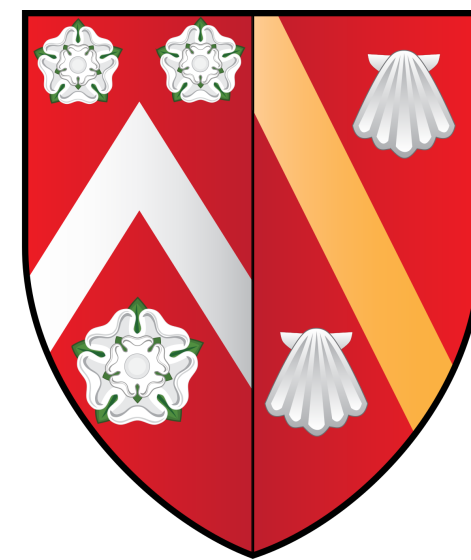


IR-safe flavoured jet algorithms for the precision era

LoopFest XXI, SLAC, June 27 2023

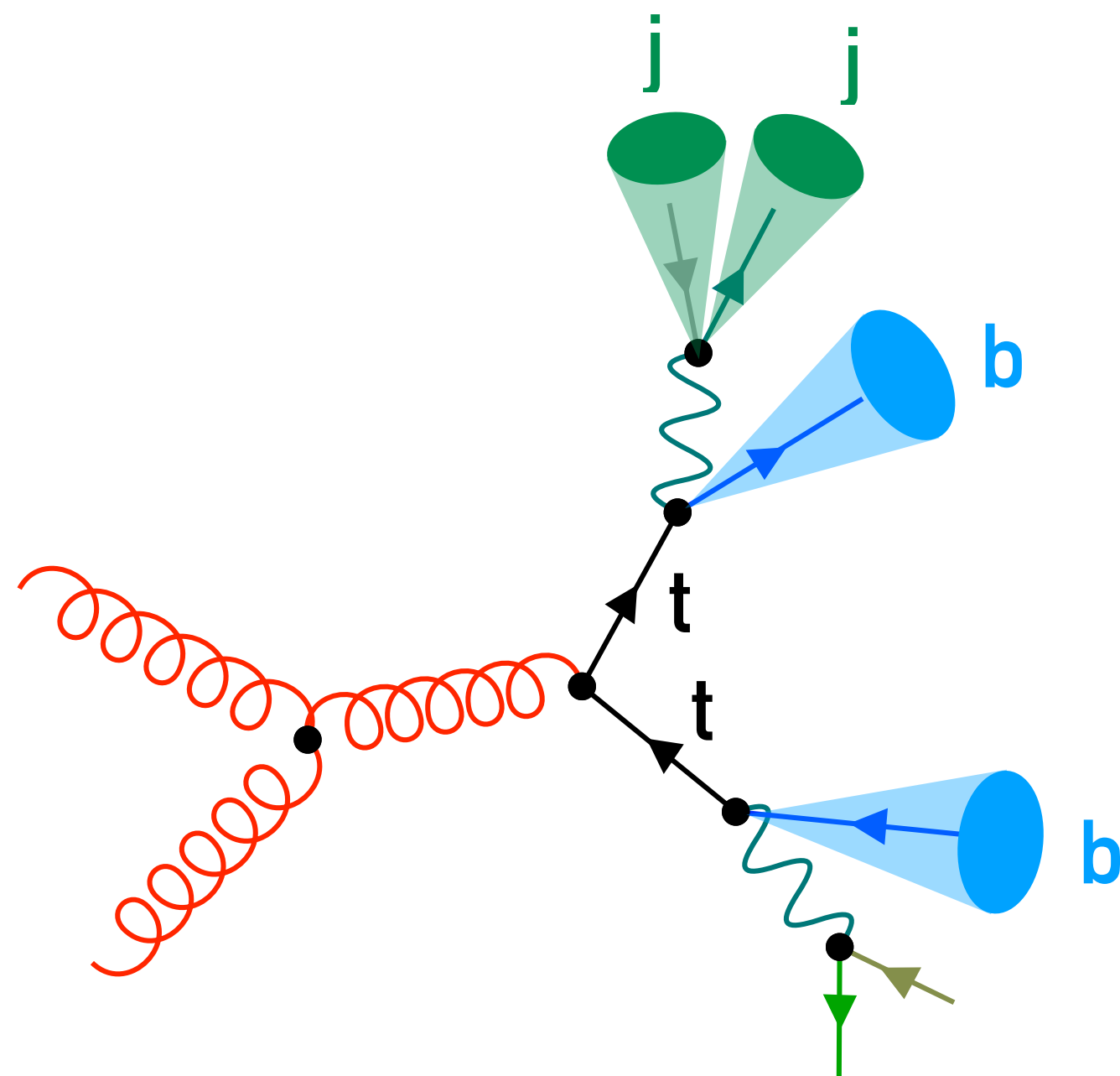


Fabrizio Caola, with R. Grabarczyk, M. Hutt, G. Salam, L. Scyboz and J. Thaler, arXiv:2306.07314

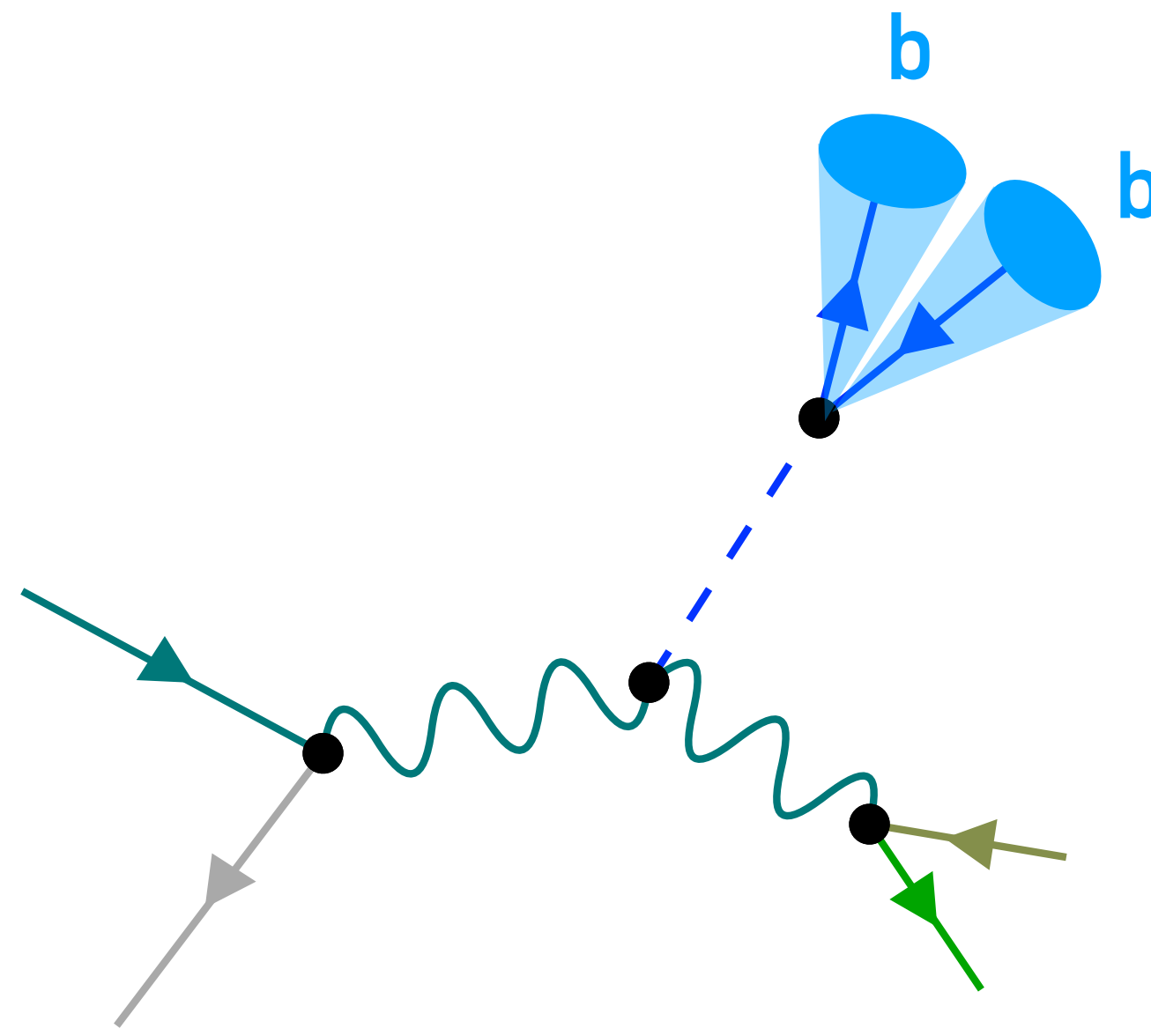
Rudolf Peierls Centre for Theoretical Physics & Wadham College

Tracking the flavour of a jet: more and more important

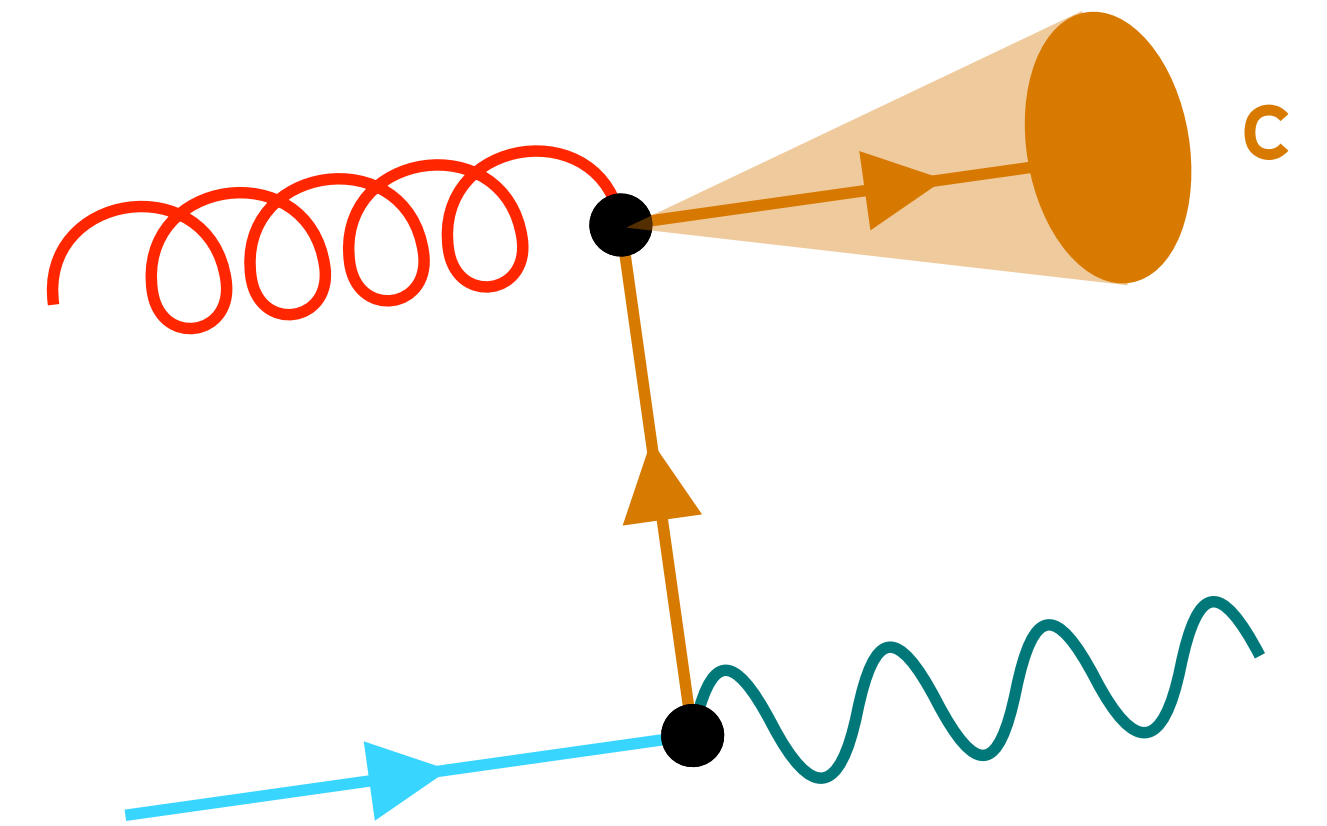
Top, ttH, ttV, ...



VH, HH,
H → bb



Wc



Quark/gluon
discrimination,
substructure...

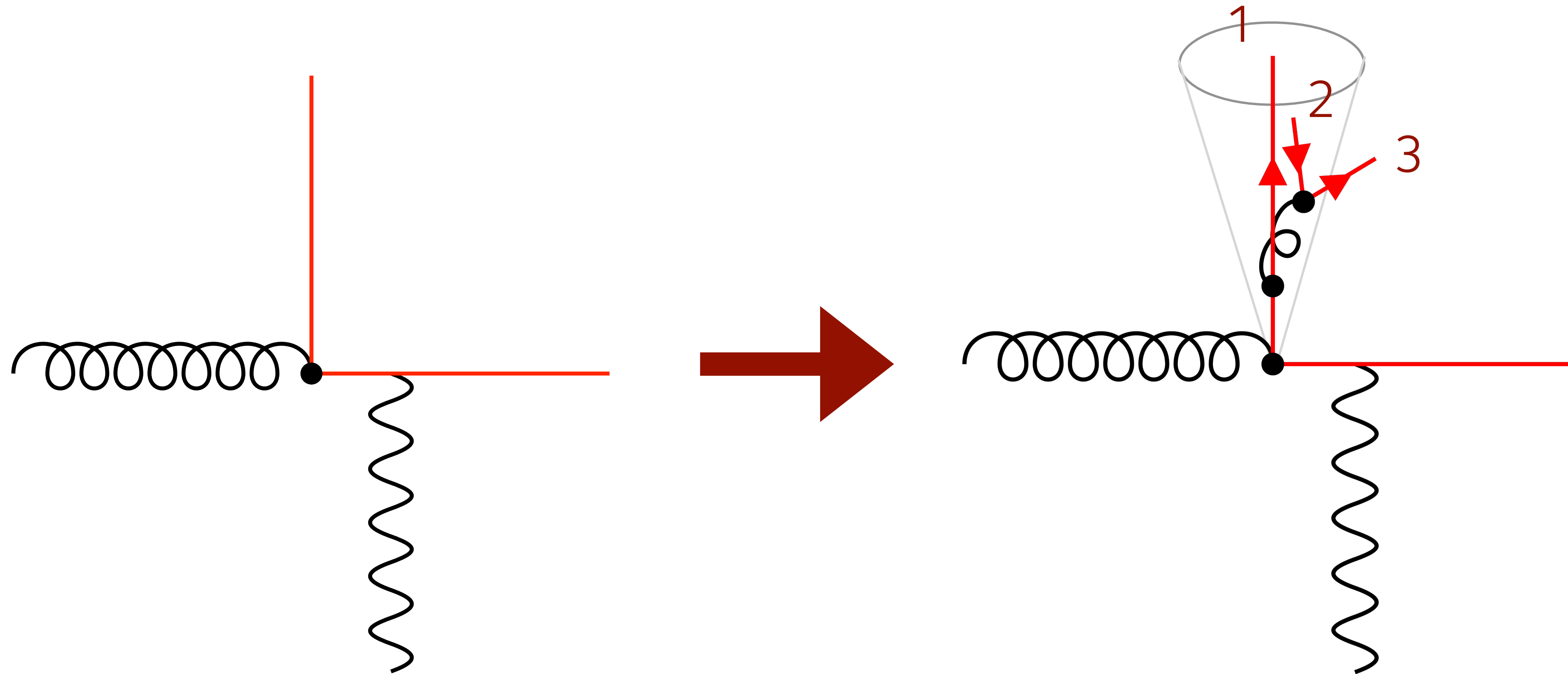
- More and more precise measurements
- More and more accurate predictions
- Apple-to-apple comparison difficult without suitable definition of “jet flavour”

The problem of jet flavour: IR-unsafe at higher orders

- Experimentally: anti- k_t , b-tagging
- Theory:

$$d_{ij} = \min(p_{t,i}^{-2}, p_{t,j}^{-2}) \frac{\Delta R_{ij}^2}{R^2} \quad \Delta R_{ij}^2 = (y_i - y_j)^2 + (\phi_i - \phi_j)^2$$

$$d_{iB} = p_{t,i}^{-2}$$



$$\min(d_{ij}, d_{iB}) = d_{12}$$

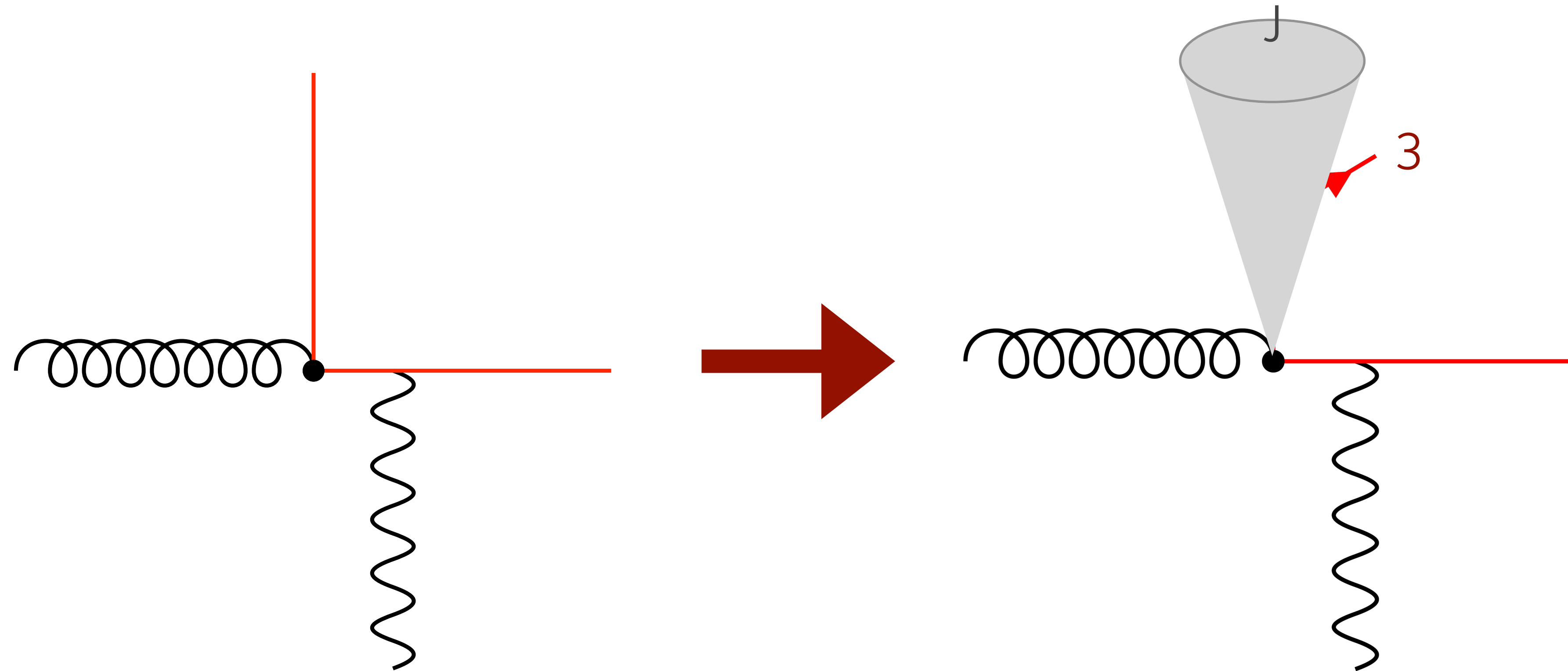
- Cluster 1+2
- $b + \bar{b} \rightarrow$ no flavour

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$$d_{ij} = \min(p_{t,i}^{-2}, p_{t,j}^{-2}) \frac{\Delta R_{ij}^2}{R^2} \quad \Delta R_{ij}^2 = (y_i - y_j)^2 + (\phi_i - \phi_j)^2$$

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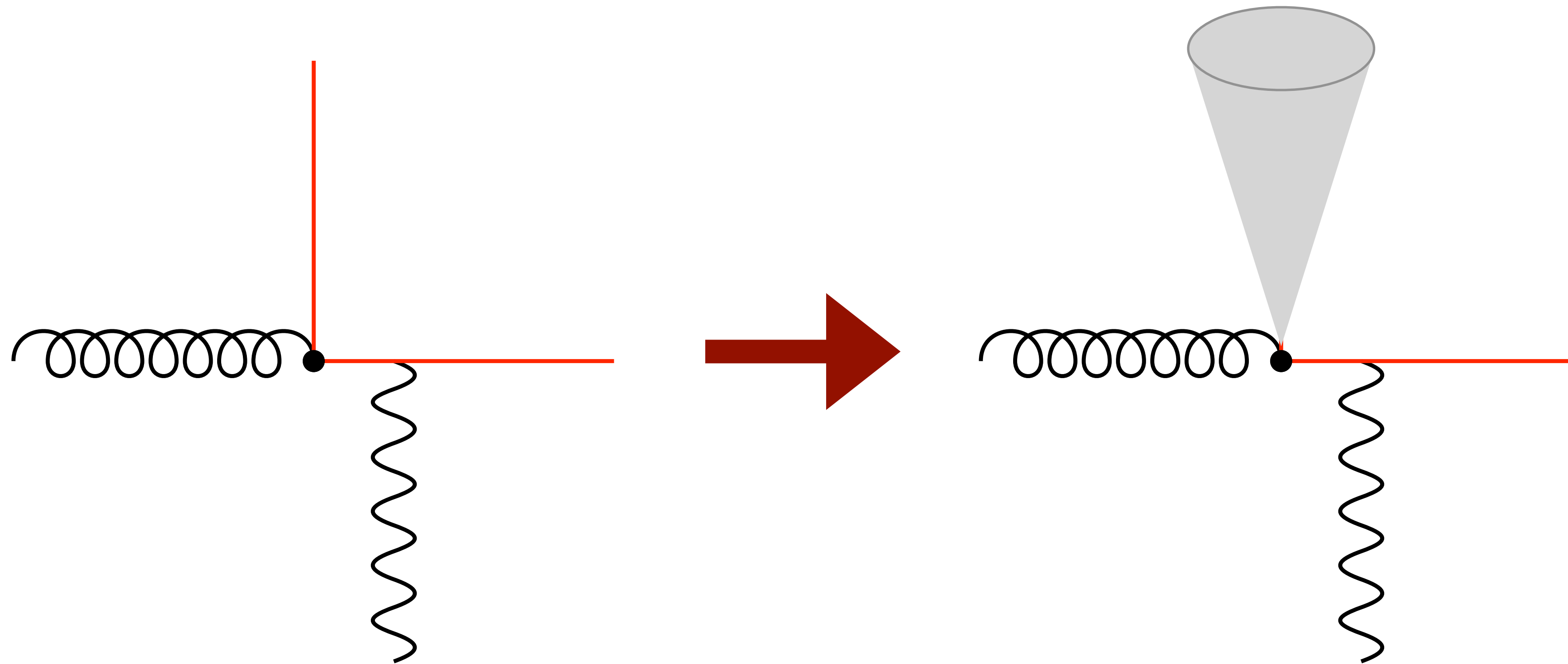
$$\min(d_{ij}, d_{iB}) = d_{jB}$$

- J and 3 jets
- 3: soft \rightarrow removed by $p_{t,\min}$ requirement

The problem of jet flavour: IR-unsafe at higher orders

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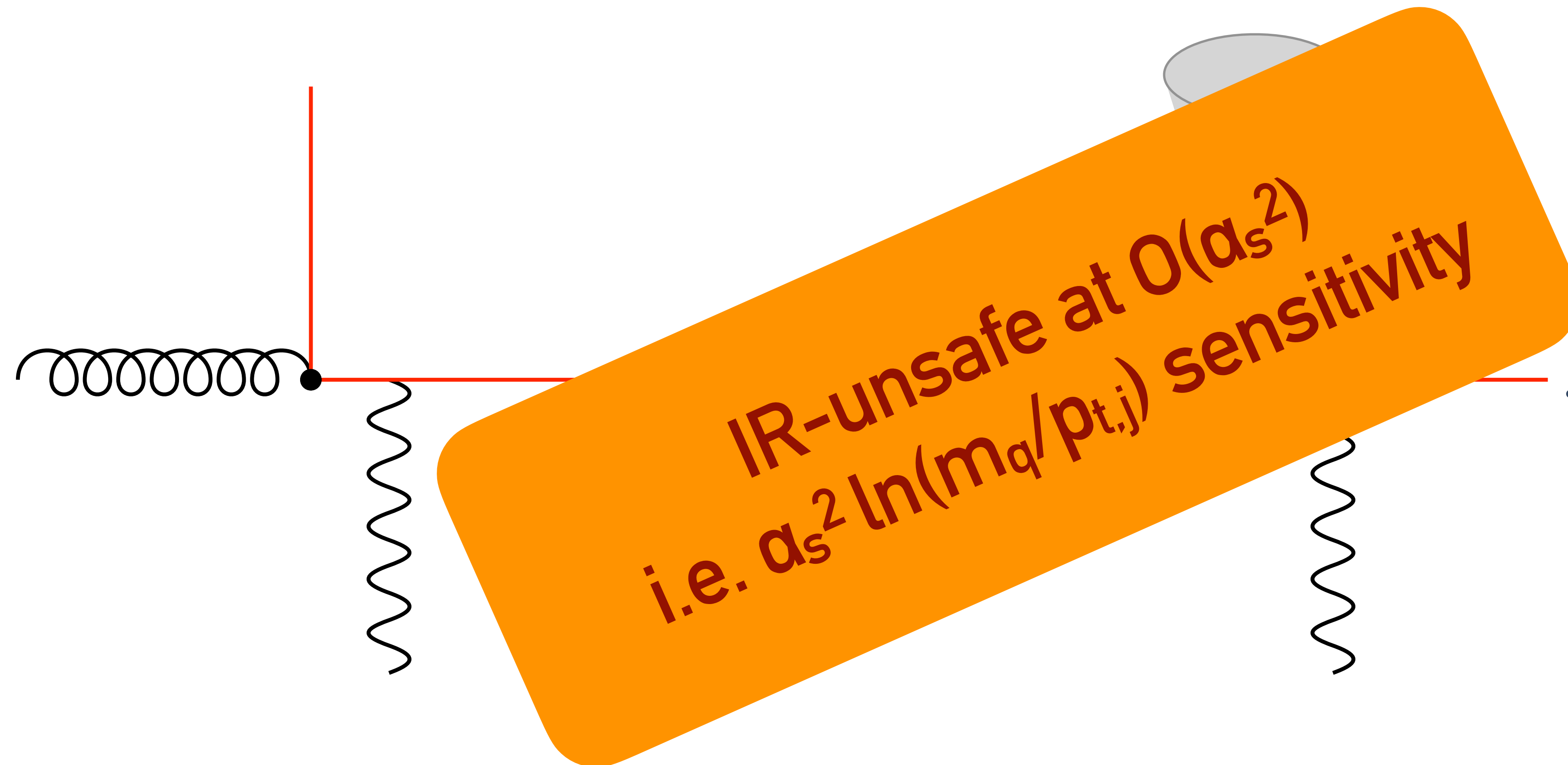
- Flavour contamination
- Log sensitivity to quark mass, $\ln(m_q/p_{t,j})$

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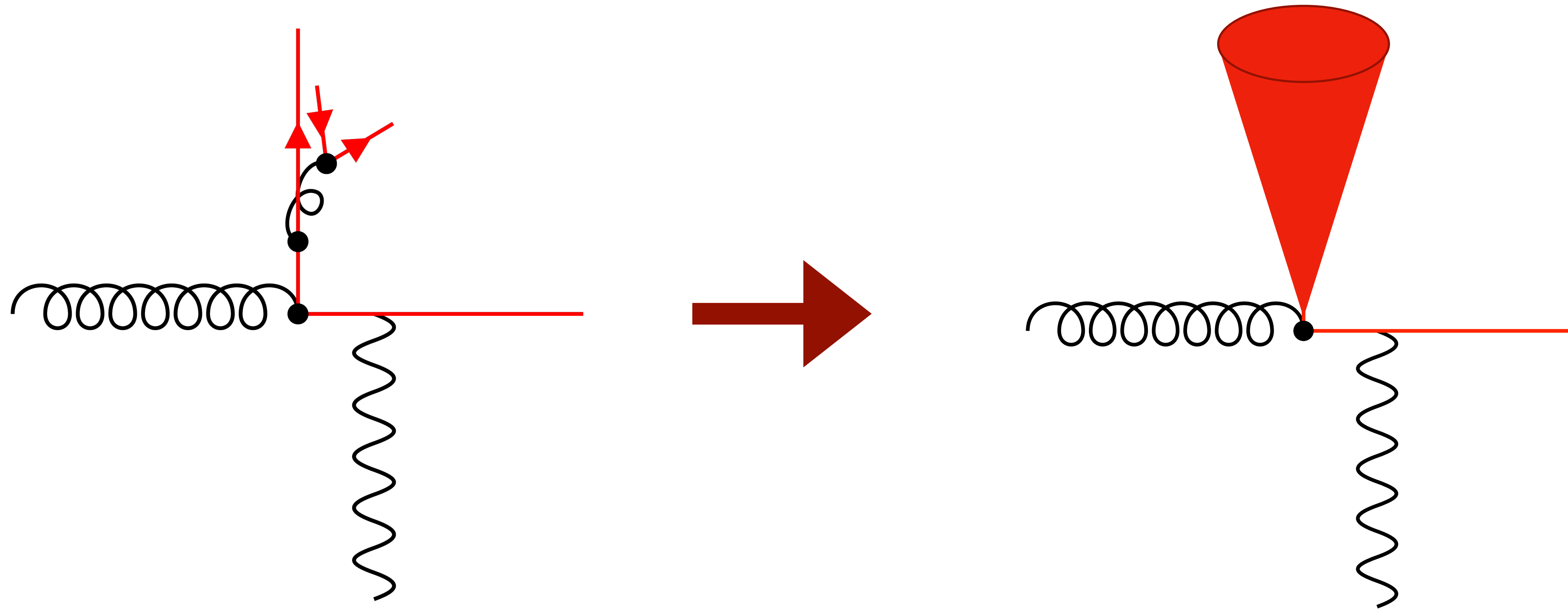


- Flavour contamination
- Log sensitivity to quark mass, $\ln(m_q/p_{t,j})$

The “old” solution: flavour- k_t

Flavour- k_t [Banfi, Salam, Zanderighi (2006)]:

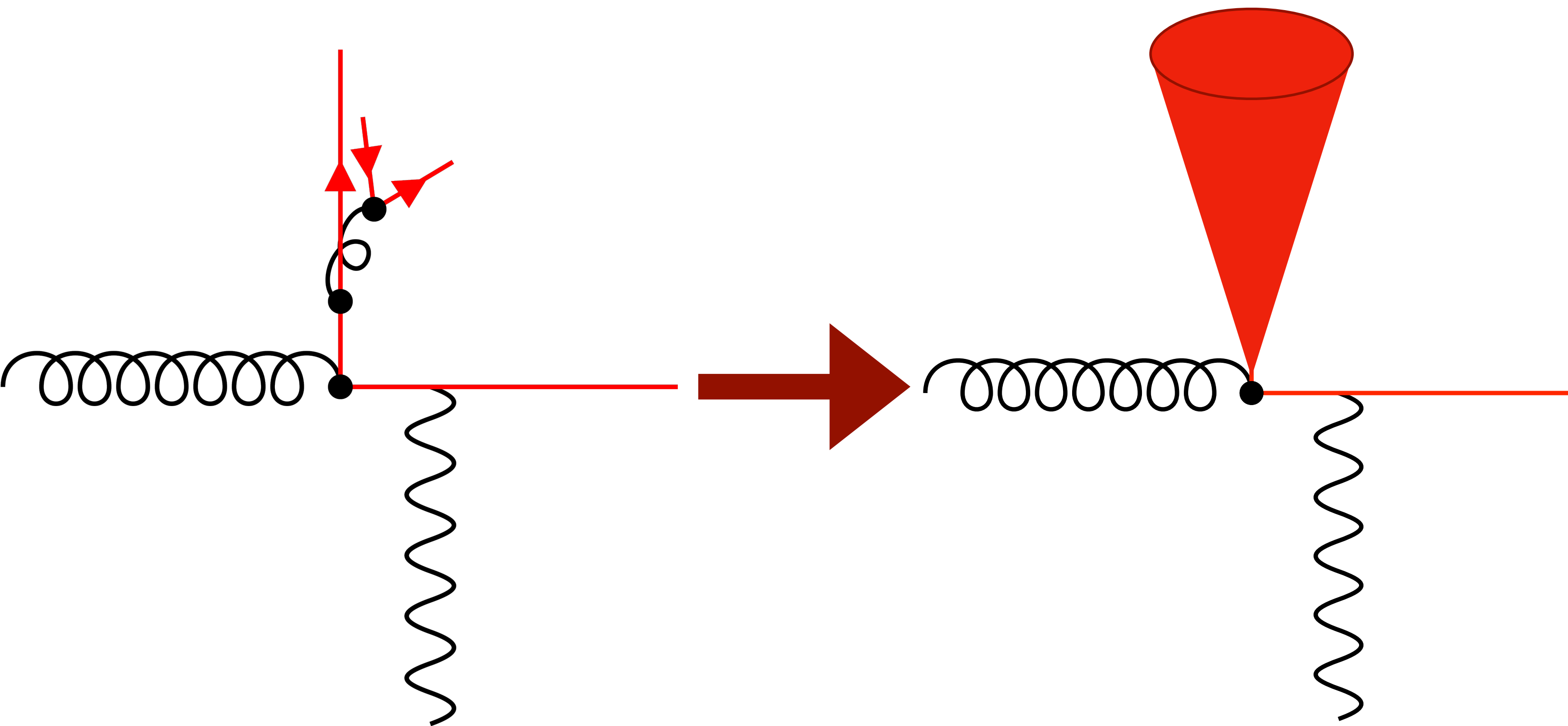
modify d_{ij} , d_{iB} to ensure that soft flavoured objects are clustered first



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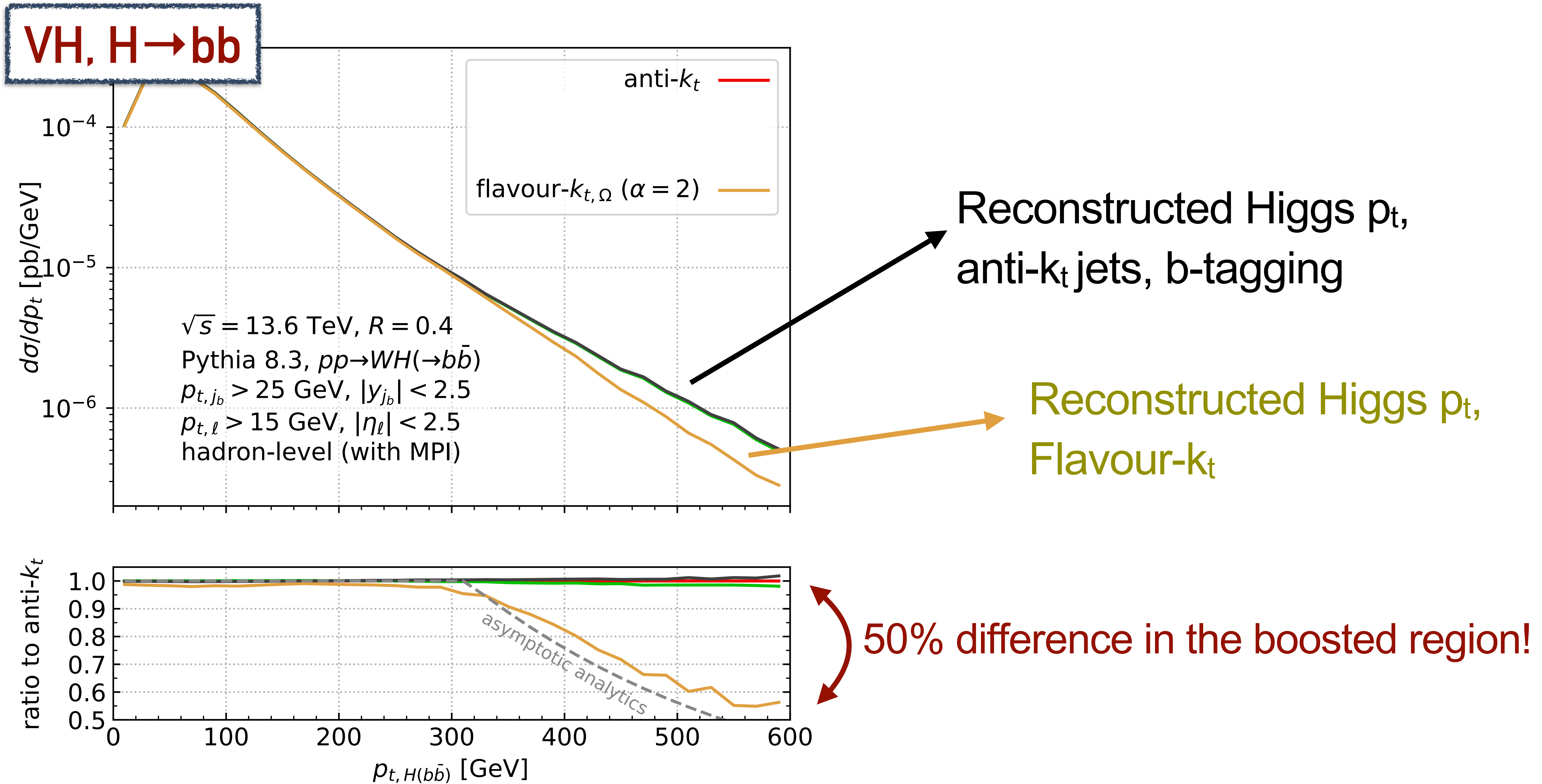
modify d_{ij} , d_{iB} to ensure that soft flavoured objects are clustered first



✓: remove the contamination

✗: different d_{ij} → different recombination → different kinematics w.r.t. anti- k_t !

The "old" solution: flavour- k_t



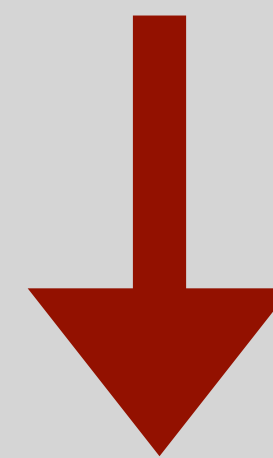
The “old” solution: flavour- k_t

VH, H→bb

$d\sigma/dp_t$ [pb/GeV]
 10^{-4}
 10^{-5}
 10^{-6}
ratio to anti- k_t
1.0
0.9
0.8
0.7
0.6
0.5
0

Flavour- $k_t \neq$ anti- k_t

For a long time: flavour- k_t only option
for higher-order (NNLO) calculations



Precise calculations, but applies to
oranges comparisons!

$p_{t, H(bb)}$ [GeV]

Recently: a flurry of activity

- Caletti, Larkoski, Marzani, Reichelt (2022): “Practical jet flavour through NNLO”

Fix the problem at NNLO, ignoring higher-order issues

- Czakon, Mitov, Poncelet [CMP] (2022): “Infrared-safe anti- k_t jets”

All-orders, modify the anti- k_t distance, but only close to “dangerous” configurations → similar kinematics to anti- k_t

- Gauld, Huss, Stagnitto [GHS] (2022): “A dress of flavour to suit any jet”

All-orders, separate kinematics and flavour recombination

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All-orders, separate kinematics and flavour recombination

What are the features of an ideal flavoured-jet algorithm?

Flavoured-jet algorithms: wish-list

A good jet flavour algorithm should:

- allow for reliable data-theory comparisons, at high precision → exact anti- k_t kinematics
 - *Flavour- k_t* : ✗
 - *CMP*: ~
 - *GHS*: ✓
- allow for reliable jet substructure studies → track the flavour along the clustering sequence, Cambridge/Aachen
- be IR-safe to all-orders

Achieving this is more difficult than it may sound

Our proposal: Interleaved Flavour Neutralisation (IFN)

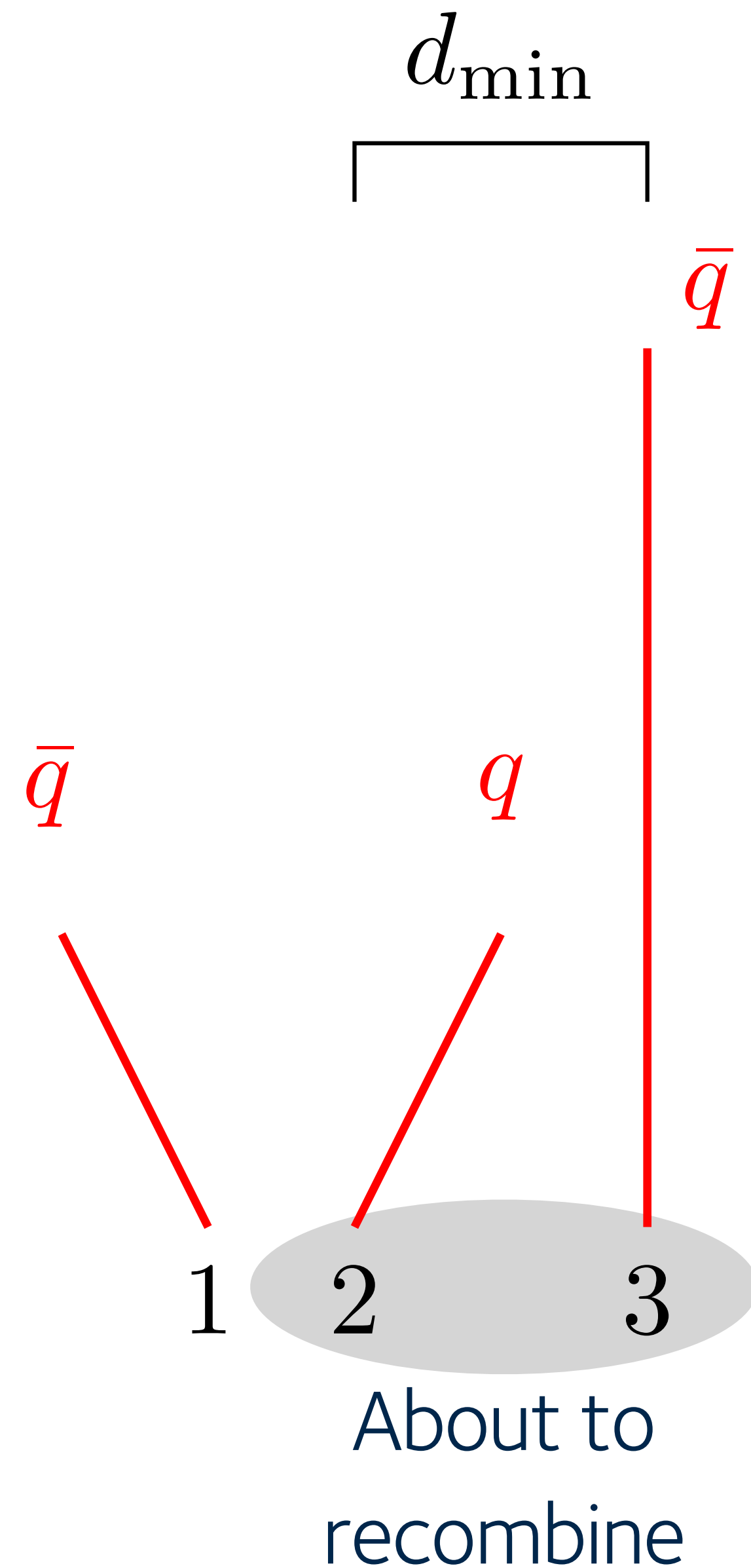
The main idea:

keep the standard clustering procedure (anti- k_t , C/A), but modify flavour-recombination at each step of the clustering sequence

By construction then:

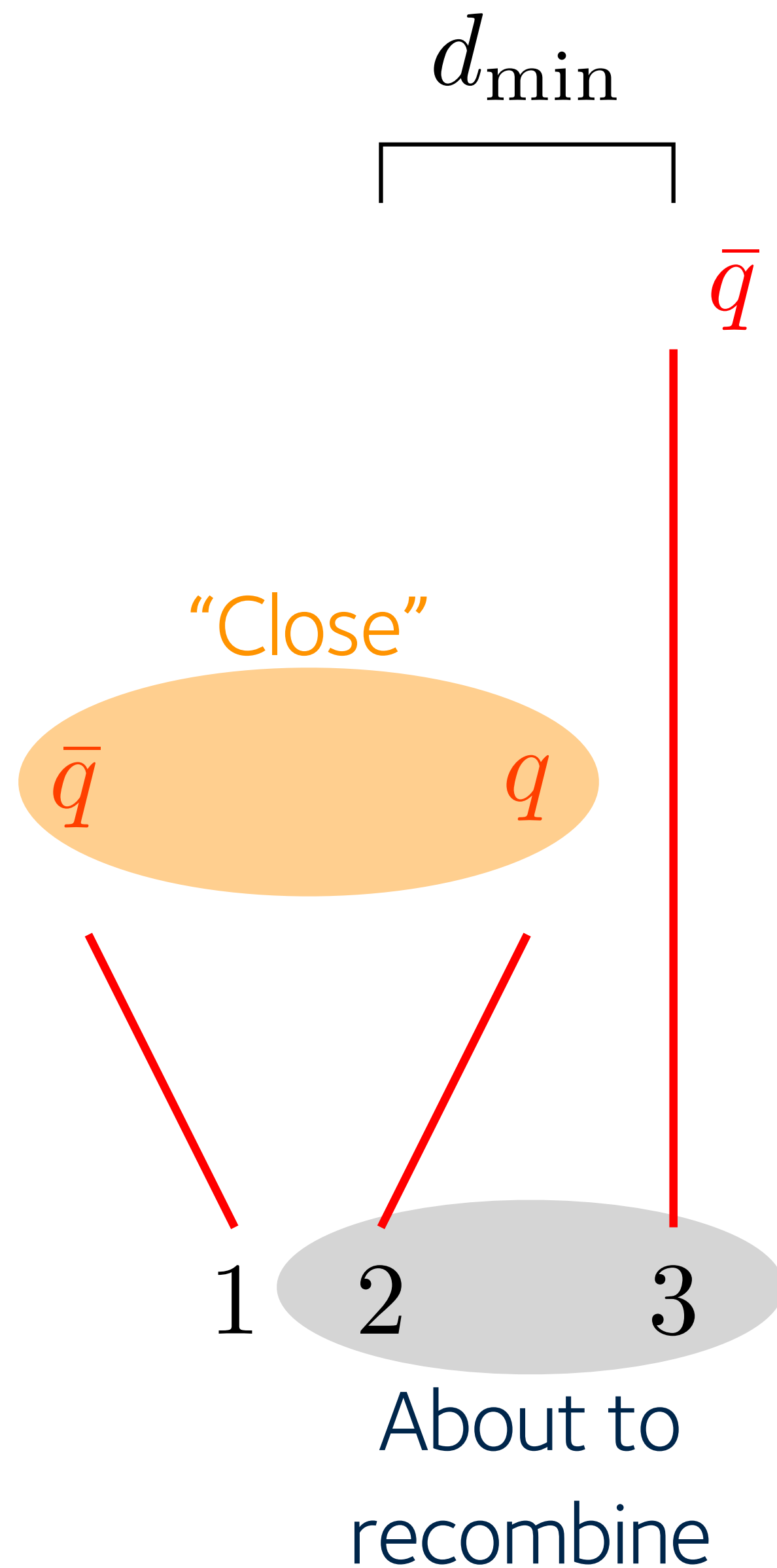
- same identical kinematics of anti- k_t , C/A
- at each stage of the recombination: IR-safe (sub)-jets \rightarrow substructure friendly

Integrated Flavour Neutralisation (IFN): a cartoon



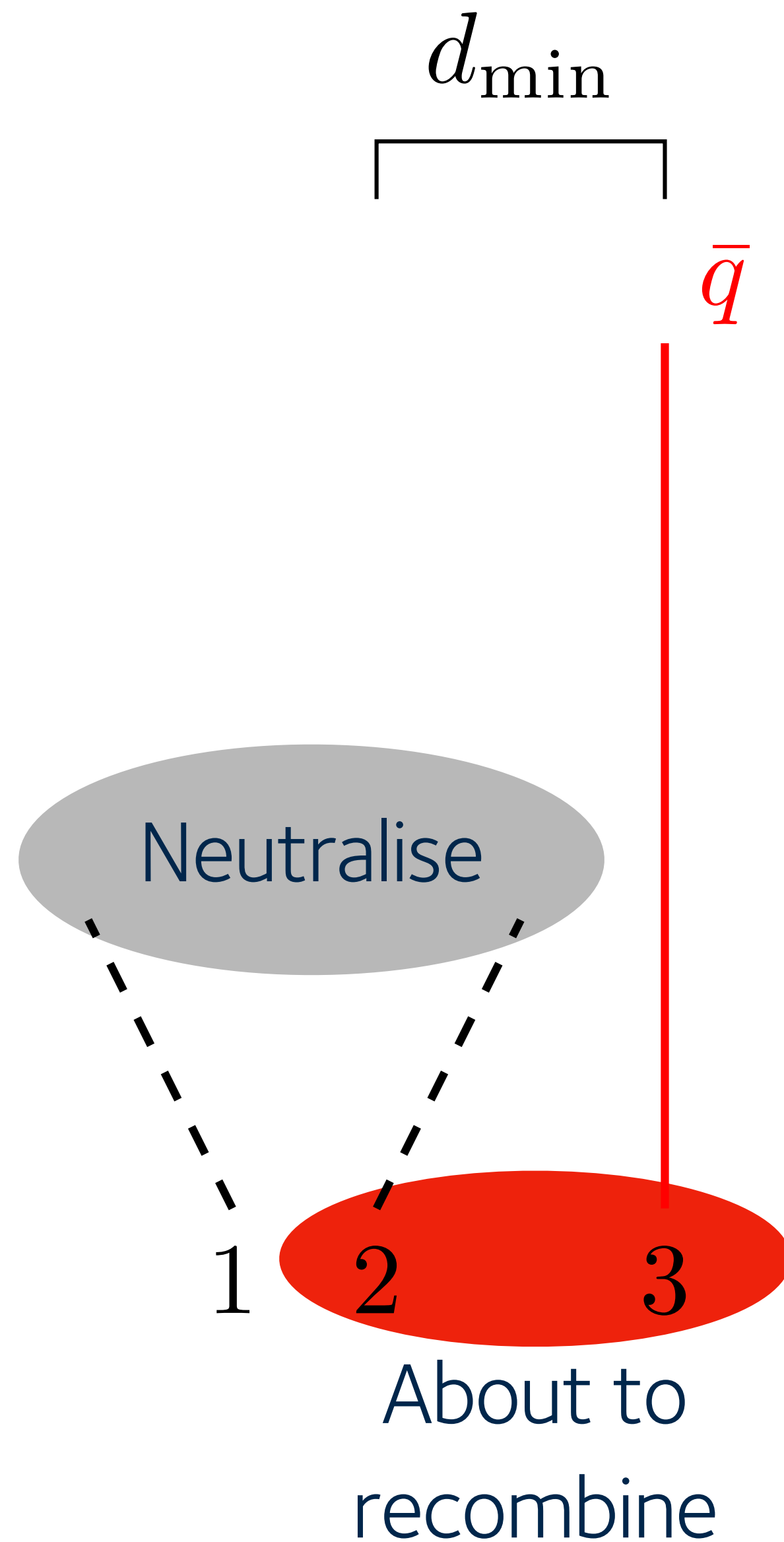
- soft flavoured object (2) about to kinematically recombined per (anti- k_t /CA...) → trigger a “flavour neutralisation” search

Integrated Flavour Neutralisation (IFN): a cartoon



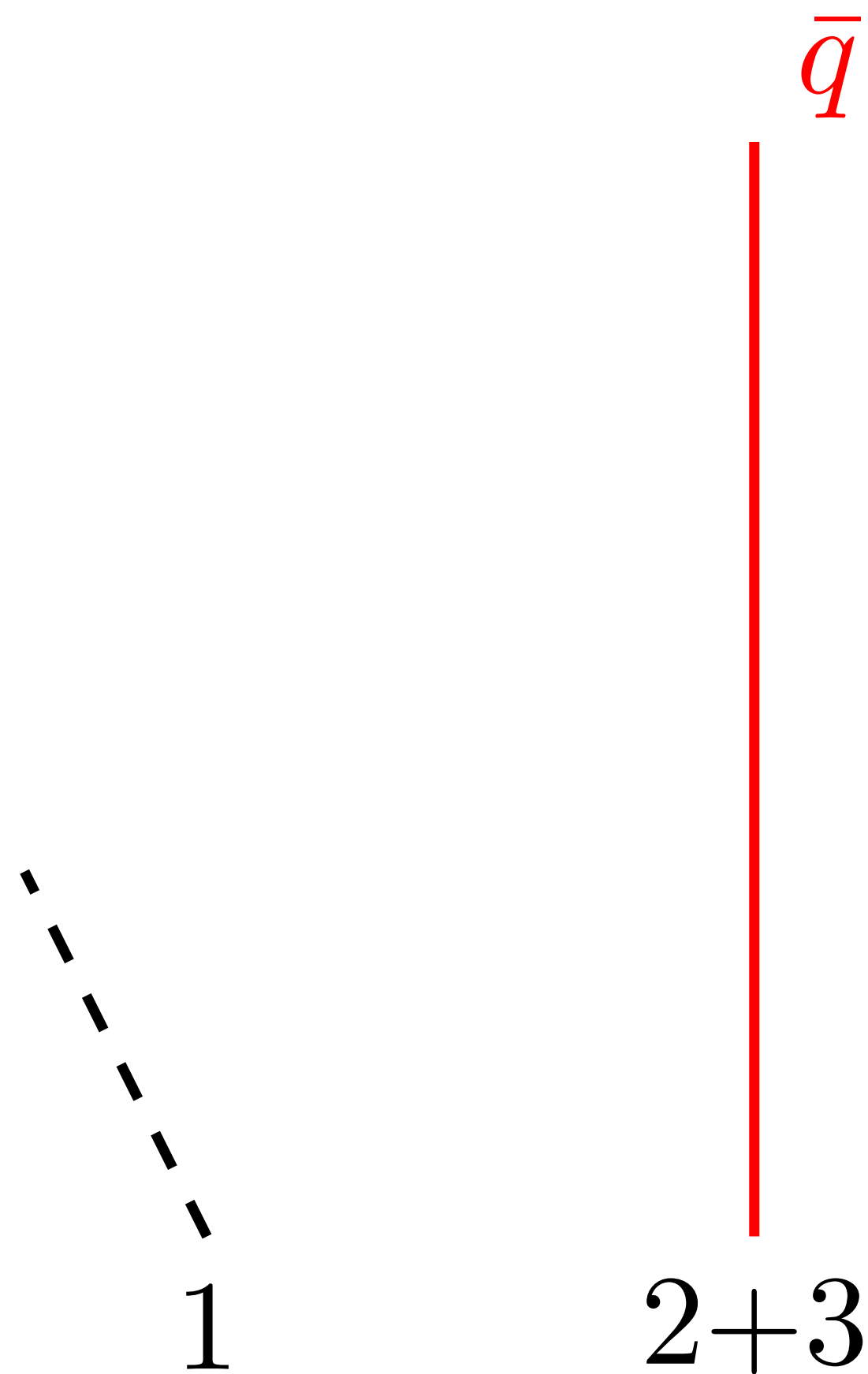
- soft flavoured object (2) about to kinematically recombined per (anti- k_t /CA...) \rightarrow trigger a "flavour neutralisation" search
- look globally in the event for objects that should neutralise \rightarrow identify 1

Integrated Flavour Neutralisation (IFN): a cartoon



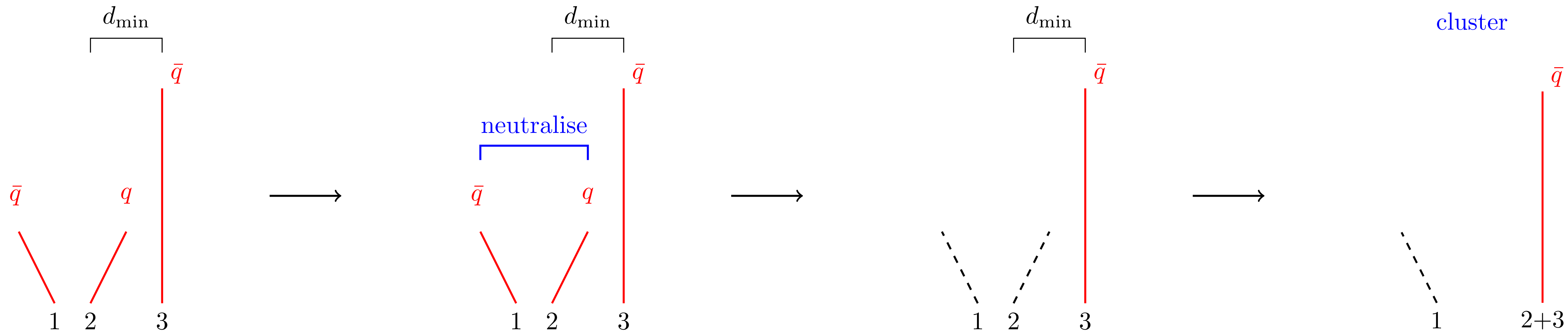
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Integrated Flavour Neutralisation (IFN): a cartoon



- soft flavoured object (2) about to kinematically recombined per (anti- k_t /CA...) \rightarrow trigger a “flavour neutralisation” search
- look globally in the event for objects that should neutralise \rightarrow identify 1
- neutralise 1 and 2, then recombine
- Flavoured jets with anti- k_t /CA kinematics

Integrated Flavour Neutralisation (IFN): a cartoon



Flavoured jets with anti- k_t /CA kinematics

Crucial for IR-safety + good behaviour

- proper choice of a “flavour distance”
- making sure neutralising partner is not “stolen” from more suitable candidate (\rightarrow recursion)

Integrated Flavour Neutralisation (IFN): a cartoon

The neutralisation distance

$$u_{ik} \equiv [\max(p_{ti}, p_{tk})]^\alpha [\min(p_{ti}, p_{tk})]^{2-\alpha} \times \Omega_{ik}^2,$$

~ flavour- k_t , soft objects are close

Angular distance.

Critical: able to compare objects
event-wide → far apart

Integrated Flavour Neutralisation (IFN): a cartoon

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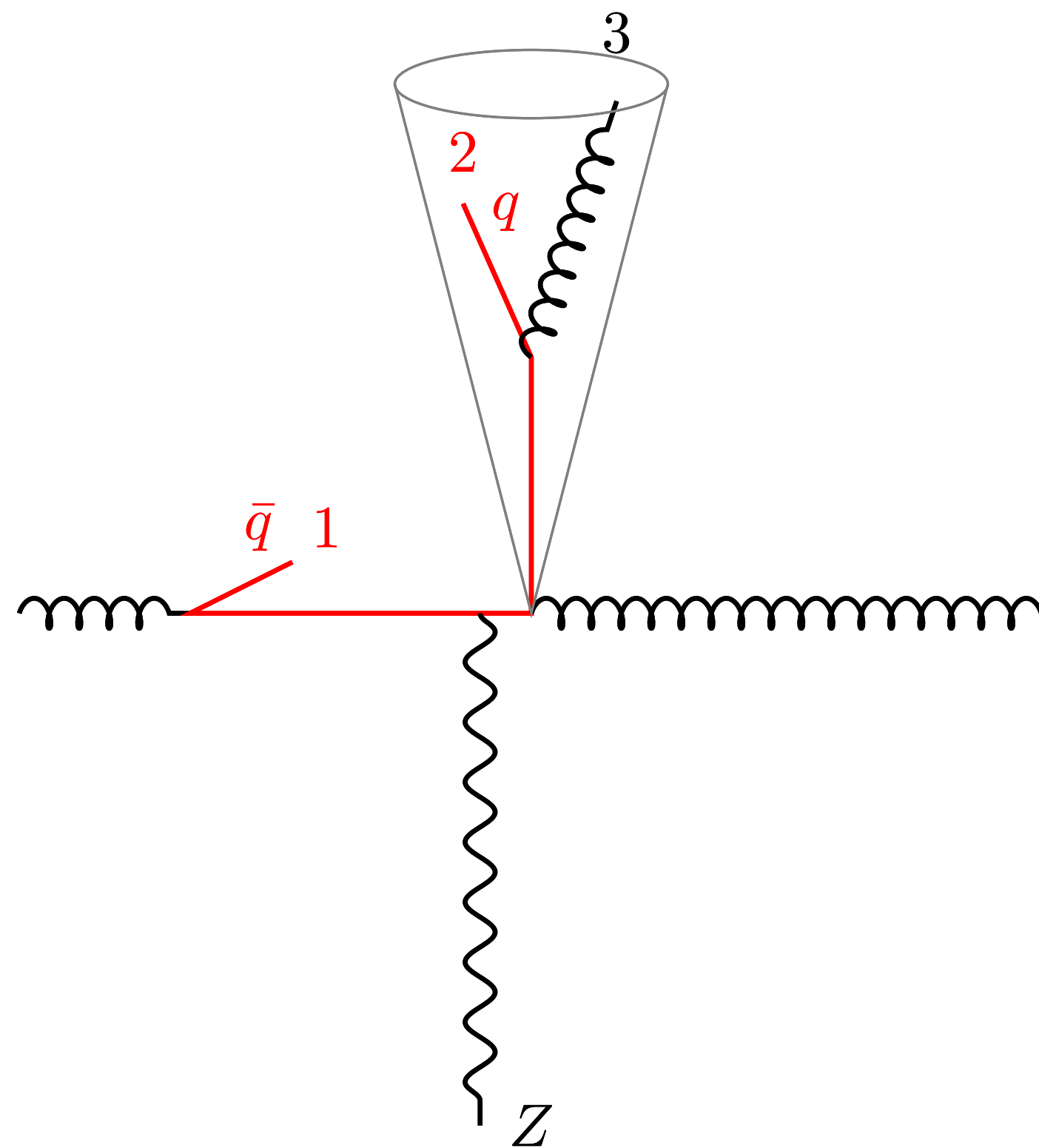
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Integrated Flavour Neutralisation (IFN): a cartoon

The neutralisation distance

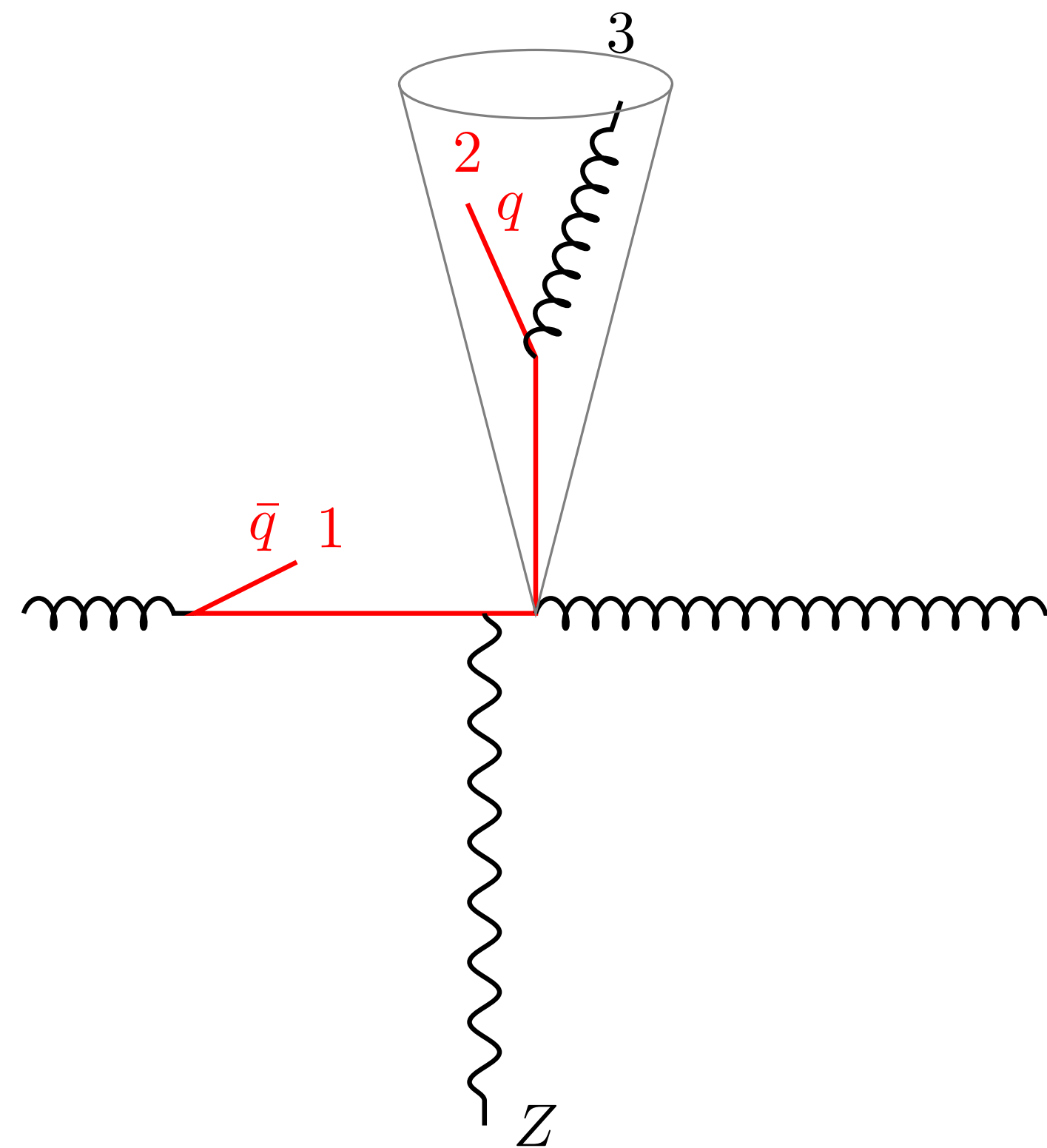
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~ flavour- k_t , soft objects are close

Angular distance.

1 and 2 must be "far" \longrightarrow

Critical: able to compare objects event-wide \rightarrow far apart



$$\Omega_{ik}^2 \equiv 2 \left[\frac{1}{\omega^2} (\cosh(\omega \Delta y_{ik}) - 1) - (\cos \Delta \phi_{ik} - 1) \right]$$

- $\Omega \sim \Delta R$ for small distances
- Exponentially large distance if far apart in rapidity

IR-safety: $0 < \alpha \leq 2$; $\alpha + \omega > 2$

Testing IR safety

9 Fri 4 Dec

You are viewing Gavin Salam's screen

$Z^2 + (1-Z)^2$
 $\int dl_1 e^{-l_1} dl_2 e^{-l_2}$
 $S(l-l_1-l_2)$
 $\hookrightarrow L$

$dz \times \omega t \Rightarrow \frac{dz}{z} \times z$

Participants Chat Share Screen Record Reactions

Fabrizio Caola

jet flavour discussions - Google x

https://jamboard.google.com/d/1rVvtF07iWD7iCUG14wNvUm2k35spWEZJ.../edit

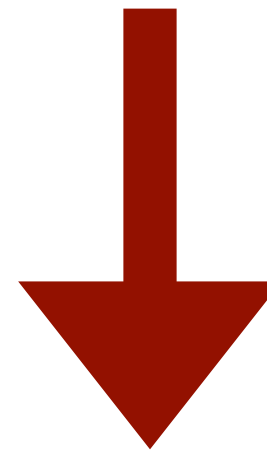
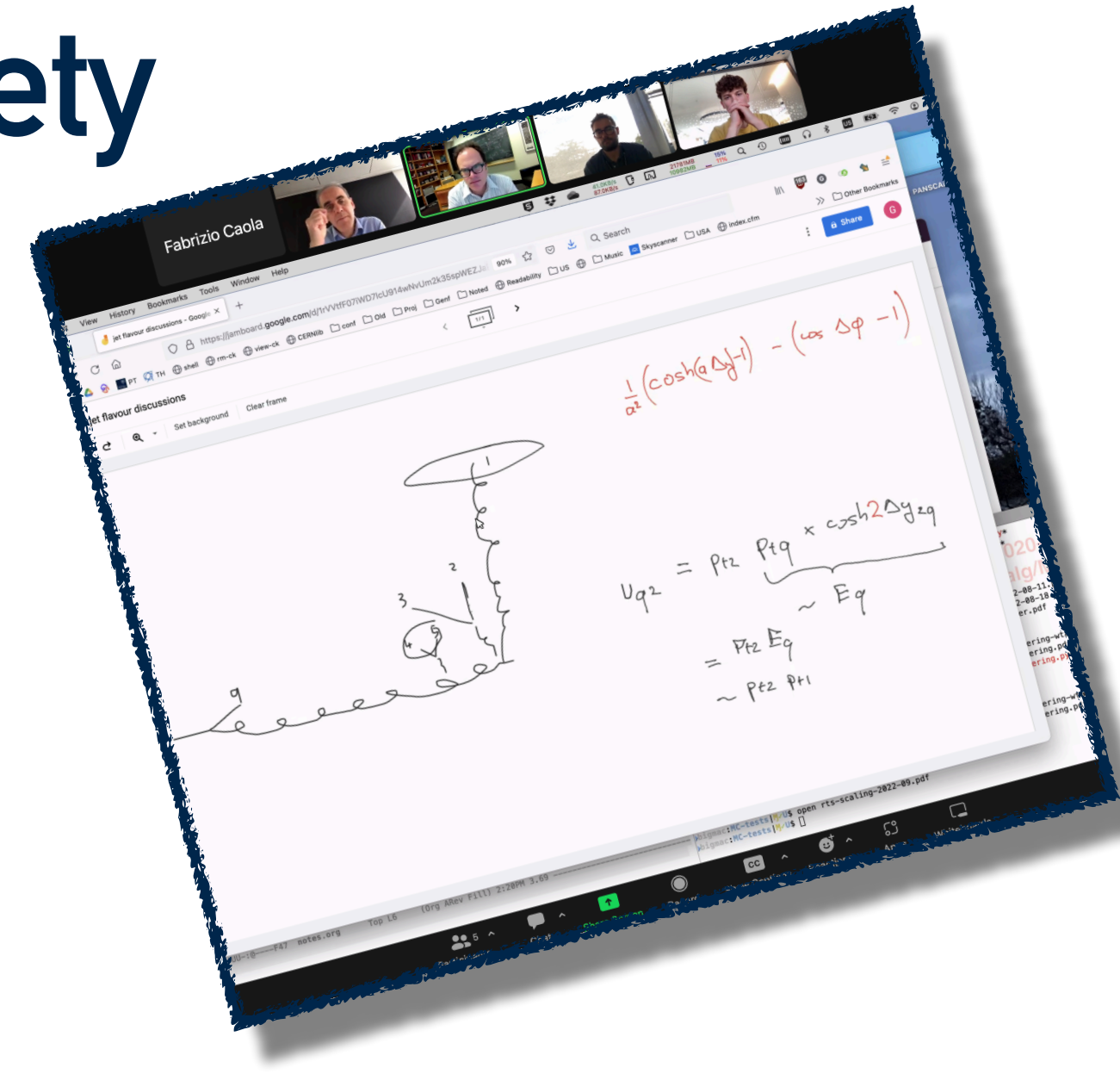
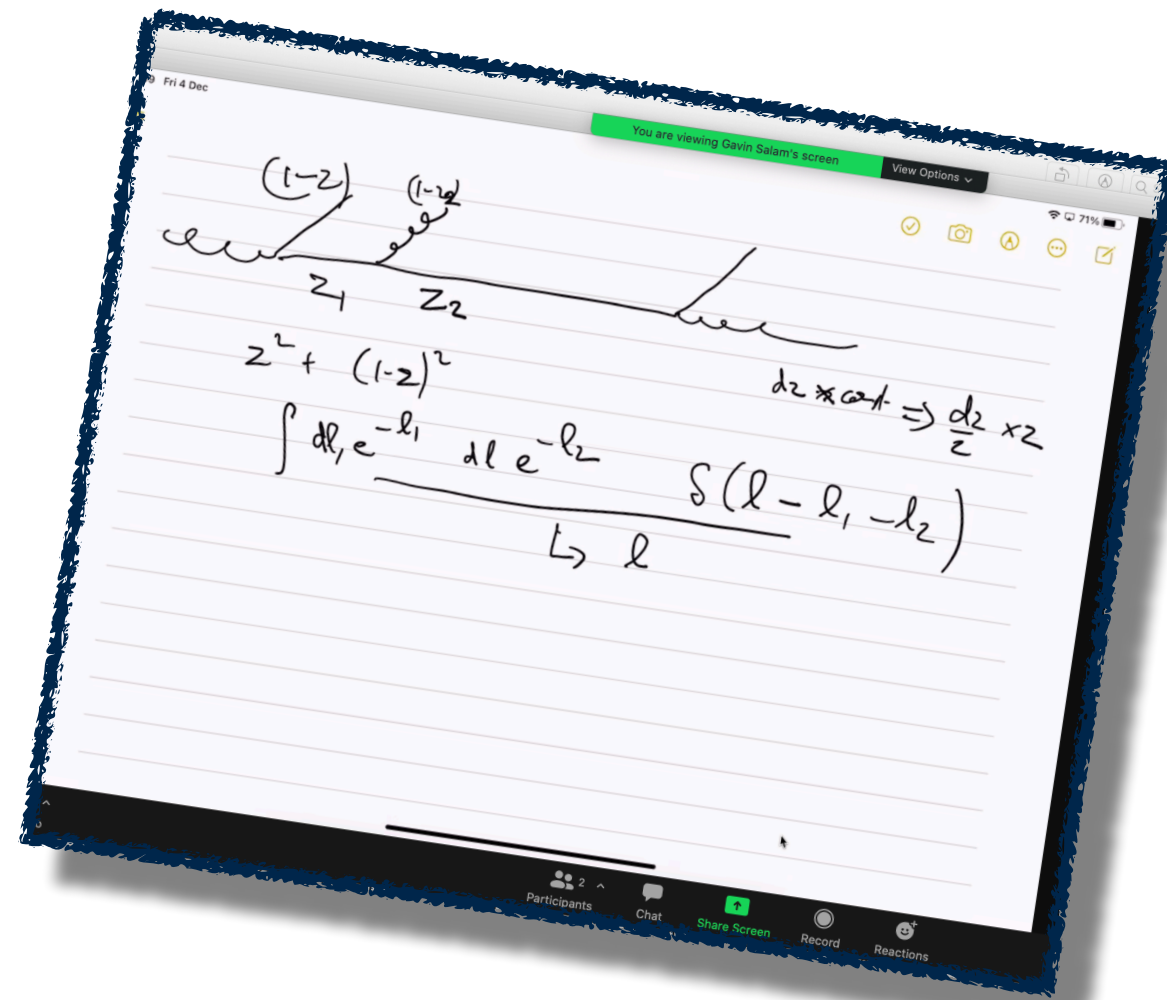
jet flavour discussions

$\frac{1}{a^2} (\cosh(a\Delta y) - (\cos \Delta \varphi - 1))$

$U_{q2} = P_{t2} P_{tq} \times \cosh 2\Delta y z_9$
 $\sim E_q$
 $= P_{t2} E_q$
 $\sim P_{t2} P_{t1}$

Participants Chat Share Screen Record Show Captions Reactions

Testing IR safety



Configuration:

where $b,1$ is an input-state splitting, $2,3$ is a wide-angle SFT pair and 4 is a hard gluon at wide scattering angle to the beam.

Approximations:

- 4 is emitted at large angle so $y_4 \approx 0$
- 1 is an input-state collinear splitting so $E_1 \sim E_q$ but since $y_4 \approx 0$ then must have $p_{t1} \sim p_{t4} e^{-|y_1|} \Rightarrow y_1 \sim \ln(\frac{p_{t4}}{p_{t1}})$
- $\Delta\phi \leq \pi$ in all cases so normally $\Delta y \gg \Delta\phi$ so approx. $\Delta R_{ij} \sim \Delta y_{ij}$.
- $y_2 \sim y_3$ as from same SFT pair: $\Delta R_{12} \sim \Delta R_{14} \sim \Delta R_{23} \sim \Delta R_{24} \sim \Delta R_{34} \sim \Delta R_{13}$ if pore contamination risk to 4 .

$\Rightarrow \Delta R_{12} = \Delta R_{14} = \Delta R_{24} \sim y_1$

Let $z = p_{t2}/p_{t23}$ and $z < 1/2$ st. 2 is softer than 3 .

Criterion for automation: Take measure $U_{ij} = \left(\frac{\max(p_{ti}, p_{tj})}{\min(p_{ti}, p_{tj})} \Delta R_{ij} \right)^2$

$U_{12} = \frac{\max(p_{t1}, p_{t23})}{\min(p_{t1}, p_{t23})} \Delta R_{12}$ $\sim p_{t1}$ is exponentially small.

$= \frac{z p_{t23}}{p_{t1}} \Delta R_{12} \sim \frac{z p_{t23}}{p_{t1}} \ln(\frac{p_{t4}}{p_{t1}})$

$U_{23}^{1/2} = \frac{\max(z p_{t23}, (1-z) p_{t23})}{\min(z p_{t23}, (1-z) p_{t23})} \Delta R_{23} \sim z < 1/2$

$= \frac{1-z}{z} \Delta R_{23} \sim \left(\frac{1}{z} - 1\right) O(R) \sim \frac{1}{z}$

$\Rightarrow U_{12} < U_{23} \Rightarrow \frac{1}{z} > \frac{z p_{t23}}{p_{t1}} \ln(\frac{p_{t4}}{p_{t1}})$

Cross-sec. for automation:

$\sigma \sim \int \frac{dp_{t1} dp_{t23} dz}{p_{t1} p_{t23}} \theta\left(\frac{1}{z} > \frac{z p_{t23}}{p_{t1}} \ln\left(\frac{p_{t4}}{p_{t1}}\right)\right)$

where $z = p_{t2}/p_{t23}$. Define $L = \ln(p_{t4}/p_{t1})$, $l = \ln(p_{t23}/p_{t1})$ Change variables

$J = \frac{\partial(L, l)}{\partial(p_{t1}, p_{t23})} = \begin{vmatrix} \partial L / \partial p_1 & \partial L / \partial p_2 \\ \partial l / \partial p_1 & \partial l / \partial p_2 \end{vmatrix}$

$= \begin{vmatrix} -1/p_1 & 0 \\ -1/p_1 & 1/p_{23} \end{vmatrix}$

$\Rightarrow |J| = \frac{1}{p_1 p_{23}}$

$\Rightarrow \sigma \sim \int dl \int dz \int dt \theta\left(\frac{1}{z} > z e^{L-l}\right)$

$\sim \int_0^L dl \int_0^L dz \int_0^L dt \theta\left(\frac{1}{z} > e^{L-l}\right)$ \leftarrow has $z \in [0,1]$ in slide?

$\Rightarrow z < e^{-l/2}$ (for +ve solns at $z > 0$).

$\int_0^L dz \theta(z < z_0) = \begin{cases} L-a, & x_0 > b \\ x_0-a, & x_0 \in [a,b] \\ 0, & x_0 < a \end{cases}$

$\frac{e^{-l/2}}{\sqrt{l}} = \left[\frac{p_{t23}}{p_{t1}} \ln\left(\frac{p_{t4}}{p_{t1}}\right) \right]^{-1/2}$

Note: p_{t1} is exponentially small: $p_{t23}/p_{t1} \gg 1, p_{t4}/p_{t1} \gg 1$

$\Rightarrow \frac{e^{-l/2}}{\sqrt{l}} \ll 1 \Rightarrow \frac{e^{-l/2}}{\sqrt{l}} \in [0, 1/2]$ \downarrow $l, L \gg 1$

$\Rightarrow \sigma \sim \int_0^L dl \int_0^L dz \frac{e^{-l/2}}{\sqrt{l}} = \int_0^L \frac{dl}{\sqrt{l}} (-2) [e^{-l/2} - 1]$

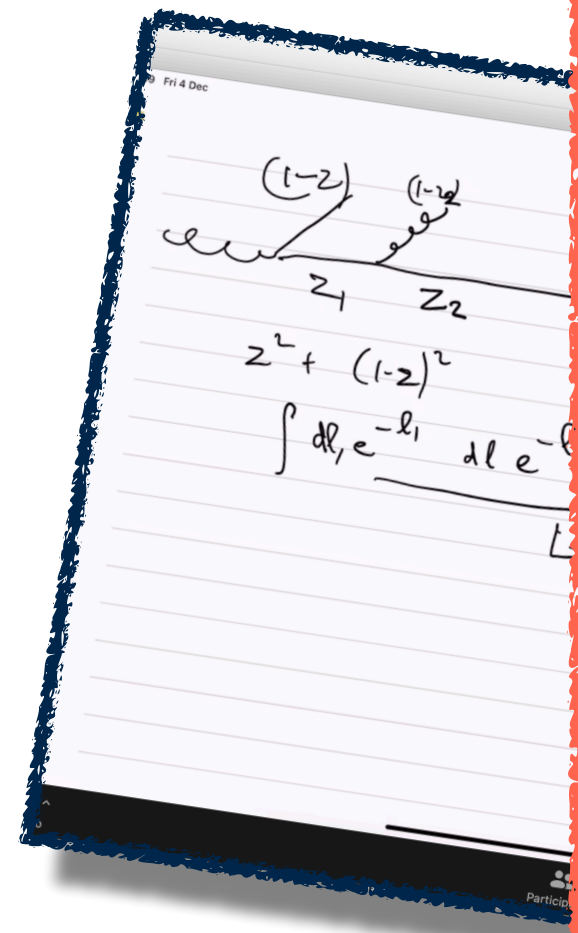
$= \int_0^L dl \frac{1}{\sqrt{l}} (1 - e^{-l/2})$

$\rightarrow \infty$

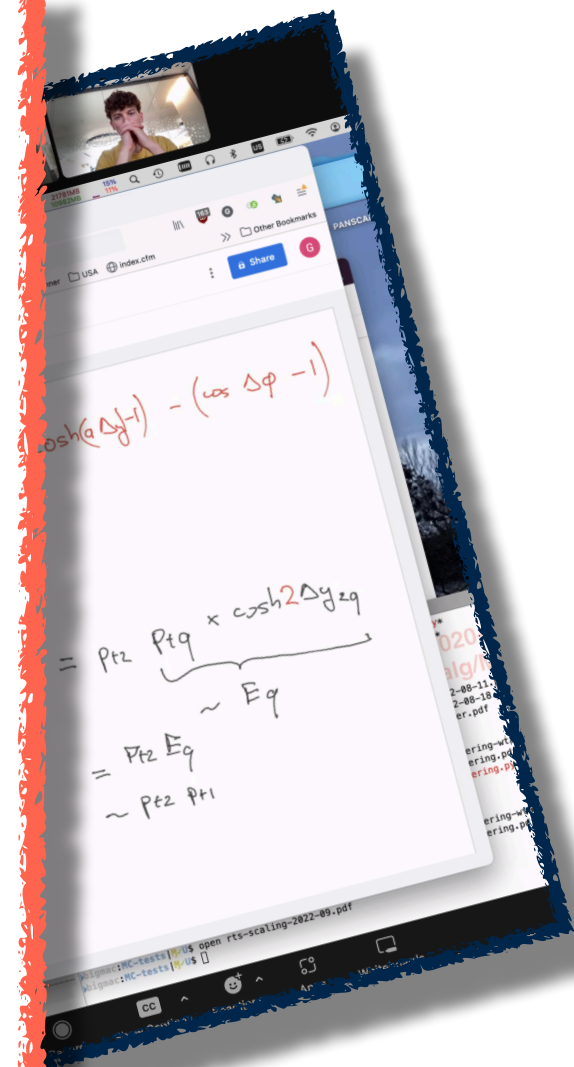
...

$$= 2\sqrt{2} \cdot \left(\frac{2\sqrt{\pi} \Gamma(5/4)}{\Gamma(3/4)} - \frac{\pi}{2} \right)$$

\rightarrow finite!



GIVEN THE PACE OF TECHNOLOGY, I PROPOSE WE LEAVE MATH TO THE MACHINES AND GO PLAY OUTSIDE.



...

$$\left(\frac{2\sqrt{\pi} \Gamma(5/4)}{\Gamma(3/4)} - \frac{\pi}{2} \right)$$

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Criterion for automation: Take

$$U_{12}^{1/2} = \frac{\max(p_{t1}, p_{t23})}{\min(p_{t1}, p_{t23})} \Delta R_{12} \sim \frac{z p_{t23}}{p_{t1}} \Delta R_{12} \sim \frac{z p_{t23}}{p_{t1}} \Delta R_{12}$$

$$U_{23}^{1/2} = \frac{\max(z p_{t23}, (1-z) p_{t23})}{\min(z p_{t23}, (1-z) p_{t23})} \Delta R_{23} \sim \frac{1-z}{z} \Delta R_{23} \sim \left(\frac{1}{z}\right) \Delta R_{23}$$

$\Rightarrow U_{12} < U_{23} \Rightarrow \frac{1}{z} > \frac{z p_{t23}}{p_{t1}}$

Cross-sec. for automation:

$$\sigma \sim \int \frac{dp_{t1} dp_{t23} dz}{p_{t1} p_{t23}} \theta(\dots)$$

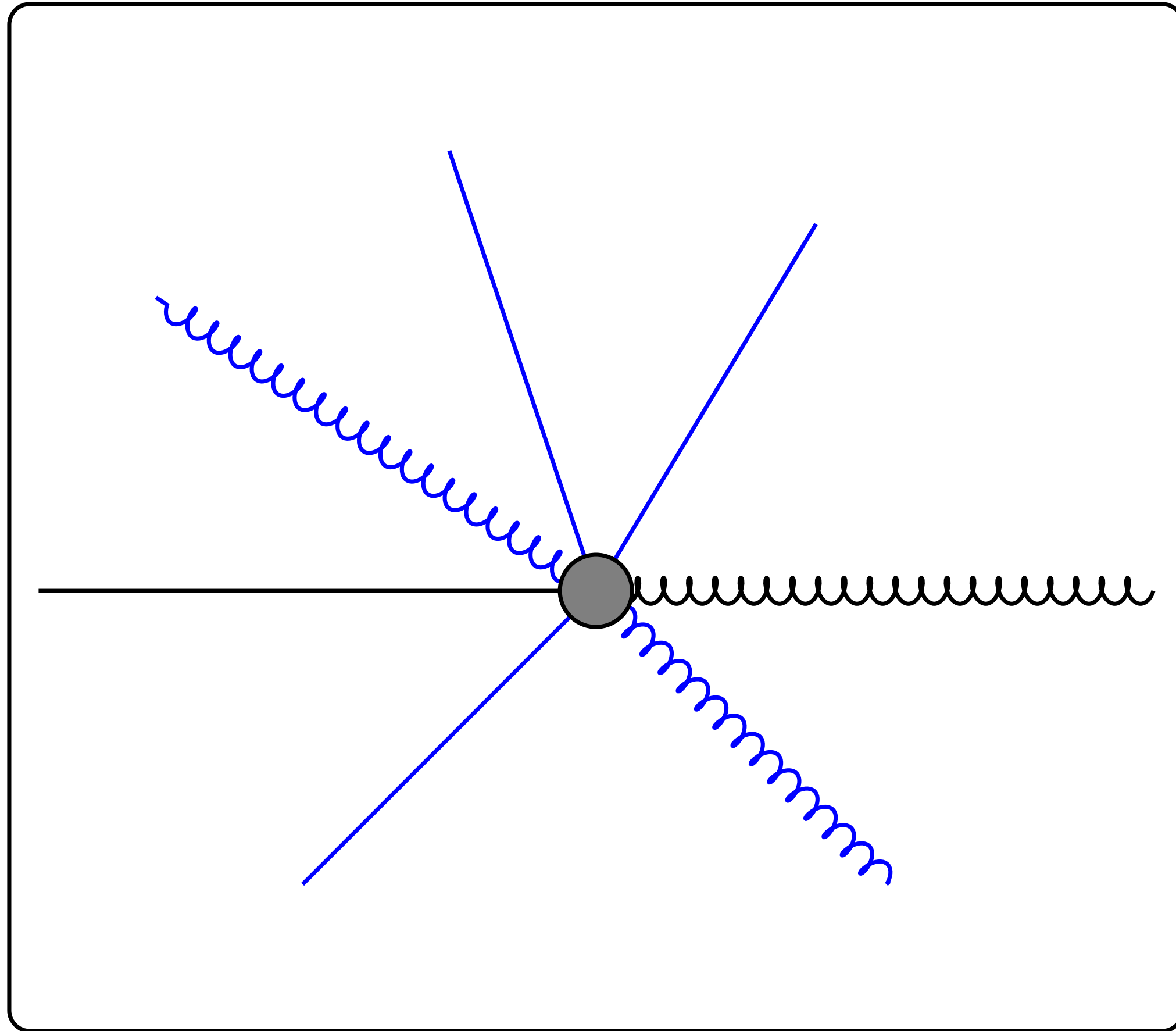
where $z = p_{t2}/p_{t23}$. Define $l = \ln(p_{t23}/p_{t1})$. Change v

$$J = \frac{\partial(L, l)}{\partial(p_{t1}, p_{t23})} = \begin{vmatrix} \partial L / \partial p_{t1} & \partial L / \partial p_{t23} \\ \partial l / \partial p_{t1} & \partial l / \partial p_{t23} \end{vmatrix} = \begin{vmatrix} -1/p_{t1} & 1/p_{t23} \\ -1/p_{t1} & 0 \end{vmatrix} = \frac{1}{p_{t1} p_{t23}}$$

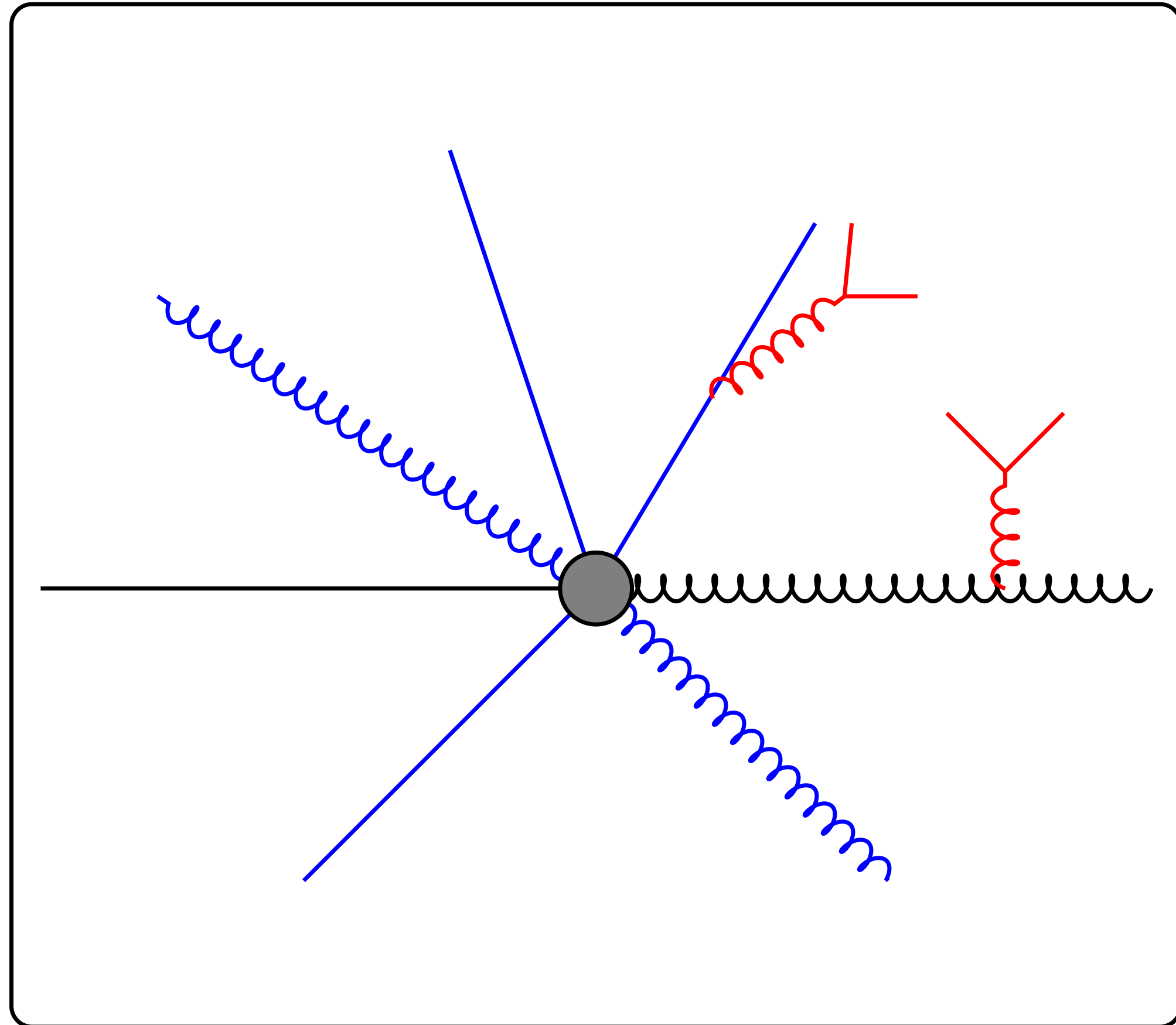
$\Rightarrow |J| = \frac{1}{p_{t1} p_{t23}}$

A framework for IR-safety tests

- Consider a hard underlying event

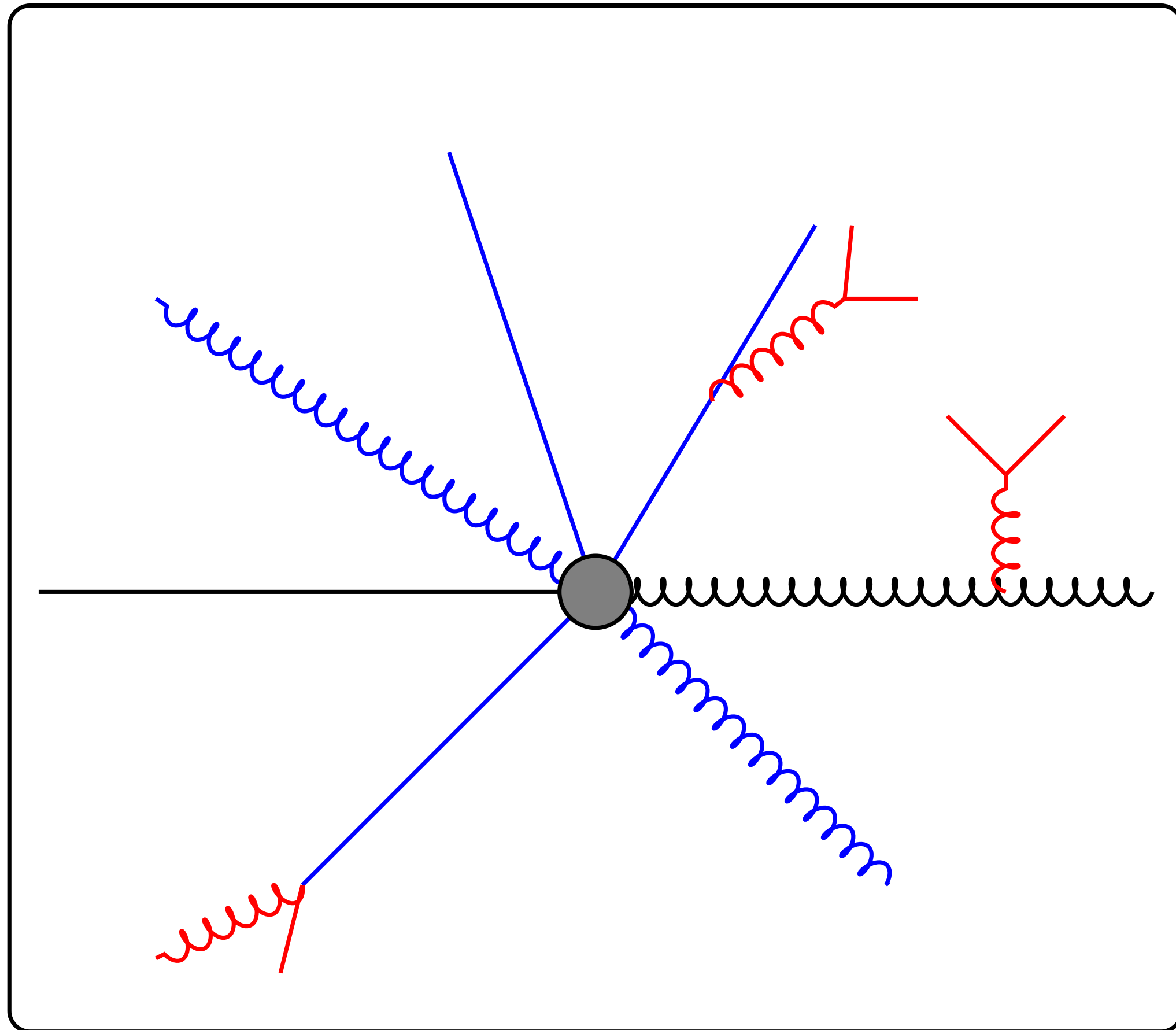


A framework for IR-safety tests



- Consider a hard underlying event
- Dress with
 - (IR/FS) DS

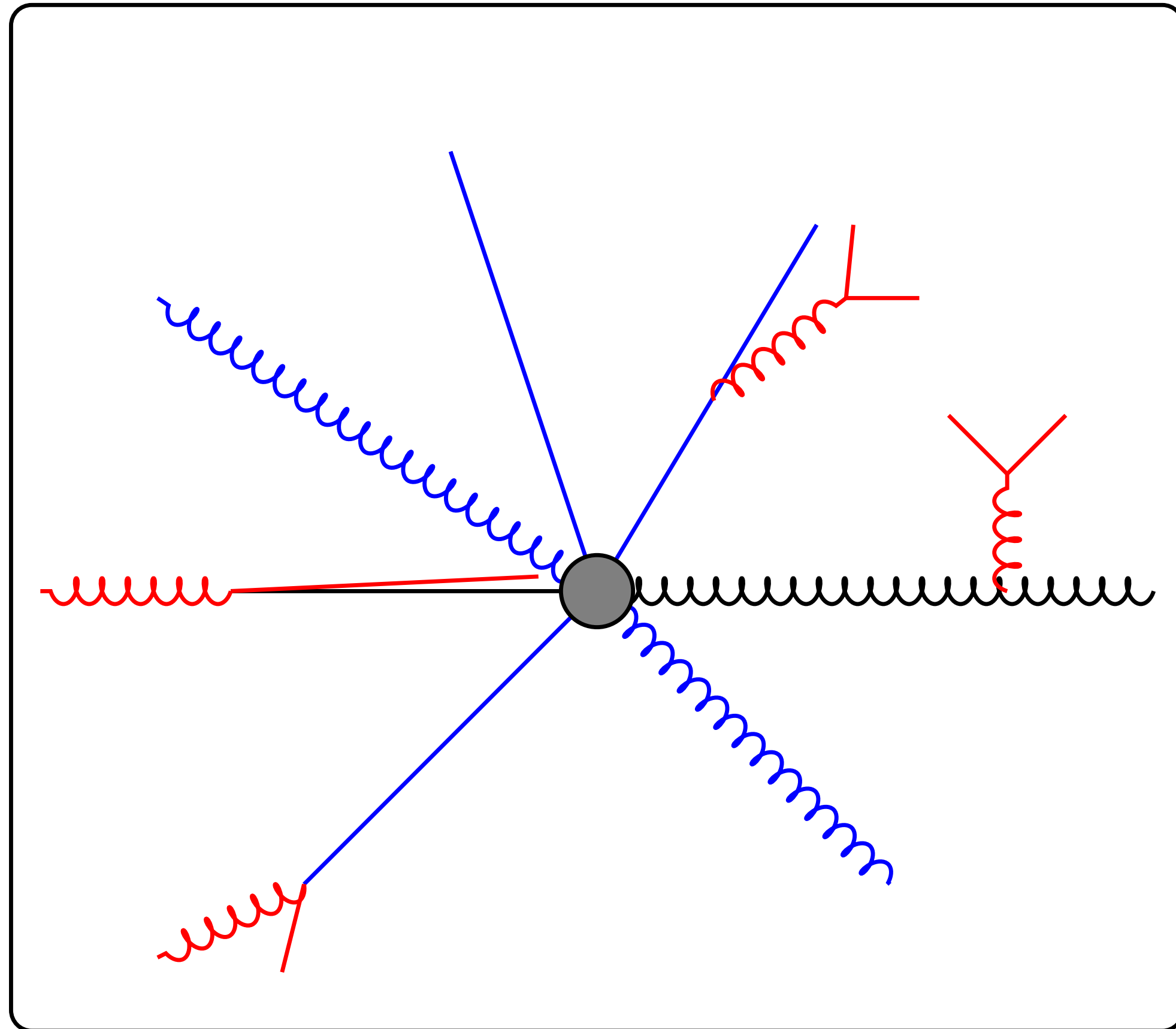
A framework for IR-safety tests



- Consider a hard underlying event
- Dress with
 - (IR/FS) DS
 - FS hard collinear (FHC)

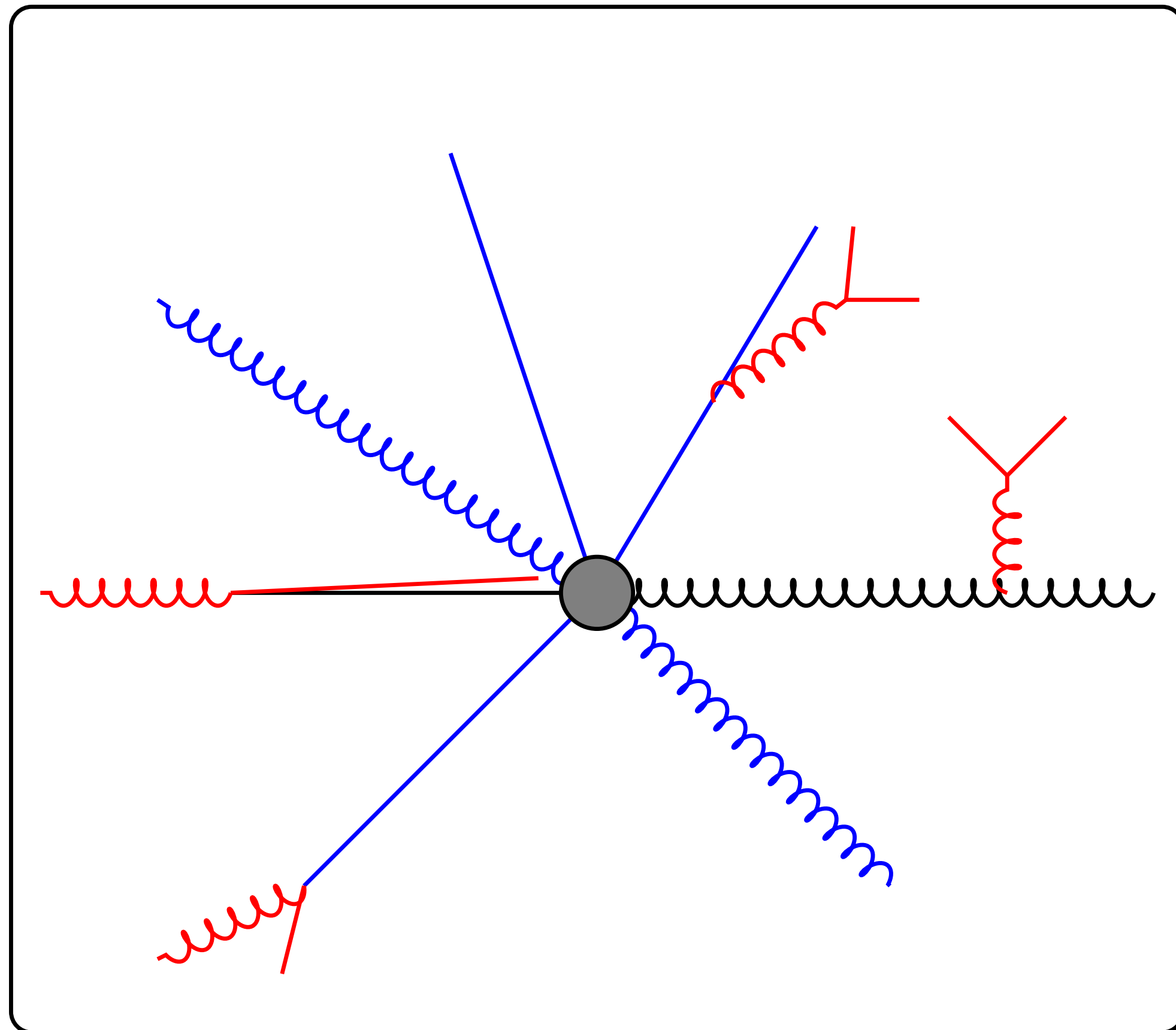
A framework for IR-safety tests

- Consider a hard underlying event
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 - FS hard collinear (FHC)
 - IS hard collinear (IHC)

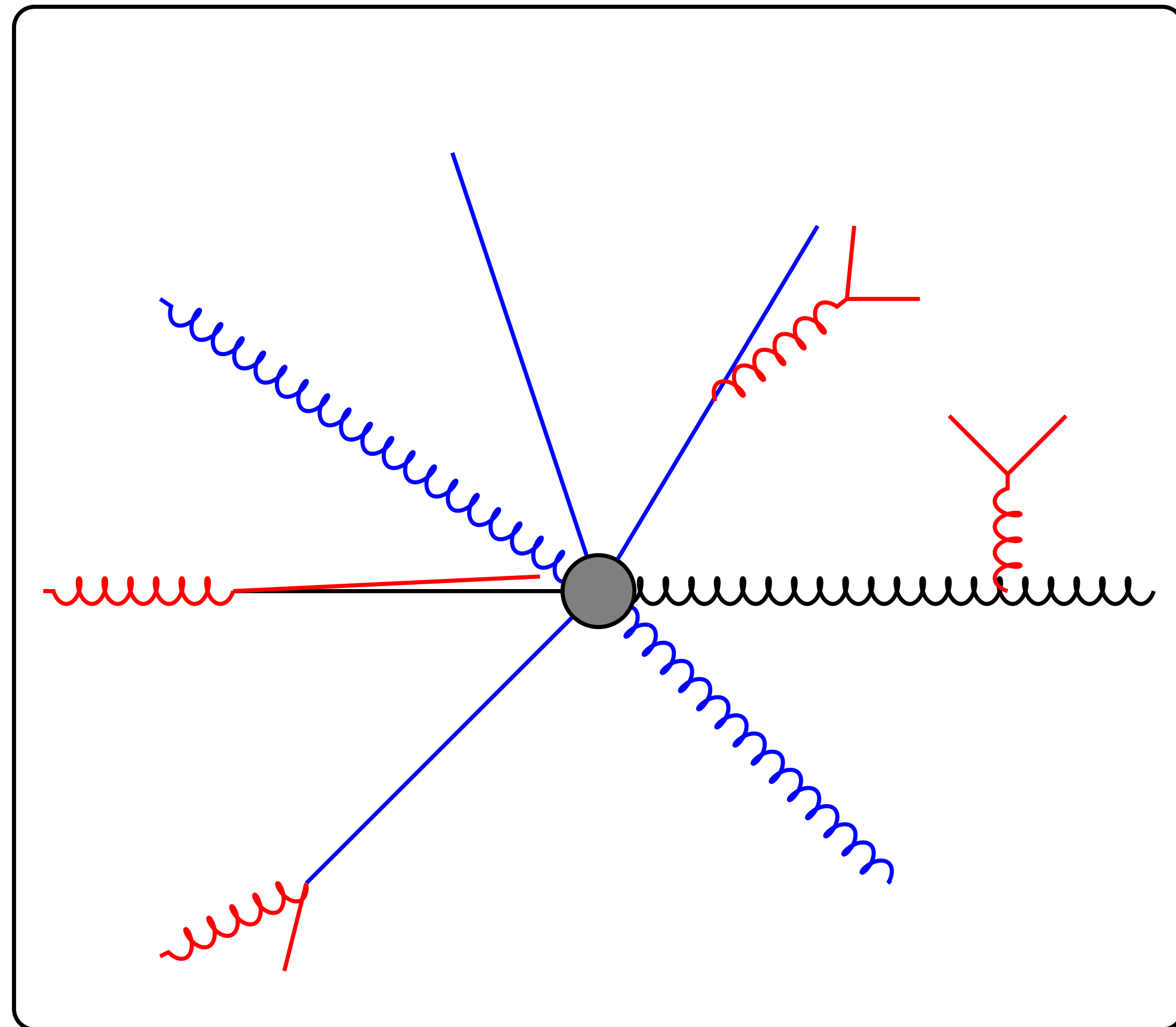


A framework for IR-safety tests

- Consider a hard underlying event
- Dress with
 - (IR/FS) DS
 - FS hard collinear (FHC)
 - IS hard collinear (IHC)
- Possibly nested

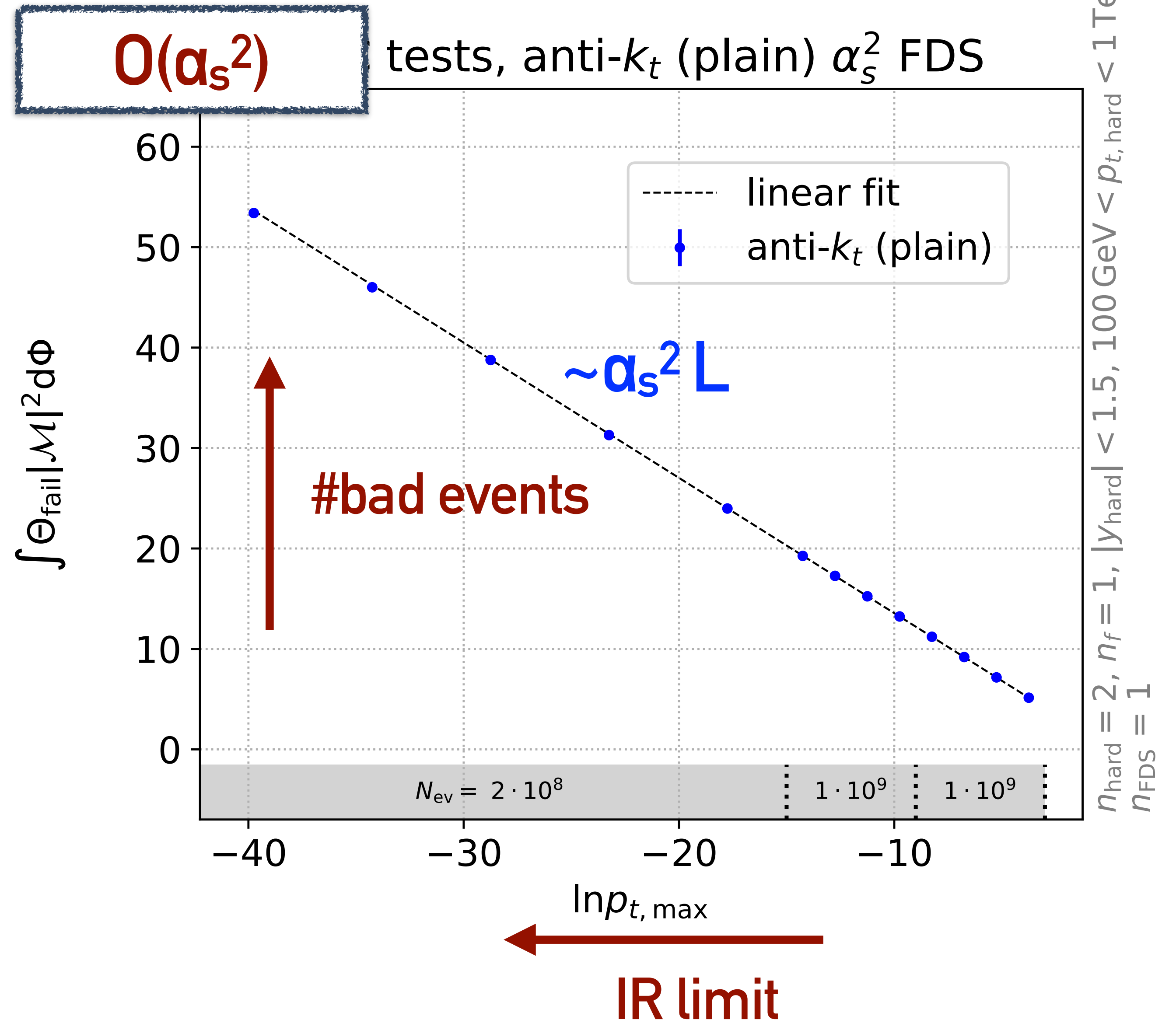
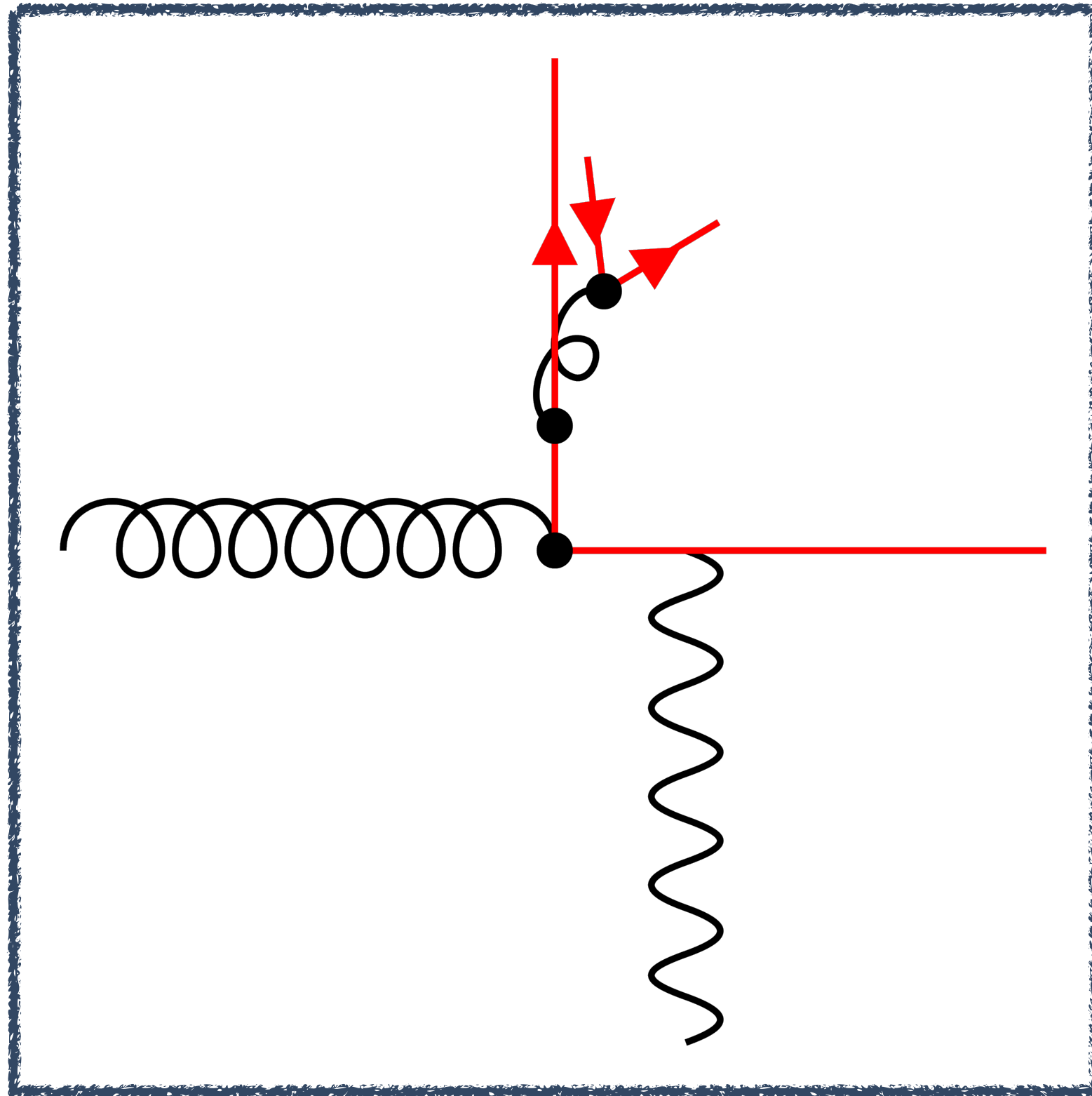


A framework for IR-safety tests



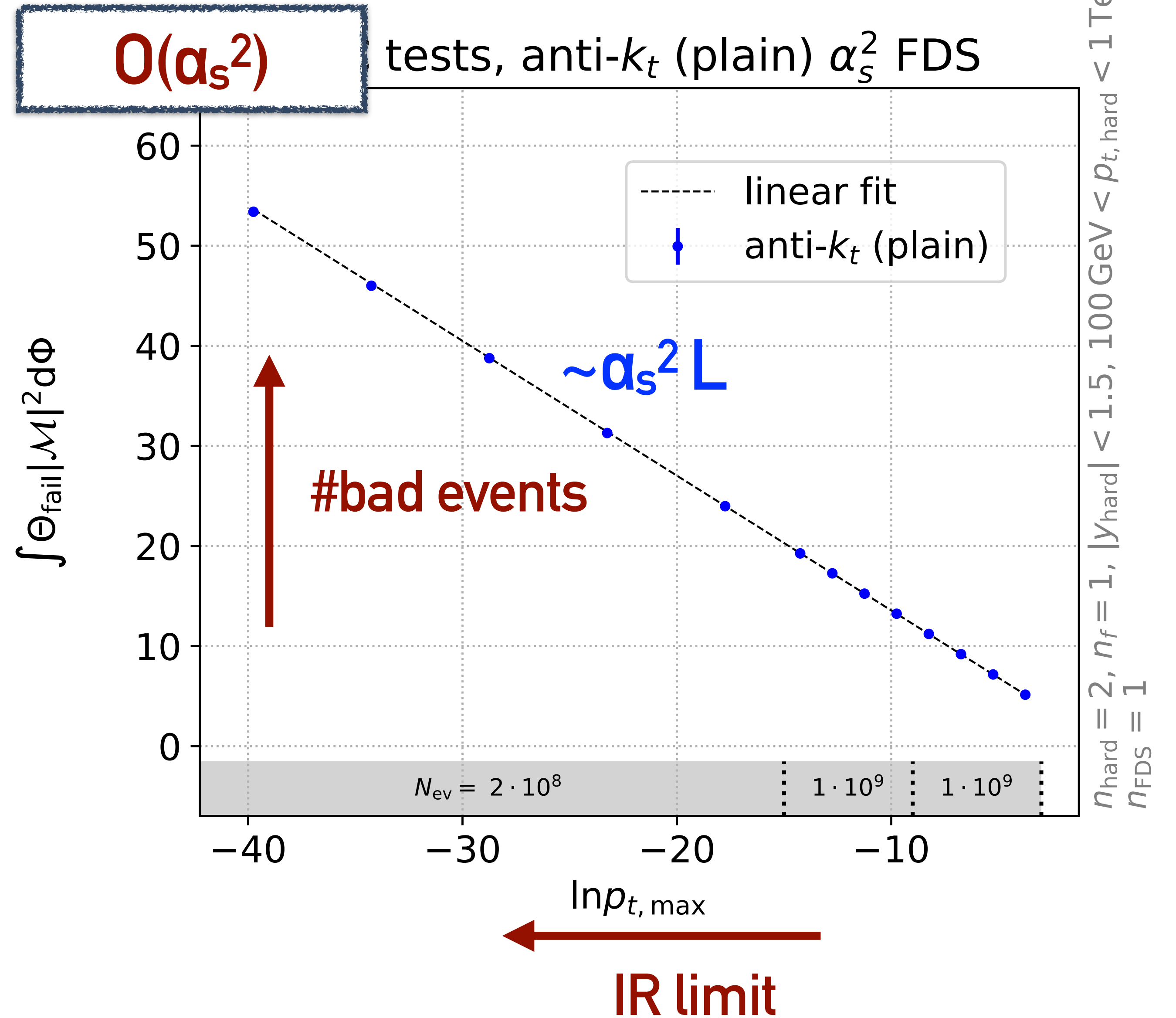
- Consider a hard underlying event
- Dress with
 - (IR/FS) DS
 - FS hard collinear (FHC)
 - IS hard collinear (IHC)
 - Possibly nested
- As extra radiation becomes unresolved:
 $\text{Hard} + \text{IR} \rightarrow \text{Hard}$

Example: plain anti- k_t +DS



Example: plain anti- k_t +DS

order relative to Born		anti- k_t
α_s	FHC	✓
	IHC	✓
α_s^2	FDS	✗ IIB
	IDS	✗ IIB
	FHC×IHC	✓
	IHC ²	✓
	FHC ²	✓



IR-safety tests: results

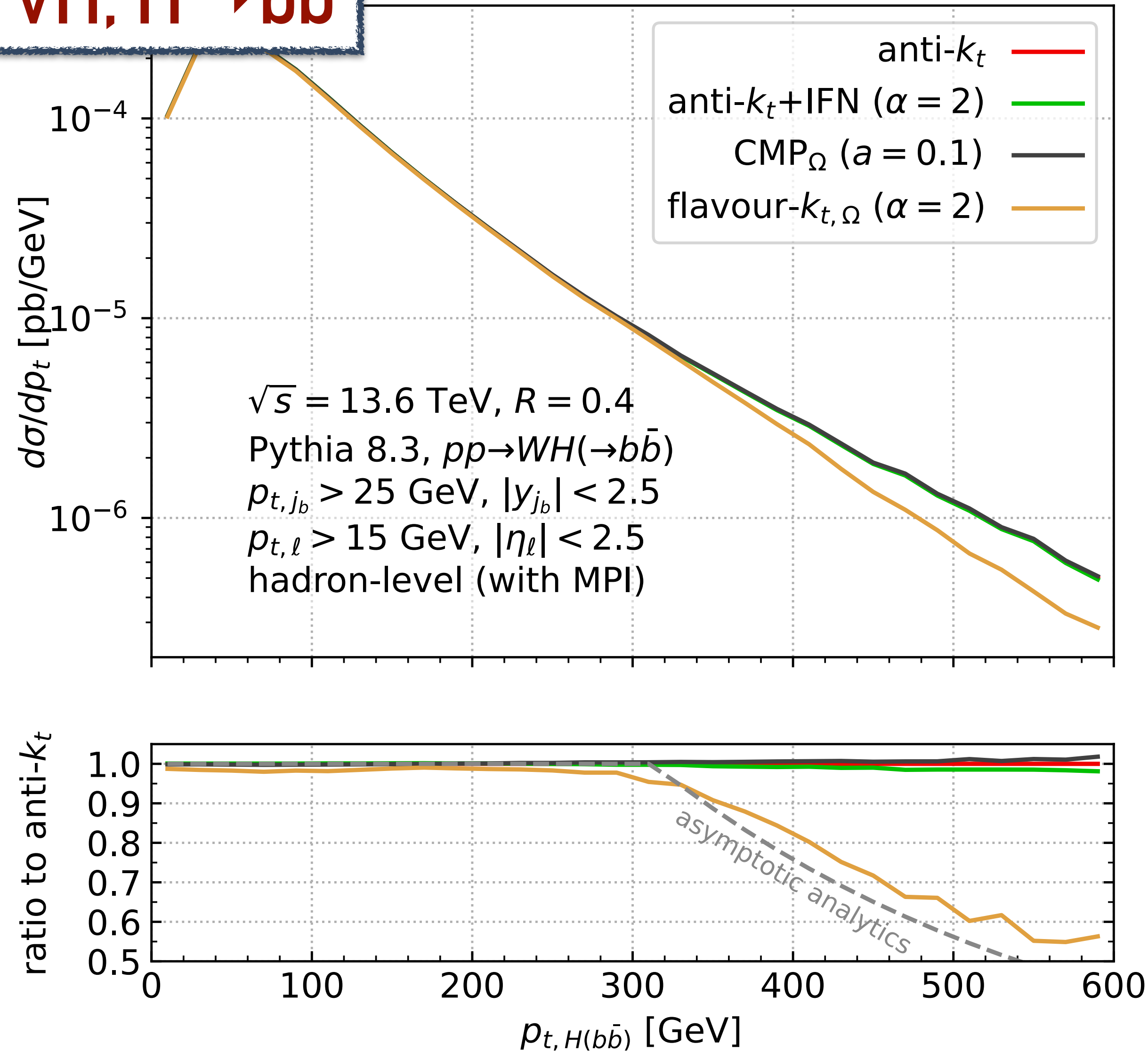
order relative to Born		anti- k_t	flav- k_t ($\alpha = 2$)	CMP	GHS $_{\alpha,\beta}$ (2, 2)	anti- $k_t + \text{IFN}_\alpha$	C/A + IFN_α
α_s	FHC	✓	✓	✓	✓	✓	✓
	IHC	✓	✓	✓	✓	✓	✓
α_s^2	FDS	✗ IIB	✓	✓	✓	✓	✓
	IDS	✗ IIB	✓	✓	✓	✓	✓
	FHC × IHC	✓	✓	✓	✓	✓	✓
	IHC ²	✓	✓	✗ C2	✓	✓	✓
	FHC ²	✓	✓	✓	✗ C4	✓	✓
α_s^3	IHC × IDS		~C1	✗ C3	~C1	✓	✓
	rest					✓	✓
α_s^4	IDS × FDS				✗ C5	✓	✓
	rest					✓	✓
α_s^5						✓	✓
α_s^6						✓	✓

Easy to fix

IFN passes IR-safety tests, up to highest order we were able to probe

Pheno results: Pythia8, Hadron-level + MPI

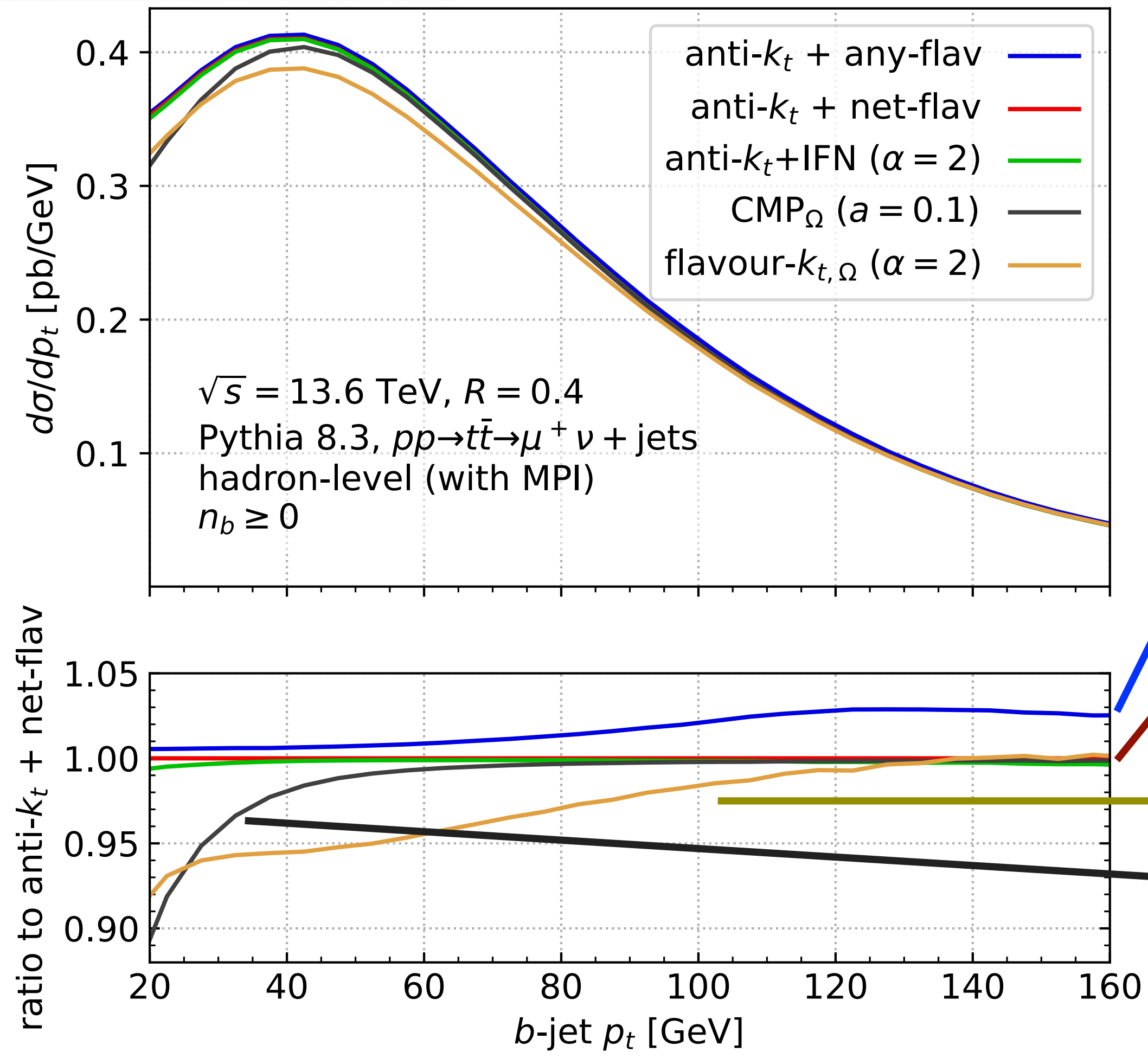
VH, $H \rightarrow b\bar{b}$



Both IFN & CMP_Ω indistinguishable
from plain anti- k_t

Pheno results: Pythia8, Hadron-level + MPI

Semileptonic tt



Plain anti- k_t , at least 1 b-tag

Plain anti- k_t , net flavour \approx IFN

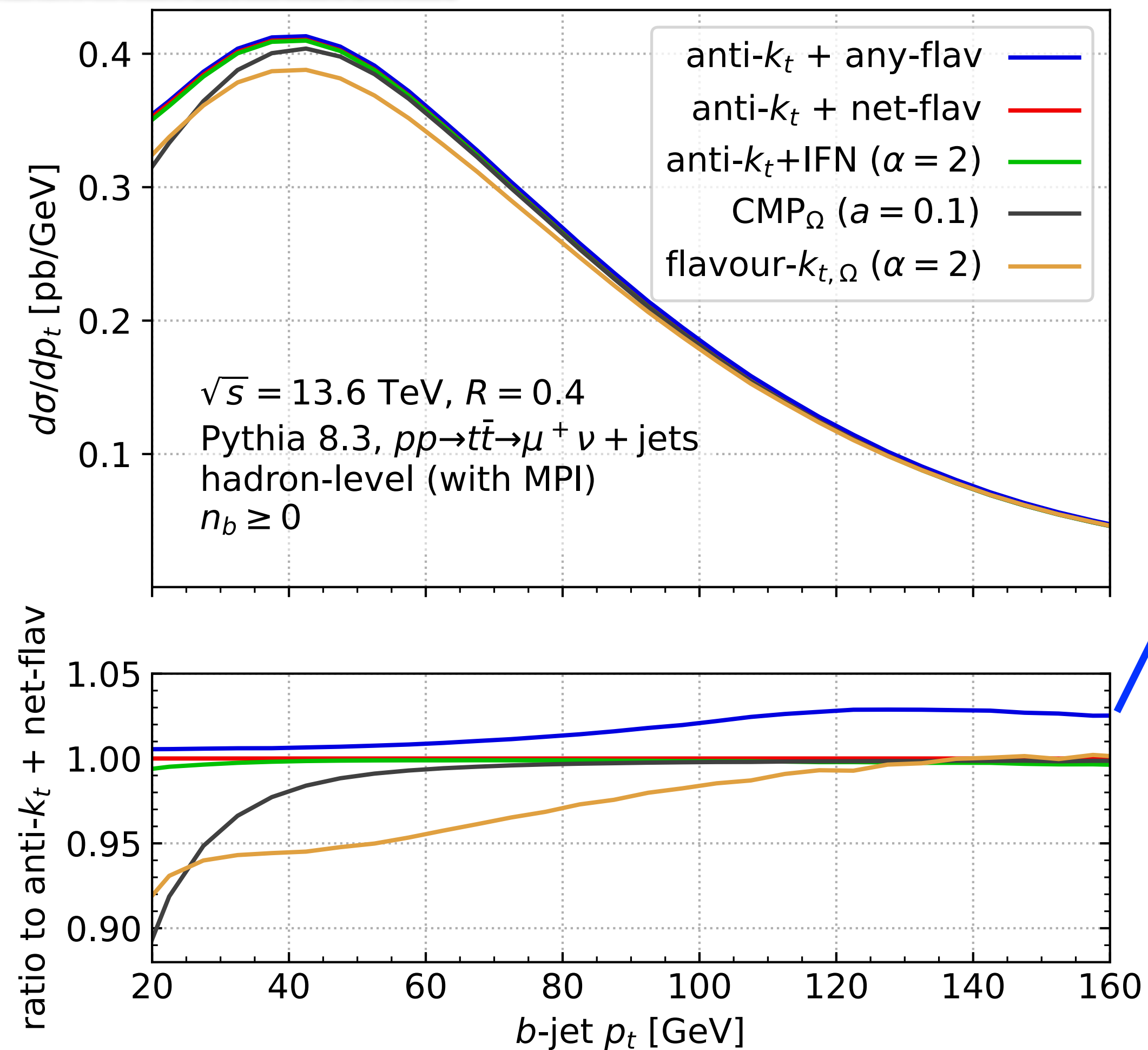
Flavour- $k_{t,\Omega}$
 CMP_Ω

~10% differences at low p_t

Inclusive b-jet p_t

Pheno results: Pythia8, Hadron-level + MPI

Semileptonic $t\bar{t}$



Inclusive $b\text{-jet } p_t$

Plain anti- k_t , at least 1 b-tag

Plain anti- k_t , net flavour \approx IFN

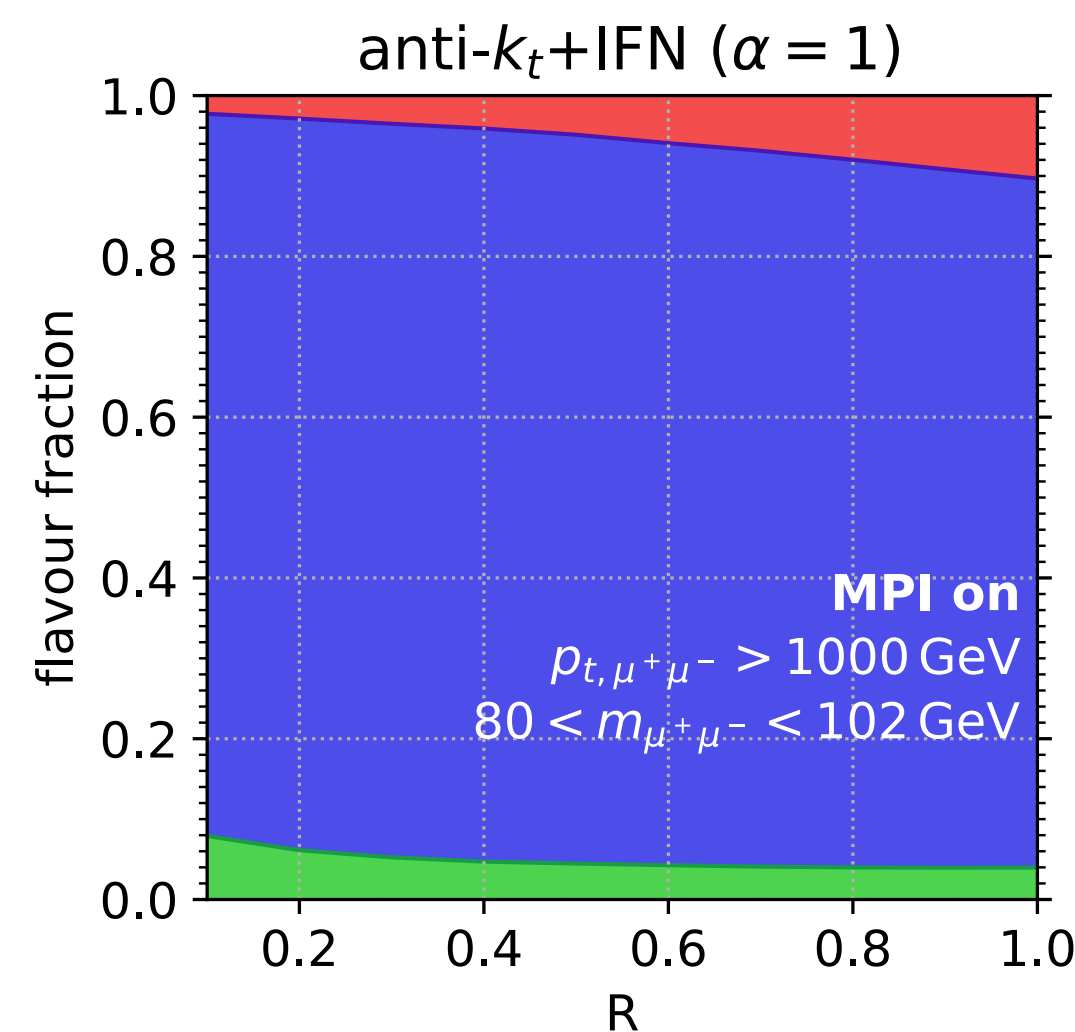
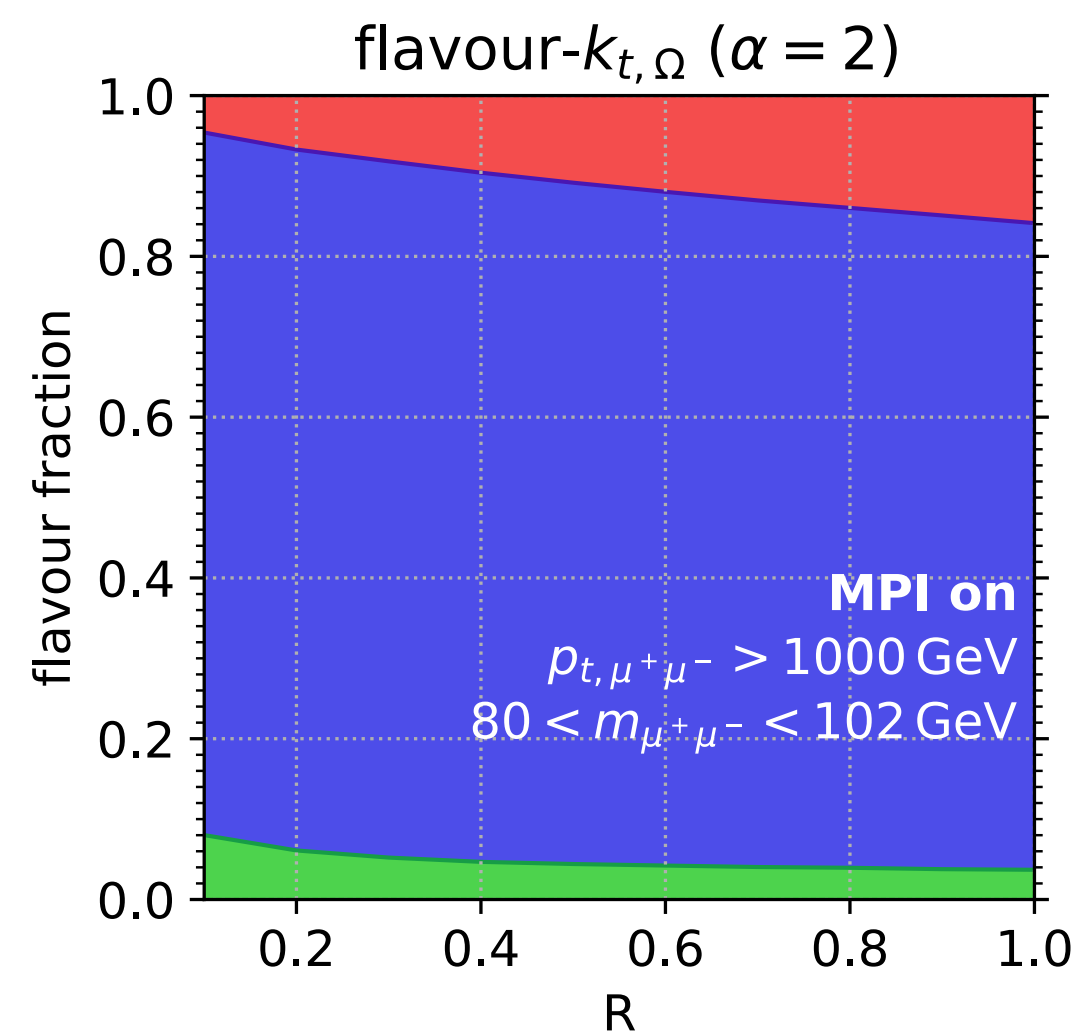
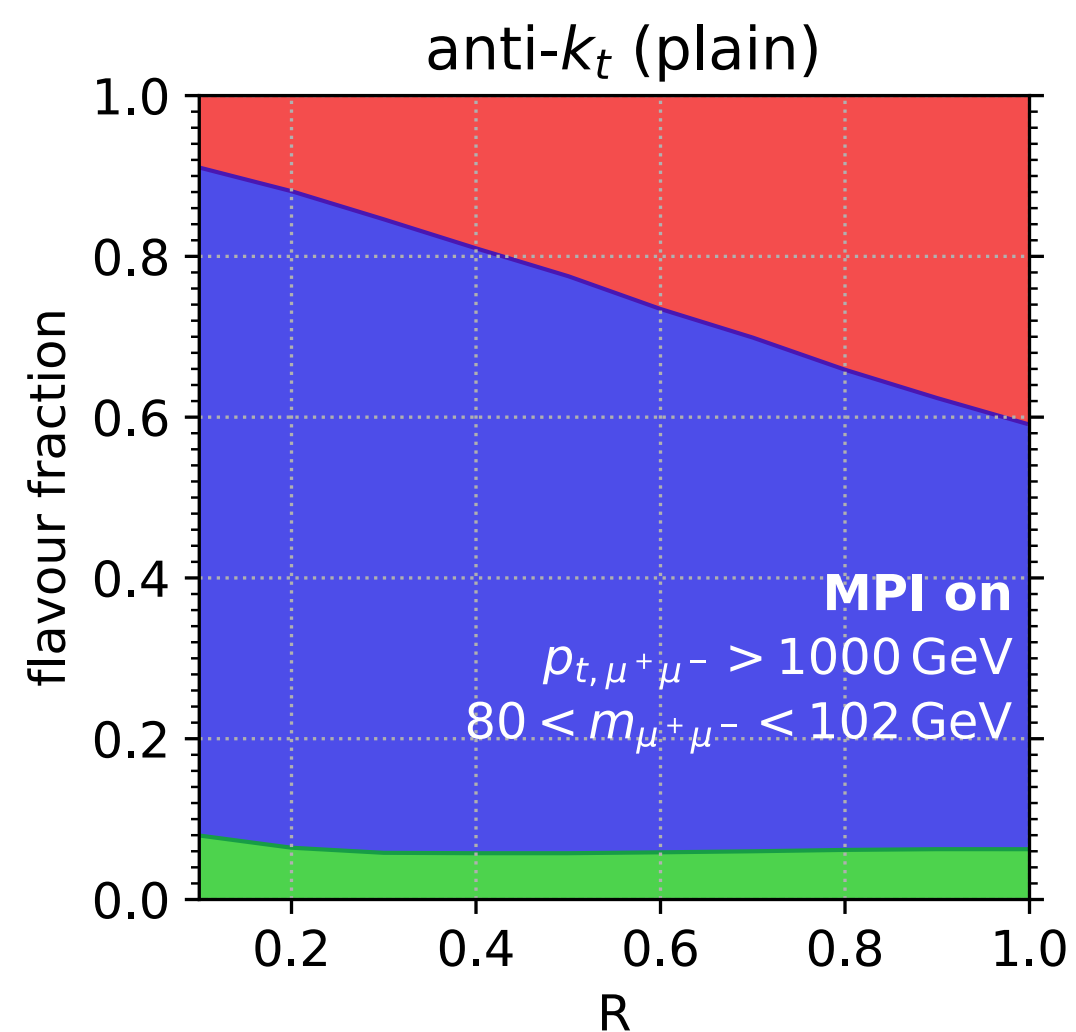
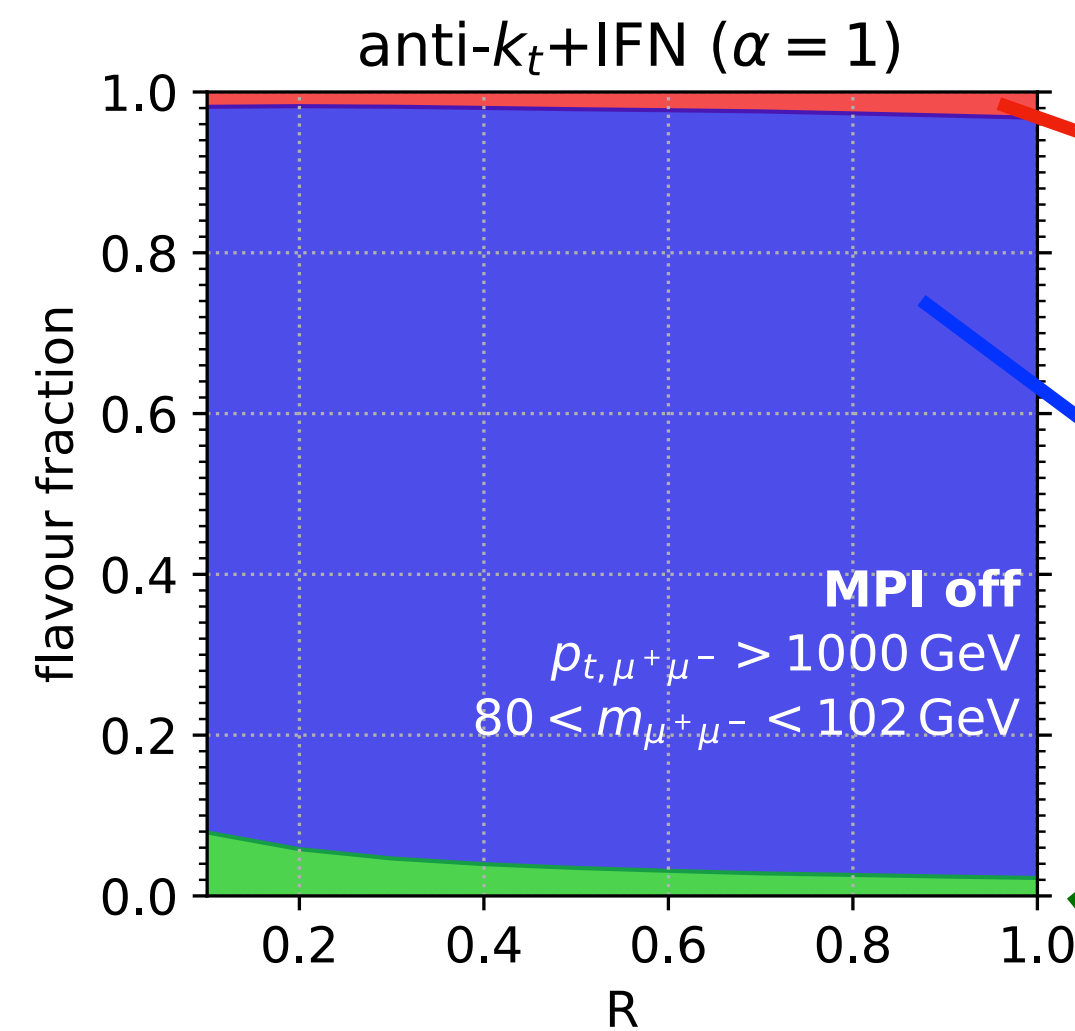
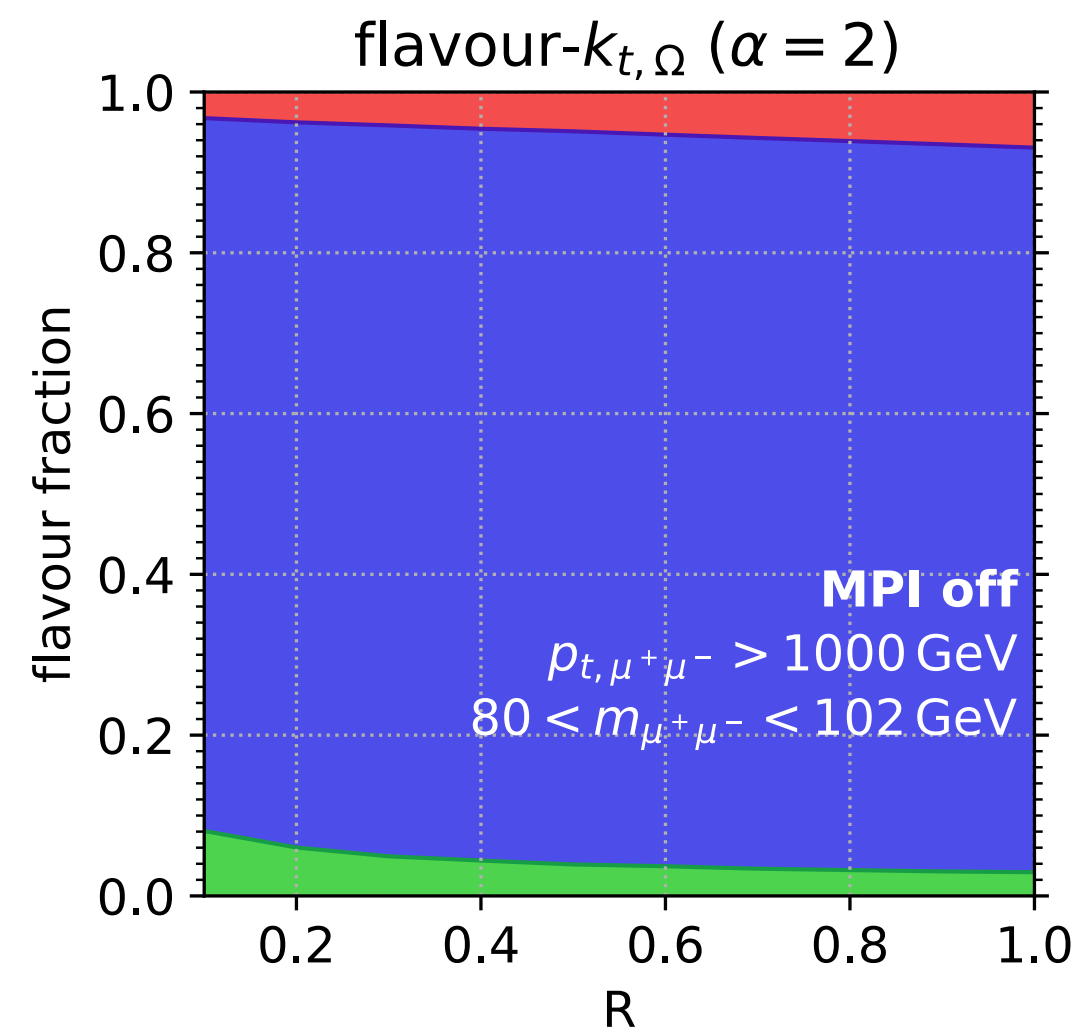
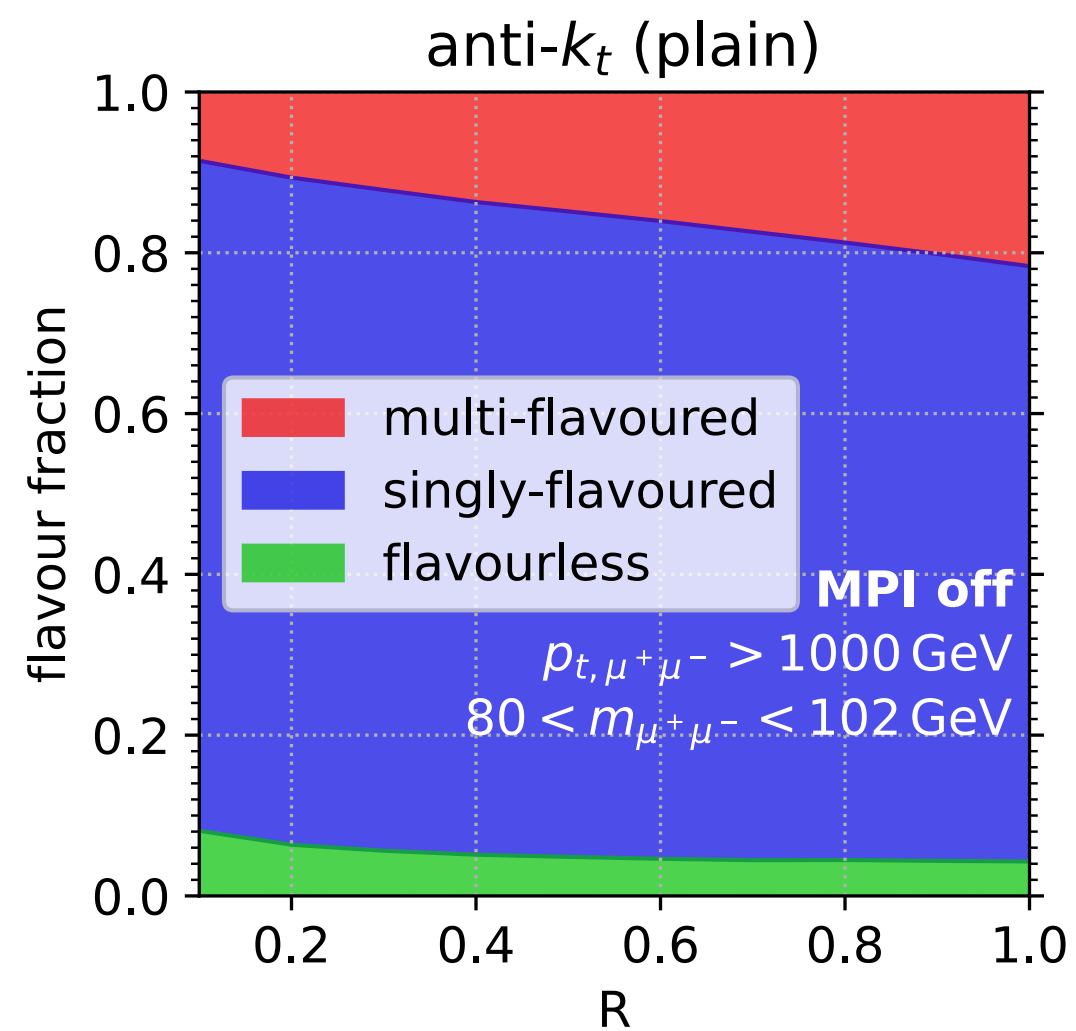
- IFN behaves very well
- Less than 5% off w.r.t. proxy for current analysis \rightarrow **unfold?**

Pheno results: Z+q results

Plain anti- k_t

Flav- $k_{t,\Omega}$

IFN



Multi-flavour

Single-flavour

Flavourless

Conclusions

- A proper definition of jet flavour is **non-trivial**
- Multiple attempt in the past to define IR-safe algorithms with kinematics identical or very similar to anti- k_t . Subtle IR-safety issues
- Our proposal: **Interleaved Flavour Neutralisation**
 - Definition of flavour interleaved but distinct from kinematics clustering
 - Kinematics unchanged, neutralisation based on suitable flavour distance
 - Passed non-trivial IR-safety tests
- Promising phenomenology → interesting investigations ahead + experimental feasibility

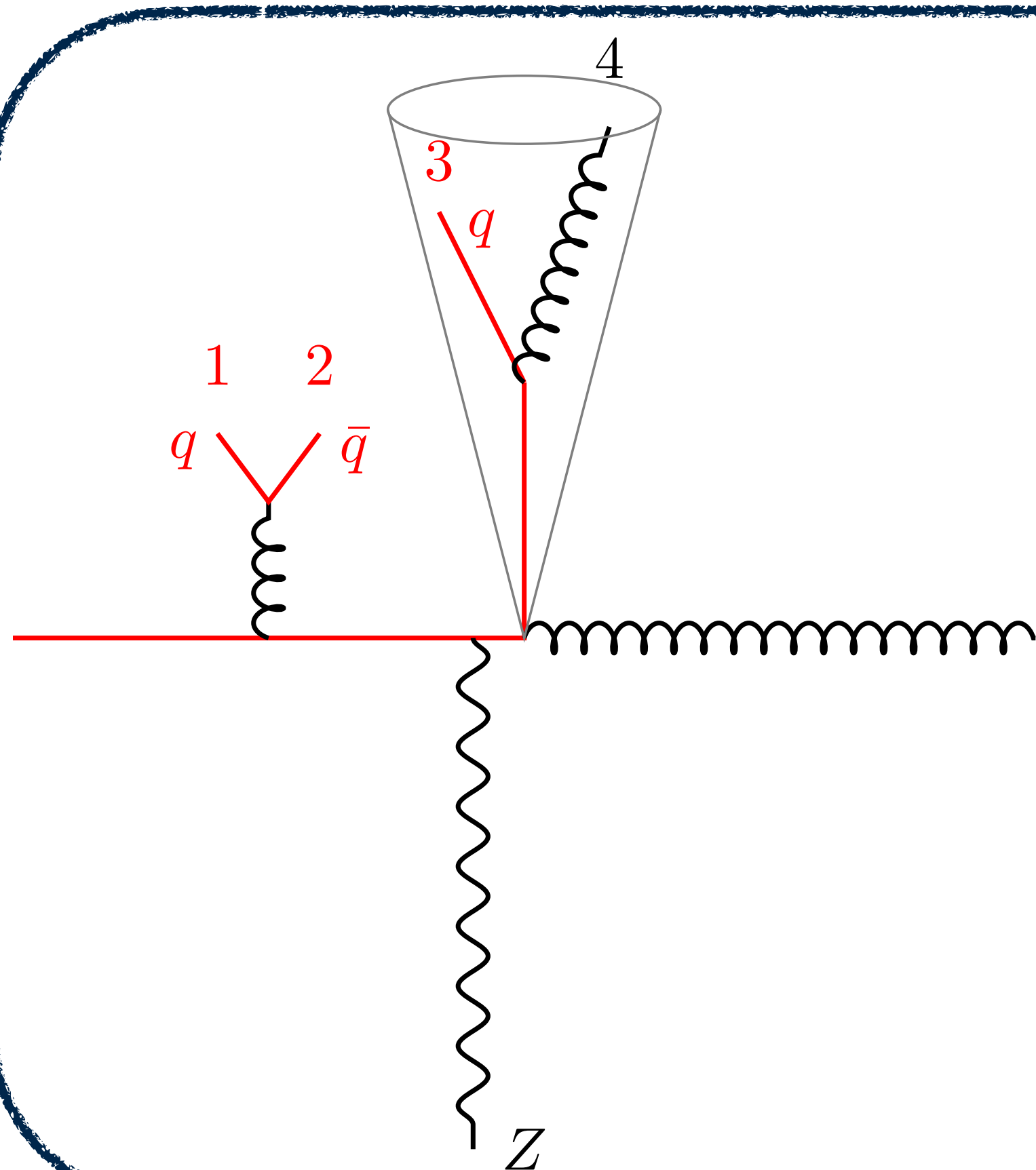


Thank you very much for your attention!

Integrated Flavour Neutralisation (IFN): a cartoon

Crucial for IR-safety + good behaviour

- proper choice of a “flavour distance”
- making sure neutralising partner is not “stolen” from more suitable candidate

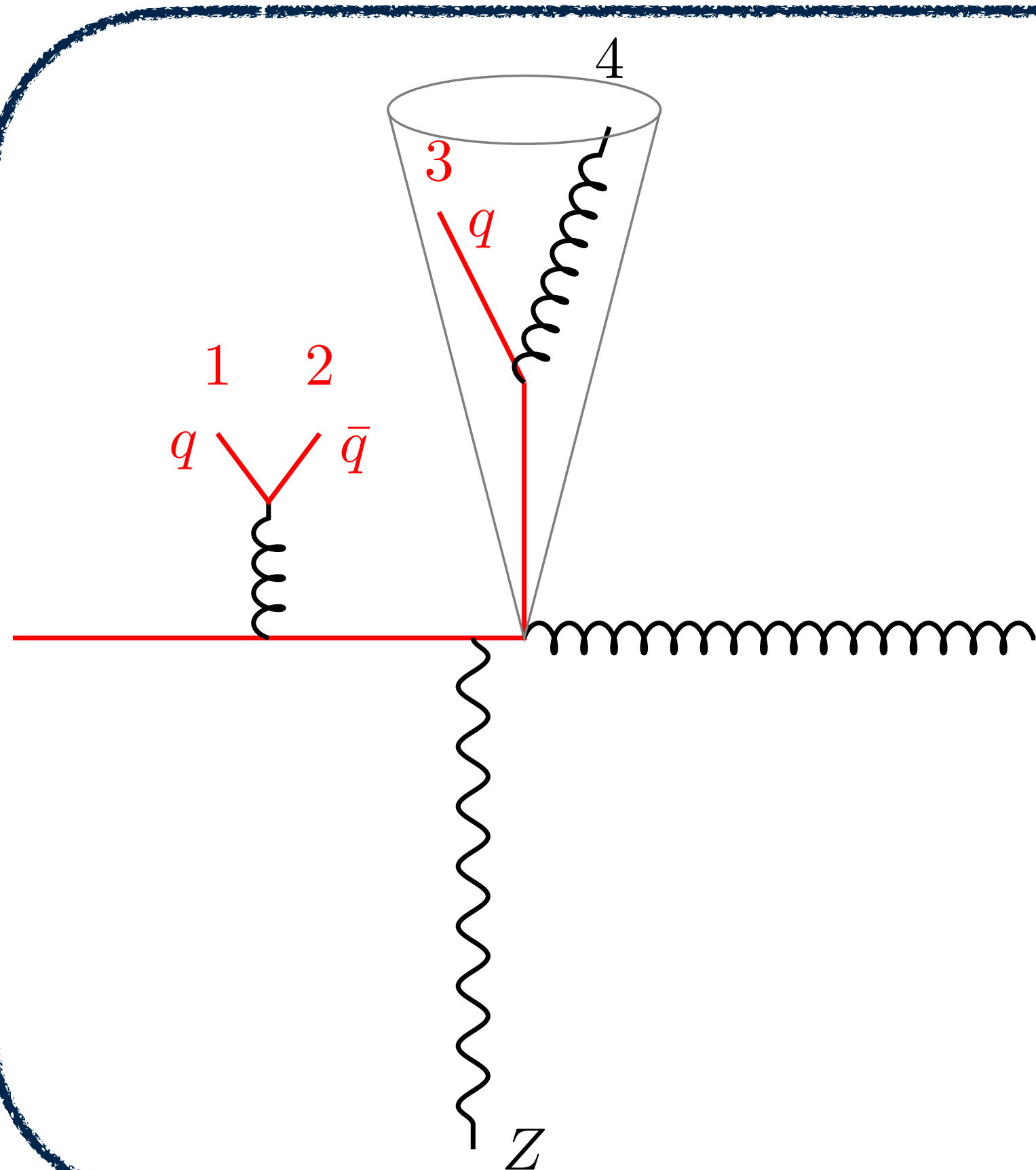


- (34) recombination \rightarrow trigger neutralisation search
- find 2 as a potential candidate
- if used: neutralised hard jet + soft flavoured jet 😞

Integrated Flavour Neutralisation (IFN): a cartoon

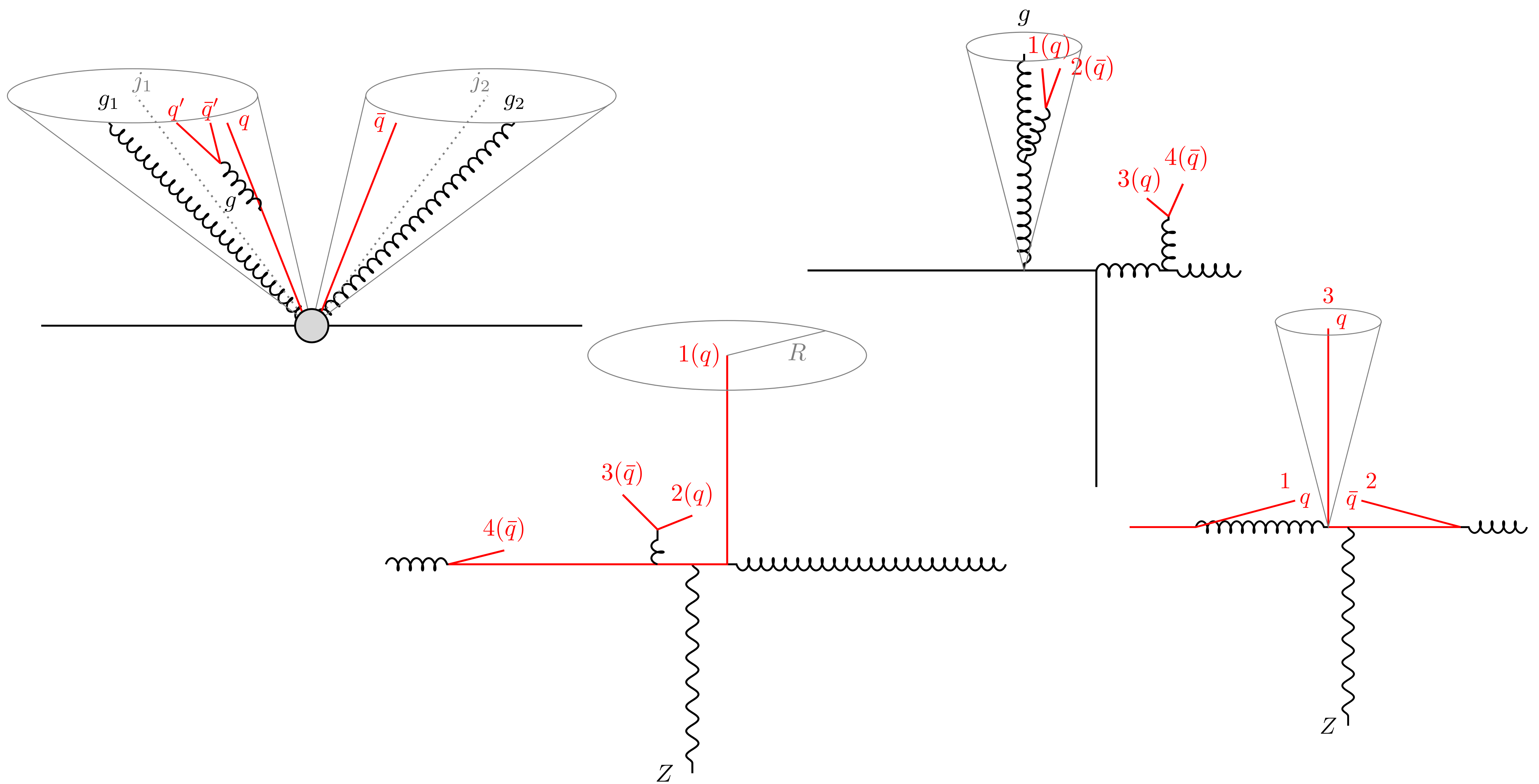
Crucial for IR-safety + good behaviour

- proper choice of a “flavour distance”
- making sure neutralising partner is not “stolen” from more suitable candidate



- (34) recombination \rightarrow trigger neutralisation search
- find 2 as a potential candidate
- way out: recursion
- before (23) neutralisation, look elsewhere to neutralise 2 \rightarrow find 1 and neutralise
- \Rightarrow Hard (34) flavour jet, soft (1 2) gluon jet \checkmark

Problematic configurations



Flavour- k_t and CMP distances

$$d_{ij}^{\text{flav-}k_t} = [\max(p_{ti}, p_{tj})]^\alpha [\min(p_{ti}, p_{tj})]^{2-\alpha} \frac{\Delta R_{ij}^2}{R^2},$$

if softer of i and j is flavoured,

$$d_{iB}^{\text{flav-}k_t} = [\max(p_{ti}, p_{tB}(y_i))]^\alpha [\min(p_{ti}, p_{tB}(y_i))]^{2-\alpha},$$

$$d_{ij}^{\text{flav-anti-}k_t} = d_{ij}^{\text{anti-}k_t} \times \mathcal{S}_{ij},$$

if i and j are oppositely flavoured,

where

$$\mathcal{S}_{ij} = 1 - \Theta(1 - \kappa) \cos\left(\frac{\pi}{2}\kappa\right), \quad \kappa \equiv \frac{1}{a} \frac{p_{ti}^2 + p_{tj}^2}{2p_{t,\max}^2},$$