RI-(S)MOM - $\overline{\text{MS}}$ Matching for B_K at Two-Loop Order

Sandra Kvedaraitė
in collaboration with Sebastian Jäger and Martin Gorbahn

LoopFest XXI, June 2023

University of Cincinnati

Overview

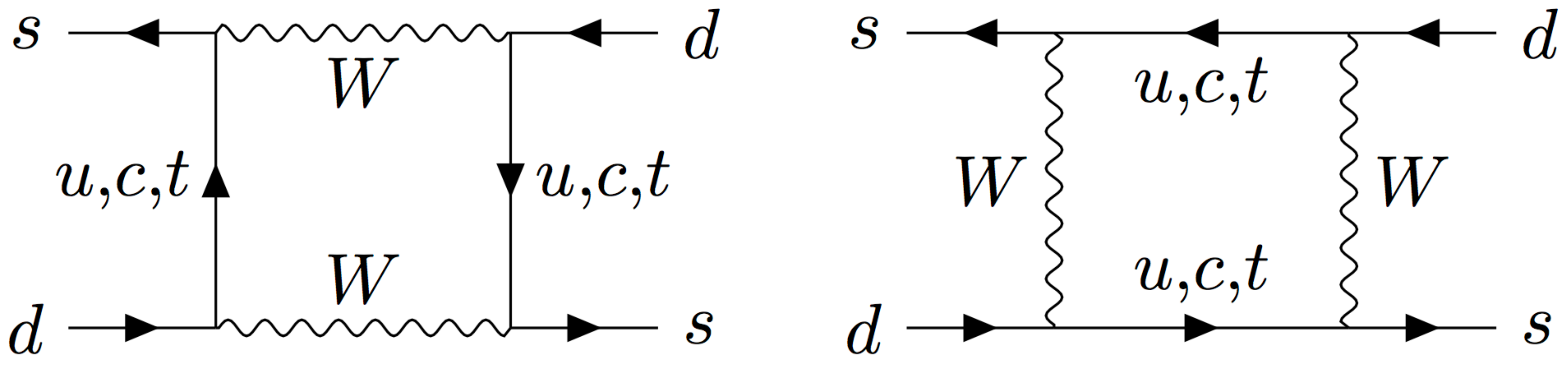
- Introduction
- Momentum space subtraction schemes
- Calculation of the two-loop amplitude
- Results

Introduction - Kaon bag parameter B_K

- Parametrizes matrix element of local ΔS=2 operator

$$Q_{S2} = \left(\bar{s}_L^{\alpha}\gamma_\mu d_L^{\alpha}\right) \otimes \left(\bar{s}_L^{\beta}\gamma^\mu d_L^{\beta}\right)$$

- Arises via Kaon mixing



- Enters dominant contribution to indirect CP violation ϵ_K.

- Wilson coefficients known up to NNLL in QCD and NLL in QED in $\overline{\text{MS}}$ NDR scheme (*JHEP* 12 (2021) 198 (2108.00017) and *JHEP* 12 (2022) 014 (2207.07669)).

Introduction - Kaon bag parameter B_K

- Can be computed non-perturbatively on the lattice, renormalized using some intermediate scheme *A*.

- For phenomenological applications have to transform to renormalization-group-invariant (RGI) bag parameter

$$\hat{B}_K = \hat{C}^{RGI\to A}(\mu)\frac{\langle \bar{K}^0 | Q^A_{S2} | K^0\rangle(\mu)}{\frac{8}{3} f_K^2 m_K^2}$$

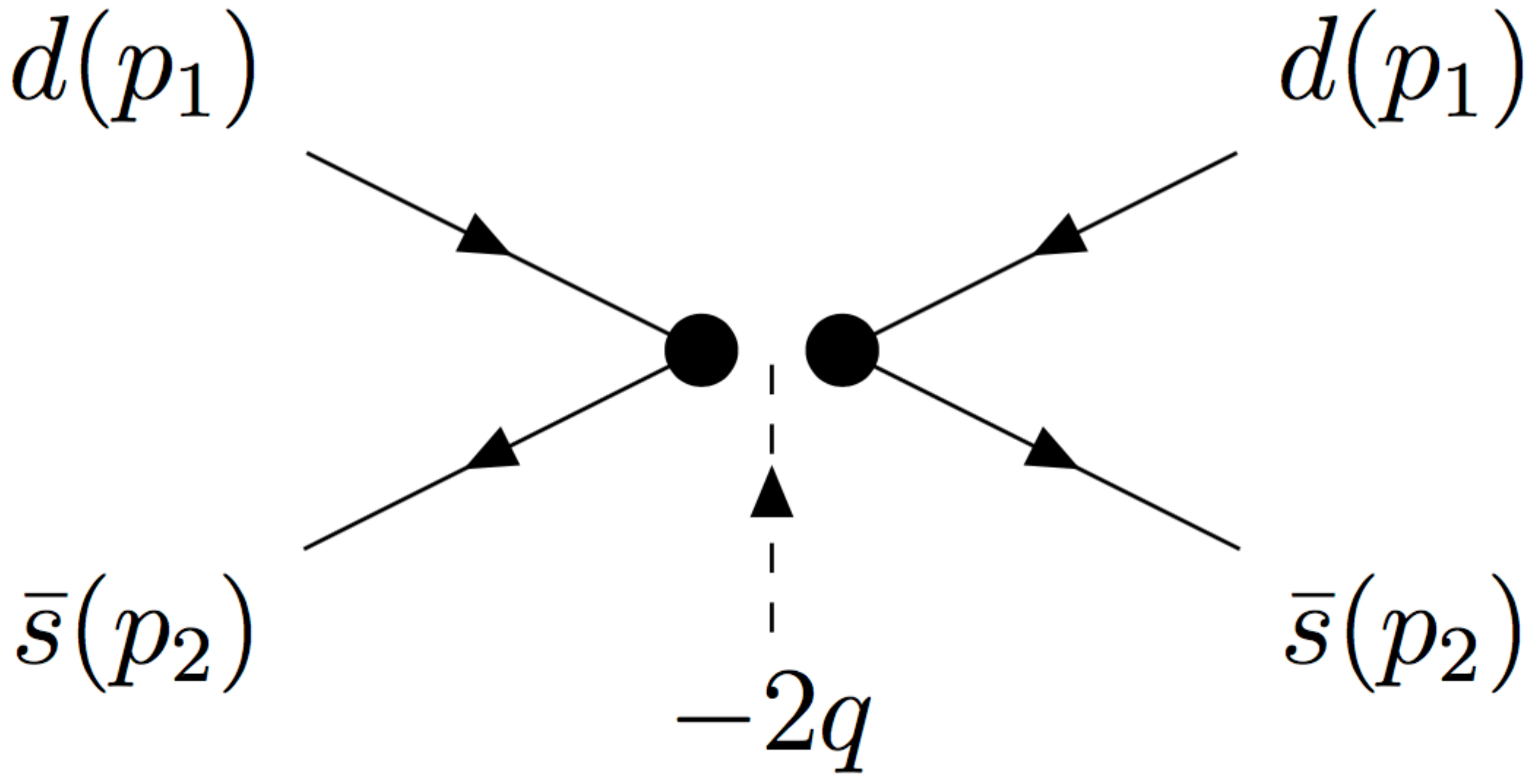
- Current value: $\hat{B}_K = 0.7625(97)$ ($N_f = 2 + 1$ FLAG global average)
(*Eur. Phys. J. C* 82 (2022) 10, 869)

- Intermediate schemes - momentum space subtraction schemes (S)MOM.

Momentum space subtraction schemes

RI scheme kinematics:

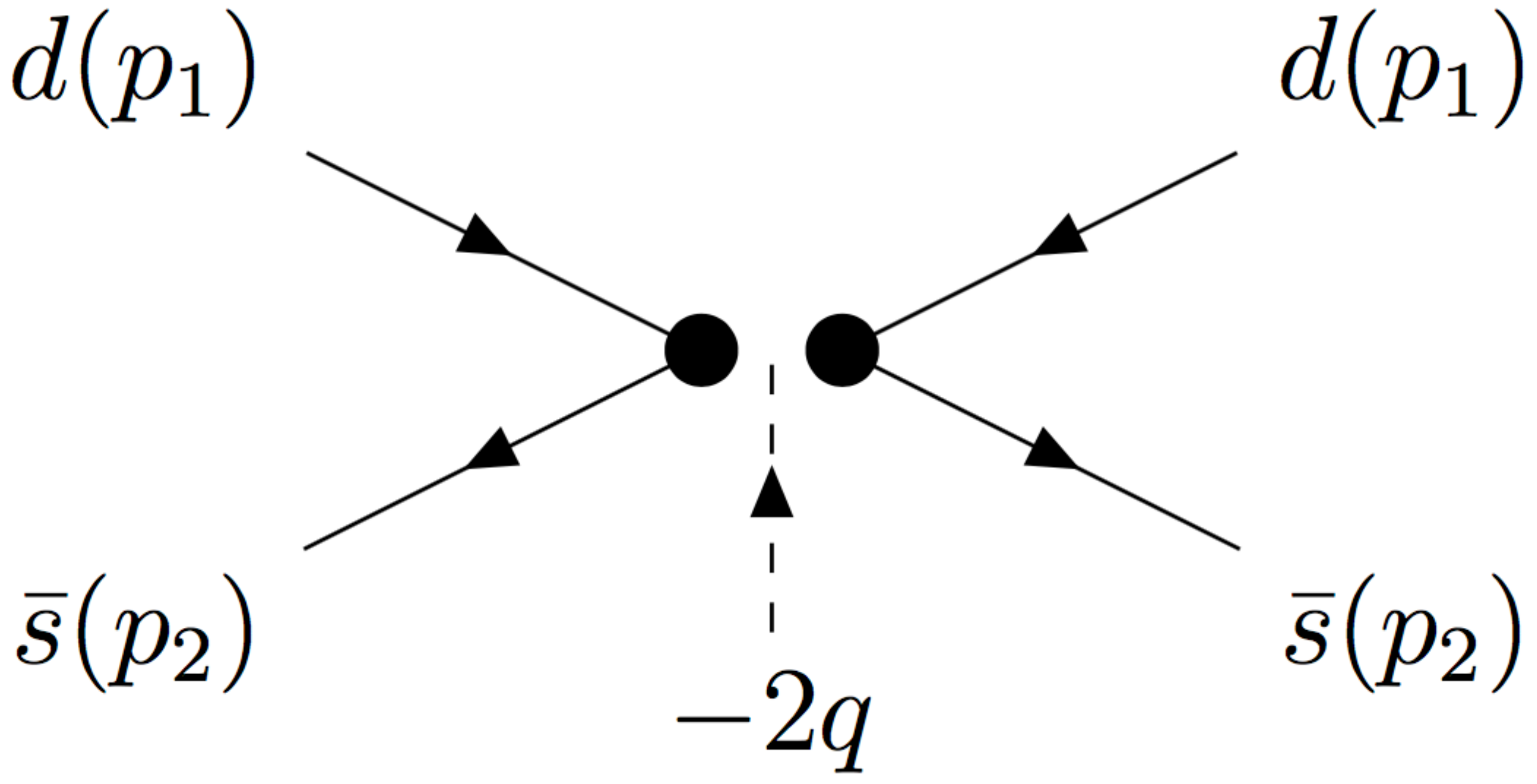
- MOM $p_1^2 = p_2^2 = -\mu^2 \qquad q^2 \equiv (p_1 - p_2)^2 = 0$
- SMOM $p_1^2 = p_2^2 = q^2 = -\mu^2$



Momentum space subtraction schemes

RI scheme kinematics:

- MOM $p_1^2 = p_2^2 = -\mu^2 \qquad q^2 \equiv (p_1 - p_2)^2 = 0$
- SMOM $p_1^2 = p_2^2 = q^2 = -\mu^2$



Amp. of two-point functions S^A:

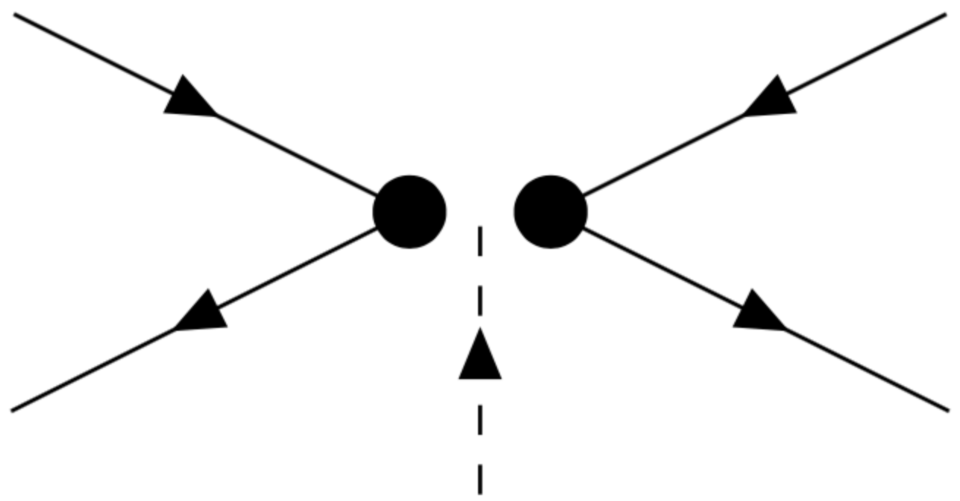
$$\sigma^{(A,\gamma)} \equiv \frac{1}{16}\mathrm{Tr}\left[\gamma^\mu \frac{\partial (S^A)^{-1}(p)}{\partial p^\mu}\right] \qquad \sigma^{(A,\not{p})} \equiv \frac{1}{4\,p^2}\mathrm{Tr}\left[(S^A)^{-1}(p)\not{p}\right]$$

$$\sigma^{(A,\not{p})} = 1$$

$$\sigma^{(A,\gamma)} = 1$$

Momentum space subtraction schemes

RI scheme kinematics:



- MOM $\quad p_1^2 = p_2^2 = -\mu^2 \qquad q^2 \equiv (p_1 - p_2)^2 = 0$
- SMOM $\quad p_1^2 = p_2^2 = q^2 = -\mu^2$

Amp. of two-point functions S^A:

$$\sigma^{(A,\gamma)} \equiv \frac{1}{16}\mathrm{Tr}\left[\gamma^\mu \frac{\partial (S^A)^{-1}(p)}{\partial p^\mu}\right] \qquad \sigma^{(A,\not p)} \equiv \frac{1}{4\,p^2}\mathrm{Tr}\left[(S^A)^{-1}(p)\not p\right]$$

$$\sigma^{(A,\not p)} = 1$$

$$\sigma^{(A,\gamma)} = 1$$

Amp. of four-point functions Λ^A:

$$P^{ij,kl}_{(\gamma_\mu),\alpha\beta,\gamma\delta} = \frac{(\gamma^\nu)_{\beta\alpha}(\gamma_\nu)_{\delta\gamma} + (\gamma^\nu\gamma^5)_{\beta\alpha}(\gamma_\nu\gamma^5)_{\delta\gamma}}{256 N_c(N_c+1)}\delta_{ij}\delta_{kl}$$

$$\lambda^{(A,\mathrm{RI})} \equiv \Lambda^{A,ijkl}_{\alpha\beta\gamma\delta}\mathscr{P}^{ijkl}_{\mathrm{RI}\alpha\beta,\gamma\delta} \qquad \Lambda^{(\mathrm{tree}),ijkl}_{\alpha\beta\gamma\delta}\mathscr{P}^{ijkl}_{\mathrm{RI}\alpha\beta,\gamma\delta} = 1$$

$$P^{ij,kl}_{(\not q),\alpha\beta,\gamma\delta} = \frac{(\not q)_{\beta\alpha}(\not q)_{\delta\gamma} + (\not q\gamma^5)_{\beta\alpha}(\not q\gamma^5)_{\delta\gamma}}{64 q^2 N_c(N_c+1)}\delta_{ij}\delta_{kl}$$

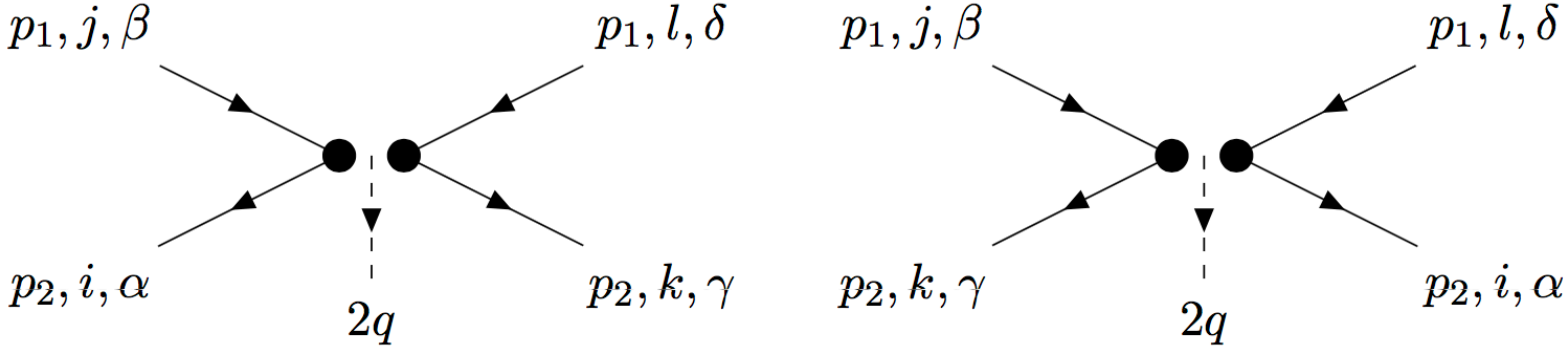
Matching between schemes

- Any two renormalization schemes *A* and *B* are related by

$$O^A = \mathscr{C}_O^{A\to B} O^B$$

- RI-SMOM $\qquad \mathcal{C}_O^{A\to(l,s)} = \lambda^{A,l}\left(\sigma^{A,s}\right)^2, \qquad s,l \in \{\not{q}, \gamma\}$
- RI-MOM $\qquad \mathcal{C}_O^{A\to \text{RI-MOM}} = \lambda^{A,\gamma}\left(\sigma^{A,\gamma}\right)^2$
- RI'-MOM $\qquad \mathcal{C}_O^{A\to \text{RI}'\text{-MOM}} = \lambda^{A,\gamma}\left(\sigma^{A,\not{q}}\right)^2$

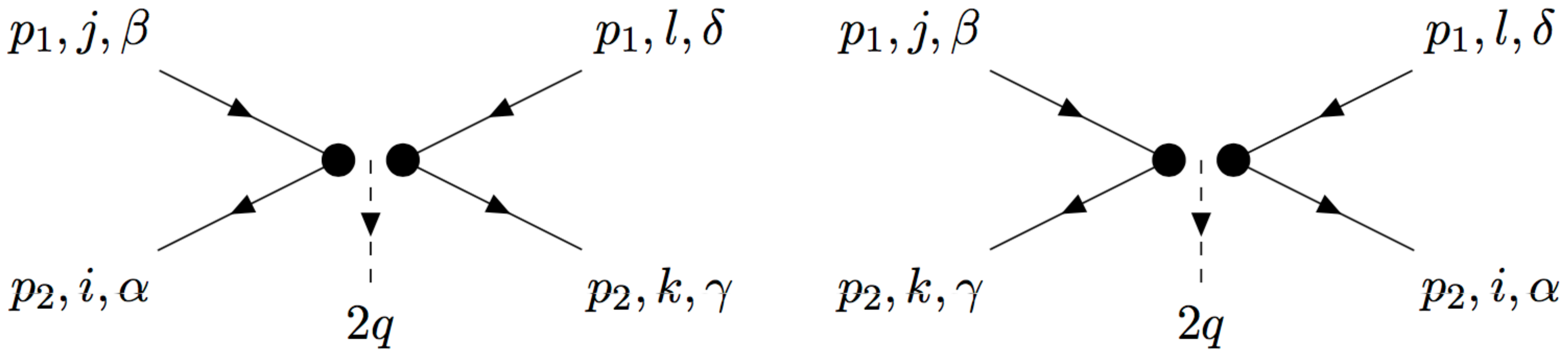
LO amplitude



$$\langle Q \rangle \equiv \Lambda^{ijkl}_{\alpha\beta\gamma\delta}(Q) = 2\left((\gamma^\mu P_L)_{\alpha\beta}(\gamma_\mu P_L)_{\gamma\delta}\delta^{ij}\delta^{kl} - (\gamma^\mu P_L)_{\alpha\delta}(\gamma_\mu P_L)_{\gamma\beta}\delta^{il}\delta^{kj}\right)$$

$$\equiv 2\gamma^\mu P_L \otimes \gamma_\mu P_L \; 1 \otimes 1 - 2\,\gamma^\mu P_L \,\tilde{\otimes}\, \gamma_\mu P_L \; 1 \,\tilde{\otimes}\, 1 = \langle Q \rangle_2 + \langle Q \rangle_1$$

LO amplitude



$$\langle Q \rangle \equiv \Lambda^{ijkl}_{\alpha\beta\gamma\delta}(Q) = 2\left((\gamma^\mu P_L)_{\alpha\beta}(\gamma_\mu P_L)_{\gamma\delta}\delta^{ij}\delta^{kl} - (\gamma^\mu P_L)_{\alpha\delta}(\gamma_\mu P_L)_{\gamma\beta}\delta^{il}\delta^{kj} \right)$$

$$\equiv 2\gamma^\mu P_L \otimes \gamma_\mu P_L \; 1 \otimes 1 - 2\,\gamma^\mu P_L \,\tilde{\otimes}\, \gamma_\mu P_L \; 1 \,\tilde{\otimes}\, 1 = \langle Q \rangle_2 + \langle Q \rangle_1$$

$$P_{\alpha\beta,\gamma\delta} \propto X_{\beta\alpha} Y_{\delta\gamma} \delta_{ij} \delta_{kl}$$

$P\langle Q \rangle_1 \propto \mathrm{Tr}(X\gamma^\mu P_L Y \gamma_\mu P_L)$ "direct"

$P\langle Q \rangle_2 \propto \mathrm{Tr}(X\gamma^\mu P_L)\mathrm{Tr}(Y\gamma_\mu P_L)$ "crossed"

$\overline{\text{MS}}$ NDR scheme and γ_5

- Define traces over γ_5 consistently

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5) = 4i\epsilon^{\mu\nu\rho\sigma} \qquad \text{4 dimensions}$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5) = 0 \qquad D \text{ dimensions}$$

- Evanescent operators, e.g. $E_1 = (\bar{s}^i \gamma^{\mu_1\mu_2\mu_3} P_L d^j)(\bar{s}^k \gamma_{\mu_1\mu_2\mu_3} P_L d^l) - (16 - 4\epsilon - 4\epsilon^2)Q$

$\overline{\text{MS}}$ NDR scheme and γ_5

- Define traces over γ_5 consistently

$$\text{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) = 4i\epsilon^{\mu\nu\rho\sigma}$$ 4 dimensions

$$\text{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) = 0$$ *D* dimensions

- Evanescent operators, e.g. $E_1 = (\bar{s}^i\gamma^{\mu_1\mu_2\mu_3}P_L d^j)(\bar{s}^k\gamma_{\mu_1\mu_2\mu_3}P_L d^l) - (16 - 4\epsilon - 4\epsilon^2)Q$
- For "single" trace use "Greek projections" to project out the evanescent part of the result (A. J. Buras and P. H. Weisz (1989), N. Tracas and N. Vlachos (1982)).

$$\gamma^\mu P_L\gamma^\tau\gamma_\mu P_L = (2-d)\gamma^\tau P_L \qquad\qquad \Gamma\gamma^\tau\Gamma' = 0$$

- No way to define the "double" traces.

“Direct” amplitude method

For a set of operators Q_i and $\tilde{Q}_i$, where tilde means $\otimes \leftrightarrow \tilde{\otimes}$ for color, and “greek” projectors Π_i

Full amplitude

$$\Lambda = \sum_{i=1}^{4} \left(A_i \langle Q_i \rangle + \tilde{A}_i \langle \tilde{Q}_i \rangle \right) + \text{evanescent}$$

“Direct” amplitude method

For a set of operators Q_i and $\tilde{Q}_i$, where tilde means $\otimes \leftrightarrow \tilde{\otimes}$ for color, and “greek” projectors Π_i

Full amplitude

$$\Lambda = \sum_{i=1}^{4} (A_i \langle Q_i \rangle + \tilde{A}_i \langle \tilde{Q}_i \rangle) + \text{evanescent}$$

“Direct” amplitude

$$\Lambda_1 = \sum_{i=1}^{4} (A_i \langle Q_i \rangle_1 + \tilde{A}_i \langle \tilde{Q}_i \rangle_1) + \text{evanescent}$$

"Direct" amplitude method

For a set of operators Q_i and $\tilde{Q}_i$, where tilde means $\otimes \leftrightarrow \tilde{\otimes}$ for color, and "greek" projectors Π_i

Full amplitude

$$\Lambda = \sum_{i=1}^{4} (A_i\langle Q_i\rangle + \tilde{A}_i\langle \tilde{Q}_i\rangle) + \text{evanescent}$$

"Direct" amplitude

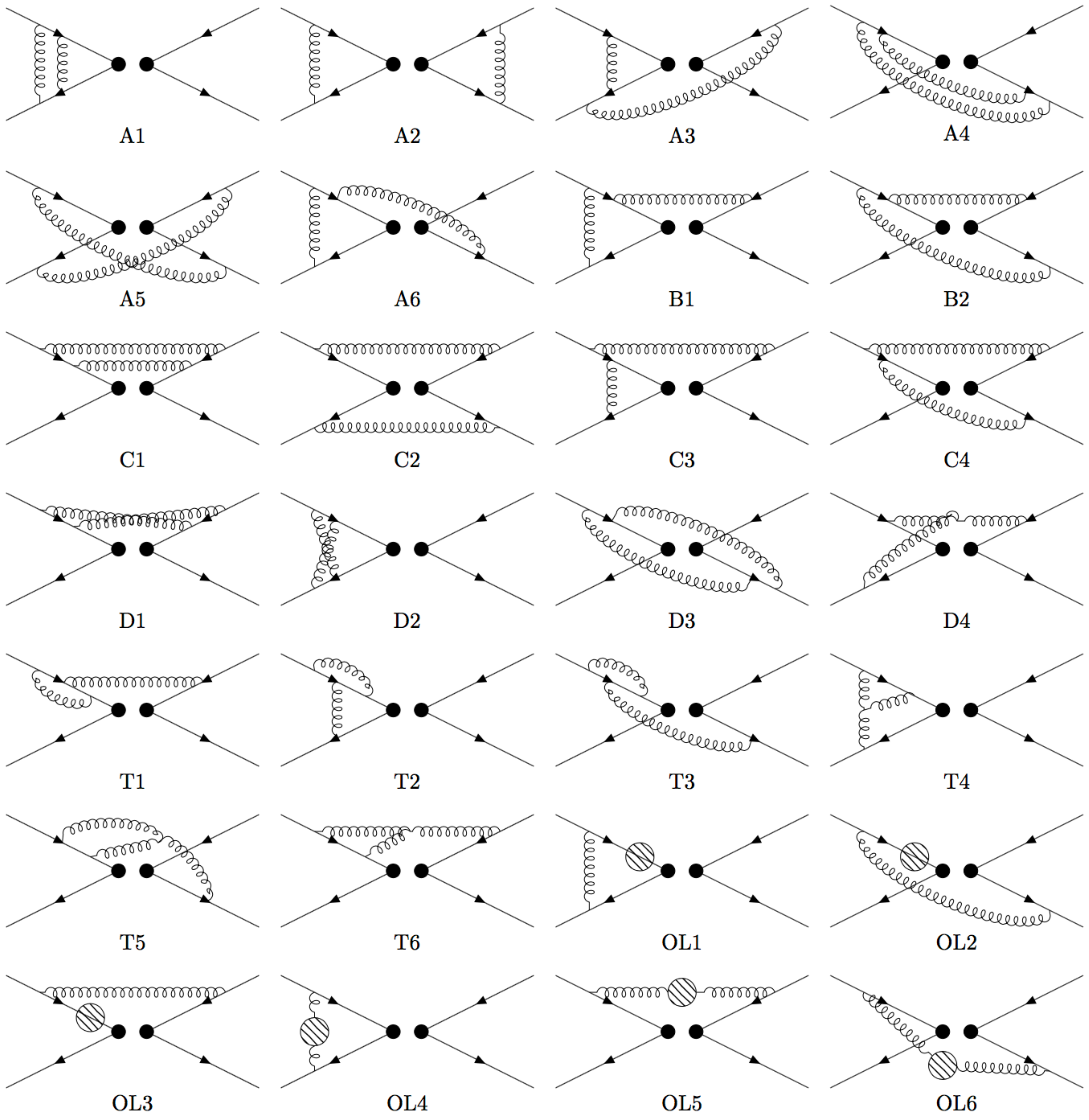
$$\Lambda_1 = \sum_{i=1}^{4} (A_i\langle Q_i\rangle_1 + \tilde{A}_i\langle \tilde{Q}_i\rangle_1) + \text{evanescent}$$

Can compute

$$\Pi_i(\Lambda_1) = \sum_{j=1}^{4} \left(A_j\Pi_i(\langle Q_j\rangle_1) + \tilde{A}_j\Pi_i(\langle \tilde{Q}_j\rangle_1)\right)$$

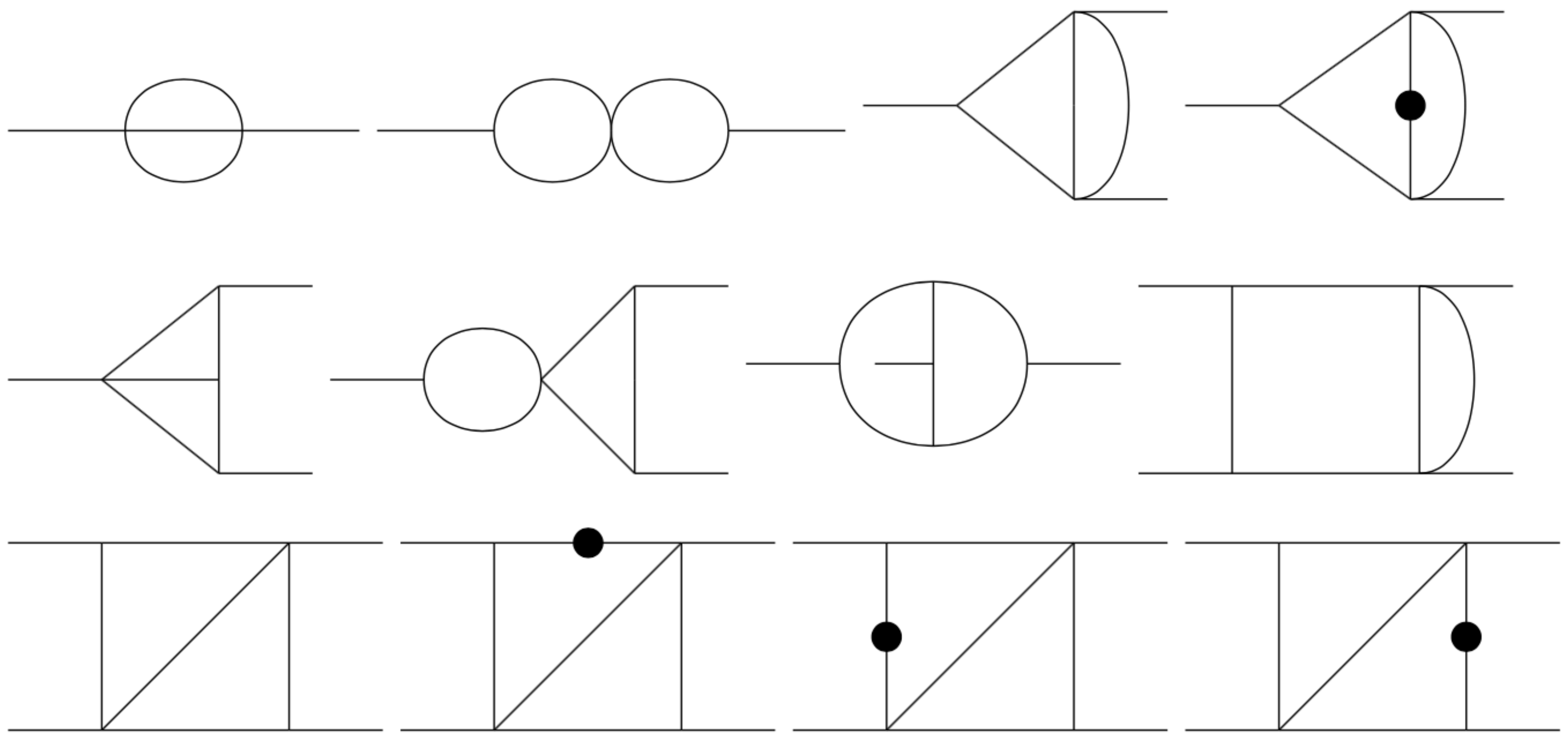
$$= \sum_{j=1}^{4} B_{ij}\left(A_j 1 \,\tilde{\otimes}\, 1 + \tilde{A}_j 1 \otimes 1\right) = \sum_{j=1}^{4} \left(C_i 1 \,\tilde{\otimes}\, 1 + \tilde{C}_i 1 \otimes 1\right)$$

NNLO amplitude



- Direct part - 103 diagrams
- Trace over γ_5 in D dimensions
- Tensor reduction - via IBP
- Reduction into a set of master integrals
- Partially numerical evaluation
- Renormalization

Master integrals and evaluation



- All bubble and triangle diagrams have been calculated analytically by N. I. Ussyukina and A. I. Davydychev (1994).
- Box diagrams are obtained via sector decomposition method using PySecDec.

Results

NNLO (S)MOM to $\overline{\text{MS}}$ NDR conversion factors

- $N_c = 3$
- $\mu^2 = \nu^2 = 3$ GeV

S	$C^S_{B_K,\text{NLO}}$	$C^S_{B_K,\text{NNLO}}$
$(\gamma_\mu, \not{q})$	$8\log 2 - 8 = -2.45482\ldots$	$3.88N_f + 21.05 \pm 0.08$
(γ_μ, γ_μ)	$8\log 2 - 16/3 = 0.211844\ldots$	$-0.42N_f + 86.41 \pm 0.08$
$(\not{q}, \not{q})$	$8\log 2 - 6 = -0.454823\ldots$	$0.90N_f + 52.78 \pm 0.09$
$(\not{q}, \gamma_\mu)$	$8\log 2 - 10/3 = 2.21184\ldots$	$-3.39N_f + 123.47 \pm 0.09$
RI′-MOM	$8\log 2 - 14/3 = 0.878511\ldots$	$0.17N_f + 61.71 \pm 0.07$
RI-MOM	$8\log 2 - 14/3 = 0.878511\ldots$	$-4.49N_f + 94.04 \pm 0.07$

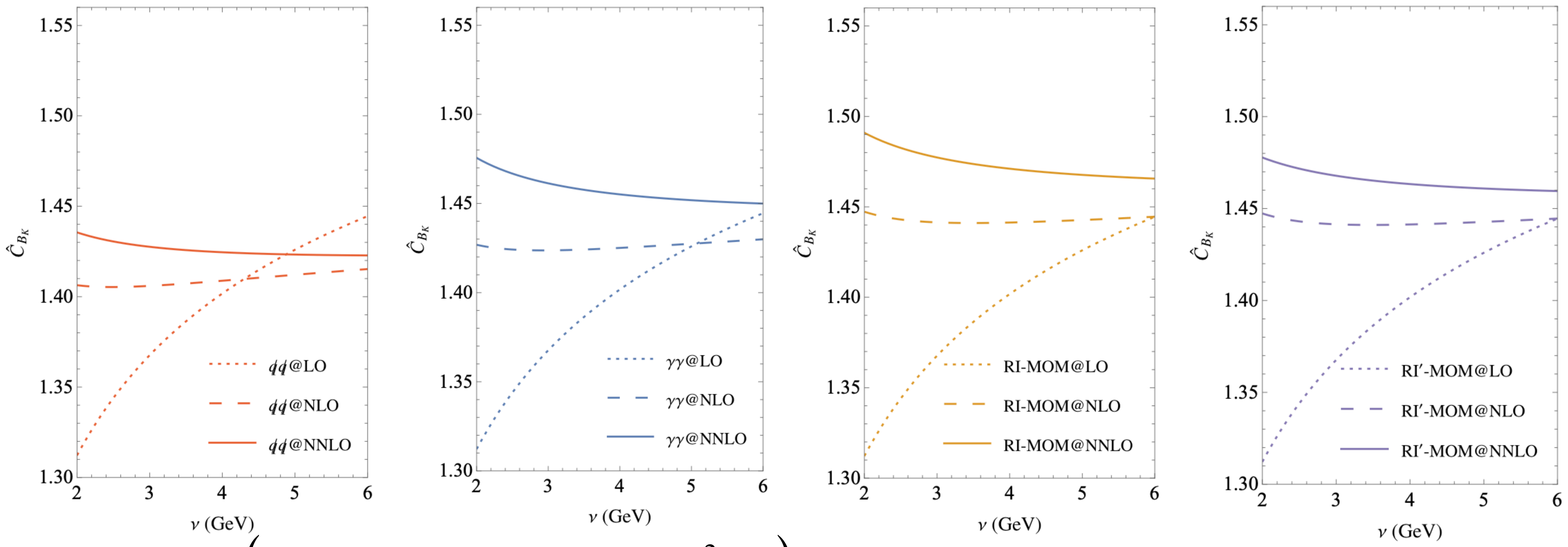
NNLO (S)MOM to $\overline{\text{MS}}$ NDR conversion factors

- $N_c = 3$
- $\mu^2 = \nu^2 = 3$ GeV
- $L(\mu, \nu) = \log(\nu/\mu)$

S	$C^S_{B_K,\text{NLO}}$	$C^S_{B_K,\text{NNLO}}$
$(\gamma_\mu, \not{q})$	$8\log 2 - 8 = -2.45482...$	$3.88 N_f + 21.05 \pm 0.08$
(γ_μ, γ_μ)	$8\log 2 - 16/3 = 0.211844...$	$-0.42 N_f + 86.41 \pm 0.08$
$(\not{q}, \not{q})$	$8\log 2 - 6 = -0.454823...$	$0.90 N_f + 52.78 \pm 0.09$
$(\not{q}, \gamma_\mu)$	$8\log 2 - 10/3 = 2.21184...$	$-3.39 N_f + 123.47 \pm 0.09$
RI′-MOM	$8\log 2 - 14/3 = 0.878511...$	$0.17 N_f + 61.71 \pm 0.07$
RI-MOM	$8\log 2 - 14/3 = 0.878511...$	$-4.49 N_f + 94.04 \pm 0.07$

$$C^S_{B_K}(\mu,\nu) = 1 + \frac{\alpha_s(\nu)}{4\pi}\left[C^S_{B_K,\text{NLO}} - 4L(\mu,\nu)\right] + \frac{\alpha_s^2(\nu)}{16\pi^2}\left[C^S_{B_K,\text{NNLO}}\right.$$
$$\left. + C^S_{B_K,\text{NLO}} L(\mu,\nu)(18 - \tfrac{3}{9}f) + L(\mu,\nu)\left\{(7 - \tfrac{4}{9}f) - L(\mu,\nu)(72 - \tfrac{12}{9}f)\right\}\right]$$

Renormalization scale dependence of conversion factors

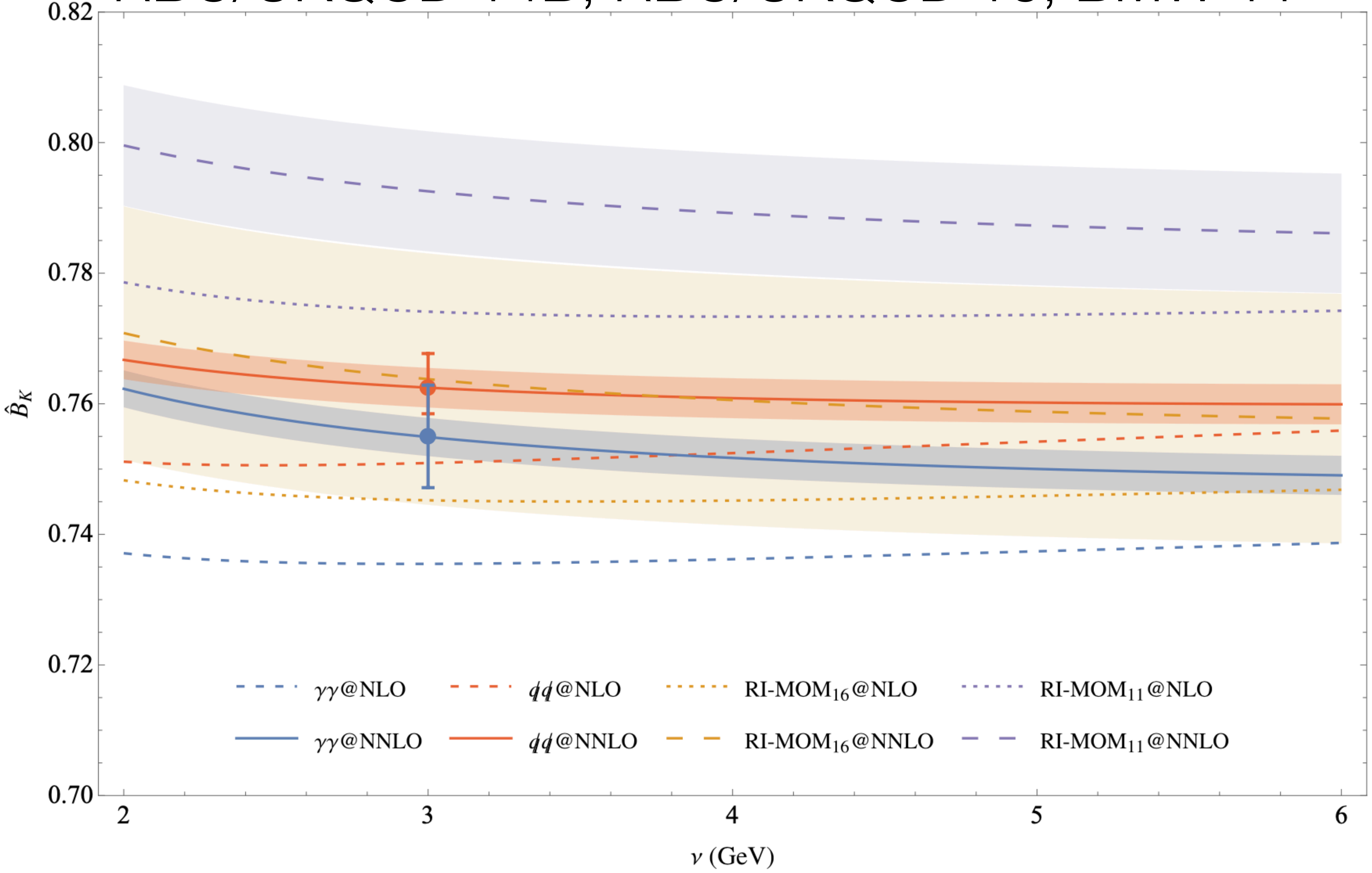


$$\hat{C}^S_{B_K} = U^{(0)}_{(f)}(\nu)\left(1 + \frac{\alpha_s^{(f)}(\nu)}{4\pi}J^{(1)}_{(f)} + \left(\frac{\alpha_s^{(f)}(\nu)}{4\pi}\right)^2 J^{(2)}_{(f)}\right) C^S_{B_K}(\nu)$$

$\mu = 3$ GeV and $f = 3$

Renormalization scale dependence of B_K

RBC/UKQCD 14B, RBC/UKQCD 16, BMW 11



B_K average

fl.	Ref.	Scheme	μ (GeV)	$B_K^{(\text{Scheme})}(\mu)$	$\hat{B}_{K,\text{NLO}}^{(f)}$
4	ETM 15	RI′-MOM	3	0.498(16)	0.717(24)
3	SWME 15A	$\overline{\text{MS}}$	3	0.519(26)	0.735(36)
3	RBC/	$(\not{q}, \not{q})$	3	0.5341(18)	0.7499(152)
	UKQCD 14B	(γ_μ, γ_μ)	3	0.5166(18)	-
3	Laiho 11	$\overline{\text{MS}}$	2	0.5572(152)	0.7628(208)
3	BMW 11	RI-MOM	3.5	0.5308(61)	0.7727(117)

B_K average

fl.	Ref.	Scheme	μ (GeV)	$B_K^{(\text{Scheme})}(\mu)$	$\hat{B}_{K,\text{NLO}}^{(f)}$	$\hat{B}_K^{(f=3)}$	$\hat{B}_K^{(f=4)}$
4	ETM 15	RI′-MOM	3	0.498(16)	0.717(24)	0.733(25)	0.745(25)
3	SWME 15A	$\overline{\text{MS}}$	3	0.519(26)	0.735(36)	0.735(36)*	0.747(37)
3	RBC/	$(\not{q}, \not{q})$	3	0.5341(18)	0.7499(152)	0.7625(52)	0.7748(78)
	UKQCD 14B	(γ_μ, γ_μ)	3	0.5166(18)	-	0.7549(79)	0.7650(98)
3	Laiho 11	$\overline{\text{MS}}$	2	0.5572(152)	0.7628(208)	0.7628(208)*	0.7750(219)
3	BMW 11	RI-MOM	3.5	0.5308(61)	0.7727(117)	0.791(13)	0.804(15)

B_K average - preliminary

fl.	Ref.	Scheme	μ (GeV)	$B_K^{(\text{Scheme})}(\mu)$	$\hat{B}_{K,\text{NLO}}^{(f)}$	$\hat{B}_K^{(f=3)}$	$\hat{B}_K^{(f=4)}$
4	ETM 15	RI′-MOM	3	0.498(16)	0.717(24)	0.733(25)	0.745(25)
3	SWME 15A	$\overline{\text{MS}}$	3	0.519(26)	0.735(36)	0.735(36)*	0.747(37)
3	RBC/	$(\not{q}, \not{q})$	3	0.5341(18)	0.7499(152)	0.7625(52)	0.7748(78)
	UKQCD 14B	(γ_μ, γ_μ)	3	0.5166(18)	-	0.7549(79)	0.7650(98)
3	Laiho 11	$\overline{\text{MS}}$	2	0.5572(152)	0.7628(208)	0.7628(208)*	0.7750(219)
3	BMW 11	RI-MOM	3.5	0.5308(61)	0.7727(117)	0.791(13)	0.804(15)

$N_f = 2+1:$ $\hat{B}_K = 0.7625(57)$ (prev. $\hat{B}_K = 0.7625(97)$) $\chi^2/\text{dof} = 1.941$

$N_f = 2+1+1:$ $\hat{B}_K = 0.745(25)$ (prev. $\hat{B}_K = 0.717(24)$)

Conclusion

- Conversion factors between momentum space subtraction schemes and $\overline{\text{MS}}$ NDR and RGI values.

- Conversion between different numbers of flavours including threshold corrections.

- Uncertainty on the Kaon bag parameter down from 1.3% to 0.75% (exp. $|\epsilon_K| = 2.228(11) \times 10^{-3} \approx 0.5\,\%$ uncertainty).

- Complimentary to existing two-loop QCD and QED corrections to ϵ_K as well as the upcoming three-loop QCD corrections.