NON-PLANAR TWO-LOOP QCD CORRECTIONS TO $q\bar{q} \to \gamma\gamma\gamma$:

FINITE REMAINDERS IN THE SPINOR-HELICITY FORMALISM

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in collaboration with:
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based on: arXiv:2305.17056

LoopFest XXI

PSI SLAC

Find these slides at gdelaurentis.github.io/slides/loopfestxxi_june2023

INTRODUCTION

State-of-the-Art of $\mathcal{A}_n^{(2-\text{loop})}$

Five-point massless amplitudes in full color:

- $pp \to \gamma jj$ — Badger, Czakon, Hartanto, Moodie, Peraro, Poncelet, Zoia ('23)
- $pp \to \gamma\gamma j$ — Agarwal, Buccioni, von Manteuffel, Tancredi ('21)

  Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia ('21)
- $pp \to \gamma\gamma\gamma$ — Abreu, GDL, Ita, Klinkert, Page, Sotnikov ('23); This talk!
- $pp \to jjj$ (?) — Next talk by Federico

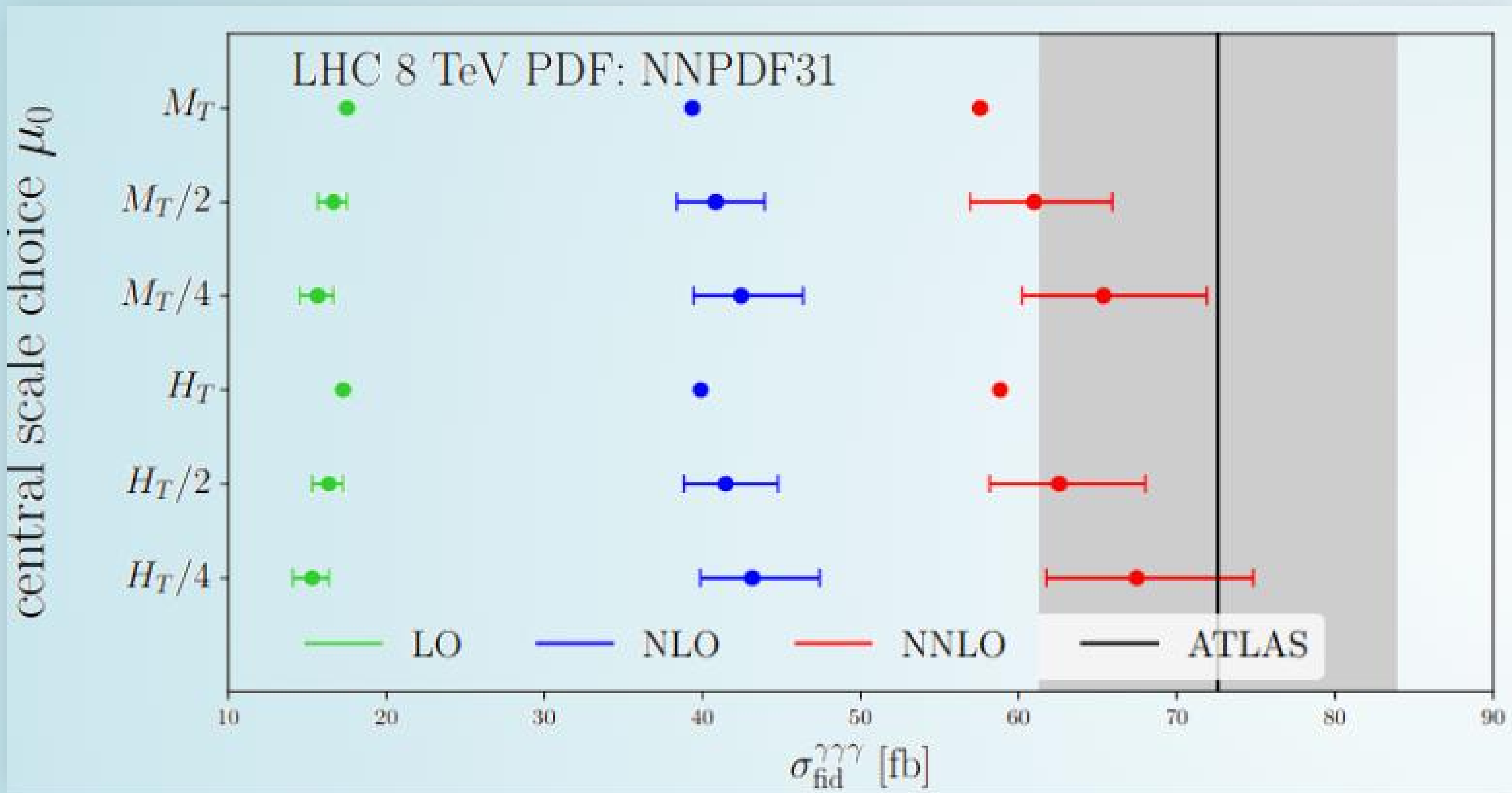
Five-point one-mass amplitudes at leading color:

- $pp \to Wb\bar{b}$ — Badger, Hartanto, Zoia ('21)
- $pp \to Hb\bar{b}$ — Badger, Hartanto, Kryś, Zoia ('21)
- $pp \to Wjj$ — Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov ('21); A bit about this towards the end of this talk!
- $pp \to W\gamma j$ — Badger, Hartanto, Kryś, Zoia ('22)

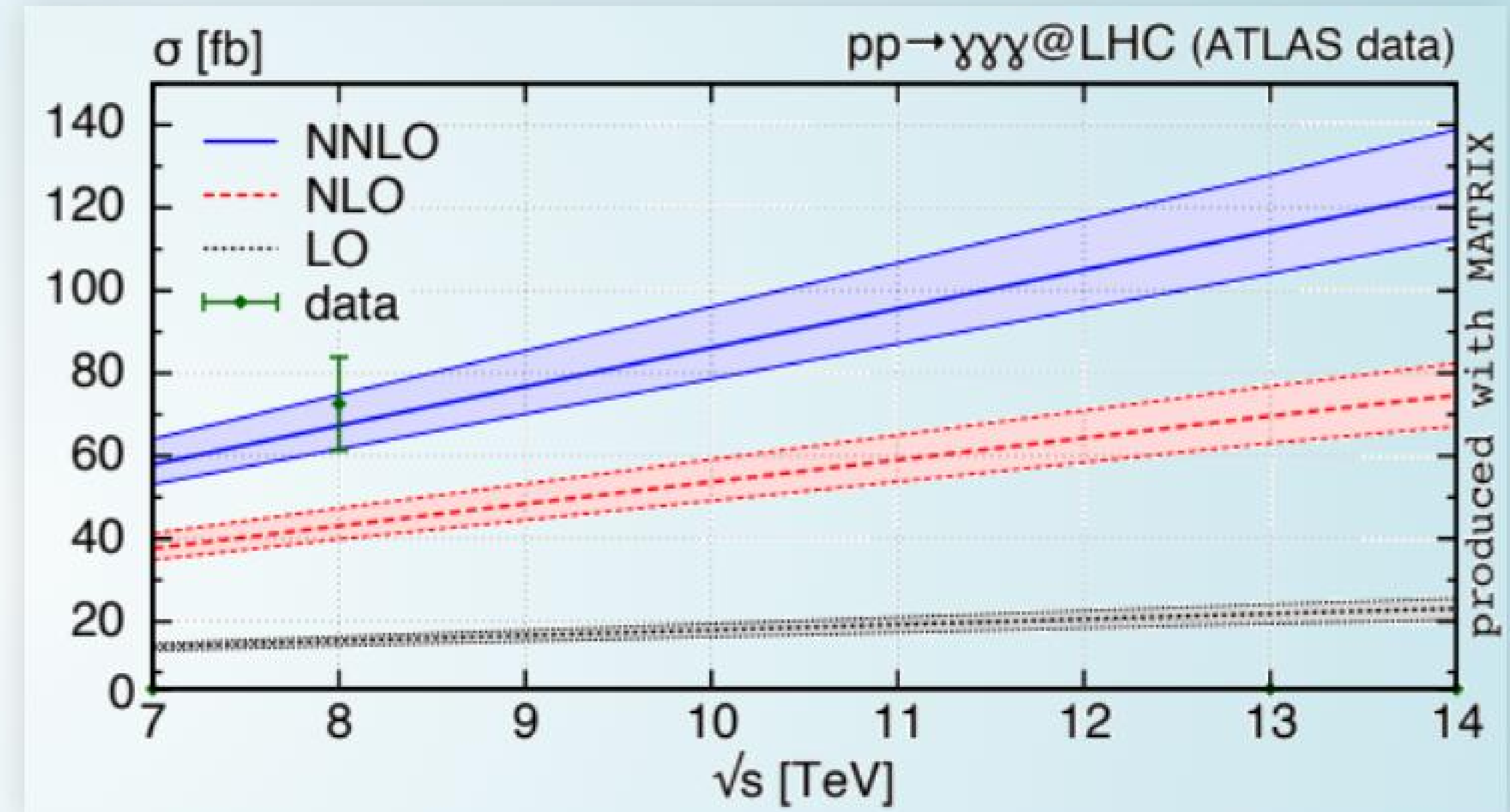
Two-loop five-point amplitudes remain a challenge, but are now very much feasible.

SIZABLE NNLO CORRECTIONS TO $q\bar{q} \rightarrow \gamma\gamma\gamma$

- NNLO cross-sections computed with leading-color double-virtual amplitudes



Chawdhry, Czakon, Mitov, Poncelet ('19)



Kallweit, Sotnikov, Wiesemann ('20)

- Analytic two-loop amplitudes in limit $N_c \rightarrow \infty\,,\; N_c/N_f = \text{const.}$

Chawdhry, Czakon, Mitov, Poncelet ('20) Abreu, Page, Pascual, Sotnikov ('20)

Question: are the subleading-color contributions trully negliegible?

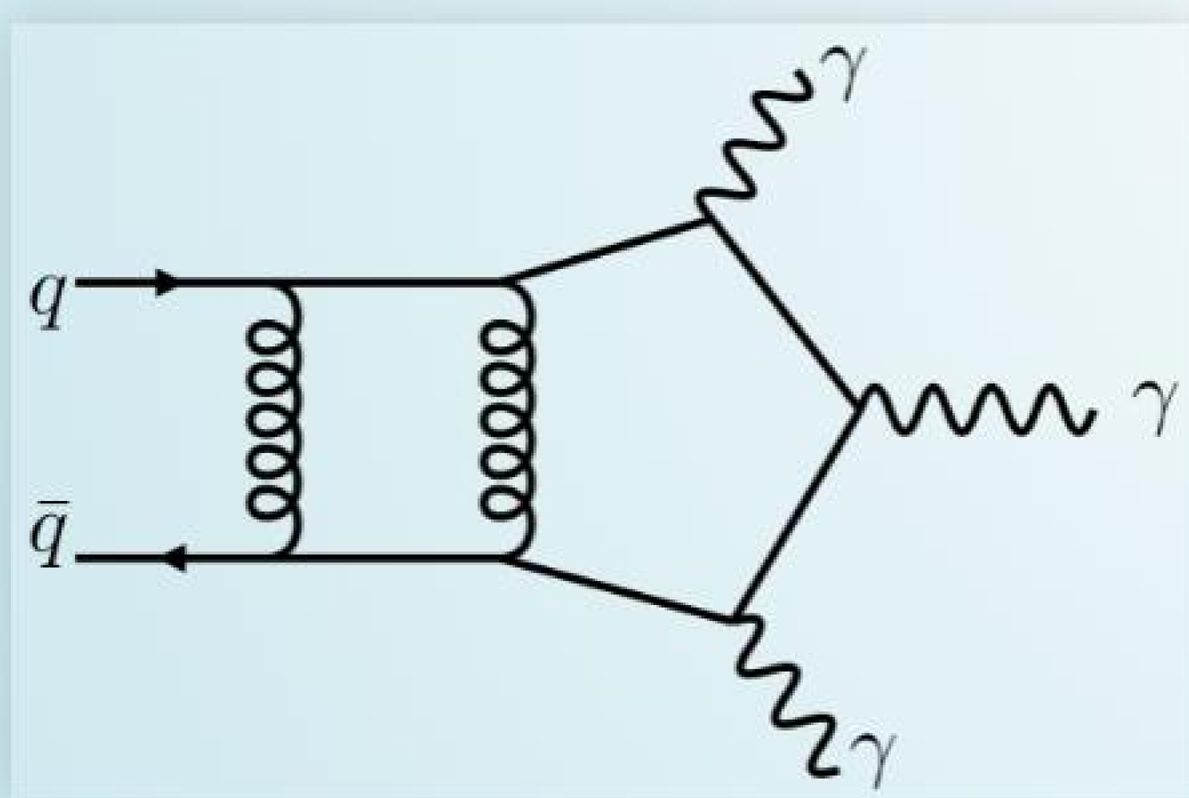
Gauge-Invariant Subamplitudes

$$\mathcal{A}^{(2)}_{2q3\gamma} = \frac{N_c^2}{4}\left(A^{(2,0)}_{2q3\gamma} - \frac{1}{N_c^2}\left(A^{(2,0)}_{2q3\gamma} + A^{(2,1)}_{2q3\gamma}\right) + \frac{1}{N_c^4} A^{(2,1)}_{2q3\gamma}\right)$$

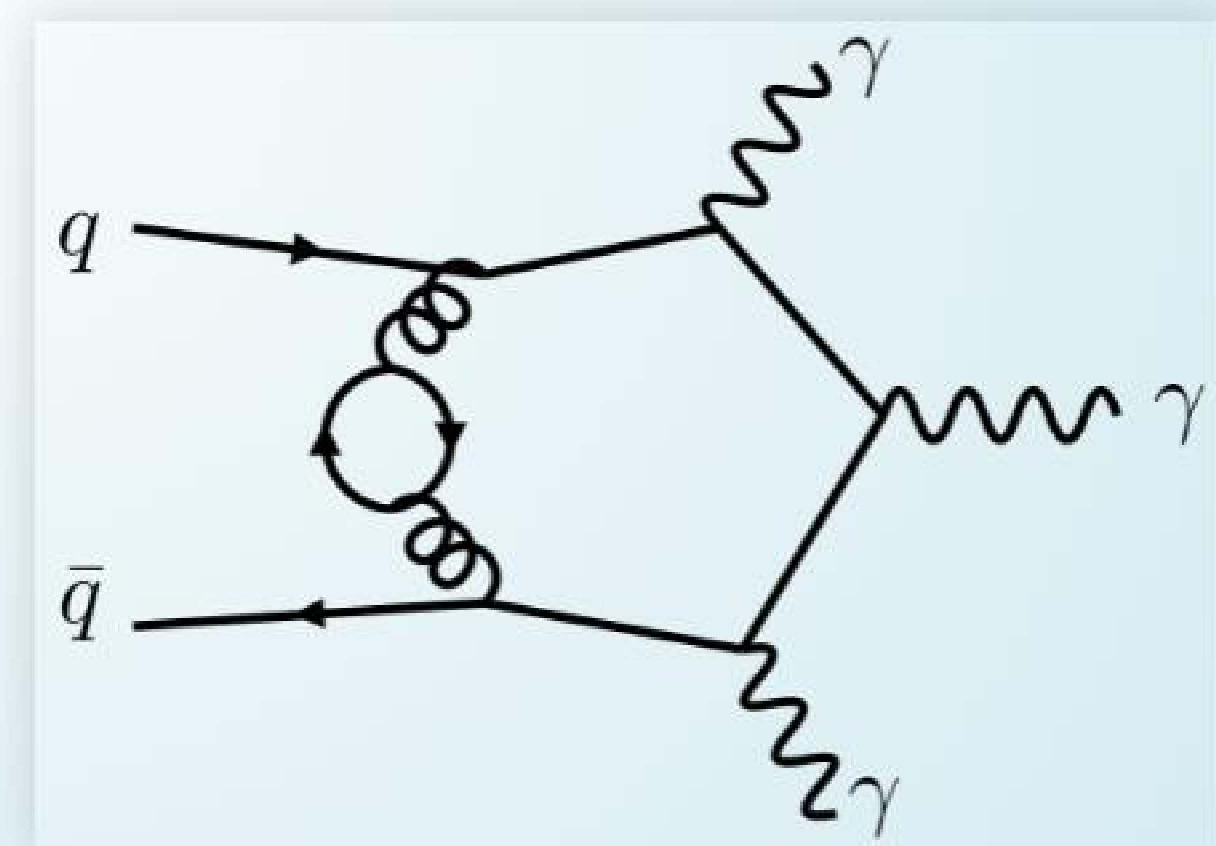
$$+ C_F T_F N_f A^{(2,N_f)}_{2q3\gamma} + \underbrace{C_F T_F \left(\sum_{f=1}^{N_f} Q_f^2\right)}_{\text{trully suppressed?}} A^{(2,\tilde{N}_f)}_{2q3\gamma}\,,$$

- Example diagram for each amplitude:

$A^{(2,0)}_{2q3\gamma}$:

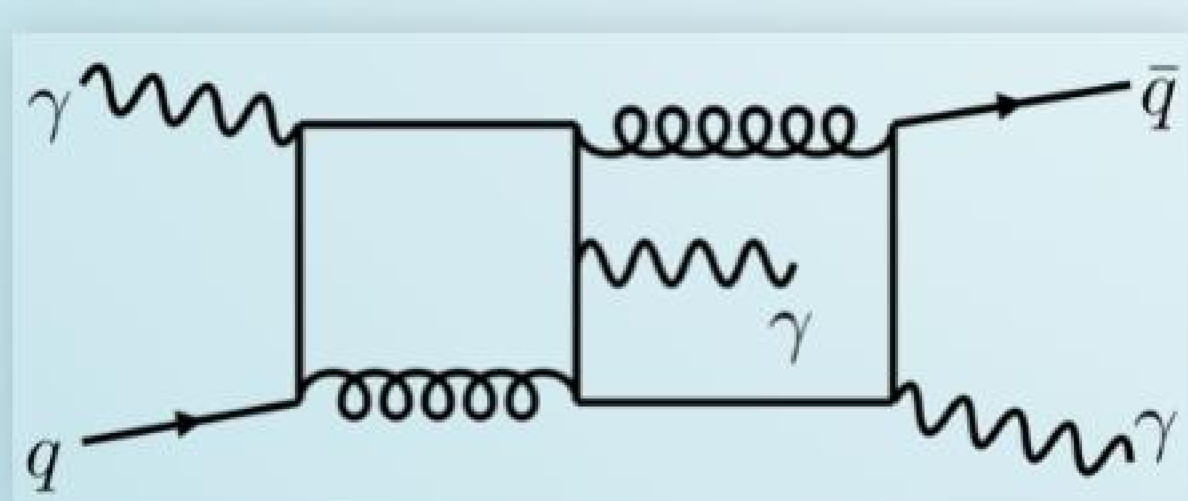


$A^{(2,N_f)}_{2q3\gamma}$:

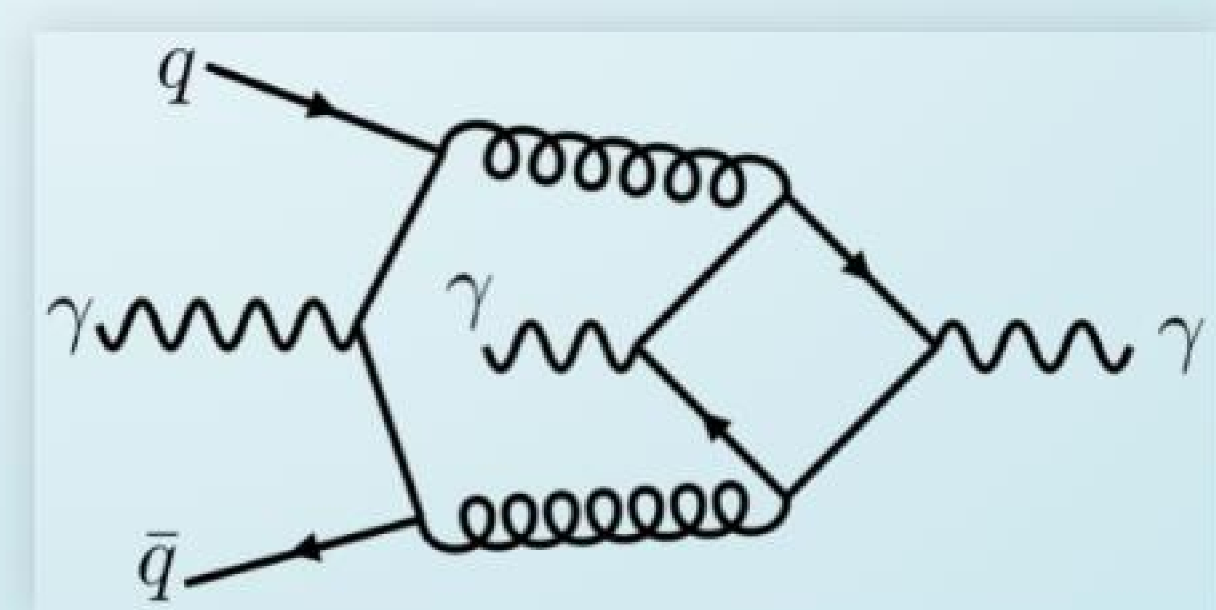


Previously known

$A^{(2,1)}_{2q3\gamma}$:



$A^{(2,\tilde{N}_f)}_{2q3\gamma}$:



New in this work

ORGANIZATION OF THE COMPUTATION

Generalized Unitarity

- Loop integrands can be written as ($\lambda = |\bullet\rangle, \tilde{\lambda} = [\bullet|, \lambda\tilde{\lambda} = \not{p}$)

$$A(\lambda, \tilde{\lambda}, \ell) = \sum_{\Gamma} \sum_{i \in M_\Gamma \cup S_\Gamma} c_{\Gamma,i}(\lambda, \tilde{\lambda}) \frac{m_{\Gamma,i}(\lambda\tilde{\lambda}, \ell)}{\prod_j \rho_{\Gamma,j}(\lambda\tilde{\lambda}, \ell)}$$

- Generalized unitarity relates cuts of loop amplitudes to products of trees

$$\sum_{\text{states}} \prod_{\text{trees}} A^{\text{tree}}(\lambda, \tilde{\lambda}, \ell)\Big|_{\text{cut}} = \sum_{\substack{\Gamma' \geq \Gamma, \\ i \in M'_\Gamma \cup S'_\Gamma}} c_{\Gamma',i}(\lambda, \tilde{\lambda}) \frac{m_{\Gamma',i}(\lambda\tilde{\lambda}, \ell)}{\prod_{j \in P_{\Gamma'}/P_\Gamma} \rho_j(\lambda\tilde{\lambda}, \ell)}\Bigg|_{\text{cut}}$$

$$\left.\begin{array}{r} \underline{\text{Master integrands}} : \int d^D\ell \, \frac{m_{i \in M_\Gamma}}{\prod_j \rho_j} \neq 0 \\ \underline{\text{Surface terms}} : \int d^D\ell \, \frac{m_{i \in S_\Gamma}}{\prod_j \rho_j} = 0 \end{array}\right\} \text{Equivalent to IBP reduction}$$

Ita ('15)

C++ code

Caravel

Abreu, Dormans, Febres Cordero, Ita Kraus, Page, Pascual, Ruf, Sotnikov ('20)

New Features of the Reduction

○ Master / surface decomposition for non-planar topologies

IBP-generating vectors: $$\int d^D\ell \frac{\partial}{\partial \ell_a^\mu} \frac{v_a^\mu(\ell)}{\rho_1 \dots \rho_N} = 0 \quad \text{(in dim. reg.)}$$

No propagator doubling: $$\sum_{a,\mu} v_a^\mu(\ell) \frac{\partial \rho_i}{\partial \ell_a^\mu} - f_i(\ell)\rho_i = 0$$

(v_a^μ, f_i) form a *syzygy module*, solved for in *embedding space* using `Singular` + linear algebra.

○ Semi-numerical surface terms: $m_{i\in S_\Gamma}(\ell \leftarrow \text{analytical}, s_{ij} \leftarrow \text{numerical})$

⋆ dependance on external kinematics (s_{ij}) obtained from sparse linear systems

○ Little group information retained throughout the computation

⋆ genuine $c_{\Gamma,i}(\lambda, \tilde{\lambda})$ instead of $c_{\Gamma,i}(\lambda\tilde{\lambda})$ + conventions for the polarization states.

Finite remainders & the Rational / Transcendental split

- In general, in $D = 4 - 2\epsilon$, with *pure* master integrals $I_{\Gamma,i}$ we have

$$A_n^{\ell-loop}(\lambda, \tilde{\lambda}) = \sum_{\Gamma} \sum_{i \in M_\Gamma} \frac{c_{\Gamma,i}(\lambda, \tilde{\lambda}, \epsilon)\, I_{\Gamma,i}(\lambda\tilde{\lambda}, \epsilon)}{\prod_j (\epsilon - a_{ij})}\,, \quad \text{with} \quad a_{ij} \in \mathbb{Q}$$

- For NNLO applications, we are interested in the *finite remainder*

$$\mathcal{A}_R^{(2)} = \underbrace{\mathcal{R}}_{\text{finite remainder}} + \underbrace{I^{(1)}\mathcal{A}_R^{(1)} + I^{(2)}\mathcal{A}_R^{(0)}}_{\text{divergent + convention-dependent finite part}} + \mathcal{O}(\epsilon)$$

- Finite remainder as a weighted sum of *pentagon functions*

Chicherin, Sotnikov ('20);

$$\mathcal{R}(\lambda, \tilde{\lambda}) = \textstyle\sum_i r_i(\lambda, \tilde{\lambda})\, h_i(\lambda\tilde{\lambda})$$

Reconstruct $r_i(\lambda, \tilde{\lambda})$ from $\mathbb{F}_p$ samples

von Manteuffel, Schabinger ('14)
Peraro ('16)

ANALYTIC RECONSTRUCTION

The Least Common Denominator

- The $r_i(\lambda,\tilde\lambda)$ belong to the field of fractions over a poly. quotient ring, $FF(R_5)$

GDL, Page ('22);
Campbell, GDL, Ellis ('22)

$$r_i(\lambda,\tilde\lambda) = \frac{\text{Num. poly}(\lambda,\tilde\lambda)}{\text{Denom. poly}(\lambda,\tilde\lambda)} = \frac{\text{Num. poly}(\lambda,\tilde\lambda)}{\prod_j W_j^{q_{ij}}(\lambda,\tilde\lambda)}$$

- The denominator factors W_j are conjectured to be restricted to the letters of the symbol alphabet

Abreu, Dormans, Febres Cordero, Ita, Page ('18)

$$\{W_j\} = \bigcup_{\sigma \,\in\, \mathrm{Aut}(R_5)} \sigma \circ \big\{\langle 12\rangle, \langle 1|2+3|1]\big\}$$ Identical to 1-loop!

- Why bother with (redundant) spinor variables:
  - ⋆ the LCD is **not** little group invariant: the degree is lower in spinors;
  - ⋆ no (arbitrary) split into parity even and odd: half sampling requirement;
  - ⋆ in LCD form we would need 29 059 evaluations instead of 117 810 (with s_{ij}) for $\mathcal{R}^{(2)}_{2q3\gamma}$.

The Numerator Ansatz

- The numerator Ansatz takes the form

GDL, Maître ('19)

$$\text{Num. poly}(\lambda, \tilde{\lambda}) = \sum_{\vec{\alpha},\vec{\beta}} c_{(\vec{\alpha},\vec{\beta})} \prod_{j=1}^{n} \prod_{i=1}^{j-1} \langle ij \rangle^{\alpha_{ij}} [ij]^{\beta_{ij}}$$

subject to constraints on $\vec{\alpha}, \vec{\beta}$ due to: 1) mass dimension; 2) little group; 3) linear independence.

- Construct the Ansatz via the algorithm from Section 2.2 of GDL, Page ('22)

Linear independence = irreducibility by the Gröbner basis of a specific ideal.

- Efficient implementation using open-source software only

Singular

Gröbner bases $\rightarrow$ constrain $\vec{\alpha}, \vec{\beta}$

Decker, Greuel, Pfister, Schönemann

Google OR-Tools

Integer programming $\rightarrow$ enumerate sols. $\vec{\alpha}, \vec{\beta}$

Perron and Furnon (Google optimization team)

- All linear systems solved with CUDA over $\mathbb{F}_{p\leq 2^{31}-1}$ on a laptop ($t_{\text{solving}} \ll t_{\text{sampling}}$)

Taming the Algebraic Complexity

- Instead of the common denominator form, perform a partial fraction decomposition

$$r_i(\lambda,\tilde\lambda) = \frac{\mathcal{N}(\lambda,\tilde\lambda)}{\prod_j W_j^{q_{ij}}(\lambda,\tilde\lambda)} = \sum_k \frac{\mathcal{N}_k(\lambda,\tilde\lambda)}{\prod_j W_j^{q_{ijk}}(\lambda,\tilde\lambda)} = \sum_k r_{ik} \quad \text{with} \quad q_{ijk} \leq q_{ij}$$

- Use insights from physics, e.g. no denominator in $\mathcal{R}^{(2)}_{2q3\gamma}$ contains more than a single $\langle i|j+k|i]$

- As by now standard, we pick a set of independent r_i to reconstruct: $r_i \notin \text{span}(r_{j\neq i})$.
  However, generally $r_{ik} \in \text{span}(r_{j\neq i})$ for some, but not all, k. Thus, write:

$$r_i = \textstyle\sum_{j\neq i} r_j + \sum_{k'\subset\{k\}} r_{ik'}$$

Sampling requirement reduced from 29 059 to 4 003 points.

- For example, a posteriori, we find that for the most complicated r_i, we only needed

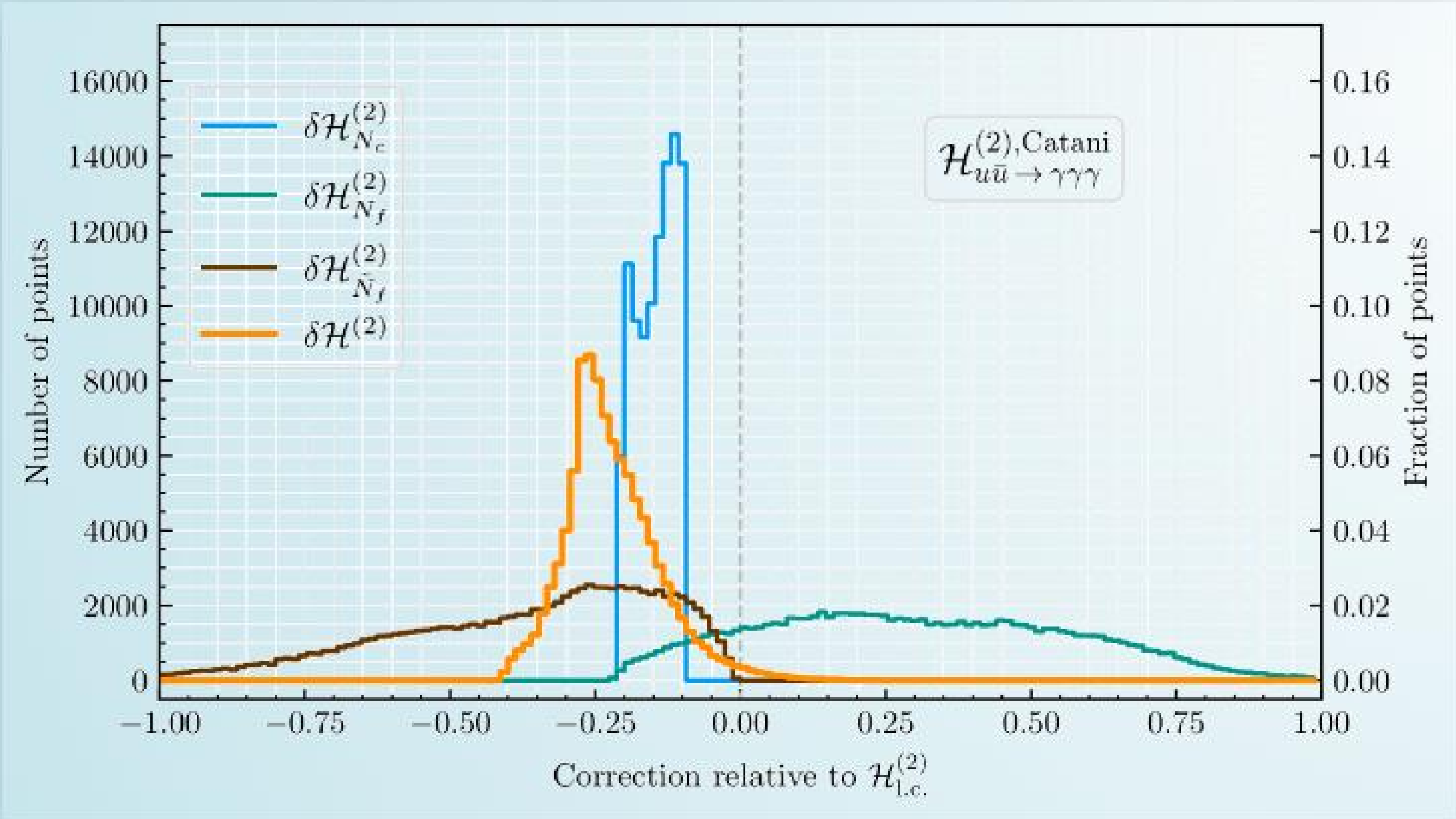
$$\sum_{k'\subset\{k\}} r_{ik'} = \frac{\langle 13\rangle[14]^2\langle 24\rangle\langle 34\rangle[45]}{\langle 45\rangle\langle 4|1+3|4]^3} - \frac{[14]\langle 25\rangle\langle 34\rangle^2[45]}{\langle 45\rangle^2\langle 4|1+3|4]^2} - \frac{[14]\langle 24\rangle\langle 34\rangle\langle 35\rangle}{\langle 45\rangle^3\langle 4|1+3|4]}$$

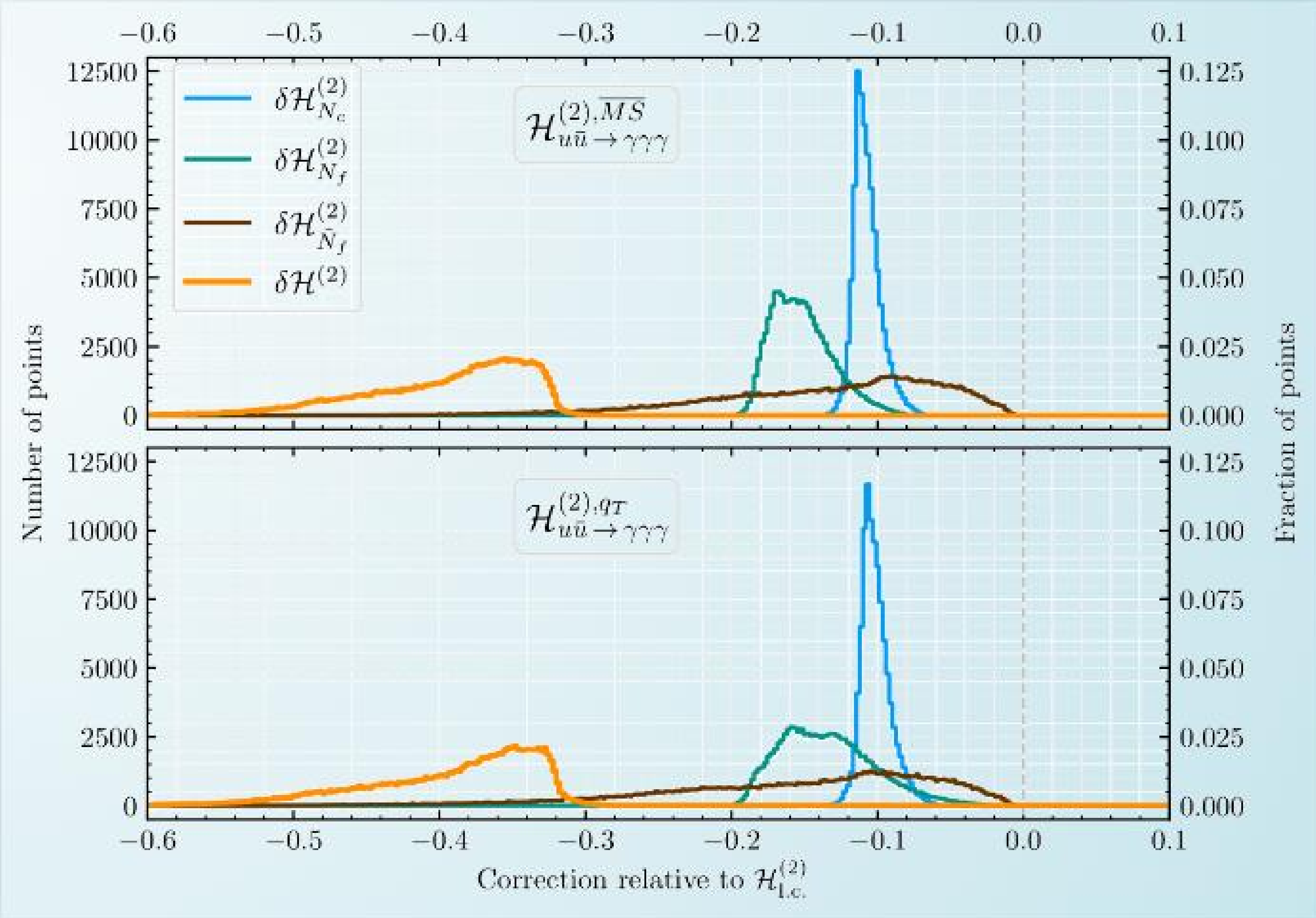
TOWARDS PHENOMENOLOGY

SLC Corrections to the Hard Functions

- Full-color 2-loop remainders & 1-loop amplitudes implemented in an open-source `C++ Program`
- To estimate the impact of the subleading-color contributions, consider the *2-loop hard functions*

$$\mathcal{H}^{(2)} = \sum_h |\mathcal{R}_h|^2 \Big/ \sum_h |\mathcal{A}_h^{(0)}|$$



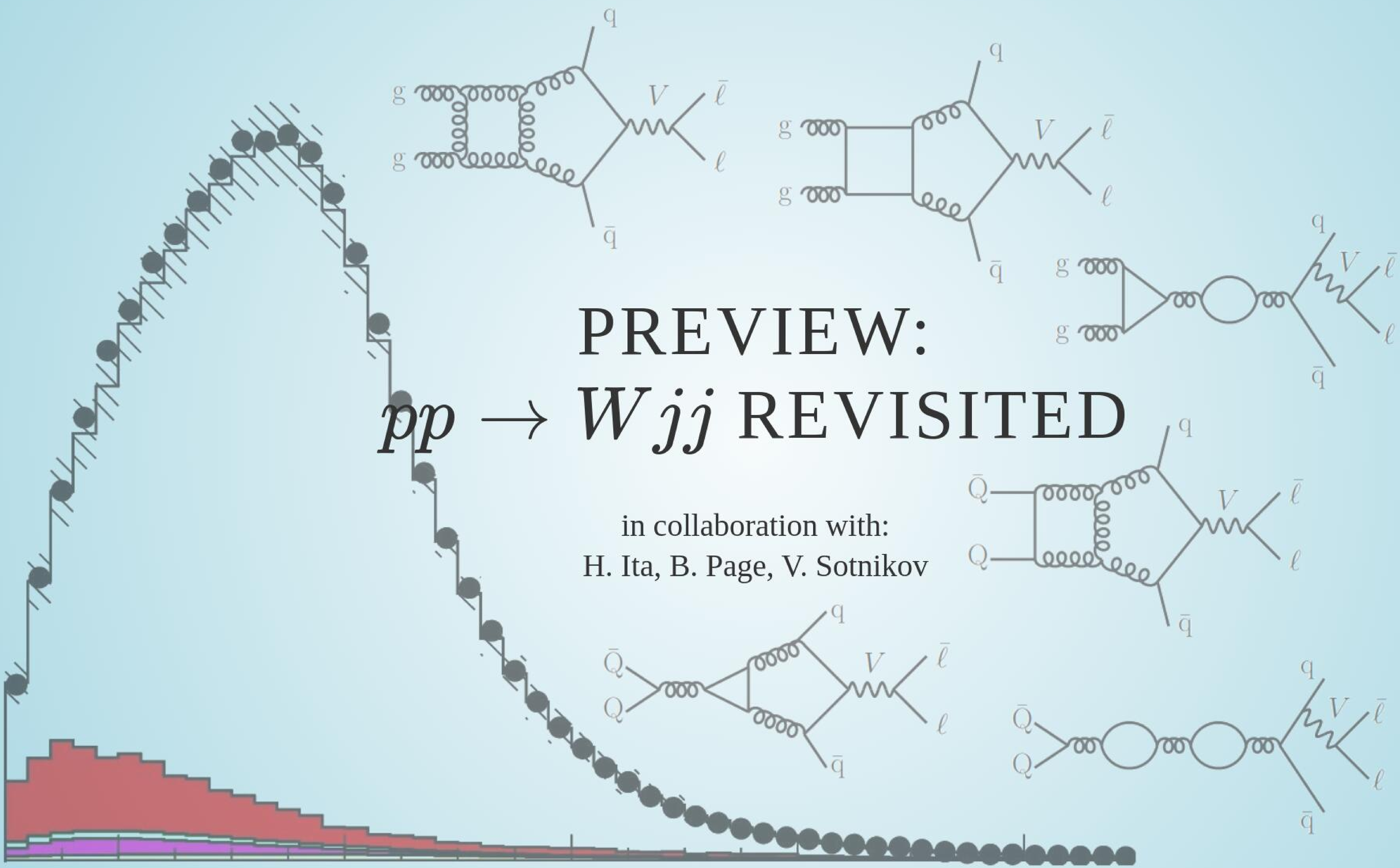


About $25\% - 35\%$ correction to $\mathcal{H}^{(2)}_{\text{l.c.}}$. The correction to $\sigma^{\text{NNLO}}_{q\bar{q}\to\gamma\gamma\gamma}$ will be much smaller.

PREVIEW: $pp \to Wjj$ REVISITED

in collaboration with:

H. Ita, B. Page, V. Sotnikov



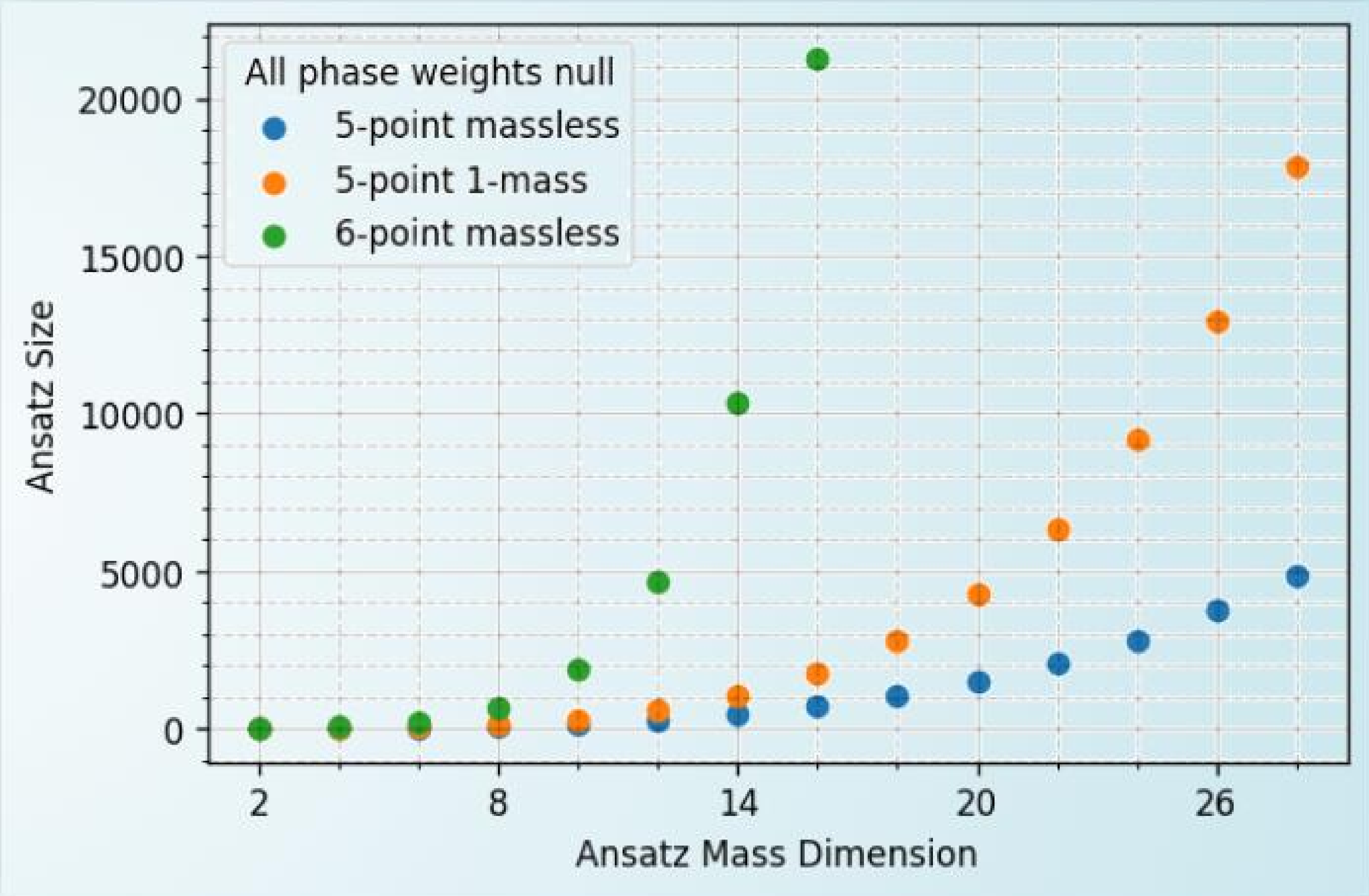
Bottlneck for $pp \to Wjj$ at NNLO

- No pheno study yet, despite the amplitudes have been available for almost 2 years!

- The algebraic complexity—think Ansatz size—
grows quickly with multiplicity (m)
and mass dimension (d):

$$\left(\!\!\binom{m(m-3)/2}{d/2}\!\!\right)$$

is a lower bound. GDL, Maître ('20)



- The anlytic expressions of Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov ('21) are 1.2GB.

Having more control on the analytic structure starts to become important!

Simplification strategy

1. Script to split up the expressions, and compile them ($\sim$ 20GB of `C++`) for evaluation over $\mathbb{F}_p$;
2. Recombine the 3 projections $p_V \parallel p_1, p_V \parallel p_2, p_V \parallel p_3$ and reintroduce the little group factors to build 6-point spinor-helicity amplitudes (subject to degree bounds on $|5\rangle, [5|, |6\rangle, [6|$);
3. Perform partial fraction decompositions* based on expected structures and fit the Ansatze.

Comparison of $q\bar{q} \to \gamma\gamma\gamma$ (in full color) to $pp \to Wjj$ (at leading color):

Kinematics	**# Poles (W)**	**LCD Ansatz**	**Partial-Fraction Ansatz**	**Rational Functions**
5-point massless	30	29k	4k	~300 KB
5-point 1-mass	>200	>5M	~40k	~25 MB

$$\{W_j\} = \bigcup_{\sigma \in \mathrm{Aut}(R_6)} \sigma \circ \big\{\langle 12\rangle, \langle 1|2+3|1], \langle 1|2+3|4], \Delta_{12|34|56}, \langle 3|2|5+6|4|3] - \langle 2|1|5+6|4|2]\big\}$$

* sometimes it's actually a bit more than a partial fraction decomposition, see next slide.

Analytic Structures of 2-loop 5-point 1-mass Amplitudes

- Compact residues for the new 2-loop (spurious?) pole, $\langle k|j|\not{p}_V|l|k] - \langle j|i|\not{p}_V|l|j]$, e.g.:

$$r^{(5\text{ of }54)}_{\bar{u}^+g^+g^+d^-(V\to\ell^+\ell^-)} = \frac{[12][23]\langle 24\rangle\langle 46\rangle^2\langle 1|2+3|4]\langle 2|1+3|4]}{\langle 12\rangle\langle 23\rangle\langle 56\rangle(\langle 3|2|5+6|4|3] - \langle 2|1|5+6|4|2])^2}$$

- The three mass Grams, $\Delta_{12|34|p_V}$, $\Delta_{14|23|p_V}$, behave analogously to one-loop amplitudes, e.g.:

$$r^{(73\text{ of }120)}_{\bar{u}^+g^-g^+d^-(V\to\ell^+\ell^-)} = \frac{105}{128}\frac{\langle 2|1+4|3]\langle 4|2+3|1]\langle 6|1+4|5]s_{14}s_{23}s_{56}(s_{124}-s_{134})(s_{123}-s_{234})(s_{25}+s_{26}+s_{35}+s_{36})}{\langle 3|1+4|2]\Delta^4_{23|14|56}} + \left[-6\frac{[12]^2\langle 13\rangle[25]\langle 34\rangle\langle 36\rangle\langle 56\rangle[56](s_{124}-s_{134})}{\langle 3|1+4|2]^5}\right] + \Big[\Big]_{1234\to\overline{4321}} + \mathcal{O}\left(\frac{1}{\langle 3|1+4|2]^4\Delta^3_{23|14|56}}\right)$$

but the pole orders have been doubled, see Bern, Dixon, Kosower ('97)

- $\Delta_{23|14|56}$ behaves as a perfect square on the surface where $\langle 3|1+4|2]$ vanishes:

$$\sqrt{\Big\langle \langle 3|1+4|2], \Delta_{23|14|56}\Big\rangle_{R_6}} = \Big\langle \langle 3|1+4|2], s_{124}-s_{134}\Big\rangle_{R_6}$$

GDL, Page ('22);
Campbell, GDL, Ellis ('22)

Thank you for your attention!

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