Hyperfunction method for two loop calculations for PRadII experiment

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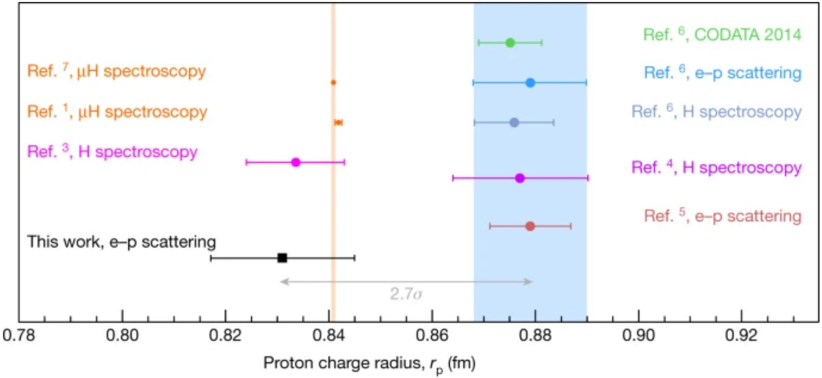
Outline

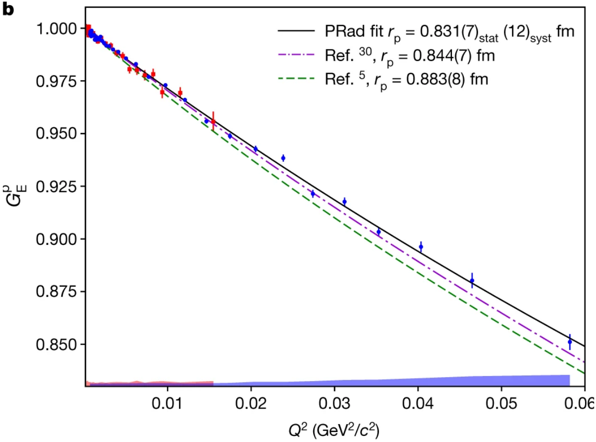
- Topological formulation of the problem
- Computation of the relavant cohomology groups
- Analytic structure of the amplitudes
- Determinantal complexes and Landau variaties
- Cut amplitudes as subbundles
- Some numerical results for PRadII

Proton radius puzzle



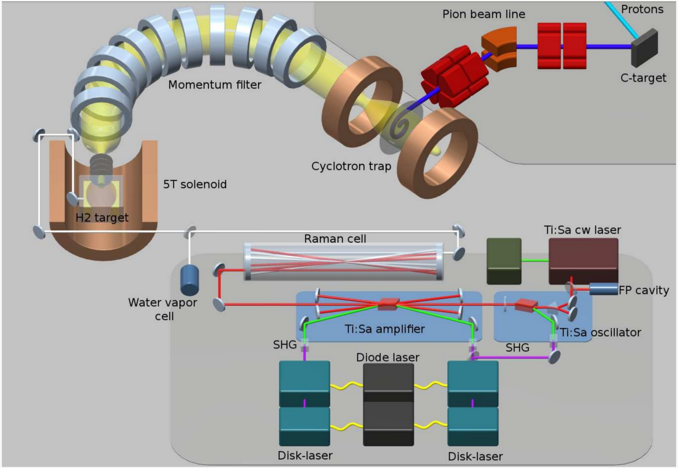
"A small proton charge radius from an electron–proton scattering experiment", Xiong, W and Gasparian, A and Gao, H et. al. Nature , 575, 2019

r_p is extracted from the slope of electric form factor



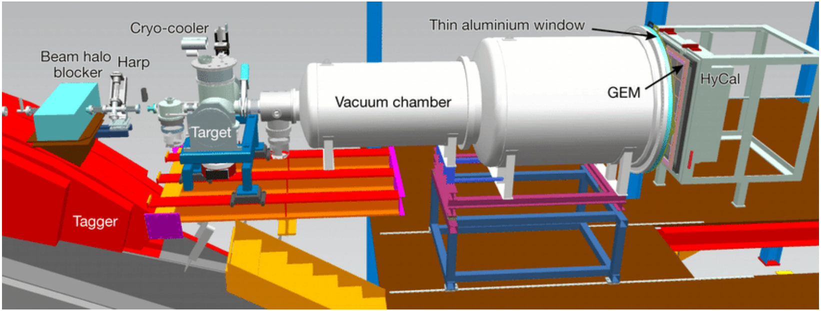
Muonic hydrogen experiment

p-beam
(1 MW)
↓
π (MeV)
↓
μ (MeV)
↓
μ (keV)
↓
μp(2S)
↓
Laser
↓
X-ray



940 nm
(2 kW)
↓
1030 nm
↓
515 nm
↓
708 nm
↓
1.0 μm
↓
1.6 μm
↓
6.0 μm

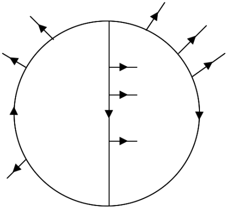
PRadII experiment at JLab



Two loop integrals

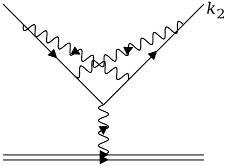
$$J(p_{1,i}, p_{2,i}, p_{3,i}, m_{a,i}) = \int d^d q_q d^d q_2 \prod \frac{1}{(q_1 + p_{1,i})^2 + m_{1,i}^2} \times \quad (1)$$

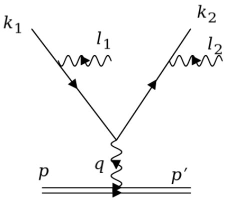
$$\prod \frac{1}{(q_1 + p_{1,i})^2 + m_{1,i}^2} \prod \frac{1}{(q_1 + p_{1,i})^2 + m_{1,i}^2} \quad (2)$$



Inclusive cross sections

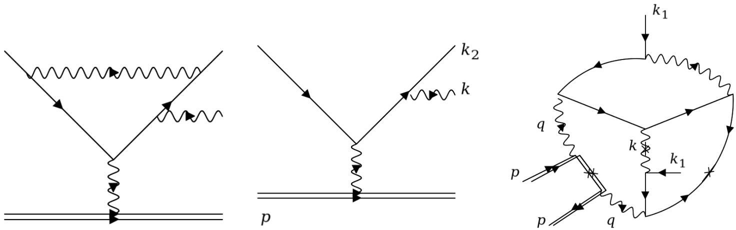
$$\int dk_2 \delta(k_2^2 - m^2) \int \frac{dl_1 dl_2}{\prod k_i l_j} \tag{3}$$





Cut amplitudes and real emission

Lemma Inclusive 2loop amplitudes are equivalent to cut 3loop amplitudes.



Analytic continuation of amplitudes

$$\int (d^d q) \frac{1}{\prod (q_i + p_i)^2 + m_i^2} \to \int_\xi \omega^{(dL,0)} R(q,p,m) \tag{4}$$

$$\xi \in H_{dL}(\mathbb{CP}^{dL} - \cup\{D_i = 0\}; \mathbb{CP}^{dL-1} - \cup\{D_i^\infty = 0\}) \tag{5}$$

We compactify the space of momenta to $\mathbb{CP}^{dL}$ and integrate over chains in the complement to the Feynman quadrics.

Obs The quadrics are highly singular. The singularity locus of the quadric is a plane of dimension $d(L-1)$.

Obs Dim Reg is obtained by compactifying to $\mathbb{C}^{dL}$ and adding factor ξ_0^ϵ in the homogeneous coordinates $q_{a,\mu} = \xi_{a,\mu}/\xi_0$

Analytic structure of amplitudes: flat bundles

Two approaches: p-space, Feynman param

$$J = \int_\xi (d^d q_a) \prod d_i(q,p,m) \quad (6)$$

$$\xi \in H_{dL}(\mathbb{CP}^{dL} - \cup\{D_i = 0\}; \mathbb{CP}^{dL-1} - \cup\{D_i^\infty = 0\}) \quad (7)$$

$$J = \int_\eta (ND + p_i p_j L_i \hat{M} L_j)^\alpha D^{-dL/2-\alpha} x^m d^I x \quad (8)$$

$$D = det(M),\ M_{ab} = \sum x_i l_{i,a} l_{i,b},\ \alpha = -I + dL/2 \quad (9)$$

$$\eta \in H_L(\mathbb{CP}^L - \{P=0\} \cup \{D=0\}; \cup H_i - \cup(H_i \cap (\{P=0\} \cup \{D=0\}))) \quad (10)$$

Highly singular hypersurfaces.
(Stratification is not fully known even in 2loop case; see however works of Francis Brown! 2L seems feasible)

The moduli space of 2-loop virtual amplitides

$$J = \int_{\eta} d^l x (a_{ijk}x_i x_j x_k + b_{ij}x_i x_j + c_i x_i + d)^{\alpha} x^m \qquad (11)$$

$$J = \int d^l x (\sum s_i P_i(x))^{\alpha} D^{\beta} \qquad (12)$$

Obs The degeneration of cubic hypersurface can have at most quadratic degeneracy.

Problem The singularity locus can be high dimensional, even in QFT case.

Th (four-fold complete classification) (R. Laza) Singularities are:

- elliptic curve of deg 4 and a rational normal curve deg 1
- two isolated E_8 singularities
- isolated E_7 singularity and elliptic curve of deg 2
- three isolated E_6
- rational normal curve of deg 4
- elliptic normal curve of deg 6

Conj (the picture)

Cut diagrams as sub-bundles

2d toy example :

$$J = \int_{\zeta} P^{\alpha} \frac{d^2 q}{dQ} \;\; J = \int_{\tau(\zeta)} P^{\alpha} Q^{-1} d^2 q \tag{13}$$

Actual function

$$J = \int_{\xi} \prod' \frac{1}{D_i(q,p,m)} \frac{d^{dL} q}{\wedge_j dD_j(q,p,m)} \tag{14}$$

$$J = \int_{\tau(\xi)} \prod \frac{1}{D_i(q,p,m)} d^{dL} q \tag{15}$$

Th Cut amplitudes are sub-bundles of the corresponding uncut amplitudes.

Corr The singularities are the same (the same Landau poly). Gauss Manin connection has the same denominators.

Rem GM connection has the form

$$\partial_s J = \sum \frac{A_{s,l}}{L_l} J \tag{16}$$

Vanishing cycles, hypercohomology, and higher discriminantal complexes

By Alexander duality, we reduce the cohomology computation to

$$H_{dL-1}(\cup\{D_i = 0\}) \tag{17}$$

This is computed by Mayer Vietoris spectral sequence. It reduces to computation of

$$H_{dL-|I|}(\cap_{i\in I} D_i) \tag{18}$$

Th All singularities in J are associated with degeneracy of normals of the intersections of propagators.
Th (Complete classification of singularities at finite distance) For each tuple of q_a, there is a set of singularities that result in degeneracy among normals of a tuple of D_i in q_a-subspace.
If there are fewer than L pinched q_a, the factor in

$$J = L^A F + reg \tag{19}$$

F is also an integral.

Higher discriminantal complexes

Th The complete list of singularity loci is obtained from elimination of q_a from degeneracy equations.
This is particular case of solving overdetermined systems

$$P_0 = ... = P_d = 0 \tag{20}$$

General technique: the elimination process can be formulated as taking determinant of a complex of vector space (Reidemeister torsion! classical topic see Milnor's works).
More details : A.Morozov, Gelfand-Kapranov-Zaslavski, Miller

Toric geometry and MIs

The monomials x^m in $\int p^a x^m d^d x$ transfrom under the Borel subgroup of $GL(|\Omega|, \mathbb{C})$. This group acts term by term on the IBP Koszul complex.

Th There exists an equivariant basis of integrals organized according to GIT invariants of the Borel group action.

Rk This action is sensitive to the polytope Ω

Hyperfunction approach to amplitudes

Perturbative correlation functions are hyperfunctions in the sense of microlocal analysis (huge mathematical literature!)
def Hyperfunction on $\mathbb{C}^n$ is a section of a bundle that satisfies a holonomic d-module with regular coefficients.
Th(Deligne-Mostow, GZK,Sturmfels,...,Cih, Sadykov) After dim reg, the hyperfunction admits expansion of the form

$$J_s = L^{\alpha_s} R_1(x_{\|}) + R_2(L, x_{\|}) \tag{21}$$

where R_i are regular.
The coefficients of this expansion are meromorphic in d.

Kapranov uniformization and lift of the hyperfuntion to universal covering space

Th(Kapranov uniformization) The discriminant of the system of algebraic equations is uniformized by the images of projective space under the inverse Gauss map.
The picture:

- The singularity strata of deformation

$$J(P,D) = \int (dx)(P)^{\alpha} D^{\beta} \tag{22}$$

  are uniformized by projective spaces.
- Extend this uniformization onto the generic point of the space P, D. We get complement to arrangement of hyperplanes.
- Lift bundle defined by J to this uniformization. This defines "moduli independent" bundle J'.

Applications to PRadII

- 3 types of diagrams
- dim reg (Frautschi Suura Yennie , practically)
- PDEs
- Vanishing cycle analysis

Example : The mechanism of formation of singularities: Vanishing cycles at 2 loops

At two loops, there are 3 families of singularities:

- In q_1-space
- In q_2-space
- In q_1, q_2-space

Ex:

$$q_1^2 + m_1^2 = 0,\ (q_1 + p_1)^2 + m_2^2 = 0 \tag{23}$$

$$q_1 = \alpha p_1,\ L = .. = 0 \tag{24}$$

$$J = L^{-2+d/2} \prod{}' \frac{1}{(Q_1 + p_{1,i})^2 + m_{1,i}^2} \int dq_2 \prod \frac{1}{(Q_1 + q_2 + p_{3,i})^2 + m_{3,i}^2} \tag{25}$$

$$\prod \frac{1}{(q_1 + p_{2,i})^2 + m_{2,i}^2} \tag{26}$$

Lemma The above list is exhaustive. All singularites are of Morse

vanishing cycles, pinch point, direct sum and direct product of singularites

Th If the pinch points $(Q_1^{(3,1)}, Q_1^{(3,2)})$ and $(Q_2^{(3,1)}, Q_2^{(3,2)})$ are distinct, then the singularity of j is thhe direct sum

$$J = S_1((Q_1^{(3,1)}, Q_1^{(3,2)})) + S_2((Q_2^{(3,1)}, Q_2^{(3,2)})) + reg \tag{27}$$

Th In the case of the pinch points $(Q^{(1)}, 0)$ and $(0, Q_1^{(3,2)})$, the integral is decomposed into direct product

$$J = S_1((Q^{(1)}, 0)) \times S_2((0, Q^{(2)})) + reg \tag{28}$$

Th In the case of two pinch points $(Q_1^{(1)}, 0)$ and $(Q_2^{(1)}, 0)$, J is direct sum of singularities unless $Q_1^{(1)} = Q_1^{(1)}$.
Ex In the last case, for the 2-pinch $(0, p-1)$ and 3-pinch $(0, p_1, p_2)$, the coincidence manifold is of codimension 2:

$$Q_1 = \alpha p_1 \tag{29}$$

$$Q_2 = \beta p_1 + \gamma p_2 \tag{30}$$

PDEs

Lemma: MIs for 1real+1virtual are

$$J = \int d^d k_2 \delta(k_2^2 - m^2)\delta(k_1 k_2 - c) \int d^d k \delta(k^2)\delta((p+q-k)^2 - M^2) \tag{31}$$

$$\int d^d l \frac{1}{l^2((l+k_1)^2 + m^2)((l+k_2)^2 + m^2)((l+k_1+k)^2 + m^2)} \tag{32}$$

and single derivatives of the δ.

PDEs for virtual corrections

$$F_i(\epsilon, Q^2) = \mu^{2(2-d)} \int \frac{\{1, k_1 k_2\} d^d k_1 d^d k_2}{\prod k_i^2((k_i + p_i)^2 + m^2) \prod (k_1 + k_2 + p_i)^2 + m^2} \tag{33}$$

$$\frac{dF_1}{dQ^2} = -(1+2\epsilon)(\frac{1}{Q^2} + \frac{1-2\epsilon}{Q^2+2m^2})F_1 - \tag{34}$$

$$-\frac{2\epsilon}{m^2}(\frac{1}{Q^2} - \frac{1-2\epsilon}{Q^2+2m^2})F_2 + \Omega_1 \tag{35}$$

$$\frac{dF_2}{dQ^2} = \epsilon F_1 - \frac{1-2\epsilon}{2}(\frac{1}{Q^2} + \frac{1}{Q^2+4m^2})F_2 + \Omega_2 \tag{36}$$

"Vertex diagrams for the QED form factors at the 2-loop level"
Bonciani, Roberto and Mastrolia, Pierpaolo and Remiddi, Ettore

Discussion

- we discussed several general principles including
- vanishing cycles, reduction to direct sum, spectral sequences
- IBP and toric geometry
- uniformization of the complement of the singularity loci and pullback of the Gauss Manin connection
- applied this machinery to 2loop calculations for PRadII experiment