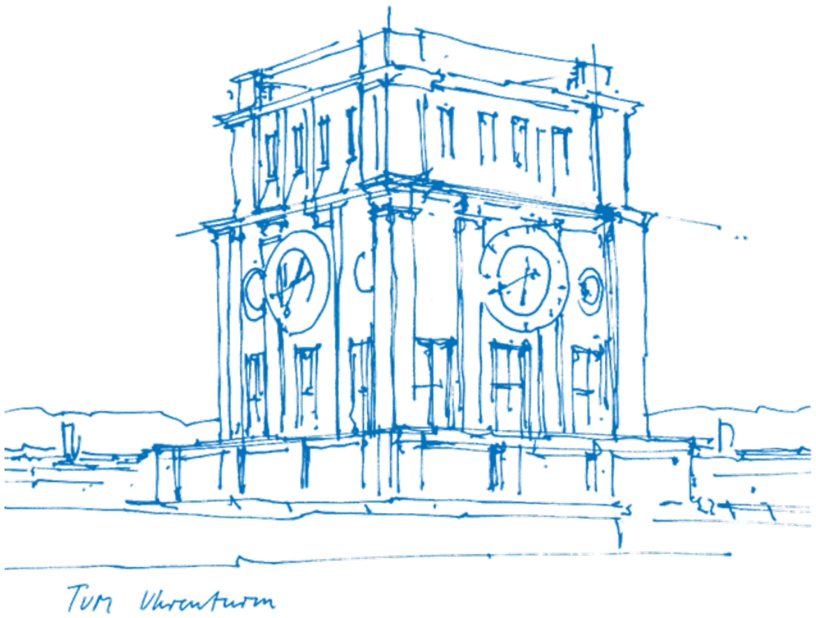
On phase-space integrals with Heaviside functions

In collaboration with: D. Baranowski, K. Melnikov, A. Pikelner and C.-Y. Wang
Based on: 2111.13594 and 2204.094559

Maximilian Delto
Loopfest XXI
27th of June, 2023



Fully differential predictions for LHC

[Chen et al. '22]

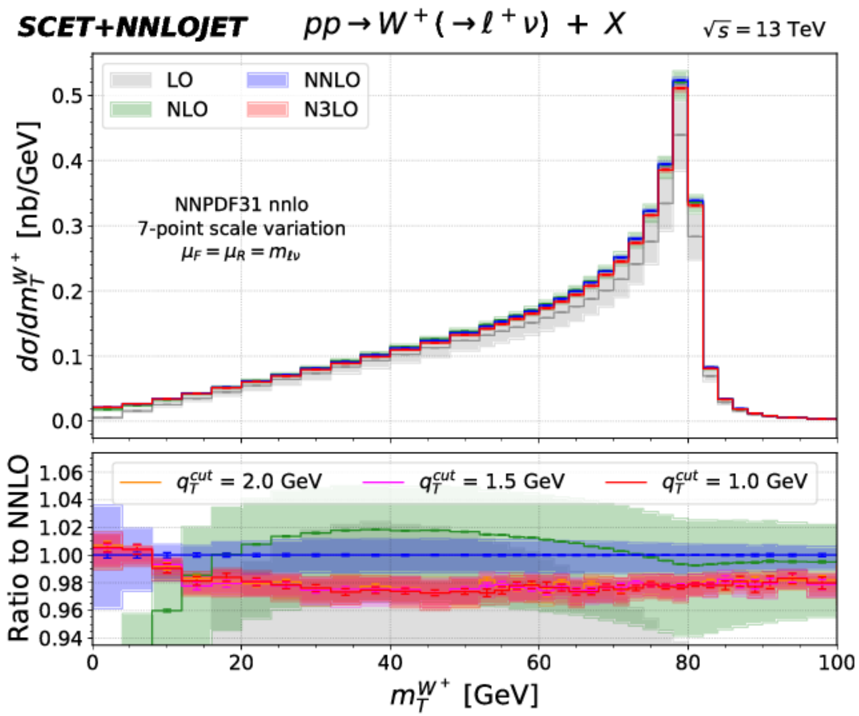
Fully-differential description of LHC cross sections for color singlet production has reached third order in perturbative QCD.

- ⇒ Higgs production [Chen et al. '21]
- ⇒ Drell-Yan [Chen et al. '22], [Chen et al. '22]

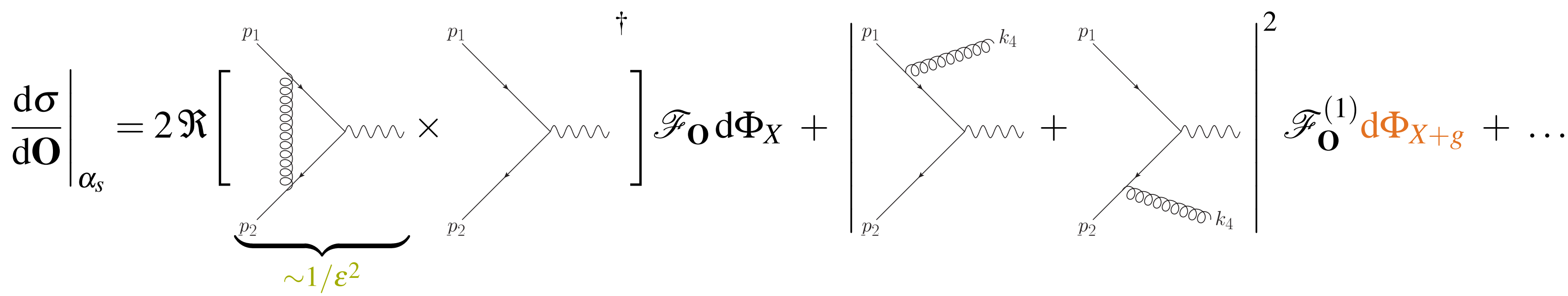
...as well as re-computation including resummation

- ⇒ Drell-Yan $\mathrm{N^4LL}_p + \mathrm{N^3LO}$ [Neumann et al. '22]

These computations are vital for precise studies at the LHC.



One (of many) challenges: explicit vs. implicit IR singularities



Handling IR divergences

Subtraction schemes

⇒ subtract and add back a term that approximates the divergent behavior

$$\mathscr{I} = \frac{f(0)}{n\varepsilon} + \int_0^1 \mathrm{d}x \frac{f(x) - f(0)}{x} + \mathscr{O}(\varepsilon)$$

☺ local regularisation of divergences

☺ numerically stable

☹ complex construction, tedious to implement

for example

- CoLoRFull [Somogyi et al. ’05]
- Antenna [Gehrmann-De Ridder et al. ’05]
- STRIPPER [Czakon ’10]
- Nested soft-collinear subtraction [Caola et al. ’17]
- Local Analytic Sector Subtraction [Magnea et al. ’18]
- . . .

Slicing schemes

⇒ slice phase space into divergent and finite region

$$\mathscr{I} = \left[\frac{1}{n\varepsilon} + \ln(\delta_{\text{cut}})\right] f(0) + \int_{\delta_{\text{cut}}}^1 \frac{\mathrm{d}x}{x} f(x) + \mathscr{O}(\delta, \varepsilon)$$

☹ small cut-off leads to large numerical cancellations and requires precise control of f

☺ achievable at N3LO

for example

- q_T-slicing [Catani et al. ’07]
- N-jettiness slicing [Boughezal et al. ’15]
- . . .

Other schemes

- projection-to-Born [Cacciari et al. ’15]
- local unitarity [Capatti et al. ’20]
- . . .

Zero-jettiness slicing

Introduce [see Riccardo's talk]

$$\tau = \sum_j \min_{q \in \{n,\bar{n}\}} \{q \cdot k_j\} \; , \; q \cdot k_j = k_j^0 (1 \mp \cos\theta_j) \text{ for } q = n, \bar{n}$$

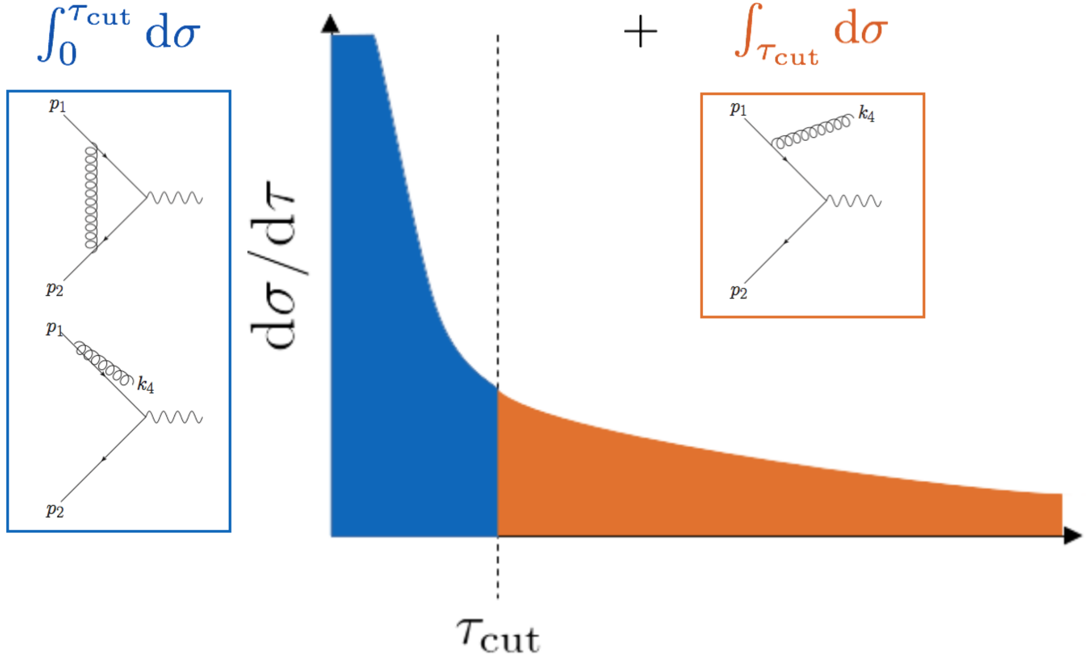
where $\tau = 0 \Leftrightarrow$ *all* emitted partons unresolved (soft/collinear to incoming partons).

Write cross-section as

$$\sigma_{\rm NLO} = \int \frac{d\sigma}{d\tau} \, d\tau$$

and slice phase-space

$$\sigma_{\rm NLO} = \underbrace{\int_0^{\tau_{\rm cut}} d\sigma}_{\text{fully unresolved emission}} + \underbrace{\int_{\tau_{\rm cut}} d\sigma}_{\geq 1 \text{ resolved emission}}$$



Zero-jettiness factorization

The below-cut region factorizes [Stewart et al. '09]

$$d\sigma = B \otimes B \otimes S \otimes H \otimes d\sigma_{\text{Born}} + \mathcal{O}(\tau)$$

Beam function

- collinear emission off incoming partons
- known through N3LO [Behring et al '19] [Ebert et al. '20] [Baranowski et al '22]

Soft function

- soft, non-collinear emission
- known through N2LO [Kelley et al. '11] [Monni et al. '11]
- RRV N3LO [Chen et al. '22]

Hard function

- contains virtual corrections

⇒ factorisation allows for explicit computation and cancellation of $1/\varepsilon$ IR poles

⇒ remaining $\mathcal{O}(\varepsilon^0)$ coefficient yields below-cut cross section

⇒ potential linear power corrections, see e.g. [Alekhin et al. '21]

Missing ingredient is the triple-real contribution to the soft function.

$$S^{(3)} = \int d\Phi_3 \left|\boldsymbol{J}_{ggg}\right|^2 + \int d\Phi_3 \left|\boldsymbol{J}_{gq\bar{q}}\right|^2$$

Three-gluon-emission contribution

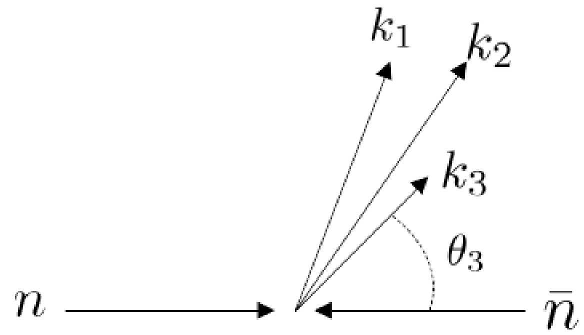
The maximally non-abelian $3g$ contribution

$$S^{ggg}\Big|_{C_a C_A^2} = 2\int \mathrm{d}\Phi^{nnn}_{\theta\theta\theta}\,|\boldsymbol{J}(k_1,k_2,k_3)|^2 \;+\; 6\int \mathrm{d}\Phi^{nn\bar{n}}_{\theta\theta\theta}\,|\boldsymbol{J}(k_1,k_2,k_3)|^2$$

- triple-soft eikonal function $|\boldsymbol{J}|^2$ [Catani et al. '19] contains $\{q\cdot k_i\ ,\ q\cdot k_{ij}\ ,\ q\cdot k_{123}\ ,\ k_i\cdot k_j\ ,\ k_{123}^2\}$, $q=n,\bar{n}$
- zero-jettiness $\tau = \sum_j \min_{q\in\{n,\bar{n}\}}\{q\cdot k_j\}$

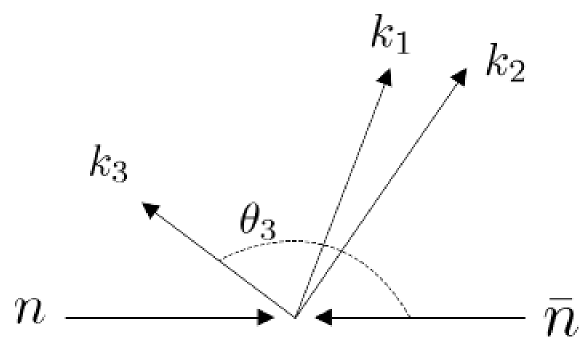
has two distinguishable configurations

1) $\mathrm{d}\Phi^{nnn}_{\theta\theta\theta} = \widetilde{[\mathrm{d}k]}\,\delta[\tau - k_{123}\cdot n]\prod_{i=1}^{3}\theta[k_i\cdot\bar{n} - k_i\cdot n]$



2) $\mathrm{d}\Phi^{nn\bar{n}}_{\theta\theta\theta} = \widetilde{[\mathrm{d}k]}\,\delta[\tau - k_{12}\cdot n - k_3\cdot\bar{n}]\prod_{i=1}^{2}\theta[k_i\cdot\bar{n} - k_i\cdot n]$

$f_3(k_3\cdot n - k_3\cdot\bar{n})$



where $\widetilde{[\mathrm{d}k]} = \prod_{i=1}^{3}\mathrm{d}^{\mathrm{d}}k_i\,\delta\left[k_i^2\right]$

Integration-by-parts relations

$$0 = \int \prod_{l=1}^{L} \mathrm{d}^{\mathrm{d}} k_l \frac{\partial}{\partial k_i^{\mu}} \left[q^{\mu} \prod_{n=1}^{N} D_n^{-\alpha_n} \right] , \quad q \in \{k_1, .., k_L, p_1, \ldots, p_E\} , \quad \alpha_n \in \mathbb{Z} ,$$ [Chetyrkin,Tkachov '81]

- set of linearly independent propagators $\{D_n\}$ defines topology uniquely
- IBP relations relate different integrals within a given topology

$\Rightarrow$ reduce given problem to minimal set of "master integrals" [AIR,FIRE,Reduze,LiteRed,Kira,..]

Integration-by-parts relations

$$0 = \int \prod_{l=1}^{L} \mathrm{d}^d k_l \frac{\partial}{\partial k_i^\mu} \left[q^\mu \prod_{n=1}^{N} D_n^{-\alpha_n} \right], \quad q \in \{k_1, .., k_L, p_1, \dots, p_E\}, \quad \alpha_n \in \mathbb{Z},$$ [Chetyrkin,Tkachov '81]

- set of linearly independent propagators $\{D_n\}$ defines topology uniquely
- IBP relations relate different integrals within a given topology

$\Rightarrow$ reduce given problem to minimal set of "master integrals" [AIR,FIRE,Reduze,LiteRed,Kira,..]

---

1. δ-function: reverse unitarity [Anastasiou,Melnikov '02]

$$(-2\pi \mathrm{i})\, \delta\left[q^2 - m^2\right] = \lim_{\sigma \to 0^+} \left[\frac{1}{q^2 - m^2 + \mathrm{i}\sigma} - \frac{1}{q^2 - m^2 - \mathrm{i}\sigma} \right] = \frac{1}{\left[q^2 - m^2\right]_c}$$

Integration-by-parts relations

$$0 = \int \prod_{l=1}^{L} \mathrm{d}^d k_l \frac{\partial}{\partial k_i^\mu} \left[q^\mu \prod_{n=1}^{N} D_n^{-\alpha_n} \right], \quad q \in \{k_1,..,k_L,p_1,\dots,p_E\}, \quad \alpha_n \in \mathbb{Z},$$ [Chetyrkin,Tkachov '81]

- set of linearly independent propagators $\{D_n\}$ defines topology uniquely
- IBP relations relate different integrals within a given topology

$\Rightarrow$ reduce given problem to minimal set of "master integrals" [AIR,FIRE,Reduze,LiteRed,Kira,..]

---

1. δ-function: reverse unitarity [Anastasiou,Melnikov '02]

$$(-2\pi \mathrm{i})\, \delta\left[q^2 - m^2\right] = \lim_{\sigma \to 0^+} \left[\frac{1}{q^2 - m^2 + \mathrm{i}\sigma} - \frac{1}{q^2 - m^2 - \mathrm{i}\sigma} \right] = \frac{1}{[q^2 - m^2]_c}$$

2. θ-function:

Integration-by-parts relations

$$0 = \int \prod_{l=1}^{L} \mathrm{d}^{\mathrm{d}} k_l \, \frac{\partial}{\partial k_i^{\mu}} \left[q^{\mu} \prod_{n=1}^{N} D_n^{-\alpha_n} \right], \quad q \in \{k_1, .., k_L, p_1, \ldots, p_E\}, \quad \alpha_n \in \mathbb{Z},$$ [Chetyrkin,Tkachov '81]

- set of linearly independent propagators $\{D_n\}$ defines topology uniquely
- IBP relations relate different integrals within a given topology

$\Rightarrow$ reduce given problem to minimal set of "master integrals" [AIR,FIRE,Reduze,LiteRed,Kira,..]

---

1. δ-function: reverse unitarity [Anastasiou,Melnikov '02]

$$(-2\pi\mathrm{i})\, \delta\left[q^2 - m^2\right] = \lim_{\sigma \to 0^+} \left[\frac{1}{q^2 - m^2 + \mathrm{i}\sigma} - \frac{1}{q^2 - m^2 - \mathrm{i}\sigma} \right] = \frac{1}{\left[q^2 - m^2\right]_c}$$

2. θ-function:

Parametric representation

[Caola et al. '18] [Angeles-Martinez et al. '18]
[Bizon et al. '20] [Baranowski '20]

$$\theta[b-a] = \int_0^1 \mathrm{d}z\, \delta[z \cdot b - a]\, b \quad a, b > 0$$

[Chen '20]

$$(2\pi\mathrm{i})\, \theta[D] = \int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{e^{\mathrm{i}xD}}{x + \mathrm{i}0^+}$$

Integration-by-parts relations

$$0 = \int \prod_{l=1}^{L} \mathrm{d}^d k_l \frac{\partial}{\partial k_i^\mu} \left[q^\mu \prod_{n=1}^{N} D_n^{-\alpha_n} \right], \quad q \in \{k_1, .., k_L, p_1, \ldots, p_E\}, \quad \alpha_n \in \mathbb{Z},$$ [Chetyrkin,Tkachov '81]

- set of linearly independent propagators $\{D_n\}$ defines topology uniquely
- IBP relations relate different integrals within a given topology

$\Rightarrow$ reduce given problem to minimal set of "master integrals" [AIR,FIRE,Reduze,LiteRed,Kira,..]

---

1. δ-function: reverse unitarity [Anastasiou,Melnikov '02]

$$(-2\pi \mathrm{i})\, \delta\left[q^2 - m^2\right] = \lim_{\sigma \to 0^+} \left[\frac{1}{q^2 - m^2 + \mathrm{i}\sigma} - \frac{1}{q^2 - m^2 - \mathrm{i}\sigma} \right] = \frac{1}{[q^2 - m^2]_c}$$

2. θ-function:

Parametric representation

[Caola et al. '18] [Angeles-Martinez et al. '18]
[Bizon et al. '20] [Baranowski '20]

$$\theta[b-a] = \int_0^1 \mathrm{d}z\, \delta[z \cdot b - a]\, b \quad a, b > 0$$

[Chen '20]

$$(2\pi \mathrm{i})\, \theta[D] = \int_{-\infty}^{+\infty} \mathrm{d}x \frac{e^{\mathrm{i}xD}}{x + \mathrm{i}0^+}$$

Extended notion of IBPs

[Baranowski et al. '21]

$$\frac{\partial}{\partial k} \{g(k)\, \theta[f(k)]\} = \overbrace{g'\theta}^{\text{hom}} + \overbrace{g\,\delta f'}^{\text{inhom}}$$

☹ proliferation of topologies
☹ manual generation of IBP identities
☺ no auxiliary integration

(see also [Luo et al. '19])

Integration-by-parts - example @ NNLO

$$\int \frac{\mathrm{d}\Phi_{\theta\theta}^{nn}}{(k_2 n)\,(k_1 \bar{n})\,(k_{12}\bar{n})}\,, \qquad \mathrm{d}\Phi_{\theta\theta}^{nn} = \widetilde{[\mathrm{d}k]}\,\delta[\tau - k_{12}n]\,\theta[k_1\bar{n} - k_1 n]\,\theta[k_2\bar{n} - k_2 n]$$

Integration-by-parts - example @ NNLO

$$\int \frac{\mathrm{d}\Phi^{nn}_{\theta\theta}}{(k_2 n)\,(k_1 \bar{n})\,(k_{12}\bar{n})}\,, \qquad \mathrm{d}\Phi^{nn}_{\theta\theta} = \widetilde{[\mathrm{d}k]}\,\delta[\tau - k_{12}n]\,\theta[k_1\bar{n} - k_1 n]\,\theta[k_2\bar{n} - k_2 n]$$

- reverse unitarity

$$\int \frac{\mathrm{d}^d k_1\,\mathrm{d}^d k_2\,\theta(k_1\bar{n} - k_1 n)\theta(k_2\bar{n} - k_2 n)}{\left[(k_1^2)(k_2^2)(\tau - k_{12}n)\right]_c\,(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})} = \mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$

Integration-by-parts - example @ NNLO

$$\int \frac{\mathrm{d}\Phi^{nn}_{\theta\theta}}{(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})}\,, \qquad \mathrm{d}\Phi^{nn}_{\theta\theta} = \widetilde{[\mathrm{d}k]}\,\delta[\tau - k_{12}n]\,\theta[k_1\bar{n} - k_1 n]\,\theta[k_2\bar{n} - k_2 n]$$

- reverse unitarity

$$\int \frac{\mathrm{d}^{\mathrm{d}}k_1\,\mathrm{d}^{\mathrm{d}}k_2\,\theta(k_1\bar{n} - k_1 n)\theta(k_2\bar{n} - k_2 n)}{\left[(k_1^2)(k_2^2)(\tau - k_{12}n)\right]_c (k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})} = \mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$

- IBP relation

$$0 = \int \mathrm{d}^{\mathrm{d}}k_1\,\mathrm{d}^{\mathrm{d}}k_2\,\frac{\partial}{\partial k_1^{\mu}} k_1^{\mu} \ldots$$

Integration-by-parts - example @ NNLO

$$\int \frac{\mathrm{d}\Phi^{nn}_{\theta\theta}}{(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})}\,, \quad \mathrm{d}\Phi^{nn}_{\theta\theta} = \widetilde{[\mathrm{d}k]}\,\delta[\tau - k_{12}n]\,\theta[k_1\bar{n} - k_1 n]\,\theta[k_2\bar{n} - k_2 n]$$

- reverse unitarity

$$\int \frac{\mathrm{d}^{\mathrm{d}}k_1\,\mathrm{d}^{\mathrm{d}}k_2\,\theta(k_1\bar{n} - k_1 n)\theta(k_2\bar{n} - k_2 n)}{\left[(k_1^2)(k_2^2)(\tau - k_{12}n)\right]_c\,(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})} = \mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$

- IBP relation

$$0 = \int \mathrm{d}^{\mathrm{d}}k_1\,\mathrm{d}^{\mathrm{d}}k_2\,\frac{\partial}{\partial k_1^\mu}k_1^\mu \ldots$$

yields

$$0 = \left[(d-4) - \mathbf{7}^+\mathbf{6}^- - \mathbf{3}^+\mathbf{5}^- + \mathbf{3}^+\right]\mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$
$$+ \int \mathrm{d}^{\mathrm{d}}k_1\,\mathrm{d}^{\mathrm{d}}k_2\left[k_1^\mu(\bar{n} - n)_\mu\,\delta(k_1\bar{n} - k_1 n)\right] \times \frac{\theta(k_2\bar{n} - k_2 n)}{(k_1^2)(k_2^2)(\tau - k_{12}n)(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})}$$

Integration-by-parts - example @ NNLO

$$\int \frac{\mathrm{d}\Phi^{nn}_{\theta\theta}}{(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})}\,, \qquad \mathrm{d}\Phi^{nn}_{\theta\theta} = \widetilde{[\mathrm{d}k]}\,\delta[\tau - k_{12}n]\,\theta[k_1\bar{n} - k_1 n]\,\theta[k_2\bar{n} - k_2 n]$$

- reverse unitarity

$$\int \frac{\mathrm{d^d}k_1\,\mathrm{d^d}k_2\,\theta(k_1\bar{n} - k_1 n)\theta(k_2\bar{n} - k_2 n)}{\left[(k_1^2)(k_2^2)(\tau - k_{12}n)\right]_c\,(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})} = \mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$

- IBP relation

$$0 = \int \mathrm{d^d}k_1\,\mathrm{d^d}k_2\,\frac{\partial}{\partial k_1^\mu}k_1^\mu \ldots$$

yields

$$0 = \left[(d-4) - \mathbf{7}^+\mathbf{6}^- - \mathbf{3}^+\mathbf{5}^- + \mathbf{3}^+\right]\mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$
$$+ \int \mathrm{d^d}k_1\,\mathrm{d^d}k_2\left[k_1^\mu(\bar{n} - n)_\mu\,\delta(k_1\bar{n} - k_1 n)\right] \times \frac{\theta(k_2\bar{n} - k_2 n)}{(k_1^2)(k_2^2)(\tau - k_{12}n)(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})}$$

vanishes $\sim x\,\delta[x]$

Integration-by-parts - example @ NNLO

$$\int \frac{\mathrm{d}\Phi^{nn}_{\theta\theta}}{(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})}\,, \quad \mathrm{d}\Phi^{nn}_{\theta\theta} = \widetilde{[\mathrm{d}k]}\,\delta[\tau - k_{12}n]\,\theta[k_1\bar{n} - k_1 n]\,\theta[k_2\bar{n} - k_2 n]$$

- reverse unitarity

$$\int \frac{\mathrm{d}^{\mathrm{d}}k_1\,\mathrm{d}^{\mathrm{d}}k_2\,\theta(k_1\bar{n} - k_1 n)\theta(k_2\bar{n} - k_2 n)}{\left[(k_1^2)(k_2^2)(\tau - k_{12}n)\right]_c (k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})} = \mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$

- IBP relation

$$0 = \int \mathrm{d}^{\mathrm{d}}k_1\,\mathrm{d}^{\mathrm{d}}k_2\,\frac{\partial}{\partial k_1^{\mu}} k_1^{\mu} \ldots$$

vanishes $\sim x\,\delta[x]$

yields

$$0 = \left[(d-4) - \mathbf{7}^{+}\mathbf{6}^{-} - \mathbf{3}^{+}\mathbf{5}^{-} + \mathbf{3}^{+}\right]\mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$
$$+ \int \mathrm{d}^{\mathrm{d}}k_1\,\mathrm{d}^{\mathrm{d}}k_2 \left[k_1^{\mu}(\bar{n} - n)_{\mu}\,\delta(k_1\bar{n} - k_1 n)\right] \times \frac{\theta(k_2\bar{n} - k_2 n)}{(k_1^2)(k_2^2)(\tau - k_{12}n)(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})}$$

$\Rightarrow$ “conventional” IBP relation

Integration-by-parts - example @ NNLO

$$\int \frac{\mathrm{d}\Phi^{nn}_{\theta\theta}}{(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})}\,, \quad \mathrm{d}\Phi^{nn}_{\theta\theta} = \widetilde{[\mathrm{d}k]}\,\delta[\tau - k_{12}n]\,\theta[k_1\bar{n} - k_1 n]\,\theta[k_2\bar{n} - k_2 n]$$

- reverse unitarity

$$\int \frac{\mathrm{d}^{\mathrm{d}}k_1\,\mathrm{d}^{\mathrm{d}}k_2\,\theta(k_1\bar{n} - k_1 n)\theta(k_2\bar{n} - k_2 n)}{\left[(k_1^2)(k_2^2)(\tau - k_{12}n)\right]_c (k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})} = \mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$

- IBP relation

$$0 = \int \mathrm{d}^{\mathrm{d}}k_1\,\mathrm{d}^{\mathrm{d}}k_2\,\frac{\partial}{\partial k_1^{\mu}} n^{\mu} \ldots$$

Integration-by-parts - example @ NNLO

$$\int \frac{\mathrm{d}\Phi^{nn}_{\theta\theta}}{(k_2 n)\,(k_1 \bar{n})\,(k_{12}\bar{n})}\,, \qquad \mathrm{d}\Phi^{nn}_{\theta\theta} = \widetilde{[\mathrm{d}k]}\,\delta[\tau - k_{12}n]\,\theta[k_1\bar{n} - k_1 n]\,\theta[k_2\bar{n} - k_2 n]$$

- reverse unitarity

$$\int \frac{\mathrm{d}^{\mathrm{d}}k_1\,\mathrm{d}^{\mathrm{d}}k_2\,\theta(k_1\bar{n} - k_1 n)\theta(k_2\bar{n} - k_2 n)}{\left[(k_1^2)(k_2^2)(\tau - k_{12}n)\right]_c\,(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})} = \mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$

- IBP relation

$$0 = \int \mathrm{d}^{\mathrm{d}}k_1\,\mathrm{d}^{\mathrm{d}}k_2\,\frac{\partial}{\partial k_1^{\mu}} n^{\mu} \ldots$$

yields

$$0 = n\bar{n}\left[\mathbf{7}^+ - \mathbf{6}^+ + \mathbf{1}^+\mathbf{6}^- - \mathbf{1}^+\right]\mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$
$$+ n\bar{n}\int \mathrm{d}^{\mathrm{d}}k_1\,\mathrm{d}^{\mathrm{d}}k_2\,\delta[k_1\bar{n} - k_1 n] \times \frac{\theta(k_2\bar{n} - k_2 n)}{(k_1^2)(k_2^2)(\tau - k_{12}n)(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})}$$

Integration-by-parts - example @ NNLO

$$\int \frac{\mathrm{d}\Phi^{nn}_{\theta\theta}}{(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})}\,, \quad \mathrm{d}\Phi^{nn}_{\theta\theta} = \widetilde{[\mathrm{d}k]}\,\delta[\tau - k_{12}n]\,\theta[k_1\bar{n} - k_1 n]\,\theta[k_2\bar{n} - k_2 n]$$

- reverse unitarity

$$\int \frac{\mathrm{d}^d k_1\,\mathrm{d}^d k_2\,\theta(k_1\bar{n} - k_1 n)\theta(k_2\bar{n} - k_2 n)}{\left[(k_1^2)(k_2^2)(\tau - k_{12}n)\right]_c\,(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})} = \mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$

- IBP relation

$$0 = \int \mathrm{d}^d k_1\,\mathrm{d}^d k_2\,\frac{\partial}{\partial k_1^\mu} n^\mu \ldots$$

yields

$$0 = n\bar{n}\left[\mathbf{7}^+ - \mathbf{6}^+ + \mathbf{1}^+\mathbf{6}^- - \mathbf{1}^+\right]\mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$
$$+\, n\bar{n}\int \mathrm{d}^d k_1\,\mathrm{d}^d k_2\,\frac{1}{(k_1\bar{n} - k_1 n)} \times \frac{\theta(k_2\bar{n} - k_2 n)}{(k_1^2)(k_2^2)(\tau - k_{12}n)(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})}$$

$\Rightarrow$ additional inhomogeneous term that requires partial fraction decomposition

Integration-by-parts - example @ NNLO

$$\underbrace{\mathcal{T}^{\text{ex}}_{1,1,1,0,1,1,1} = \int \frac{d\Phi^{nn}_{\theta\theta}}{(k_2 n)\,(k_1 \bar{n})\,(k_{12}\bar{n})}}_{\text{topology with } 2\theta\text{-functions}}, \quad d\Phi^{nn}_{f_1 f_2} = \widetilde{[dk]}\,\delta[\tau - k_{12}n] f_1[k_1\bar{n} - k_1 n] f_2[k_2\bar{n} - k_2 n]$$

$$0 = \left[(d-4) - \mathbf{7}^+\mathbf{6}^- - \mathbf{3}^+\mathbf{5}^- + \mathbf{3}^+\right] \mathcal{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$

$$0 = n\bar{n}\left[\mathbf{7}^+ - \mathbf{6}^+ + \mathbf{1}^+\mathbf{6}^- - \mathbf{1}^+\right] \mathcal{T}^{\text{ex}}_{1,1,1,0,1,1,1} + \underbrace{\int \frac{d\Phi^{nn}_{\delta\theta}}{(k_{12}\bar{n})}\left[\frac{1}{(k_1 n)} + \frac{1}{(k_2 n)}\right]}_{\text{topology with } 1\theta\text{-function}}$$

- because of inhomogeneous terms ($\sim \partial_k \theta$), we need new type of topologies:
  - come from all possible ways of replacing $\theta \to \delta$
  - fall into a hierarchical structure: $\{\theta\theta\} \overset{\partial_k\theta}{\to} \{\delta\theta, \theta\delta\} \overset{\partial_k\theta}{\to} \{\delta\delta\}$
  - relations for $\delta\delta$ topologies have no inhomogeneous terms
- feed relations to `Kira⊕FireFly` [Maierhöfer et al. '17][Klappert et al. '19] ("user-defined equations")

Integration-by-parts - example @ NNLO

$$\underbrace{\mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1} = \int \frac{\mathrm{d}\Phi^{nn}_{\theta\theta}}{(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})}}_{\text{topology with } 2\theta\text{-functions}}, \quad \mathrm{d}\Phi^{nn}_{f_1 f_2} = \widetilde{[\mathrm{d}k]}\,\delta[\tau - k_{12}n] f_1[k_1\bar{n} - k_1 n] f_2[k_2\bar{n} - k_2 n]$$

$$0 = \left[(d-4) - \mathbf{7}^+\mathbf{6}^- - \mathbf{3}^+\mathbf{5}^- + \mathbf{3}^+\right] \mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$

$$0 = n\bar{n}\left[\mathbf{7}^+ - \mathbf{6}^+ + \mathbf{1}^+\mathbf{6}^- - \mathbf{1}^+\right] \mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1} + \underbrace{\int \frac{\mathrm{d}\Phi^{nn}_{\delta\theta}}{(k_{12}\bar{n})}\left[\frac{1}{(k_1 n)} + \frac{1}{(k_2 n)}\right]}_{\text{topology with } 1\theta\text{-function}}$$

- because of inhomogeneous terms ($\sim \partial_k \theta$), we need new type of topologies:
  - come from all possible ways of replacing $\theta \to \delta$
  - fall into a hierarchical structure: $\{\theta\theta\} \overset{\partial_k\theta}{\to} \{\delta\theta, \theta\delta\} \overset{\partial_k\theta}{\to} \{\delta\delta\}$
  - relations for $\delta\delta$ topologies have no inhomogeneous terms
- feed relations to `Kira⊕FireFly` [Maierhöfer et al. '17][Klappert et al. '19] ("user-defined equations")

Reduction ✓ Master integrals X

Master integral computation - example @ NNLO

- consider

$$I^{nn}_{4,\delta\theta} = \int \frac{\mathrm{d}\Phi^{nn}_{\delta\theta}}{(k_1 k_2)(k_2 n)(k_{12}\bar{n})}$$

Master integral computation - example @ NNLO

- consider

$$I^{nn}_{4,\delta\theta} = \int \frac{\mathrm{d}\Phi^{nn}_{\delta\theta}}{(k_1k_2)(k_2n)(k_{12}\bar{n})}$$

- employ Sudakov variables $k_i^\mu = \frac{1}{2}\left[\alpha_i n^\mu + \beta_i \bar{n}^\mu + 2\sqrt{\alpha_i\beta_i}\, e_i^{\mu,\perp}\right]$

$$I^{nn}_{4,\delta\theta}\Big|_{\tau\to 1} = 2\mathcal{N}_\varepsilon^2 \int \mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\mathrm{d}\alpha_2\, \frac{\beta_1^{-2\varepsilon}(\alpha_2\beta_2)^{-\varepsilon}\,\delta[1-\beta_1-\beta_2]\,\theta[\alpha_2-\beta_2]}{\beta_1(\alpha_2+\beta_2-4\sqrt{\alpha_2\beta_2}\boldsymbol{e}_1^\perp\boldsymbol{e}_2^\perp)\beta_2(\beta_1+\alpha_2)}$$

$$\overset{\beta_2\to\alpha_2\xi_2}{=} -2\mathcal{N}_\varepsilon^2 \frac{\Gamma^2(1-2\varepsilon)}{\varepsilon\,\Gamma(1-4\varepsilon)} \int_0^1 \mathrm{d}\xi_2\, \xi_2^{-1-\varepsilon}(1-\xi_2)^{-1-2\varepsilon} F(\xi_2)$$

Master integral computation - example @ NNLO

- consider

$$I^{nn}_{4,\delta\theta} = \int \frac{\mathrm{d}\Phi^{nn}_{\delta\theta}}{(k_1 k_2)(k_2 n)(k_{12}\bar{n})}$$

- employ Sudakov variables $k_i^\mu = \frac{1}{2}\left[\alpha_i n^\mu + \beta_i \bar{n}^\mu + 2\sqrt{\alpha_i \beta_i}\, e_i^{\mu,\perp}\right]$

$$I^{nn}_{4,\delta\theta}\Big|_{\tau\to 1} = 2\mathcal{N}_\varepsilon^2 \int \mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\mathrm{d}\alpha_2\, \frac{\beta_1^{-2\varepsilon}(\alpha_2\beta_2)^{-\varepsilon}\,\delta[1-\beta_1-\beta_2]\,\theta[\alpha_2-\beta_2]}{\beta_1(\alpha_2+\beta_2-4\sqrt{\alpha_2\beta_2}\boldsymbol{e}_1^\perp\boldsymbol{e}_2^\perp)\beta_2(\beta_1+\alpha_2)}$$

$$\overset{\beta_2\to\alpha_2\xi_2}{=} -2\mathcal{N}_\varepsilon^2 \frac{\Gamma^2(1-2\varepsilon)}{\varepsilon\,\Gamma(1-4\varepsilon)} \int_0^1 \mathrm{d}\xi_2\; \xi_2^{-1-\varepsilon}(1-\xi_2)^{-1-2\varepsilon} F(\xi_2)$$

- regulate both endpoint singularities

$$I^{nn}_{4,\delta\theta}\Big|_{\tau\to 1} = -2\mathcal{N}_\varepsilon^2 \frac{\Gamma^2(1-2\varepsilon)}{\varepsilon\,\Gamma(1-4\varepsilon)} \left\{ \int_0^1 \mathrm{d}\xi_2 \left[\frac{F(\xi_2)}{\xi_2^{1+\varepsilon}(1-\xi_2)^{1+2\varepsilon}} - \frac{F(0)}{\xi_2^{1+\varepsilon}} - \frac{F(1)}{(1-\xi_2)^{1+2\varepsilon}} \right] + \frac{F(0)}{-\varepsilon} + \frac{F(1)}{-2\varepsilon} \right\}$$

Master integral computation - example @ NNLO

- consider

$$I^{nn}_{4,\delta\theta} = \int \frac{\mathrm{d}\Phi^{nn}_{\delta\theta}}{(k_1k_2)(k_2n)(k_{12}\bar{n})}$$

- employ Sudakov variables $k_i^\mu = \frac{1}{2}\left[\alpha_i n^\mu + \beta_i \bar{n}^\mu + 2\sqrt{\alpha_i\beta_i}\, e_i^{\mu,\perp}\right]$

$$I^{nn}_{4,\delta\theta}\Big|_{\tau\to 1} = 2\mathcal{N}_\varepsilon^2 \int \mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\mathrm{d}\alpha_2\, \frac{\beta_1^{-2\varepsilon}(\alpha_2\beta_2)^{-\varepsilon}\,\delta[1-\beta_1-\beta_2]\,\theta[\alpha_2-\beta_2]}{\beta_1(\alpha_2+\beta_2-4\sqrt{\alpha_2\beta_2}\boldsymbol{e}_1^\perp\boldsymbol{e}_2^\perp)\beta_2(\beta_1+\alpha_2)}$$

$$\overset{\beta_2\to\alpha_2\xi_2}{=} -2\mathcal{N}_\varepsilon^2\frac{\Gamma^2(1-2\varepsilon)}{\varepsilon\,\Gamma(1-4\varepsilon)}\int_0^1 \mathrm{d}\xi_2\;\xi_2^{-1-\varepsilon}(1-\xi_2)^{-1-2\varepsilon}F(\xi_2)$$

- regulate both endpoint singularities

$$I^{nn}_{4,\delta\theta}\Big|_{\tau\to 1} = -2\mathcal{N}_\varepsilon^2\frac{\Gamma^2(1-2\varepsilon)}{\varepsilon\,\Gamma(1-4\varepsilon)}\left\{\int_0^1 \mathrm{d}\xi_2\left[\frac{F(\xi_2)}{\xi_2^{1+\varepsilon}(1-\xi_2)^{1+2\varepsilon}} - \frac{F(0)}{\xi_2^{1+\varepsilon}} - \frac{F(1)}{(1-\xi_2)^{1+2\varepsilon}}\right] + \frac{F(0)}{-\varepsilon} + \frac{F(1)}{-2\varepsilon}\right\}$$

Master integral computation - example @ NNLO

- consider

$$I^{nn}_{4,\delta\theta} = \int \frac{\mathrm{d}\Phi^{nn}_{\delta\theta}}{(k_1 k_2)(k_2 n)(k_{12}\bar{n})}$$

- employ Sudakov variables $k_i^\mu = \frac{1}{2}\left[\alpha_i n^\mu + \beta_i \bar{n}^\mu + 2\sqrt{\alpha_i\beta_i}\, e_i^{\mu,\perp}\right]$

$$I^{nn}_{4,\delta\theta}\Big|_{\tau\to 1} = 2\mathcal{N}_\varepsilon^2 \int \mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\mathrm{d}\alpha_2\, \frac{\beta_1^{-2\varepsilon}(\alpha_2\beta_2)^{-\varepsilon}\,\delta[1-\beta_1-\beta_2]\,\theta[\alpha_2-\beta_2]}{\beta_1(\alpha_2+\beta_2-4\sqrt{\alpha_2\beta_2}\boldsymbol{e}_1^\perp\boldsymbol{e}_2^\perp)\beta_2(\beta_1+\alpha_2)}$$

$$\overset{\beta_2\to\alpha_2\xi_2}{=} -2\mathcal{N}_\varepsilon^2 \frac{\Gamma^2(1-2\varepsilon)}{\varepsilon\,\Gamma(1-4\varepsilon)} \int_0^1 \mathrm{d}\xi_2\, \xi_2^{-1-\varepsilon}(1-\xi_2)^{-1-2\varepsilon} F(\xi_2)$$

- regulate both endpoint singularities

$$I^{nn}_{4,\delta\theta}\Big|_{\tau\to 1} = -2\mathcal{N}_\varepsilon^2 \frac{\Gamma^2(1-2\varepsilon)}{\varepsilon\,\Gamma(1-4\varepsilon)} \left\{ \int_0^1 \mathrm{d}\xi_2 \left[\frac{F(\xi_2)}{\xi_2^{1+\varepsilon}(1-\xi_2)^{1+2\varepsilon}} - \frac{F(0)}{\xi_2^{1+\varepsilon}} - \frac{F(1)}{(1-\xi_2)^{1+2\varepsilon}} \right] + \frac{F(0)}{-\varepsilon} + \frac{F(1)}{-2\varepsilon} \right\}$$

Master integral computation - example @ NNLO

- consider

$$I^{nn}_{4,\delta\theta} = \int \frac{\mathrm{d}\Phi^{nn}_{\delta\theta}}{(k_1 k_2)(k_2 n)(k_{12}\bar{n})}$$

- employ Sudakov variables $k_i^\mu = \frac{1}{2}\left[\alpha_i n^\mu + \beta_i \bar{n}^\mu + 2\sqrt{\alpha_i\beta_i}\, e_i^{\mu,\perp}\right]$

$$I^{nn}_{4,\delta\theta}\Big|_{\tau\to 1} = 2\mathcal{N}_\varepsilon^2 \int \mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\mathrm{d}\alpha_2\, \frac{\beta_1^{-2\varepsilon}(\alpha_2\beta_2)^{-\varepsilon}\,\delta[1-\beta_1-\beta_2]\,\theta[\alpha_2-\beta_2]}{\beta_1(\alpha_2+\beta_2-4\sqrt{\alpha_2\beta_2}\boldsymbol{e}_1^\perp\boldsymbol{e}_2^\perp)\beta_2(\beta_1+\alpha_2)}$$

$$\overset{\beta_2\to\alpha_2\xi_2}{=} -2\mathcal{N}_\varepsilon^2 \frac{\Gamma^2(1-2\varepsilon)}{\varepsilon\,\Gamma(1-4\varepsilon)} \int_0^1 \mathrm{d}\xi_2\; \xi_2^{-1-\varepsilon}(1-\xi_2)^{-1-2\varepsilon} F(\xi_2)$$

- regulate both endpoint singularities

$$I^{nn}_{4,\delta\theta}\Big|_{\tau\to 1} = -2\mathcal{N}_\varepsilon^2 \frac{\Gamma^2(1-2\varepsilon)}{\varepsilon\,\Gamma(1-4\varepsilon)} \left\{ \int_0^1 \mathrm{d}\xi_2 \left[\frac{F(\xi_2)}{\xi_2^{1+\varepsilon}(1-\xi_2)^{1+2\varepsilon}} - \frac{F(0)}{\xi_2^{1+\varepsilon}} - \frac{F(1)}{(1-\xi_2)^{1+2\varepsilon}} \right] + \frac{F(0)}{-\varepsilon} + \frac{F(1)}{-2\varepsilon} \right\}$$

Master integral computation - example @ NNLO

- consider

$$I^{nn}_{4,\delta\theta} = \int \frac{\mathrm{d}\Phi^{nn}_{\delta\theta}}{(k_1 k_2)(k_2 n)(k_{12}\bar{n})}$$

- employ Sudakov variables $k_i^\mu = \frac{1}{2}\left[\alpha_i n^\mu + \beta_i \bar{n}^\mu + 2\sqrt{\alpha_i\beta_i}\, e_i^{\mu,\perp}\right]$

$$I^{nn}_{4,\delta\theta}\Big|_{\tau\to 1} = 2\mathscr{N}_\varepsilon^2 \int \mathrm{d}\beta_1\,\mathrm{d}\beta_2\,\mathrm{d}\alpha_2\, \frac{\beta_1^{-2\varepsilon}(\alpha_2\beta_2)^{-\varepsilon}\,\delta[1-\beta_1-\beta_2]\,\theta[\alpha_2-\beta_2]}{\beta_1(\alpha_2+\beta_2-4\sqrt{\alpha_2\beta_2}\boldsymbol{e}_1^\perp\boldsymbol{e}_2^\perp)\beta_2(\beta_1+\alpha_2)}$$

$$\stackrel{\beta_2\to\alpha_2\xi_2}{=} -2\mathscr{N}_\varepsilon^2 \frac{\Gamma^2(1-2\varepsilon)}{\varepsilon\,\Gamma(1-4\varepsilon)} \int_0^1 \mathrm{d}\xi_2\, \xi_2^{-1-\varepsilon}(1-\xi_2)^{-1-2\varepsilon} F(\xi_2)$$

- regulate both endpoint singularities

$$I^{nn}_{4,\delta\theta}\Big|_{\tau\to 1} = -2\mathscr{N}_\varepsilon^2 \frac{\Gamma^2(1-2\varepsilon)}{\varepsilon\,\Gamma(1-4\varepsilon)} \left\{ \int_0^1 \mathrm{d}\xi_2 \left[\frac{F(\xi_2)}{\xi_2^{1+\varepsilon}(1-\xi_2)^{1+2\varepsilon}} - \frac{F(0)}{\xi_2^{1+\varepsilon}} - \frac{F(1)}{(1-\xi_2)^{1+2\varepsilon}} \right] + \frac{F(0)}{-\varepsilon} + \frac{F(1)}{-2\varepsilon} \right\}$$

- expand with `HypExp` [Huber et al. ’05] & integrate with `HyperInt` [Panzer ’14]

$$I^{nn}_{4,\delta\theta}\Big|_{\tau\to 1} = \frac{2}{\varepsilon^2} + \frac{\pi^2}{3} - \frac{17\pi^4\varepsilon^2}{90} + \varepsilon^3\left[-6\pi^2\zeta_3 - 26\zeta_5\right] - \varepsilon^4\left[\frac{193\pi^6}{810} + 64\zeta_3^2\right] + \mathscr{O}(\varepsilon^5)$$

Complications for three-gluon emission

General complexity

- more "level": $\theta\theta\theta$, $\delta\theta\theta$, $\theta\delta\theta$, $\theta\theta\delta$, ...
- $\mathcal{O}(400)$ topologies
- integral relations

Complications for three-gluon emission

General complexity

- more "level": $\theta\theta\theta$, $\delta\theta\theta$, $\theta\delta\theta$, $\theta\theta\delta$, $\ldots$
- $\mathcal{O}(400)$ topologies
- integral relations

Rapidity divergences

- known to appear for certrain observables (e.g. q_T) in the limit of large rapidity
- only cancel in *soft+collinear*
- obey a rapidity renormalisation-group formalism [Chiu et al.]

in case of zero jettiness

$\Rightarrow$ we find individual integrals that are not regulated dimensionally

$\Rightarrow$ require analytic regulator

Complications for three-gluon emission

General complexity

- more "level": $\theta\theta\theta$, $\delta\theta\theta$, $\theta\delta\theta$, $\theta\theta\delta$, $\ldots$
- $\mathcal{O}(400)$ topologies
- integral relations

Rapidity divergences

- known to appear for certrain observables (e.g. q_T) in the limit of large rapidity
- only cancel in *soft+collinear*
- obey a rapidity renormalisation-group formalism [Chiu et al.]

in case of zero jettiness

$\Rightarrow$ we find individual integrals that are not regulated dimensionally

$\Rightarrow$ require analytic regulator

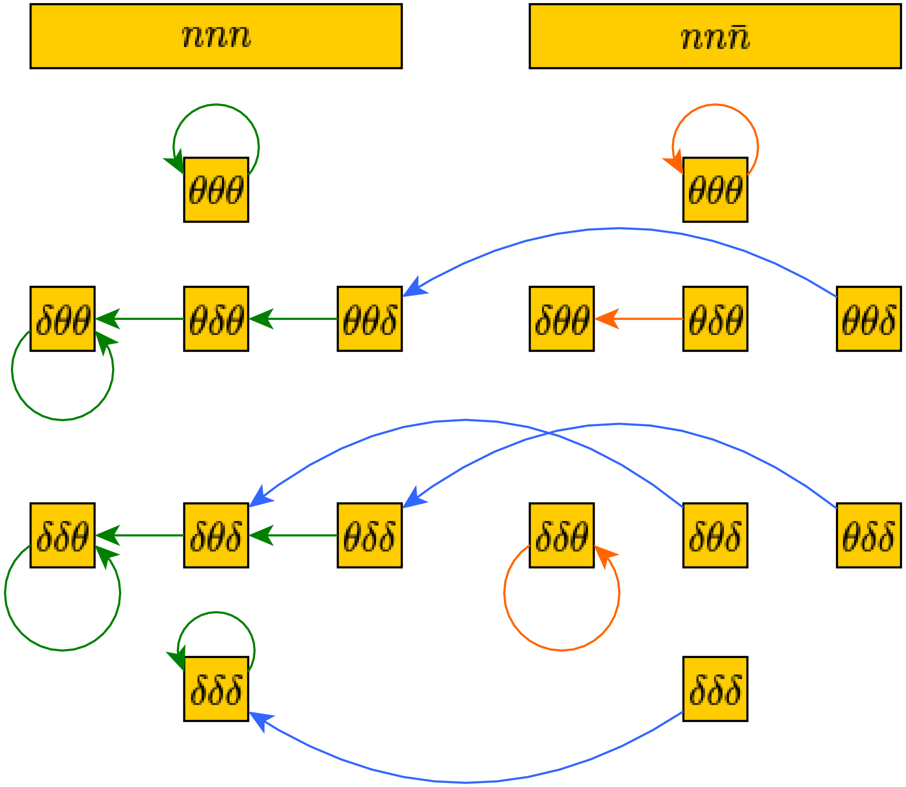
Master integrals

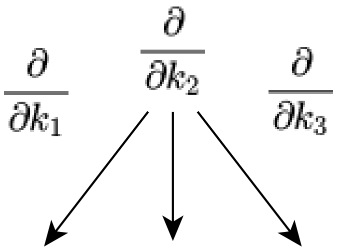
- more complicated master integrals

$\Rightarrow$ resort to numerical calculation and PSLQ reconstruction

Integral relations at N3LO

$$\mathrm{d}\Phi^{nnn}_{f_1 f_2 f_3} \sim \delta[\tau - k_1\cdot n - k_2\cdot n - k_3\cdot n]\, f_1(k_1\cdot\bar{n} - k_1\cdot n) f_2(k_2\cdot\bar{n} - k_2\cdot n) f_3(k_3\cdot\bar{n} - k_3\cdot n)$$
$$\mathrm{d}\Phi^{nn\bar{n}}_{f_1 f_2 f_3} \sim \delta[\tau - k_1\cdot n - k_2\cdot n - k_3\cdot \bar{n}]\, f_1(k_1\cdot\bar{n} - k_1\cdot n) f_2(k_2\cdot\bar{n} - k_2\cdot n) f_3(k_3\cdot n - k_3\cdot \bar{n})$$

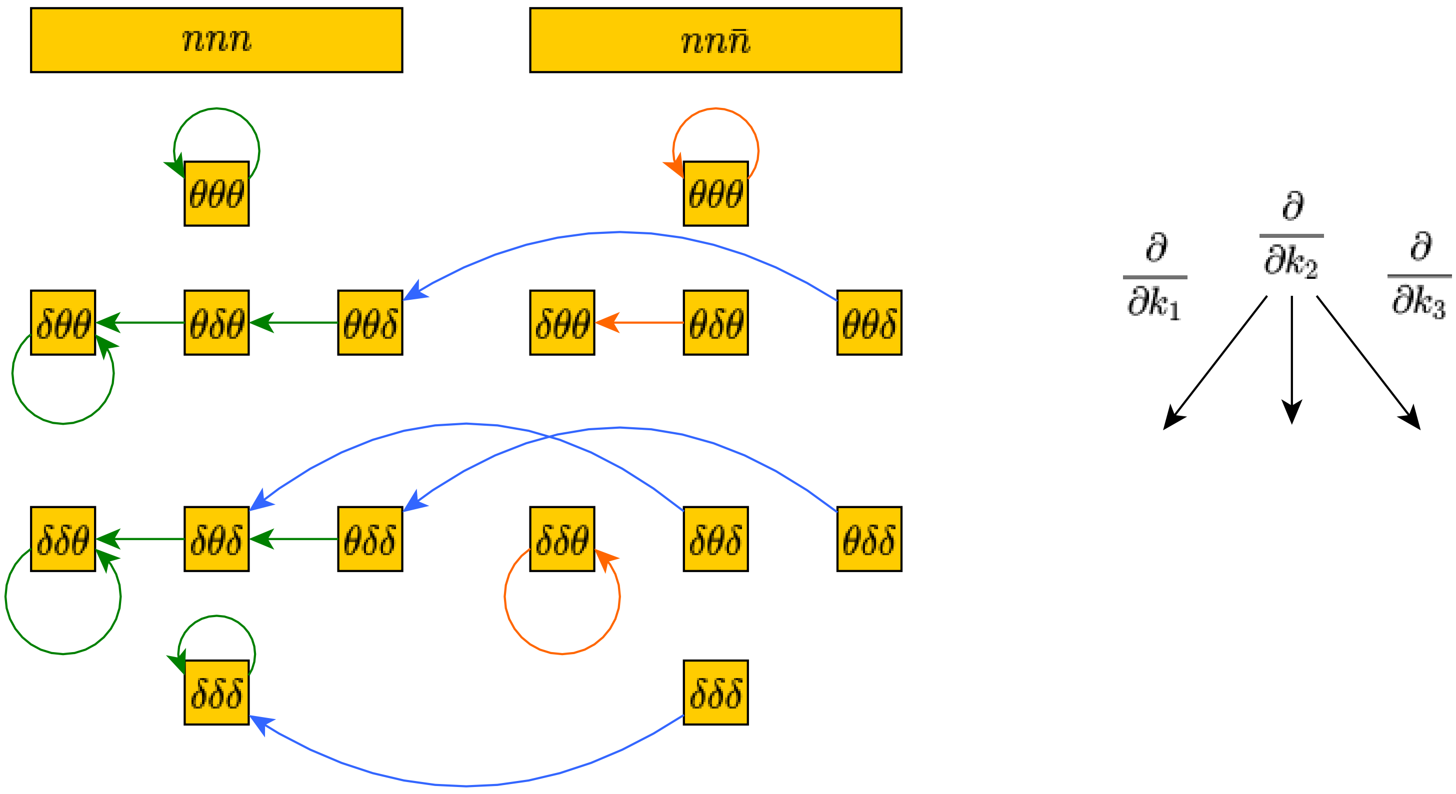




Integral relations at N3LO

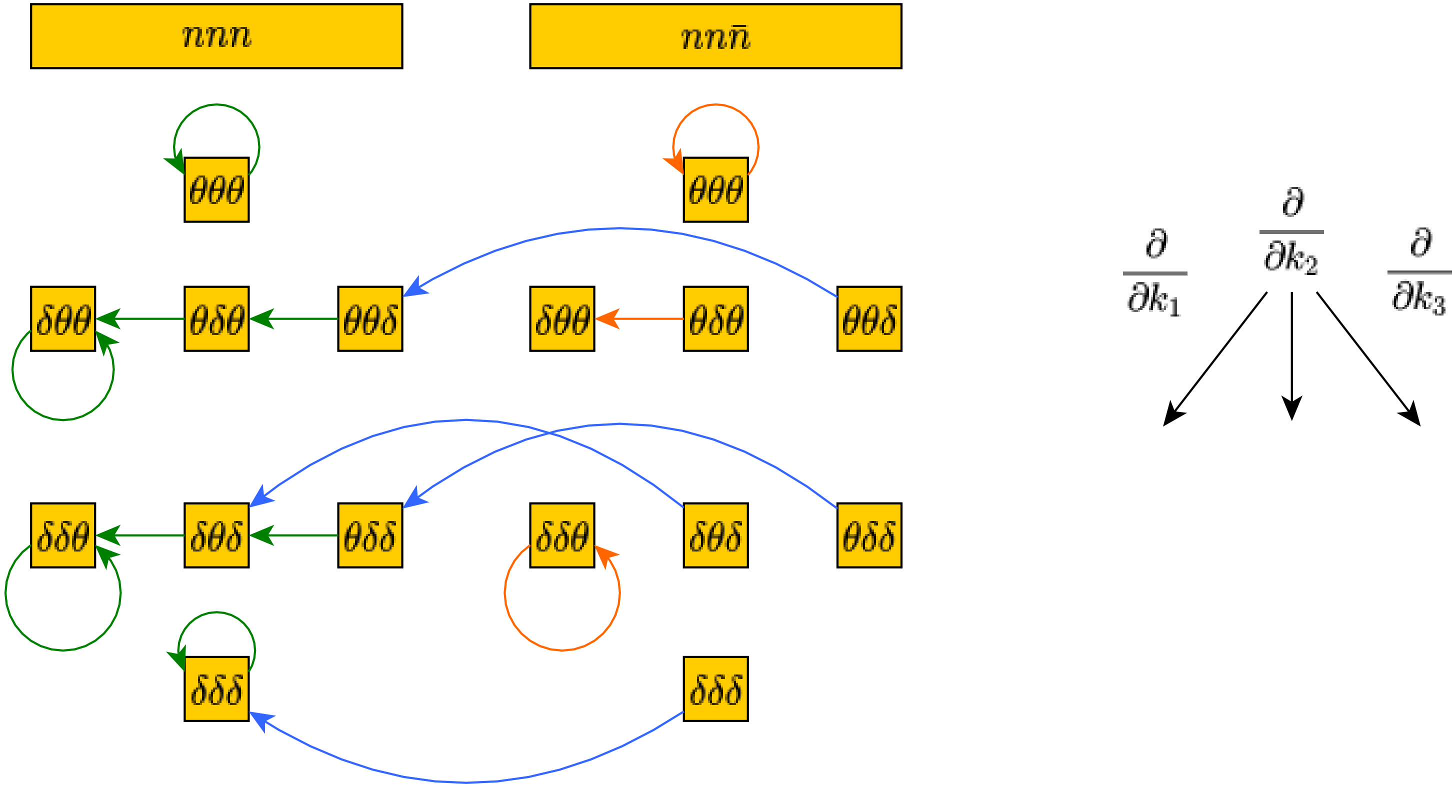
$$\mathrm{d}\Phi^{nnn}_{f_1 f_2 f_3} \sim \delta[\tau - k_1\cdot n - k_2\cdot n - k_3\cdot n]\, f_1(k_1\cdot\bar{n} - k_1\cdot n) f_2(k_2\cdot\bar{n} - k_2\cdot n) f_3(k_3\cdot\bar{n} - k_3\cdot n)$$

$$\mathrm{d}\Phi^{nn\bar{n}}_{f_1 f_2 f_3} \sim \delta[\tau - k_1\cdot n - k_2\cdot n - k_3\cdot \bar{n}]\, f_1(k_1\cdot\bar{n} - k_1\cdot n) f_2(k_2\cdot\bar{n} - k_2\cdot n) f_3(k_3\cdot n - k_3\cdot \bar{n})$$



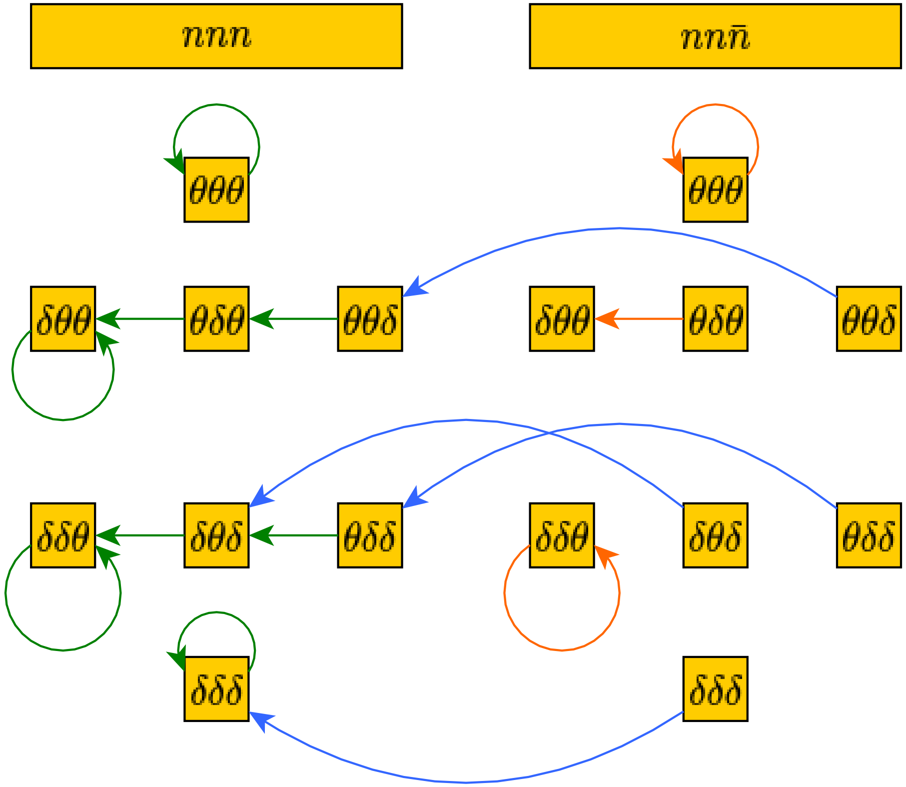
Integral relations at N3LO

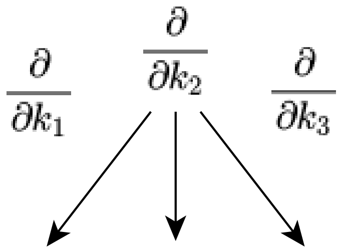
$$d\Phi^{nnn}_{f_1f_2f_3} \sim \delta[\tau - k_1\cdot n - k_2\cdot n - k_3\cdot n]\, f_1(k_1\cdot\bar{n} - k_1\cdot n) f_2(k_2\cdot\bar{n} - k_2\cdot n) f_3(k_3\cdot\bar{n} - k_3\cdot n)$$
$$d\Phi^{nn\bar{n}}_{f_1f_2f_3} \sim \delta[\tau - \boldsymbol{k_1\cdot n - k_2\cdot n} - k_3\cdot\bar{n}]\, f_1(k_1\cdot\bar{n} - k_1\cdot n) f_2(k_2\cdot\bar{n} - k_2\cdot n) f_3(k_3\cdot n - k_3\cdot\bar{n})$$



Integral relations at N3LO

$$\mathrm{d}\Phi^{nnn}_{f_1 f_2 \delta} \sim \delta[\tau - k_1\cdot n - k_2\cdot n - \boldsymbol{k_3\cdot n}]\, f_1(k_1\cdot\bar{n} - k_1\cdot n) f_2(k_2\cdot\bar{n} - k_2\cdot n)\, \delta[\boldsymbol{k_3\cdot\bar{n} - k_3\cdot n}]$$
$$\mathrm{d}\Phi^{nn\bar{n}}_{f_1 f_2 \delta} \sim \delta[\tau - k_1\cdot n - k_2\cdot n - \boldsymbol{k_3\cdot\bar{n}}]\, f_1(k_1\cdot\bar{n} - k_1\cdot n) f_2(k_2\cdot\bar{n} - k_2\cdot n)\, \delta[\boldsymbol{k_3\cdot n - k_3\cdot\bar{n}}]$$





Rapidity divergences at N3LO

Rapidity divergences at N3LO

- can in principle arise, when $k_i \cdot n \gg k_i \cdot \bar{n}$ or $k_i \cdot n \ll k_i \cdot \bar{n}$
  - for $f_i = \delta$, $k_i \cdot n = k_i \cdot \bar{n}$, gluon i confined to transverse plane $\Rightarrow$ X
  - for $f_i = \theta$, $k_i \cdot n < k_i \cdot \bar{n}$ (except $nn\bar{n}$: $k_3 \cdot \bar{n} < k_i \cdot n$) $\Rightarrow$ ✓

Rapidity divergences at N3LO

- can in principle arise, when $k_i \cdot n \gg k_i \cdot \bar{n}$ or $k_i \cdot n \ll k_i \cdot \bar{n}$
  - for $f_i = \delta$, $k_i \cdot n = k_i \cdot \bar{n}$, gluon i confined to transverse plane $\Rightarrow$ X
  - for $f_i = \theta$, $k_i \cdot n < k_i \cdot \bar{n}$ (except $nn\bar{n}$: $k_3 \cdot \bar{n} < k_i \cdot n$) $\Rightarrow \checkmark$

- break dimensional regularization and are **hard to detect**
  - original integrals with three θ functions in soft function are finite
  - master integrals are finite
  - numerical checks showed that certain reduction relations are wrong

Rapidity divergences at N3LO

- can in principle arise, when $k_i \cdot n \gg k_i \cdot \bar{n}$ or $k_i \cdot n \ll k_i \cdot \bar{n}$
  - for $f_i = \delta$, $k_i \cdot n = k_i \cdot \bar{n}$, gluon i confined to transverse plane $\Rightarrow$ X
  - for $f_i = \theta$, $k_i \cdot n < k_i \cdot \bar{n}$ (except $nn\bar{n}$: $k_3 \cdot \bar{n} < k_i \cdot n$) $\Rightarrow$ ✓

- break dimensional regularization and are **hard to detect**
  - original integrals with three θ functions in soft function are finite
  - master integrals are finite
  - numerical checks showed that certain reduction relations are wrong

Introduce analytic regulator

$$\int \mathrm{d}\Phi^{nnn}_{f_1 f_2 f_3} \longrightarrow \lim_{\nu \to 0} \int \mathrm{d}\Phi^{nnn}_{f_1 f_2 f_3} \, (k_1 n)^{\nu} \, (k_2 n)^{\nu} \, (k_3 n)^{\nu}$$

$$\int \mathrm{d}\Phi^{nn\bar{n}}_{f_1 f_2 f_3} \longrightarrow \lim_{\nu \to 0} \int \mathrm{d}\Phi^{nnn}_{f_1 f_2 f_3} \, (k_1 n)^{\nu} \, (k_2 n)^{\nu} \, (k_3 \bar{n})^{\nu}$$

- modifies IBP relations (again)

$$\frac{\partial}{\partial k_1^{\mu}} : \quad 0 = I`\theta_1 + I(\bar{n}-n)^{\mu}\delta_1 \quad \longrightarrow \quad 0 = \left[ I`\theta_1 + I(\bar{n}-n)^{\mu}\delta_1 + \nu\, n^{\mu} \frac{I\theta_1}{k_1 \cdot n} \right] (\ldots)^{\nu}$$

- IBP reduction in limit $\nu \to 0$

$$S^{ggg} = \sum_k c_k I_k + \nu \sum_k \bar{c}_k \bar{I}_k(\nu \approx 0) + \mathcal{O}(\nu)$$

Rapidity divergences at N3LO

- can in principle arise, when $k_i \cdot n \gg k_i \cdot \bar{n}$ or $k_i \cdot n \ll k_i \cdot \bar{n}$
  - for $f_i = \delta$, $k_i \cdot n = k_i \cdot \bar{n}$, gluon i confined to transverse plane $\Rightarrow$ X
  - for $f_i = \theta$, $k_i \cdot n < k_i \cdot \bar{n}$ (except $nn\bar{n}$: $k_3 \cdot \bar{n} < k_i \cdot n$) $\Rightarrow$ ✓
- break dimensional regularization and are **hard to detect**
  - original integrals with three θ functions in soft function are finite
  - master integrals are finite
  - numerical checks showed that certain reduction relations are wrong

Introduce analytic regulator

$$\int \mathrm{d}\Phi^{nnn}_{f_1 f_2 f_3} \longrightarrow \lim_{\nu\to 0} \int \mathrm{d}\Phi^{nnn}_{f_1 f_2 f_3} \,(k_1 n)^\nu\, (k_2 n)^\nu\, (k_3 n)^\nu$$

$$\int \mathrm{d}\Phi^{nn\bar{n}}_{f_1 f_2 f_3} \longrightarrow \lim_{\nu\to 0} \int \mathrm{d}\Phi^{nnn}_{f_1 f_2 f_3} \,(k_1 n)^\nu\, (k_2 n)^\nu\, (k_3 \bar{n})^\nu$$

- modifies IBP relations (again)

$$\frac{\partial}{\partial k_1^\mu}: \quad 0 = I`\theta_1 + I(\bar{n}-n)^\mu \delta_1 \longrightarrow 0 = \left[ I`\theta_1 + I(\bar{n}-n)^\mu \delta_1 + \nu\, n^\mu \frac{I\theta_1}{k_1 \cdot n} \right] (\dots)^\nu$$

"$\nu = 0$"-reduction $\qquad \sim 1/\nu$

- IBP reduction in limit $\nu \to 0$

$$S^{ggg} = \sum_k c_k I_k + \nu \sum_k \bar{c}_k \bar{I}_k(\nu \approx 0) + \mathcal{O}(\nu)$$

Rapidity divergences at N3LO

Example

- consider the following master integral

$$\bar{I}^{\nu} = \int \frac{\mathrm{d}\Phi^{nnn,\nu}_{\theta\delta\theta}}{(k_1 k_3)(k_1 n)(k_{12}\bar{n})(k_3\bar{n})}$$

Rapidity divergences at N3LO

Example

- consider the following master integral

$$\bar{I}^{\nu} = \int \frac{\mathrm{d}\Phi^{nnn,\nu}_{\theta\delta\theta}}{(k_1 k_3)(k_1 n)(k_{12}\bar{n})(k_3\bar{n})}$$

- employ Sudakov variables $k_i^{\mu} = \frac{1}{2}\left[\alpha_i n^{\mu} + \beta_i \bar{n}^{\mu} - 2\sqrt{\alpha_i\beta_i}\boldsymbol{e}_i^{\perp}\right]$
- use $\delta[\alpha_2 - \beta_2]$
- introduce fraction $\xi_i = \beta_i/\alpha_i\,, i = 1,3$

$$\bar{I}^{\nu} = \int \mathrm{d}\Omega \int_0^1 \prod \mathrm{d}\beta_i \,\mathrm{d}\xi_1\,\mathrm{d}\xi_3 \left(\frac{\beta_1^2\beta_3^2}{\xi_1\xi_3}\right)^{-\varepsilon} \frac{\beta_2^{-2\varepsilon}\,\delta[1-\beta_{123}]\,(\beta_1\beta_2\beta_3)^{\nu}}{(\xi_1+\xi_3-2\sqrt{\xi_1\xi_3}\boldsymbol{e}_1^{\perp}\boldsymbol{e}_2^{\perp})\beta_1(\beta_1+\xi_1\beta_2)\beta_3}$$

Rapidity divergences at N3LO

Example

- consider the following master integral

$$\bar{I}^{\nu} = \int \frac{\mathrm{d}\Phi^{nnn,\nu}_{\theta\delta\theta}}{(k_1 k_3)(k_1 n)(k_{12}\bar{n})(k_3\bar{n})}$$

- employ Sudakov variables $k_i^{\mu} = \frac{1}{2}\left[\alpha_i n^{\mu} + \beta_i \bar{n}^{\mu} - 2\sqrt{\alpha_i\beta_i}\boldsymbol{e}_i^{\perp}\right]$
- use $\delta[\alpha_2 - \beta_2]$
- introduce fraction $\xi_i = \beta_i/\alpha_i\,, i = 1,3$

$$\bar{I}^{\nu} = \int \mathrm{d}\Omega \int_0^1 \prod \mathrm{d}\beta_i\,\mathrm{d}\xi_1\,\mathrm{d}\xi_3 \left(\frac{\beta_1^2\beta_3^2}{\xi_1\xi_3}\right)^{-\varepsilon} \frac{\beta_2^{-2\varepsilon}\,\delta[1-\beta_{123}]\,(\beta_1\beta_2\beta_3)^{\nu}}{(\xi_1+\xi_3-2\sqrt{\xi_1\xi_3}\boldsymbol{e}_1^{\perp}\boldsymbol{e}_2^{\perp})\beta_1(\beta_1+\xi_1\beta_2)\beta_3}$$

- consider the region $\beta_1 \sim \xi_1 \sim \xi_3 \sim \lambda \to 0$

$$\bar{I}^{\nu} = \int \mathrm{d}\Omega \int \frac{\mathrm{d}\lambda}{\lambda}\left(\frac{\lambda^2}{\lambda\lambda}\right)^{-\varepsilon} \lambda^{\nu} \times \cdots \sim \frac{1}{\nu}$$

Rapidity divergences at N3LO

Example

- consider the following master integral

$$\bar{I}^{\nu} = \int \frac{\mathrm{d}\Phi^{nnn,\nu}_{\theta\delta\theta}}{(k_1 k_3)(k_1 n)(k_{12}\bar{n})(k_3\bar{n})}$$

- employ Sudakov variables $k_i^{\mu} = \frac{1}{2}\left[\alpha_i n^{\mu} + \beta_i \bar{n}^{\mu} - 2\sqrt{\alpha_i\beta_i}\boldsymbol{e}_i^{\perp}\right]$
- use $\delta[\alpha_2 - \beta_2]$
- introduce fraction $\xi_i = \beta_i/\alpha_i\,, i = 1,3$

$$\bar{I}^{\nu} = \int \mathrm{d}\Omega \int_0^1 \prod \mathrm{d}\beta_i\, \mathrm{d}\xi_1\, \mathrm{d}\xi_3 \left(\frac{\beta_1^2\beta_3^2}{\xi_1\xi_3}\right)^{-\varepsilon} \frac{\beta_2^{-2\varepsilon}\,\delta[1-\beta_{123}]\,(\beta_1\beta_2\beta_3)^{\nu}}{(\xi_1+\xi_3-2\sqrt{\xi_1\xi_3}\boldsymbol{e}_1^{\perp}\boldsymbol{e}_2^{\perp})\beta_1(\beta_1+\xi_1\beta_2)\beta_3}$$

- consider the region $\beta_1 \sim \xi_1 \sim \xi_3 \sim \lambda \to 0$

$$\bar{I}^{\nu} = \int \mathrm{d}\Omega \int \frac{\mathrm{d}\lambda}{\lambda}\left(\frac{\lambda^2}{\lambda\lambda}\right)^{-\varepsilon} \lambda^{\nu} \times \cdots \sim \frac{1}{\nu}$$

$\Rightarrow$ describes a situation where $g(k_2)$ is emitted in the transverse plane $k_2 n = k_2\bar{n}$ and gluons $g(k_1)$, $g(k_3)$ are very forward

Rapidity divergences at N3LO

Example

- consider the following master integral

$$\bar{I}^{\nu} = \int \frac{\mathrm{d}\Phi^{nnn,\nu}_{\theta\delta\theta}}{(k_1 k_3)(k_1 n)(k_{12}\bar{n})(k_3\bar{n})}$$

- employ Sudakov variables $k_i^{\mu} = \frac{1}{2}\left[\alpha_i n^{\mu} + \beta_i \bar{n}^{\mu} - 2\sqrt{\alpha_i\beta_i}\boldsymbol{e}_i^{\perp}\right]$
- use $\delta[\alpha_2 - \beta_2]$
- introduce fraction $\xi_i = \beta_i/\alpha_i\,, i = 1,3$

$$\bar{I}^{\nu} = \int \mathrm{d}\Omega \int_0^1 \prod \mathrm{d}\beta_i \,\mathrm{d}\xi_1\,\mathrm{d}\xi_3 \left(\frac{\beta_1^2\beta_3^2}{\xi_1\xi_3}\right)^{-\varepsilon} \frac{\beta_2^{-2\varepsilon}\,\delta[1-\beta_{123}]\,(\beta_1\beta_2\beta_3)^{\nu}}{(\xi_1+\xi_3-2\sqrt{\xi_1\xi_3}\boldsymbol{e}_1^{\perp}\boldsymbol{e}_2^{\perp})\beta_1(\beta_1+\xi_1\beta_2)\beta_3}$$

- consider the region $\beta_1 \sim \xi_1 \sim \xi_3 \sim \lambda \to 0$

$$\bar{I}^{\nu} = \int \mathrm{d}\Omega \int \frac{\mathrm{d}\lambda}{\lambda}\left(\frac{\lambda^2}{\lambda\lambda}\right)^{-\varepsilon} \lambda^{\nu} \times \cdots \sim \frac{1}{\nu}$$

$\Rightarrow$ describes a situation where $g(k_2)$ is emitted in the transverse plane $k_2 n = k_2\bar{n}$ and gluons $g(k_1)$, $g(k_3)$ are very forward

$\Rightarrow$ $1/\nu$-pole gets multiplied with reduction coefficient $c \sim \nu$

Master integrals at N3LO

Divide according to propagator $k_{123}^2 = 2\left[k_1k_2 + k_1k_3 + k_2k_3\right]$

- without k_{123}^2: compute along the lines discussed for NNLO

$$\sim \int \frac{1}{(k_1k_2)(k_1k_3)\ldots}$$

- with k_{123}^2: difficult to compute analytically

Numerical computation of MI (see also [Liu et al. '17])

- add parameter m^2 to 'complicated' propagator see also [Henn et al. '13]

$$I = \int \frac{\mathrm{d}\Phi^{\nu}_{f_1f_2f_3}}{\left[k_{123}^2\right]\ldots} \longrightarrow J = \int \frac{\mathrm{d}\Phi^{\nu}_{f_1f_2f_3}}{\left[k_{123}^2 + m^2\right]\ldots}$$

- using the IBP technology for θ-functions, derive differential equation

$$\partial_{m^2}\boldsymbol{J} = \widehat{K}\boldsymbol{J}$$

- compute boundary conditions in $m^2 \to \infty$, where the "bad" propagator simplifies
- solve differential equation numerically, compute the limit $m \to 0$, and recover I from Taylor-branch c_{000} in

$$J = \sum c_{n_1n_2n_3}(m^2)^{n_1+n_2\varepsilon}\ln^{n_3}(m^2)$$

- result with 2000 digits, allows to reconstruct analytic expression with PSLQ

Results

$$
\begin{aligned}
S_3^{nnn} = {} & \frac{24}{\varepsilon^5} + \frac{308}{3\varepsilon^4} + \frac{1}{\varepsilon^3}\left(-12\pi^2 + \frac{3380}{9}\right) + \frac{1}{\varepsilon^2}\left(-1000\zeta_3 + \frac{440\pi^2}{9} + \frac{10048}{9}\right) \\
& + \frac{1}{\varepsilon}\left(-\frac{2377\pi^4}{45} + \frac{440\zeta_3}{3} + \frac{7192\pi^2}{27} + \frac{253252}{81}\right) \\
& + \left(-28064\zeta_5 + \frac{1972\zeta_3\pi^2}{3} - \frac{638\pi^4}{15} + 4224\,\mathrm{Li}_4\left(\frac{1}{2}\right) + 3696\zeta_3\ln(2) - 176\pi^2\ln^2(2) + 176\ln^4(2)\right. \\
& \left. + \frac{13208\zeta_3}{3} + \frac{78848\pi^2}{81} + 96\ln(2) + \frac{1925074}{243}\right) \\
& + \varepsilon\left(2304\ \zeta_{-5,-1} - 4464\zeta_5\ln(2) + 25784\zeta_3^2 - \frac{67351\pi^6}{567} - 6336G_R(0,0,r_2,1,-1)\right. \\
& - 6336G_R(0,0,1,r_2,-1) - 3168G_R(0,0,1,r_2,r_4) - 6336G_R(0,0,r_2,-1)\ln(2) + \frac{268895\zeta_5}{3} \\
& - 45056\,\mathrm{Li}_5\left(\frac{1}{2}\right) - 45056\,\mathrm{Li}_4\left(\frac{1}{2}\right)\ln(2) + 176\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\pi - 1056\zeta_3\,\mathrm{Li}_2\left(\frac{1}{4}\right) - 3982\zeta_3\pi^2 \\
& - 21824\zeta_3\ln^2(2) + 2112\zeta_3\ln(2)\ln(3) - 1584\,\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\ln(3) - \frac{4400\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\pi^3}{27} + \frac{88\pi^4\ln(2)}{45} \\
& - \frac{616\pi^4\ln(3)}{27} + \frac{11264\pi^2\ln^3(2)}{9} - \frac{22528\ln^5(2)}{15} + 8576\,\mathrm{Li}_4\left(\frac{1}{2}\right) + 7504\zeta_3\ln(2) + \frac{4174\pi^4}{27} \\
& - \frac{1072\pi^2\ln^2(2)}{3} + \frac{1072\ln^4(2)}{3} + \frac{554032\zeta_3}{27} - 32\pi^2\ln(2) + \frac{730378\pi^2}{243} - 384\ln^2(2) + 832\ln(2) \\
& \left. + \frac{1408681}{81} + \sqrt{3}\left(192\,\Im\left\{\mathrm{Li}_3\left(\frac{\exp(i\pi/3)}{2}\right)\right\} + 160\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln(2) - 16\pi\ln^2(2) - \frac{560\pi^3}{81}\right)\right) + \mathcal{O}(\varepsilon^2)
\end{aligned}
$$

- most master integrals for $nn\bar{n}$-configuration computed, checks in progress

One-jettiness @ NLO

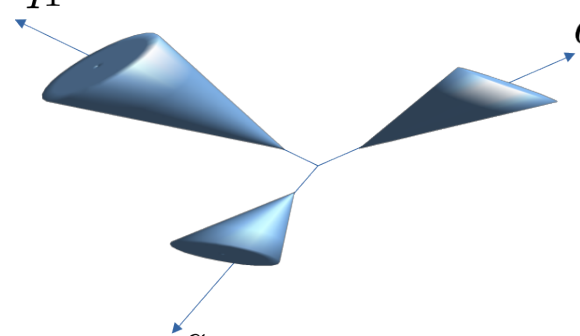
$$S = \int \mathrm{d}^{d-1}k \sum_{i,j} \boldsymbol{T}_i \boldsymbol{T}_j \frac{q_i \cdot q_j}{(q_i \cdot k)(q_j \cdot k)} \delta[\tau - k \cdot q_1]\, \theta[k \cdot (q_2 - q_1)]\, \theta[k \cdot (q_3 - q_1)]$$

function of two angular ratios

$$x_1 = \frac{y_{12}}{y_{23}}\ ,\ x_2 = \frac{y_{13}}{y_{23}}$$

where

$$y_{ij} = \frac{q_i \cdot q_j}{E_i E_j}$$



six master integrals, canonical basis [Henn ‘13]

$$\mathrm{d}\boldsymbol{J} = \varepsilon \sum_{\omega \in \mathscr{A}} \mathrm{d}\log(\omega) \boldsymbol{J}$$

$$\mathscr{A} = \Big\{ x_1, x_2, 1 - x_1, 1 - x_2, \lambda(1, x_1, x_2), \sqrt{\lambda(1,x_1,x_2)} + x_1 - x_2 - 1, \sqrt{\lambda(1,x_1,x_2)} - x_1 + x_2 - 1 \Big\}$$

boundary conditions from equal-angle limit
$y_{ij} = 1$

$$J_2 \sim \frac{\Gamma(2-2\varepsilon)}{\Gamma^2(1-\varepsilon)} + \frac{2^{-2\varepsilon}}{\pi}\, {}_2F_1 \left[\{ \tfrac{1}{2} - \varepsilon, \tfrac{1}{2} + \varepsilon \}, \{ \tfrac{3}{2} - \varepsilon \}; \tfrac{1}{4} \right]$$

(numerical computations through N2LO [Jouttenus et al. ’11] [Boughezal et al. ’15] [Campbell et al. ’17])

Conclusion

extended notion of IBP relations & reverse unitarity

- enables us to reduce phase-space integrals that contain **Heaviside functions** to master integrals
- provides the means to derive differential equations, which we solve numerically

$\Rightarrow$ with this, we computed the same-hemisphere configuration (nnn) for three-gluon emission of the zero jettiness function

Outlook for zero-jettiness

- missing master integrals with k_{123}^2-propagator in configuration $nn\bar{n}$ to finish ggg contribution
- other contributions are
  - RRR: $gq\bar{q}$-emission: only reduction, master integrals identical
  - RRV: one-loop corrections to gg- and $q\bar{q}$-emission: re-compute [Chen et al. '22]
  - RVV: two-loop correction to g-emission

General outlook

- extension to 1-jettiness under investigation
  - analytic structure, regularity conditions, . . .
- other phase-space integrals containing Heaviside functions (soft counter terms in subtraction schemes, . . .)