}+i\xi\left(\log\frac{2p}{\mu_S}-\gamma_E\right)-i(\eta-1)\frac{\pi}{2}} \frac{2\Gamma(\eta-i\xi)}{\Gamma(2\eta+1)}\sqrt{\frac{\eta-i\xi}{1-i\xi\frac{m}{E}}}\sqrt{\frac{E+\eta m}{E+m}}\sqrt{\frac{2\eta}{1+\eta}}\left(\frac{2pe^{-\gamma_E}}{\mu_H}\right)^{\eta-1}\times\left[\frac{1+M^*}{2}+\frac{1-M^*}{2}\gamma^0\right]$$

- $\eta = \sqrt{1-Z^2\alpha^2}$
- $\xi = Z\alpha/\beta$
- $M = (E+m)(1+i\xi m/E)/(E+\eta m)$

Large π^2 Factors & IR Logs

Origin Of π^2 Factors

$$\int [d^4q]\ (2\pi i)\delta(q_0) \qquad \frac{1}{2\mathbf{p}\cdot\mathbf{Q}+\mathbf{Q}^2} \qquad \frac{1}{\mathbf{Q}^2}$$

COULOMB "PINCH"

IR-LOGS

$$iZ\alpha\log(-2p/\mu) = iZ\alpha\log(2p/\mu) + Z\alpha\pi$$

Origin Of π^2 Factors

$$\mathscr{M}_{\text{EFT}}(p,\mu) \;\; S(\mu,\lambda) \to \mathscr{M}_{\text{EFT}}(p,-\mu) \;\; S(-\mu,\lambda)$$

- UV logs of soft function match IR logs of EFT matrix element.
- Soft function exponentiates. Known to all orders!

$$S(-\mu,\lambda) = e^{\pi Z\alpha/\beta} e^{i\phi_C}$$

Example With Neutron Decay

REAL HADRONS

$$n \to p\ e^-\ \bar{\nu}_e \quad \longleftrightarrow \quad N^- \to P^0\ e^-\ \bar{\nu}_e$$

FICTICIOUS HADRONS

EIKONAL ALGEBRA

- Relate amplitudes with charged particles in initial/final state.
- Example at one loop.

$$\mathrm{Re}\ \mathscr{M}^{(1)} = 27.8 \qquad\qquad \mathrm{Re}\ \mathscr{M}^{(1)} = -0.15$$

$$\mu^2 = 4\mathbf{p}_\mathbf{e}^2 \qquad\qquad \mu^2 = -4\mathbf{p}_\mathbf{e}^2$$

Anomalous Dimension

Anomalous Dimension

$$\gamma_C = \frac{dC(\mu)}{d\log\mu}$$

SOLVE DIRAC EQ'N

SYMMETRY IN MASSLESS LIMIT

$$(Z, Z-Q, Q) \longleftrightarrow (Z+Q, Z, -Q)$$

$$\gamma_C = \alpha\left(Z\gamma^{(1,1)} + \gamma^{(1,0)}\right) + \alpha^2\left(Z^2\gamma^{(2,2)} + Z\gamma^{(2,1)} + \gamma^{(2,0)}\right)$$

$$+ \ \alpha^3\left(Z^3\gamma^{(3,3)} + Z^2\gamma^{(3,2)} + Z\gamma^{(3,1)} + \gamma^{(3,0)}\right) + \ldots$$

Z=0 REDUCES TO HEAVY-LIGHT CURRENT IN HQET

TAKE FROM HQET LIT.

SOLVE DIRAC EQ'N

↕ SYMMETRY

$\gamma_{\mathcal{O}}$	α^0	α^1	α^2	α^3	α^4
Z^0	0	$\gamma^{(1,0)}$ ✓	$\gamma^{(2,0)}$ ✓	$\gamma^{(3,0)}$ ✓	$\gamma^{(4,0)}$ ✓
Z^1	–	0	$\gamma^{(2,1)}$	$\gamma^{(3,1)}$	$\gamma^{(4,1)}$
Z^2	–	–	$\gamma^{(2,2)}$ ✓	$\gamma^{(3,2)}$	$\gamma^{(4,2)}$
Z^3	–	–	–	0	$\gamma^{(4,3)}$
Z^4	–	–	–	–	$\gamma^{(4,4)}$ ✓

FRONTIER

NEW RESULTS

New Master Integrals

$$I^{(n)}_{\vec{\nu}} = \int [\mathrm{d}^4 q][\mathrm{d}^3 L_1][\mathrm{d}^3 L_2]$$

$$\omega^n \left(\frac{1}{D_1}\right)^{\nu_1}\left(\frac{1}{D_2}\right)^{\nu_2}\left(\frac{1}{D_3}\right)^{\nu_3}\left(\frac{1}{D_4}\right)^{\nu_4}\left(\frac{1}{D_5}\right)^{\nu_5}\left(\frac{1}{D_6}\right)^{\nu_6}\left(\frac{1}{D_7}\right)^{\nu_7}\left(\frac{1}{D_8}\right)^{\nu_8}$$

$$D_1 = \mathbf{L}_1^2 \qquad D_5 = \omega^2 + \mathbf{Q}^2$$

$$D_2 = \mathbf{L}_2^2 \qquad D_6 = (\mathbf{L}_1 - \mathbf{L}_2)^2 + \lambda^2$$

$$D_3 = \omega^2 + (\mathbf{L}_1 + \mathbf{Q})^2 \qquad D_7 = \mathbf{L}_1^2 + \lambda^2$$

$$D_4 = \omega^2 + (\mathbf{L}_2 + \mathbf{Q})^2 \qquad D_8 = \omega^2 + \mathbf{Q}^2 + \lambda^2$$

- Mixed dimensionality (3-d Coulomb photons) (4-d dynamical photons)
- Reference vector breaks Lorentz invariance.
- IBP relations, convolution theorem, & brute force.

Summary / Status Report

Status Of Ingredients For Amplitude

$\mu \sim 2p \rightarrow$ minimize logs

$$\mathscr{M} = C(\mu_H) \times \frac{C(\mu)}{C(\mu_H)} \mathscr{M}_{\text{EFT}}(p, \mu) \quad S(\mu, \lambda)$$

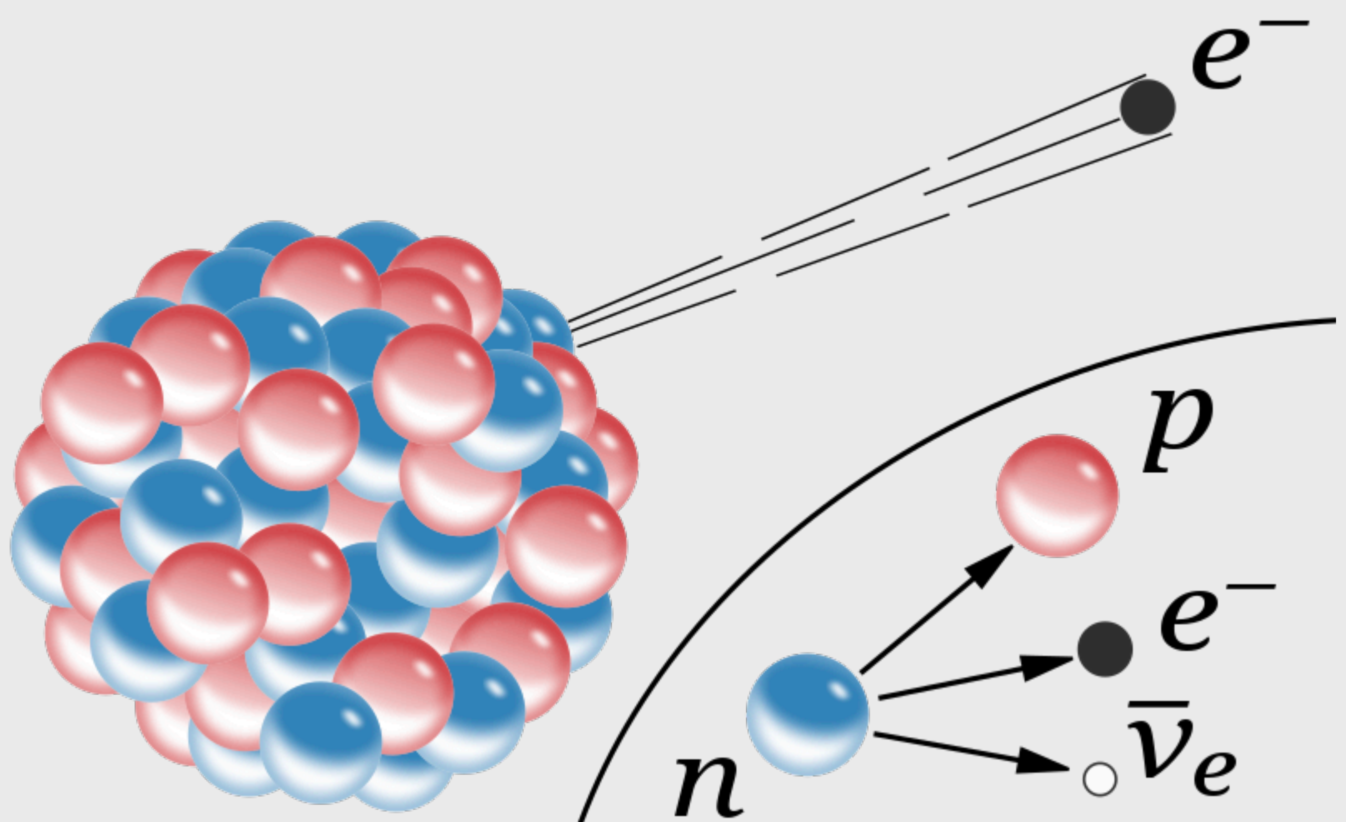
1. pQCD ✓
2. Nucleon ✓
3. Nucleus (hammer) — SENG ET. AL. TRIUMF GROUP (AB INITIO)

1. 1-loop ✓
2. 2-loops ✓
3. 3-loops ✓
4. 4-loops — 3/5 ✓, 2/5 ✗

1. 1-loop ✓
2. 2-loops (hammer) — ONGOING WITH P. VANDER GRIEND
3. 3-loop $(Z\alpha)^3$ ✓
4. 4-loop $(Z\alpha)^4$ ✓

All orders in PT ✓

Working To High Order



$Z \sim 10$ $\log(2p/m_e) \sim \log(2pR) \sim 5$

$\delta V_{\rm ud} \sim 10^{-4}$

DEFINE POWER COUNTING

$$Z \sim \mathrm{L} \sim 1/\sqrt{\alpha}$$

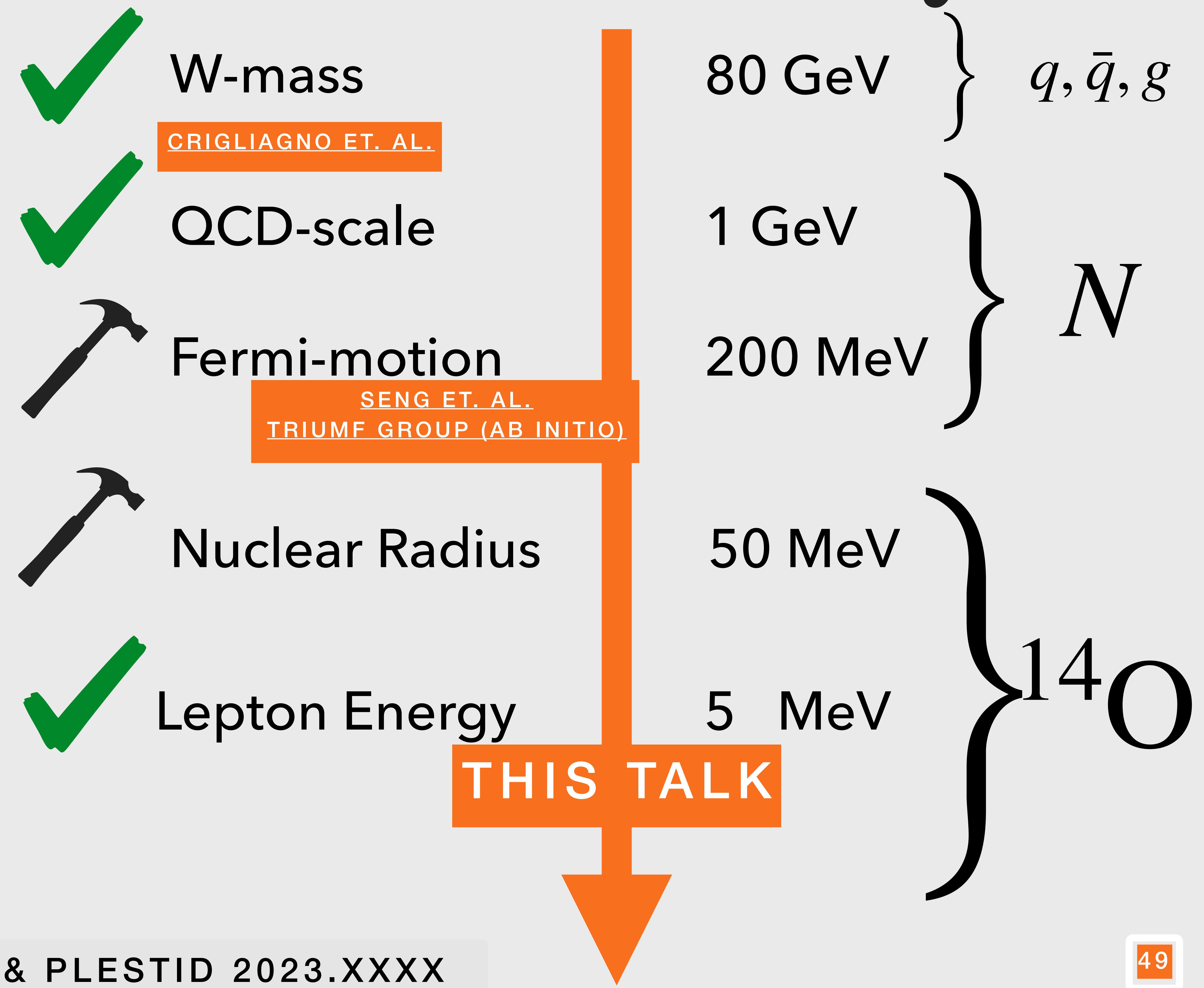
- Let us aim for $O(\alpha^{3/2})$ precision
- Conservative: ignores $(1/\pi)^n$.

✓ π^2 – enhancements under control

- 2-loops ✓
  $\alpha^2\mathrm{L}$, $(Z\alpha)\alpha$
  SIRLIN & ZUCHINI IN DIFF. SCHEME
  ONGOING WITH P. VANDER GRIEND
- 3-loops ✓
  $\alpha^3\mathrm{L}^3$, $(Z\alpha)^2\alpha\mathrm{L}$, $(Z\alpha)\alpha^2\mathrm{L}^2$, $(Z\alpha)^3$
- 4-loops ✓
  $(Z\alpha)^3\alpha\mathrm{L}^2$, $(Z\alpha)^4\mathrm{L}$
- 5-loops
  $(Z\alpha)^5\ \mathrm{L}^2$ ✓
- 6-loops
  $(Z\alpha)^6\ \mathrm{L}^3$ ✓

Necessary Inputs For Beta Decay

- Beta decay important for CKM & SMEFT.
- Low energy theory demands multi-loops.
- $\pi^2, Z,$ L enhancements
- Point-like theory under control to $O(\alpha^{3/2})$.



TO APPEAR THIS SUMMER. HILL & PLESTID 2023.XXXX