Classical Scattering at the 4-loop Order

UCLA Mani L. Bhaumik Institute for Theoretical Physics

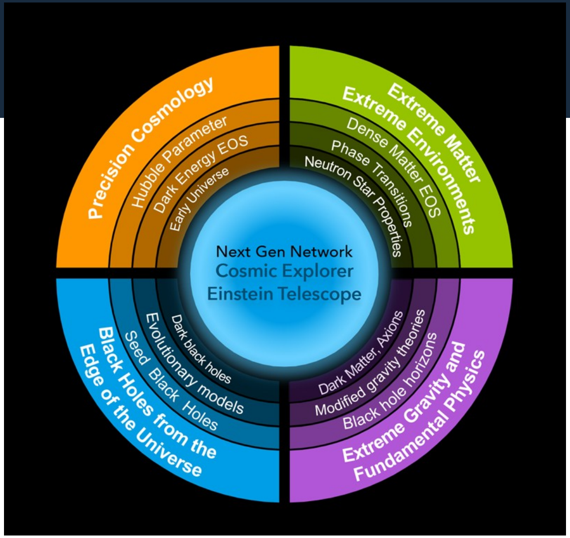
Michael Ruf
Loopfest XXI, 28. Jun 2023

based on [2305.08981] w/ Bern, Herrmann, Roiban, Smirnov, Smirnov, Zeng

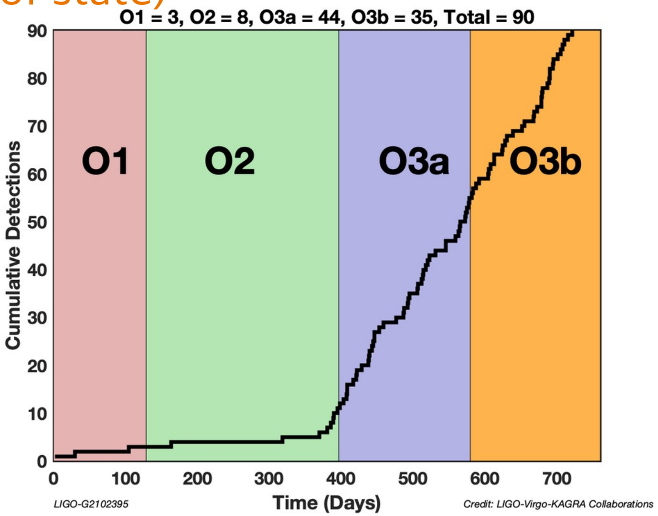
- Introduction
- GW physics as a collider problem
- Result: classical scattering in electrodynamics at O(α^5)
- Outlook: classical scattering in gravity at O(G^5)

Gravitational Waves

- Direct GW detection 2015, ~100 years after postulation
- Many sources: early Universe, supernovae, <u>binary systems</u>,…
- Physics goals:
  - Strong-field tests of GR (non-perturbative effects, horizons)
  - Cataloging black hole (BH) binaries (properties, abundance)
  - Probing ultra-dense matter (neutron stars (NS) equation of state)
  - Multi-messenger astronomy
  - …
- O(100) mergers events: BH-BH, BH-NS, NS-NS
- O4 just started: expect 1 merger/2-3 days!

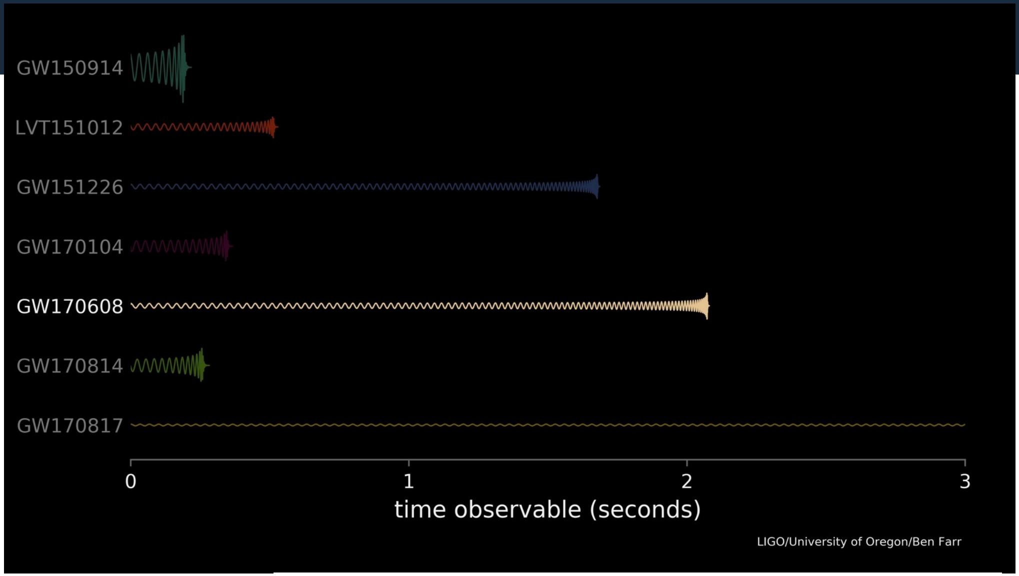


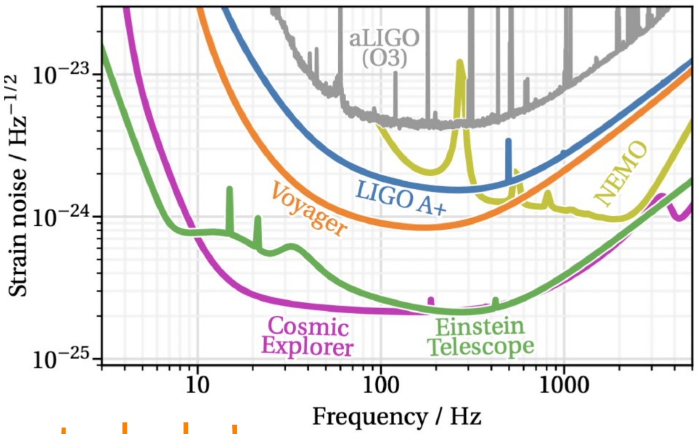
[B. S. Sathyaprakash]



Two-body systems

- GW astronomy is a precision game:
  - Relative length changes $\Delta\ell/\ell \sim 10^{-20}$
  - Observations of up to O(10^3) cycles; phase errors accumulate!
  - Parameter estimation: angular momentum, structure of constituents, new physics,. . .
- Next-gen. Experiments (ca. 2035): factor 10-100 improved S/N ratio
- Good handle on experimental errors
- High-precision theory predictions required





The particle theory community is in an unique position to help!

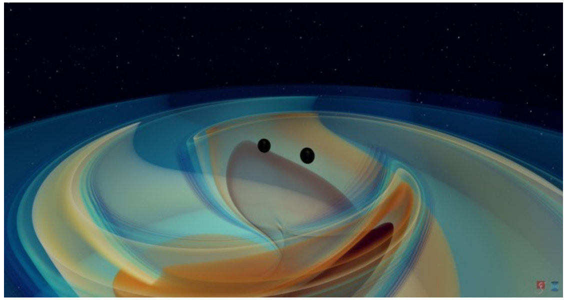
Two-body systems

- Target observables: Waveform $h_{\mu\nu}$, Periastron advance $\Delta\Phi$, rad. losses ΔE , ...
- Solve Einstein's equation (+ BND conditions):

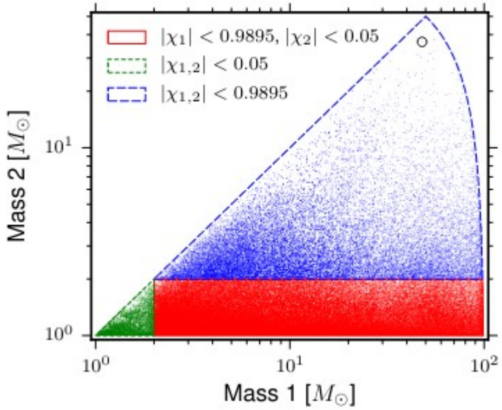
$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

- Significant resources requirements:
  - O(10^5) CPU h/NR template
  - GW150914: 250k templates
- Challenging in PS-corners: $m_1 \ll m_2,\ v \to c,\ S_i/m_i \to 1$
- Solution: analytic and hybrid models (GW150914: post-Newtonian + effective-one-body)
- Key input: perturbative corrections to Newton's potential

$$V(r) = -\frac{G\mu M}{r} + \frac{1}{c^2}\left[-\frac{3G\mu M\boldsymbol{v}^2}{2r} + \frac{G^2M^2}{r^2}\right] + \dots$$



[GW190521, LIGO]



[GW150914, LIGO]

Post-Newtonian (PN) expansion

Analysis needs 6PN/$\mathrm{N^6LO}$ two-body potential for next-gen. experiments!

$$\frac{GM}{rc^2} \sim \frac{v^2}{c^2} \ll 1$$

Virial theorem

N	1PN	2PN	3PN	4PN	5PN
1687	1938	1973	1980	2014	2020

5PN ← distinguish BH/NS

$$G \times (1 + v^2 + v^4 + v^6 + v^8 + \dots)$$

$$G^2 \times (1 + v^2 + v^4 + v^6 + \dots)$$

$$G^3 \times (1 + v^2 + v^4 + \dots)$$

$$G^4 \times (1 + v^2 + \dots)$$

$$G^5 \times (1 + \dots)$$

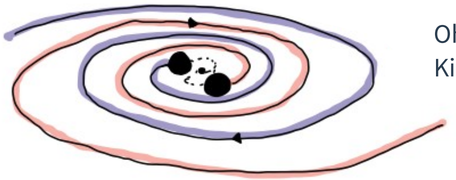
Newton

Einstein, Infeld
Hoffman

Ohta, Okamura,
Kimura, Hiida

Damour, Jaranowski,
Schafer; Blanchet, Faye

Damour,Jaranowski,Schafer



State of the art:

5PN [Blumlein, Marquard, Maier, Schafer] [Foffa, Sturani, Mastrolia, Sturm, Torres Bobadilla]

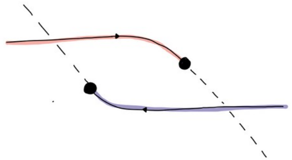
Partial 6PN [Blumlein, Marquard, Maier, Schafer]

Still some tension between results

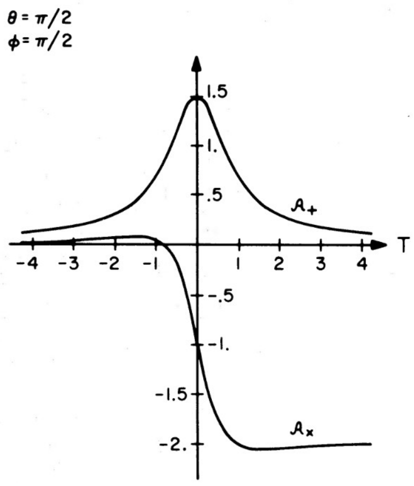
[Porto, Rothstein, Iyer, Will, Wiseman, Poisson, Cutler, Finn, Flanagan, Deruelle, Thorne, Sathyaprakash, Bini, Geralico, Goldberger, Rothstein, Buonanno, Le Tiec, Marsat, Foffa, Sturani, Mastrolia, Sturm, Torres Bobadilla, ...]

Gravitational scattering

- We will consider <u>scattering</u> instead of merger
  Q:Unlikely to be observed soon, why bother?



1) Cleaner environment (asymptotic data $\{b, p_i\}$)
   both NR and perturbative approaches
2) Part of the problem is universal (e.g. instantaneous potential; important caveat: hereditary effects)
3) Meshes naturally with amplitudes program



[Kovacs, Thorne]

- Relativistic treatment exposes additional structure, e.g. polynomiality in $\nu = \dfrac{m_1 m_2}{(m_1+m_2)^2}$

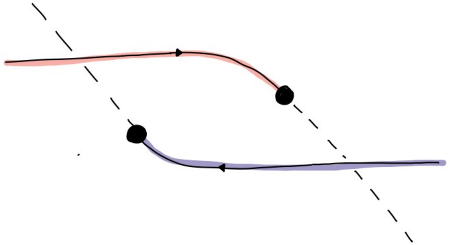
[Favata/SXS/K.Thorne]

Laboratory to improve methods, look for structures and clear conceptual issues

Post-Minkowskian (PM) expansion

$\frac{GM}{rc^2} \ll 1 \qquad v = \mathcal{O}(1)$

v-resummation of PN expansion
Scattering and eccentric motion

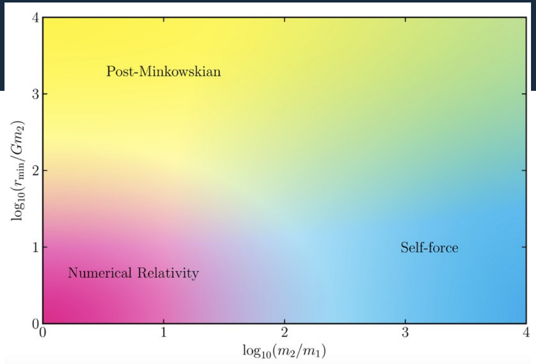


nPM = (n-1) loops

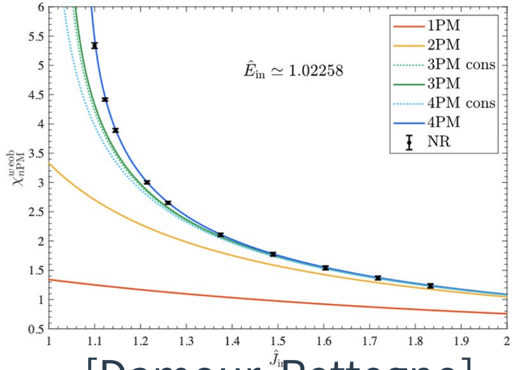
1PM	$G \times (1 + v^2 + v^4 + v^6 + v^8 + \dots)$	
2PM 1985	$G^2 \times (1 + v^2 + v^4 + v^6 + \dots)$	Westphal
3PM 2019	$G^3 \times (1 + v^2 + v^4 + \dots)$	Bern, Cheung, Roiban, Shen, Solon, Zeng (BCRSSZ)
4PM 2021	$G^4 \times (1 + v^2 + \dots)$	Bern, Parra-Martinez, Roiban, **MSR**, Shen, Solon, Zeng
5PM WIP	$G^5 \times (1 + \dots)$	

Relevant work on 4PM by [Dlapa, Kalin, Liu, Neef, Porto; Bjerrum-Bohr, Plante, Vanhove; Jakobsen, Mogull, Plefka, Sauer, Xu]

Two-body systems



- The GR community is very interested in scattering
  [Barack, Berti, Bini, Buonanno, Cardoso, Damour, East, Geralico, Gralla, Guercilena, Hinder, Hinderer, Hopper, Khalil, Lobo, Long, Nagar, Pfeiffer, Pretorious, Pretorius, Rettegno, Rezzolla, Sperhake, Steinhoff, Vines, Whittall, Yunes,. . .]
- New GR self-force $(m_1 \ll m_2)$ and NR results for scattering expected
- Fruitful interplay between different communities.
- Goal of creating new hybrid models



[Damour, Rettegno]

Self-force correction to the deflection angle in black-hole scattering: a scalar charge toy model

Leor Barack[1] and Oliver Long[1]
[1] *Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, United Kingdom*
(Dated: November 16, 2022)

Self-force effects in post-Minkowskian scattering

Samuel E. Gralla and Kunal Lobo
Department of Physics, University of Arizona, Tucson, Arizona 85721, USA

Strong-field scattering of two black holes: Numerical Relativity meets Post-Minkowskian gravity

Thibault Damour[1] and Piero Rettegno[2,3]
[1] *Institut des Hautes Etudes Scientifiques, 91440 Bures-sur-Yvette, France*
[2] *INFN Sezione di Torino, Via P. Giuria 1, 10125 Torino, Italy and*
[3] *Dipartimento di Fisica, Università di Torino, via P. Giuria 1, 10125 Torino, Italy*

A short history of the amplitudes-based program

- How can particle theorist help: Quantum S-matrix has all information on classical scattering!
- Not an entirely new idea…

> **The Two-body Problem in the Theory of the Quantized Gravitational Field †**
>
> By E. CORINALDESI ‡
> Dublin Institute for Advanced Studies
>
> *Communicated by L. Rosenfeld ; MS. received 7th June 1955 and in amended form 17th November 1955*
>
> *Abstract.* The equations of the two-body problem of general relativity are derived by a Hamiltonian method based on an expansion of the general covariant Lagrangian in powers of the gravitational constant and by employing the techniques and the viewpoint of quantum field theory. It is found that, within the approximation in which they have so far been calculated, the equations could have been obtained identically from a linear theory of gravitation.

> **Fourth-Order Gravitational Potential Based on Quantum Field Theory.**
>
> Y. IWASAKI
> *Research Institute for Fundamental Physics, Kyoto University - Kyoto*
>
> (ricevuto l'1 Marzo 1971)
>
> There have been many attempts (1) to understand the gravitational interaction in terms of quantum field theory in flat Minkowskian space-time in analogy to the electromagnetic interaction. Since in the case of the electromagnetic interaction there is excellent agreement between the quantized theory and experiment (2), we also believe that the gravitational interaction can be and should be understood by means of quantum field theory. This is the starting point of our discussions.

- Also related work on Bremsstrahlung [Feynman, Barker, Gupta, Kaskas, ⋯]
- Great idea, but hardly competitive, recomputed 1PN potential, known for at least 20 years
- To convince people compute something new!

A short history of the amplitudes-based program

- Over 50 years of improvements
  - Double copy, Gravity = (Gauge)^2 [e.g. Kawai, Lewellen, Tye; Bern, Carrasco, Johansson] 1985+
  - Generalized unitarity [Bern, Dixon, Kosower; Britto, Cachazo and Feng] [See also Fernando's talk] 1998+
  - Improved EFT understanding [Beneke, Smirnov; Goldberger, Rothstein] 1997+
  - Improvements in integration (IBP, DE etc.) [Laporta; Tkachov; Chetyrkin; Kotikov; Remiddi,
  - Gehrmann; Henn, Anastasiou, Melnikov, …] 1981+
  - Improvements in computing power
- Clear encouragement from GR community to revive the program

High-energy gravitational scattering and the general relativistic two-body problem

Thibault Damour*

Institut des Hautes Etudes Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France

(Dated: October 31, 2017)

[…] tum gravitational scattering amplitude of two particles, and we urge amplitude experts to use their novel techniques to compute the 2-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian

A short history of the amplitudes-based program

Q: Why is this a good idea? Bhabha scattering is hard at 2 loop order

A: In the classical limit the problem simplifies. Efficiency outweighs cost

The modern version of the program is highly competitive! State-of-the art results to be used in the LIGO/Virgo/Kagra pipeline

Amplitudes-based workflow

Essentially identical to workflow used for collider problems

GR EFT, QED,...

QFT

HARD ↓ — Feynman rules, double copy, gen. unitarity

Integrand

VERY HARD ↓ — Integral reduction, Evaluation of master integrals

Amplitude

EASY* ↓ — Assembling contributions, phase space integration

Observable

Integrands no bottleneck in the near future!

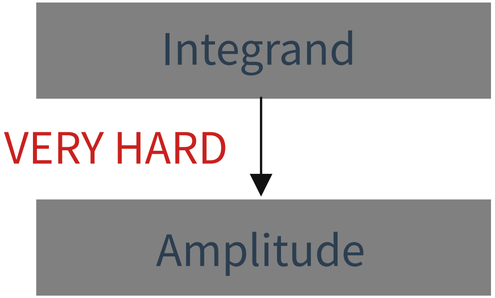
Target observables: Waveform $h_{\mu\nu}$, Periastron advance $\Delta\Phi$, rad. losses ΔE, ...

*for the conservative scattering angle

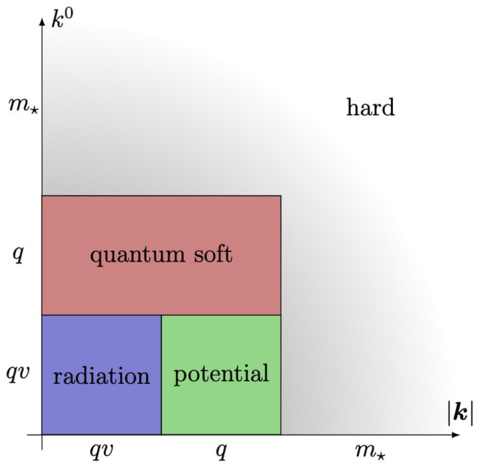
Integration

- Classical physics: Large number of soft exchanges $q = \mathcal{O}(\hbar)$

$$1 \ll J^2 \sim \frac{s}{q^2} \sim \frac{m_i^2}{q^2} \quad \rightarrow \quad q^2 \ll m_i^2 \sim s$$



- Relativistic regions: Method of Regions [Beneke, Smirnov]
  - Hard $\ell \sim m$ $\longleftarrow$ UV, quantum effects $\lambda_{\text{compton}} \sim b$
  - Soft $\ell \sim q$ $\longleftarrow$ long range $\lambda_{\text{compton}} \ll b$
- Threshold expansion: $v = |p_{\text{COM}}|/\sqrt{s}$
  - Potential (p) $(\omega, \ell) \sim (|q|v, |q|)$ $\longleftarrow$ Instantaneous
  - Radiation (r) $(\omega, \ell) \sim (|q|v, |q|v)$



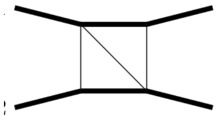
Integration

[Parra-Martinez, **MSR**, Zeng, '20]

- Many topologies trivial
  → crucial for IBP

- Integrals with eikonal propagators (HQET type)

$$\frac{1}{\ell^2 + 2p_1 \cdot \ell} = \frac{1}{2p_1 \cdot \ell} + \cdots .$$



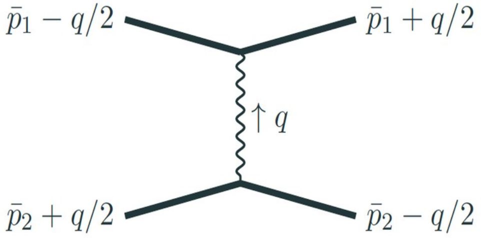
$$\sim \frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}} \log(-q^2)$$

- Specialized variables: single-variable problem $y \simeq \sigma$
  → crucial for integration and IBP

$$u_i = \bar{p}_i / |\bar{p}_i| \,, \quad u_1 \cdot u_2 = y \,, \quad u_i \cdot q = 0 \,, \quad u_i^2 = 1$$

$$y = \frac{p_1 \cdot p_1}{m_1 m_2} + \mathcal{O}(q^2) \equiv \sigma + \mathcal{O}(q^2)$$

- Simpler functions:
  - no elliptic integrals at 2 loops,
  - weight L instead of weight 2L



Classical limit

- Amplitude has no classical limit.

$$\mathcal{M}_{\text{tree}} \sim \frac{1}{\hbar} \qquad \mathcal{M}_{L-\text{loop}} \sim \frac{1}{\hbar^{L+1}}$$

Amplitude

EASY* ↓

Observable

- Typical classical targets:
  - Observables ([Kosower, Maybee, O'Connell]) - divergences cancel
  - Hamiltonian (e.g. through Lippmann-Schwinger or EFT matching) - divergences match
  - Generating functionals - divergences exponentiate
- Amplitude ↔ radial action [Bern, Parra-Martinez, Roiban, **MSR**, Shen, Solon, Zeng]

$$\mathcal{M} = \mathrm{i} \int_J (e^{\mathrm{i} I_r(J)/\hbar} - 1)\,, \quad I_r(J,E) = \int_{\text{trajectory}} p_r(J,E)\mathrm{d}r$$

$$\mathcal{M}_{\text{tree}} = \boxed{\frac{G}{\hbar} I_r^0}\,, \mathcal{M}_{1-\text{loop}} = \frac{G^2}{\hbar^2} I_r^0 \star I_r^0 + \boxed{\frac{G^2}{\hbar} I_r^1}\,, \ldots$$

Classical scattering at O(G^4)

- Setup allowed to compute the first classical observables at 3-loop/4PM order
  [Bern, Parra-Martinez, Roiban, **MSR**, Shen, Solon, Zeng, '21]
- New result, potential to improve models used in LIGO/Virgo/KAGRA analysis
- Can we push this computations to the 4-loop/5PM order?

$$\mathcal{M}_4(\boldsymbol{q}) = G^4 M^7 \nu^2 |\boldsymbol{q}| \left(\frac{\boldsymbol{q}^2}{4^{\frac{1}{3}}\tilde{\mu}^2}\right)^{-3\epsilon} \pi^2 \left[\mathcal{M}_4^{\mathrm{p}} + \nu\left(\frac{\mathcal{M}_4^{\mathrm{t}}}{\epsilon} + \mathcal{M}_4^{\mathrm{f}}\right)\right] + \int_\ell \frac{\tilde{I}_{r,1}^4}{Z_1 Z_2 Z_3} + \int_\ell \frac{\tilde{I}_{r,1}^2 \tilde{I}_{r,2}}{Z_1 Z_2} + \int_\ell \frac{\tilde{I}_{r,1}\tilde{I}_{r,3}}{Z_1} + \int_\ell \frac{\tilde{I}_{r,2}^2}{Z_1},$$

$$\mathcal{M}_4^{\mathrm{p}} = -\frac{35\left(1 - 18\sigma^2 + 33\sigma^4\right)}{8\left(\sigma^2 - 1\right)}, \qquad \mathcal{M}_4^{\mathrm{t}} = h_1 + h_2 \log\left(\tfrac{\sigma+1}{2}\right) + h_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}},$$

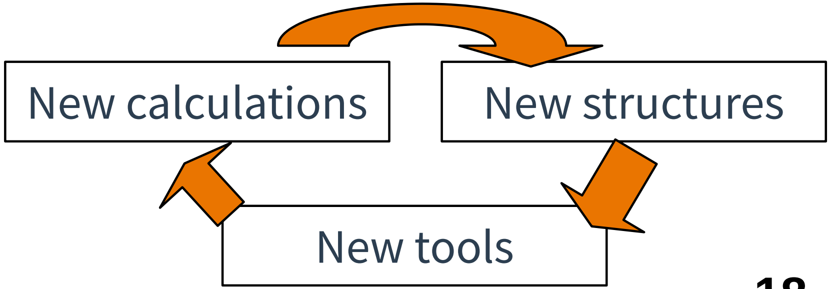
$$\sigma = \frac{p_1 \cdot p_2}{m_1 m_2} \frac{1}{\sqrt{1 - v^2}}$$

$$\begin{aligned}
\mathcal{M}_4^{\mathrm{f}} = {} & h_4 + h_5 \log\left(\tfrac{\sigma+1}{2}\right) + h_6 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + h_7 \log(\sigma) - h_2 \frac{2\pi^2}{3} + h_8 \frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1} + h_9\left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) + \frac{1}{2}\log^2\left(\tfrac{\sigma+1}{2}\right)\right] \\
& + h_{10}\left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) - \frac{\pi^2}{6}\right] + h_{11}\left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{1+\sigma}\right) - \mathrm{Li}_2\left(\tfrac{\sigma-1}{\sigma+1}\right) + \frac{\pi^2}{3}\right] + h_2 \frac{2\sigma(2\sigma^2-3)}{(\sigma^2-1)^{3/2}}\left[\mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - \mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right] \\
& + \frac{2h_3}{\sqrt{\sigma^2-1}}\left[\mathrm{Li}_2\left(1-\sigma-\sqrt{\sigma^2-1}\right) - \mathrm{Li}_2\left(1-\sigma+\sqrt{\sigma^2-1}\right) + 5\mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - 5\mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) + 2\log\left(\tfrac{\sigma+1}{2}\right)\operatorname{arccosh}(\sigma)\right] \\
& + h_{12}\mathrm{K}^2\left(\tfrac{\sigma-1}{\sigma+1}\right) + h_{13}\mathrm{K}\left(\tfrac{\sigma-1}{\sigma+1}\right)\mathrm{E}\left(\tfrac{\sigma-1}{\sigma+1}\right) + h_{14}\mathrm{E}^2\left(\tfrac{\sigma-1}{\sigma+1}\right),
\end{aligned}$$

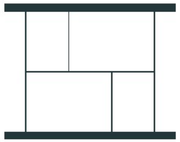
Pushing to the next order

- 5PM will have new features $$\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1\nu + \mathcal{M}_2\nu^2 \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$
  - Second order "self-force" (very important for EMRI/LISA)
- Exponential wall, while 3 loop is (relatively) easy, 4 loop is hard
- Hardest part are the IBP
  - Integrals with 22 indices - 13 propagators, 9 ISP
  - Tensors up to rank 8; doubled propagators
  - 400 families; millions of integrals

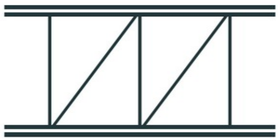
- New ideas are needed to solve this problem
- Important lesson from particle physics: Learn from computation in simpler theories e.g. with SUSY or toy models
- Virtuous circle: each new computation helps improving methods



Scattering in Electromagnetism



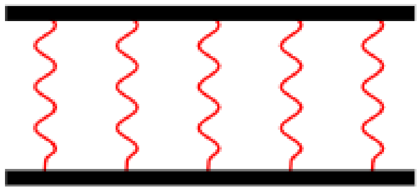
- Consider scattering of large charges in Electrodynamics (E&M)
- Simpler:
  - Fewer integral topologies (only ladders & pinches); Most complicated topologies absent
  - Lower tensor rank before expansion
  - Simple integrands
  - Instantaneous potential well-defined; No hereditary effects
- Retains complexity:
  - Soft expanded integrals have high rank tensors and doubled propagators
  - Sizable subset of GR master integrals
- Interesting by itself. Potential applications to ultraperipheral ion scattering
- The model is simple, but the computation is still involved, perhaps no surprise to collider physicists



Classical scattering at O(α^5)

- Integrand simple, e.g. R_ξ-Feynman rules (O(10^3) diagrams)

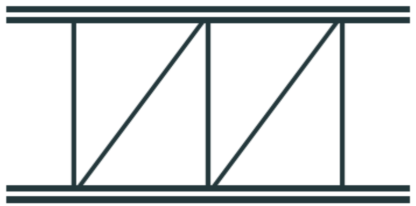
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \sum_{i=1}^{2}\left[|D_\mu\phi_i|^2 - m_i^2|\phi_i|^2\right]$$



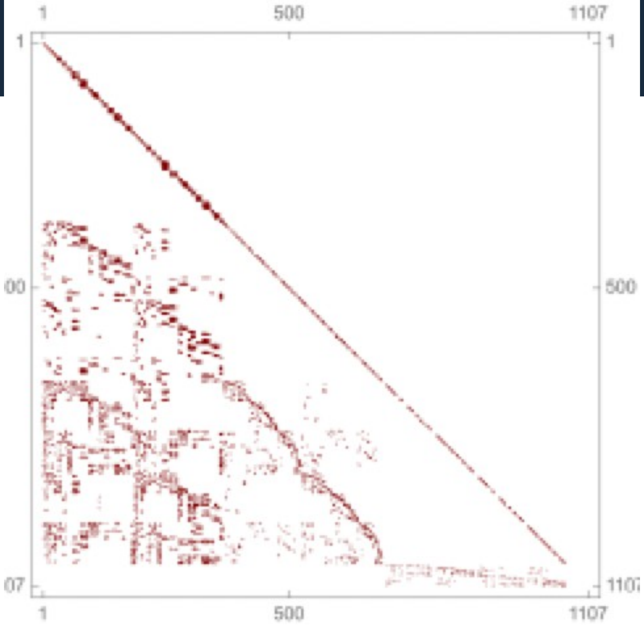
- Expand in $\hbar \to 0$, efficient using shift relations

$$\frac{1}{\ell^2 + 2p_1\cdot\ell} = \frac{1}{2p_1\cdot\ell} + \cdots + \frac{(\ell^2)^4}{(2p_1\cdot\ell)^4} + \ldots$$

- High-rank tensors (r=6) and high propagator multiplicity (d=5)
- $O(10^6)$ integrals organized in 23 families (+crossing)
- IBP: FIRE6+LiteRed on Hoffman2 cluster (~ 3 weeks, dominated by a few sectors)
- 1107 global master integrals (up to 17/sector)



Integration



- Canonical basis [Henn '13] using Lee's algorithm

$$\partial_x \vec{\mathcal{I}}(x,\epsilon) = \epsilon \sum_{w\in\mathbb{W}} w(x)\, A_w\, \vec{\mathcal{I}}(x,\epsilon) \qquad y = \frac{1+x^2}{2x}$$

- Cyclotomic kernels

$$\mathbb{W} = \left\{ \frac{1}{x}, \frac{1}{1+x}, \frac{1}{x-1}, \frac{2x}{1+x^2}, \frac{1+2x}{1+x+x^2}, \frac{2x-1}{1-x+x^2} \right\}$$

- Solutions in terms of Cyclotomic Harmonic Polylogarithms (CHPL) [Bluemlein et al.], no elliptic integrals!

$$C^{b_1,\ldots,b_n}_{a_1,\ldots,a_n}(x) = \int_0^x \mathrm{d}z\, f^{b_1}_{a_1}(z)\, C^{b_2,\ldots,b_n}_{a_2,\ldots,a_n}(z)\,,\; f^1_4 = \frac{x}{1+x^2}$$

- DE surprisingly simple (bonus relations!)

The classical scattering angle at O(α^5)

$$\chi^{5\mathrm{PL}} = \frac{\alpha^5 M^4}{30 J^5 E^4 \left(\sigma^2-1\right)^{5/2}} \times \left[r_0^{(0)} + \sum_{k=1}^{12} \left(\nu r_k^{(1)} + \nu^2 r_k^{(2)} \right) f_k \right] \qquad \nu = \frac{m_1 m_2}{M^2}$$

"Self-force" organization

- Rational coefficients

$$r_1^{(2)} = \frac{405\sigma\left(15-44\sigma^2\right)}{16\left(1-4\sigma^2\right)^2} - \frac{15\left(10\sigma^2+2\sigma-3\right)}{\sigma^3} + \ldots$$

- Transcendental functions up to weight 3

$$f_{12} = -C^{0,0,0}_{2,0,0}(x) + C^{1,0,0}_{4,0,0}(x) + \frac{21\zeta_3}{32} = \frac{1}{32}(x-1)^4 + \ldots$$

- No transcendental constants (π^2). Closely related to absence of elliptics

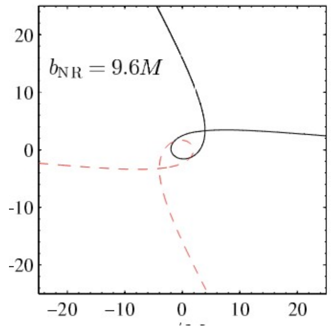
Structure of the result

- The functions are special:
  - Simple symbols, smaller alphabet $\mathbb{W} = \left\{ x, \frac{\sigma+1}{\sigma-1}, \frac{\sigma}{\sigma+1}, \frac{2\sigma+1}{2\sigma-1} \right\}$

$$\ldots, \mathcal{S}_{10}{=}x{\otimes}w_4{\otimes}x\,, \mathcal{S}_{11}{=}\frac{1}{2}x{\otimes}x{\otimes}w_2, \mathcal{S}_{12}{=}\frac{1}{2}x{\otimes}x{\otimes}w_3$$

  - Expressible through polylogs with real arguments $\mathrm{Li}_n(x^k)$
- Can we constrain function space for E&M?
- Rational functions have poles
  - $\sigma{=}1$ expected “zoom-whirl”
  - $\sigma{=}0$ present at 3 loops
  - $\sigma{=}\pm 1/2$ new at 4 loops

  } outside of region $1{<}\sigma$



- Radius of convergence $v \in [0, \sqrt{3})$

Validation

- Consistency checks:
  - R_ξ-independence and cuts
  - Cancellations of IR divergences and well-defined energy integrals in observables
  - Iterations/subtractions in amplitude-action formula
- 4PC angle from Fokker-type Lagrangian + order reduction

$$L = \sum_{a=1}^{2} m_a c^2 \left[1 - \left(1 - \frac{v_a^2}{c^2}\right)^{1/2}\right] - e_1 e_2 \sum_{s=0}^{\infty} \frac{(-)^s}{c^{2s}(2s)!} D_1^s D_2^s \left[\left(1 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2}\right) r^{2s-1}\right]$$

$$\chi_{(p)}^{5\mathrm{PL}} = \frac{\alpha_{\mathrm{eff}}^5}{J^5} \left[-\frac{2}{5v^5} + \frac{4}{3v^3} + \frac{2(8\nu - 3)}{3v} + \frac{8}{9}\nu(5 - 18\nu)v + \nu\left(\frac{80\nu^2}{3} - \frac{532\nu}{27} + \frac{226}{45}\right)v^3 + \mathcal{O}(v^5)\right] . \quad \alpha_{\mathrm{eff}} = \frac{\alpha}{J}$$

- Large-mass limit $m_1 \ll m_2$

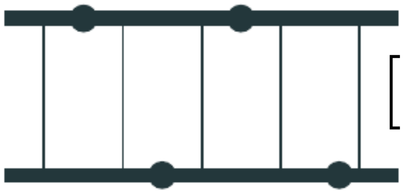
$$\chi = -\pi + \frac{2}{\sqrt{1-\alpha_{\mathrm{eff}}^2}} \arctan\left[\frac{\sqrt{\sigma^2-1}}{\sigma} \frac{\sqrt{1-\alpha_{\mathrm{eff}}^2}}{\alpha_{\mathrm{eff}}}\right] + \mathcal{O}(m_1/m_2) .$$

$$\alpha_{\mathrm{eff}} = \frac{\alpha}{J}$$

Back to gravity

- The goal is to compute classical scattering at 5PM/4-loop order in GR
- The E&M problem less complicated (both conceptually and computationally), but:
  - Integrand construction in GR under control
  - IBP complexity similar, but GR has more complicated topologies
  - Got a first glimpse at the structures at the four loop order (poles and functions)
  - Found structures that can be looked for in GR
  - Computed a significant part of the GR master integrals (~25%)
- Very optimistic about near-term progress. Ideas from particle physics will play a key role!

$$[(\ell \cdot p)^8] \subset \mathcal{M}_{\mathrm{GR}}$$



$$[(\ell^2)^4] \subset \mathcal{M}_{\mathrm{GR}}, \mathcal{M}_{\mathrm{QED}}$$

Open questions

- Is the E&M result free of elliptics at higher orders?
- Higher cyclotomies (7, 9,…)? do poles accumulate at $\sigma\!=\!1$?
- Can we bootstrap classical QED observables?
  - Constrained function space
  - Relations to lower orders (energy loss)
  - High order post-Coulombian results from Fokker action
- Can we understand the simplicity of the differential equations?
- Relation between energy loss and angle? Holds to higher order?
- Are some of the structures observed in E&M present in GR?
- Phenomenological relevance to ultraperipheral ion scattering, e.g. TOTEM experiment

Landau analysis?

Backup

Conclusions

- Classical computations recast as a collider-pheno-type computation
  - Method of regions
  - IBP reduction
  - Differential equations
- Computed the classical scattering angle at to O(α^5)
- Most challenging part is integration, IBP (22 indices, rank 8), common to all approaches
- New ideas from the from collider physics will continue to play an key role in the future
  - Intersection theory
  - Syzygies
- Most likely new functions beyond the simple elliptic integrals at 3 loops
  - Need Better understanding of functions or
  - Series expansion to high order/ numerics
- Still open conceptual issues at 4PM. Connecting back to bound problem possible? (already an issue at 3 loops)

Integration

- Boundary conditions in the static limit $x \to 1$
- Regularity fixes 814 + explicitly evaluate 293 integrals
- Ill-defined energy integrals not regulated in dim-reg. (c.f. collinear region)

$$I_{\mathrm{BND}} = \int \frac{\mathrm{d}\omega_1 \mathrm{d}\omega_2}{(\omega_1 + \mathrm{i}\varepsilon)(\omega_1 + \omega_2 + \mathrm{i}\varepsilon)}$$

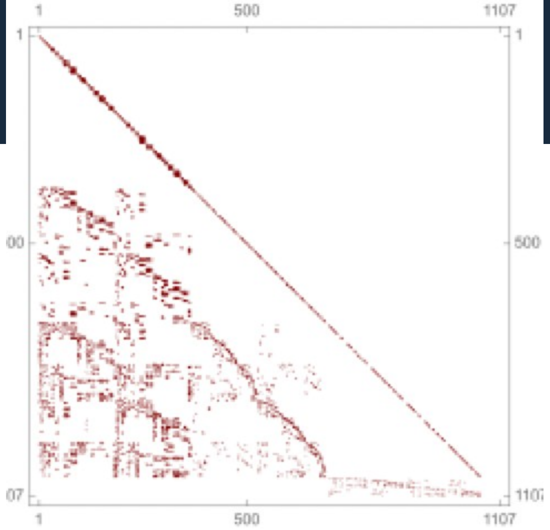
- Don't regulate. In the final result, organize to well-defined integrals:

$$\delta(\omega_1 + \ldots + \omega_n) \sum_{\text{Perms of } \omega_i} \frac{1}{\omega_1 + i\epsilon} \cdots \frac{1}{\omega_1 + \ldots + \omega_{n-1} + i\epsilon}$$
$$= (-2\pi i)^{n-1} \delta(\omega_1) \ldots \delta(\omega_n)$$

[Akhoury,Saotome, Sterman]

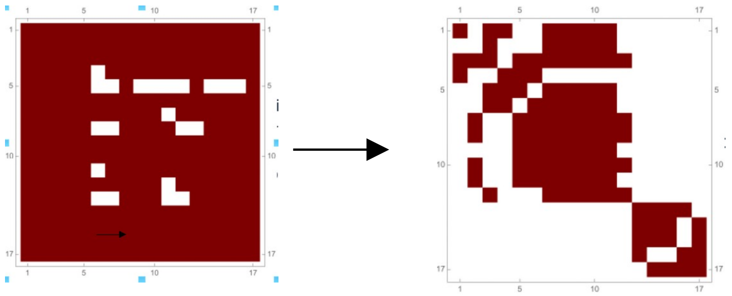
Integration

- Canonical basis is invariant under constant basis changes



$$\partial_x f(\epsilon, x) = \epsilon \sum_{\alpha \in \mathbb{W}} \log \alpha(x) A_\alpha f(\epsilon, x)$$

- Can be used to e.g. Jordan transform one A_α
- Transformation factorizes certain blocks, in particular all 17x17 blocks
- Bonus relations between potential-region integrals. Likely related to special structure of eikonal integrals
- DE is very sparse, e.g. top sectors don't talk to bottom



Lower-loop results

- Some curiosity at 3 loop order
- $O(\alpha^4)$ discontinuity related to the $O(\alpha^3)$ energy loss
- In GR there is a relation between $O(G^4)$ log(v) and $O(G^4)$ energy loss
- GR case is closely tied to the
  - Does this persist at higher orders?
  - What's the origin?
  - Does this hold in GR
  - Relation to IR-pole in GR?

[Bern et al., Buonanno et al.]

$$\chi^{1PL}_{pot} = \alpha_{eff} \frac{-2\sigma}{\sqrt{\sigma^2 - 1}},$$

$$\chi^{2PL}_{pot} = \alpha_{eff}^2 \frac{\pi}{2\sqrt{1+2\nu(\sigma-1)}},$$

$$\chi^{3PL}_{pot} = \alpha_{eff}^3 \frac{-2\sigma(2\sigma^2-3)+4\nu(\sigma-1)(\sigma^3+3\sigma^2-3)}{3\,(1+2\nu(\sigma-1))\,(\sigma^2-1)^{3/2}},$$

$$\chi^{4PL}_{pot} = \alpha_{eff}^4 \frac{3\pi}{8(1+2\nu(\sigma-1))^{3/2}} \Bigg[1 \qquad (9$$

$$+ \frac{\nu}{2(\sigma^2-1)} \Bigg\{ 3\sigma^4 - 11\sigma^3 + 3\sigma^2 + \sigma + 14 - \frac{7\sigma^2-1}{\sigma^3}$$

$$+ 2\,(3\sigma^3 - 4\sigma^2 + 9\sigma - 4) \frac{\log(x)}{\sqrt{\sigma^2-1}} + (3\sigma^2+1)\left[\frac{\log(x)}{\sqrt{\sigma^2-1}}\right]^2 \Bigg\} \Bigg]$$

$$\Delta E^{Q_1^3 Q_2^3}_{c.m.} \sim (3\sigma^3 - 4\sigma^2 + 9\sigma - 4) + (3\sigma^2+1)\frac{\log(x)}{\sqrt{\sigma^2-1}}$$

[Bern et al., Buonanno et al.]