Analytic Properties of the S-matrix

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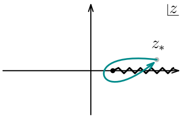
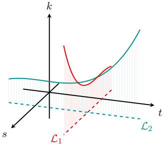




Based on work and work in progress with Sebastian Mizera, Andrew McLeod, Giulio Salvatori, Matthew Schwartz, Cristian Vergu

Motivation

How can we exploit the analyticity properties of Feynman integrals?

- Dispersion relations
- On-shell recursion relations
- Perturbative & non-perturbative bootstrap
- Generalized unitarity
- Differential equations
- Boostrap Feynman integrals

What are the constraints for Feynman-integral bootstrap?

(i) Location and types of singularities $\longrightarrow$ *letters*

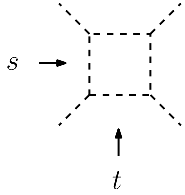
(ii) Physical-sheet constraints $\longrightarrow$ *1st entry conditions*

(iii) Asymptotic expansions $\longrightarrow$ *last entry conditions*

(iv) Discontinuities, Steinmann, extended Steinmann: $\longrightarrow$ *adjacency of letters*

Simple Example: Massless 6D box

- Symmetric in s and t
- Potential singularities at $s, t, u \in \{0, \infty\}$ ($u = -s - t$)
- Polylogarithmic, weight at most 3
- $\lim_{s \to 0} I = \frac{\log^2(s)}{2t}$
- $u = 0$ is not a singularity of I (but potentially of $\text{Disc} I$)
- Depends only on $x = \frac{s}{t}$ up to an overall mass scale



The function becomes $I = \dfrac{\log^2\left(\frac{s}{t}\right) + \pi^2}{2(s+t)}$

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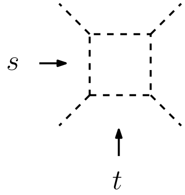


The function becomes $$I = \frac{\log^2\left(\frac{s}{t}\right) + \pi^2}{2(s+t)}$$ (π^2 ← $(i\pi)^2$)

How to encode the information?

Also a multiple polylogarithm

Algebraic functions

Multiple polylogarithms F: functions s.t. $dF = \sum_i F^{s_i} d\log s_i$.

Symbol: upgrade differential to a tensor product: $\mathcal{S}(F) = \sum_i \mathcal{S}(F^{s_i}) \otimes s_i$

Examples:

$$\mathrm{Li}_1(z) = -\log(1-z), \quad \mathrm{Li}_m(z) = \int_0^z \frac{\mathrm{Li}_{m-1}(t)}{t} dt$$

$$\mathcal{S}\big(\mathrm{Li}_m(z)\big) = -(1-z) \otimes z \otimes \cdots \otimes z$$

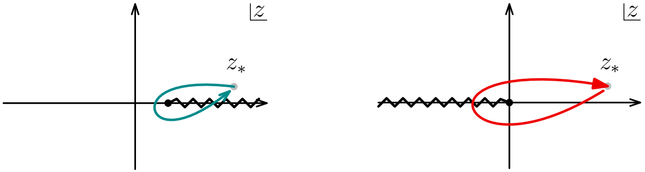
[Goncharov, Spradlin, Vergu, Volovich 2010]

[See talk by Tancredi]

Analytic structure: Symbol

Symbol makes sequential-discontinuity structure manifest

$$\mathrm{Li}_2(z) \to -(1-z) \otimes z$$



[Goncharov, Spradlin, Vergu, Volovich 2010]

Constraints on the symbol

(i) Landau singularities restrict **letters** $a_i(p)$

(ii) From the **front**: Physical sheet restricts first entries $a_1(p)$

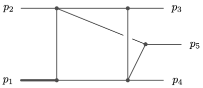
(iii) From the **back**: Asymptotic expansions restrict last entries $a_n(p)$, $a_{n-1}(p)$, ...

(iv) In the middle: Sequential discontinuities restrict **adjacent** entries

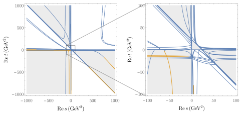
$$\mathcal{S}\big(I(p)\big) = \sum \overbrace{a_1(p) \otimes}^{\text{first discontinuity}} \overbrace{a_2(p) \otimes}^{\text{second discontinuity}} \overbrace{a_3(p) \otimes}^{\cdots} \cdots \otimes \underbrace{a_{n-2}(p) \otimes}_{\cdots} \underbrace{a_{n-1}(p) \otimes}_{\text{second derivative}} \underbrace{a_n(p)}_{\text{first derivative}}$$

Outline

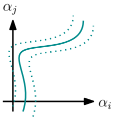
1. Singularities of amplitudes



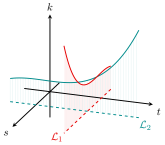
2. Physical-sheet constraints



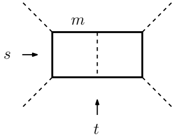
3. Asymptotic expansions



4. Sequential discontinuities



5. Bootstrapping Feynman integrals



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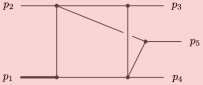

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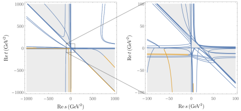

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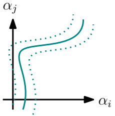

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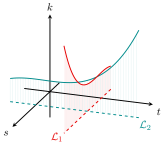
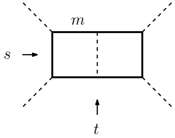


5. Bootstrapping

Feynman integrals



Find branch points: Landau equations

Loop momenta — Set N=1

$$\mathcal{A} \propto \int_0^\infty \mathrm{d}^E\alpha_e \int_\Gamma \mathrm{d}^{Ld}k\, N \exp\left[\sum \alpha_e (q_e^2 - m_e^2)\right]$$

Saddle-point analysis → branch points when

$$\alpha_e(q_e^2 - m_e^2) = 0, \qquad \text{and} \qquad \sum_{\text{loop}} \alpha_e q_e^\mu = 0$$

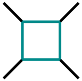
Solutions give codimension $\geqslant 1$ constraints on *external* kinematics

(As opposed to UV/IR singularities, for any kinematics)

[Bjorken 1959; Landau 1959; Nakanishi 1959; Brown 2009, Mühlbauer 2020; Klausen 2021; Mizera, Telen 2021]

Types of singularities of Feynman integrals

For massless box integral, get Landau singularities at $s = 0$, $t = 0$



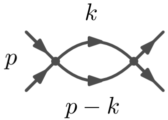
$$I_{\text{box}} = \lim_{\varepsilon\to 0^+} \frac{N}{st}\left\{\frac{2}{\epsilon^2}\left[(-s-i\varepsilon)^{-\epsilon} + (-t-i\varepsilon)^{-\epsilon}\right] - \log^2\left(\frac{-s-i\varepsilon}{-t-i\varepsilon}\right)\right\} + \mathcal{O}(\epsilon)$$

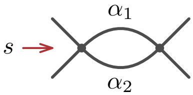
Infrared singularity

Kinematic/Landau singularities

Landau equations in momentum space

$$\alpha_e(q_e^2 - m_e^2) = 0, \qquad \text{and} \qquad \sum_{\text{loop}} \alpha_e q_e^\mu = 0$$





For bubble integral,

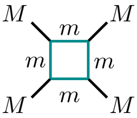
$$k^2 = m_1^2 \qquad (p-k)^2 = m_2^2 \qquad \alpha_1 k^\mu + \alpha_2 (k-p)^\mu = 0$$

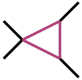
Solutions are codimension ≥ 1 constraints on *external* kinematics:

$$s = (m_1 + m_2)^2 \qquad s = (m_1 - m_2)^2$$

Landau diagrams

Correspond to any *subdiagrams* of original diagram, where lines are on shell





Box singularity:

$$st(st - 4sm^2 - 4tm^2 + 16M^2m^2 - 4M^4) = 0$$

Bubble singularities:

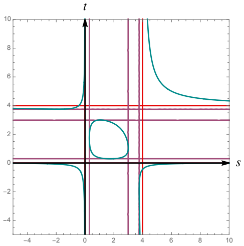
$$s(s - 4m^2) = 0, \quad t(t - 4m^2) = 0$$

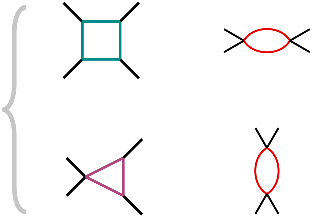
Triangle singularities:

$$s(s - 4M^2m^2 + M^4) = 0, \quad t(t - 4M^2m^2 + M^4) = 0$$

Landau diagrams

Correspond to any *subdiagrams* of original diagram, where lines are on shell





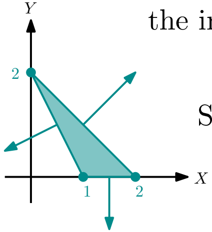
$$s(s - 4M^2m^2 + M^4) = 0, \quad t(t - 4M^2m^2 + M^4) = 0$$

$$st(st - 4sm^2 - 4tm^2 + 16M^2m^2 - 4M^4) = 0$$

$$s(s - 4m^2) = 0, \quad t(t - 4m^2) = 0$$

Massless Landau equations

For **massless particles**, use **Newton polytope** to approach the integration boundaries in Schwinger-parameter space



$$f(x, y) = x^2 + x + y^2$$

Solve the **Landau equations** at every boundary (subdiagrams not a useful classification)

[Klausen 2021; Fevola, Mizera, Telen]

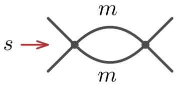
Letters $\leftrightarrow$ Landau singularities

Conjecture letters of the form $a_i = \frac{P+\sqrt{Q}}{P-\sqrt{Q}}$

Logarithmic $(P^2 - Q)$ and **algebraic** (Q) branch points are solutions to the Landau equations

Galois symmetry: $\sqrt{\bullet} \to -\sqrt{\bullet}$

Example:

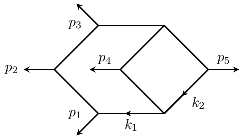


$$a = \frac{1+\sqrt{\frac{s-4m^2}{s}}}{1-\sqrt{\frac{s-4m^2}{s}}}$$

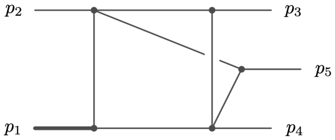
[Manteuffel, Tancredi 2017; Heller, Manteuffel, Schabinger 2019]

Letters ↔ Landau singularities

Example: Hexabox integral



[See Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser 2018]



$$\Sigma_5 = (s_{12}s_{15} - s_{12}s_{23} - s_{15}s_{45} + s_{34}s_{45} + s_{23}s_{34})^2 - 4s_{23}s_{34}s_{45}(s_{34} - s_{12} - s_{15})$$

[See Abreu, Ita, Page, Tschernow 2021]

Adding a mass gives a **new letter** at the Landau-equation solution

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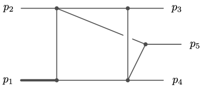

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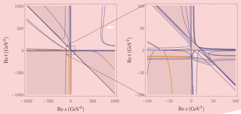
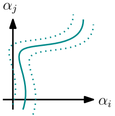
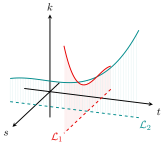


3. Asymptotic expansions



4. Sequential discontinuities



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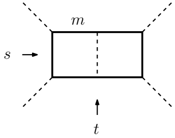
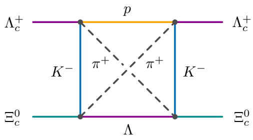
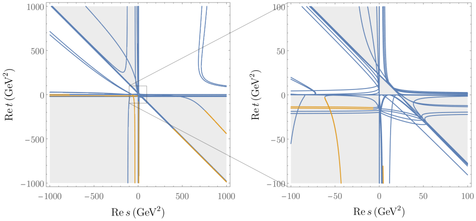


Landau singularities and Physical sheet





[Figure from HSH, Mizera '22]

$$\mathcal{A} \propto \int_0^\infty \mathrm{d}^E \alpha_e \int_\Gamma \mathrm{d}^{Ld}k\, N \exp\left[\sum \alpha_e (q_e^2 - m_e^2)\right]$$

Singularities on **physical sheet** special: Only for solutions with $\alpha_e \geq 0$

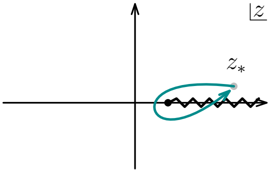
Landau singularities and Physical sheet

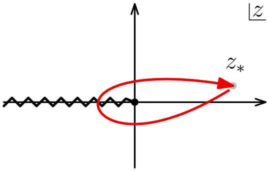
First entry condition: **α-positive** solutions only
ones that can appear for Feynman integrals

α-positive

$$\mathrm{Li}_2(z) \to -(1-z) \otimes z$$

α-positive or negative
or complex





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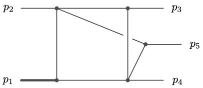
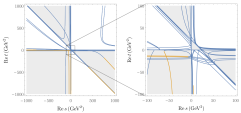


2. Physical sheet constraints



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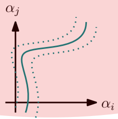
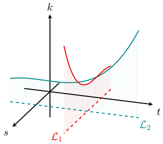


4. Sequential-discontinuities



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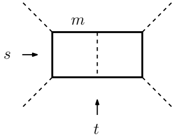


Expansion of symbol entries

We can expand polylogarithms around branch points

$$\mathrm{Li}_{q+1}(1-\varphi) \to \varphi^q \log \varphi$$

More generally, for a symbol term of the form

$$\mathcal{S}(\mathcal{A}) \supset a_1 \otimes \cdots \otimes a_k \otimes \varphi \otimes b_1 \otimes \cdots \otimes b_q \,,$$

the leading non-analytic behavior is

$$\mathcal{A} \sim \varphi^q \log \varphi$$

Asymptotic expansions

For sufficiently generic masses, we can do the asymptotic expansion:

$$\mathcal{A} \sim \begin{cases} \varphi^{\gamma} \log \varphi & \text{for } \gamma \in \mathbb{Z}_{+} \cup \{0\} \\ \varphi^{\gamma} & \text{otherwise} \end{cases}$$

Assume isolated Landau curves
Assume UV&IR finiteness

with branch point at $\varphi = 0$ (e.g. $\varphi = s - 4m^2$)

$$\gamma = \frac{Ld - E - 1}{2}$$

L: number of loops
d: dimensions
E: number of edges in Landau diagram

[Landau 1959]

Corollary on Transcendental weight

For polylogarithmic Feynman integrals with sufficiently generic masses,

$$\mathcal{A} \sim \varphi^{\gamma} \log \varphi, \qquad \text{with } \gamma = \tfrac{Ld-E-1}{2},$$

$$\mathcal{S}(\mathcal{A}) \supset a_1 \otimes \cdots \otimes a_k \otimes \varphi_0 \otimes b_1 \otimes \cdots \otimes b_q$$

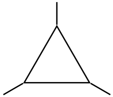
Bound on transcendental weight: $\left\lfloor \boldsymbol{\frac{Ld}{2}} \right\rfloor$

[HSH, McLeod, Schwartz, Vergu 2021]

Saddle-point expansion

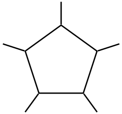
Example: expansion around **two-particle threshold**
for polygons with sufficiently generic masses:

$d = 3$



$\sim \log(s - 4m^2)$

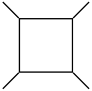
$d = 5$



$\sim (s - 4m^2) \log(s - 4m^2)$

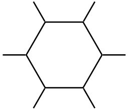
$\vdots$

$d = 4$



$\sim (s - 4m^2)^{\frac{1}{2}}$

$d = 6$



$\sim (s - 4m^2)^{\frac{3}{2}}$

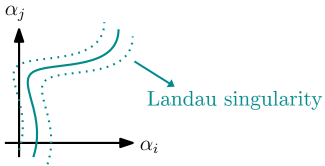
$\vdots$

Asymptotic expansions

More generally: - **solve** the Landau equations for kinematics and α

- use **method of regions** or **direct expansions** around the singular surfaces

[work in progress with G. Salvatori]



In practice: different singularities may be easy or hard to expand around

[See also Binoth, Heinrich 2000, Jantzen, Smirnov, Smirnov 2011, Heinrich, Jahn, Jones, Kerner, Langer, Magerya, Poldaru, Schlenk, Villa 2021]

General strategy for massless integrals

Study behavior of symbol near branch points

$$\mathcal{S}(\mathcal{A}) \supset a_1 \otimes \cdots \otimes a_k \otimes \underbrace{\varphi_0 \otimes \cdots \otimes \varphi_0}_{j \text{ instances}} \otimes b_1 \otimes \cdots \otimes b_q$$

$$\xrightarrow[\varphi \to 0]{} \varphi^q \log^j \varphi$$

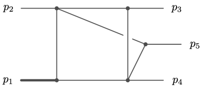
Study behavior of Feynman integral near branch points

$$\mathcal{A} \xrightarrow[\varphi \to 0]{} \varphi^\gamma \log^l \varphi$$

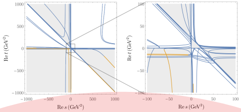
Constrain symbol

Outline

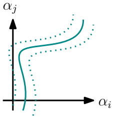
1. Singularities of amplitudes



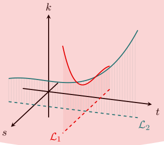
2. Physical-sheet constraints



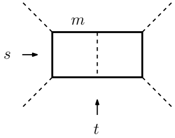
3. Asymptotic expansions



4. Sequential discontinuities

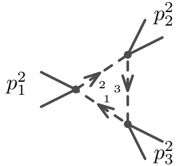


5. Bootstrapping Feynman integrals



Simple Example: Massless Triangle

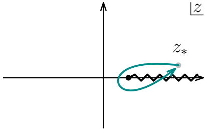
$I =$



$$\propto \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\ln(z\bar{z})\ln\left(\frac{1-z}{1-\bar{z}}\right)$$

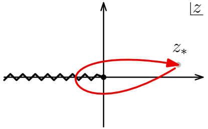
with $z\bar{z} = p_2^2/p_1^2, \quad (1-z)(1-\bar{z}) = p_3^2/p_1^2$

Dilogarithm $\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-x)}{x}dx$ has a branch cut for $z > 1$:



$$\mathrm{Disc}_{z=1}\mathrm{Li}_2(z) = 2\pi i \int_1^z \frac{1}{x}dx = 2\pi i \ln(z)$$

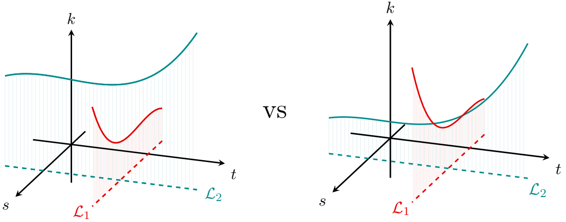
Logarithm $\ln(z) = \int_1^z \frac{1}{x}dx$ has a branch cut for $z < 0$:



$$\mathrm{Disc}_{z=0}\ln(z) = 2\pi i$$

Key point on sequential discontinuities

Whenever Landau singularities do not intersect in ***integration space***, the sequential discontinuity is zero



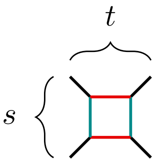
Sequential discontinuities restrict **adjacent symbol letters**

[Fotiadi, Froissart, Lascoux and Pham 1965; HSH, McLeod, Schwartz, Vergu 2022; Berghoff, Panzer 2022]

Special case: Steinmann relations

No sequential discontinuities in partially overlapping channels in the physical region

[Steinmann 1960]



Here: $\mathcal{A}$ cannot have a term of the form $\log(s - 4m^2)\log(t - 4m^2)$

Important constraint in $\mathcal{N} = 4$ bootstrap program

[Caron-Huot, Dixon, McLeod, von Hippel 2016]

Outline

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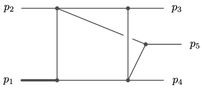
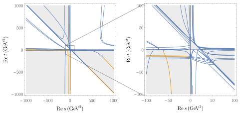
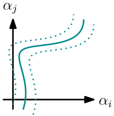


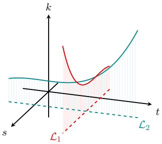
2. Physical sheet constraints



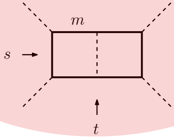
3. Asymptotic expansions



4. Sequential-discontinuities



5. Bootstrapping Feynman integrals



What are the constraints for Feynman-integral bootstrap?

(i) Location and types of singularities $\longrightarrow$ *letters*

(ii) Physical-sheet constraints $\longrightarrow$ *1st entry conditions*

(iii) Asymptotic expansions $\longrightarrow$ *last entry conditions*

(iv) Discontinuities, Steinmann, extended Steinmann: $\longrightarrow$ *adjacency of letters*

[See Caron-Huot, Chicherin, Dixon, Drummond, Dulat, Foster, Gürdoğan, Henn, von Hippel, Mitev, McLeod, Liu, Papathanasiou, Wilhelm, ...]

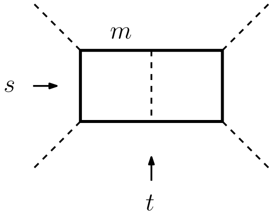
Bootstrapping the double box w/massive rung

[Caron-Huot, Henn 2014]

Letters formed using Landau singularities at

$$s = 0, \quad s = 4m^2, \quad t = 0, \quad t = 4m^2, \quad m^2 = 0,$$

$$s + t = 0, \quad st + 4sm^2 + 4tm^2 = 0$$



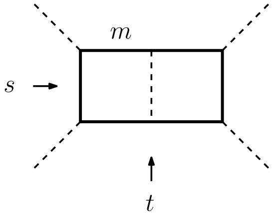
$$L_1 = u, \quad L_2 = v, \quad L_3 = 1 + u, \quad L_4 = 1 + v, \quad L_5 = u + v,$$

$$L_6 = \frac{\beta_u - 1}{\beta_u + 1}, \quad L_7 = \frac{\beta_v - 1}{\beta_v + 1}, \quad L_8 = \frac{\beta_{uv} - 1}{\beta_{uv} + 1}, \quad L_9 = \frac{\beta_{uv} - \beta_u}{\beta_{uv} + \beta_u}, \quad L_{10} = \frac{\beta_{uv} - \beta_v}{\beta_{uv} + \beta_v},$$

$$u = -\frac{4m^2}{s}, \quad v = -\frac{4m^2}{t}, \quad \beta_u = \sqrt{1 + u}, \quad \beta_v = \sqrt{1 + v}, \quad \beta_{uv} = \sqrt{1 + u + v}$$

Ansatz

$$I = \frac{1}{s^2 t\sqrt{1+u}\sqrt{1+u+v}} \underbrace{\sum c_{i_1,i_2,i_3,i_4} L_{i_1} \otimes L_{i_2} \otimes L_{i_3} \otimes L_{i_4}}_{\mathcal{S}(I)}$$



$$L_1 = u, \quad L_2 = v, \quad L_3 = 1+u, \quad L_4 = 1+v, \quad L_5 = u+v,$$

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$$u = -\frac{4m^2}{s}, \quad v = -\frac{4m^2}{t}, \qquad \beta_u = \sqrt{1+u}, \quad \beta_v = \sqrt{1+v}, \quad \beta_{uv} = \sqrt{1+u+v}$$

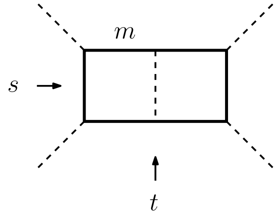
Bootstrapping the double box

integrable weight four symbols	2597
Galois symmetry	306
vanishing $s \to 0$ limit	284
only L1, L3, L6, L9, L10 in second entry after L6	230
only L2, L4, L7, L9, L10 in the second entry after L7	213
only L1, L3, L6, L9, L10 in second entry after L3	182
only L2, L4, L7, L9, L10 in the second entry after L4	160
without L2 or L3 in last entry	102
without L7 or L10 in last entry	83
without L7 or L10 in second-to-last entry	73
no L1, L2, L5, L8, or L9 in the first entry	1

Bootstrapping the double box

Constraint		
	integrable weight four symbols	2597
$\sqrt{\bullet} \to -\sqrt{\bullet}$	Galois symmetry	306
α-positive constraints	vanishing $s \to 0$ limit	284
Steinmann constraints	only L1, L3, L6, L9, L10 in second entry after L6	230
	only L2, L4, L7, L9, L10 in the second entry after L7	213
	only L1, L3, L6, L9, L10 in second entry after L3	182
	only L2, L4, L7, L9, L10 in the second entry after L4	160
Expansion constraints	without L2 or L3 in last entry	102
	without L7 or L10 in last entry	83
	without L7 or L10 in second-to-last entry	73
α-positive constraints	no L1, L2, L5, L8, or L9 in the first entry	1

Bootstrapping the double box



$$
\begin{aligned}
\mathcal{S}(I) = &-L_6 \otimes L_1 \otimes L_6 \otimes L_9 - L_6 \otimes L_1 \otimes L_9 \otimes L_6 + L_6 \otimes L_3 \otimes L_6 \otimes L_9 \\
&+ L_6 \otimes L_3 \otimes L_9 \otimes L_6 + L_6 \otimes L_6 \otimes L_1 \otimes L_9 + L_6 \otimes L_6 \otimes L_2 \otimes L_9 \\
&- L_6 \otimes L_6 \otimes L_3 \otimes L_9 - L_6 \otimes L_6 \otimes L_5 \otimes L_9 + L_6 \otimes L_6 \otimes L_8 \otimes L_6 \\
&+ L_6 \otimes L_9 \otimes L_2 \otimes L_6 - L_6 \otimes L_9 \otimes L_5 \otimes L_6 + L_6 \otimes L_9 \otimes L_8 \otimes L_9 \\
&+ L_7 \otimes L_{10} \otimes L_2 \otimes L_6 - L_7 \otimes L_{10} \otimes L_5 \otimes L_6 + L_7 \otimes L_{10} \otimes L_8 \otimes L_9 \\
&+ L_7 \otimes L_7 \otimes L_1 \otimes L_9 - L_7 \otimes L_7 \otimes L_5 \otimes L_9 + L_7 \otimes L_7 \otimes L_8 \otimes L_6
\end{aligned}
$$

Takeaway point: We can bootstrap Feynman integrals, but

What is the simplest set of constraints needed?

Takeaway point: We can bootstrap Feynman integrals, but

What is the simplest set of constraints needed?

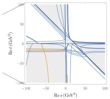
(don't need expansions around all singularities,

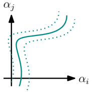
adjacency constraints for every letter, ...)

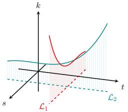
Conclusions & Outlook

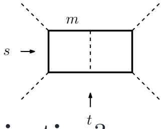
We can use **constraints on analytic structure** to determine polylogarithmic Feynman integrals

(Location and types of singularities, Physical-sheet constraints, Asymptotic expansions, Sequential Discontinuities/Steinmann)









- How to extend to **dimensional regularization**?

- How to implement for **elliptic integrals**?

- How to incorporate **numerators**?

Thanks!