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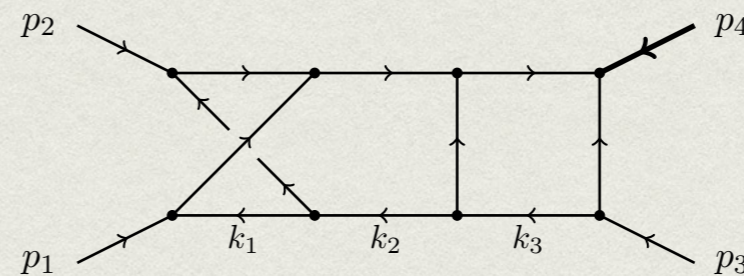
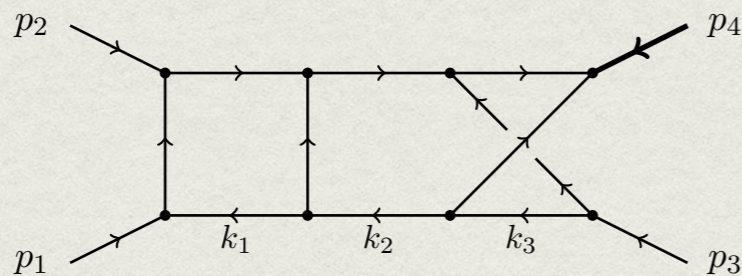
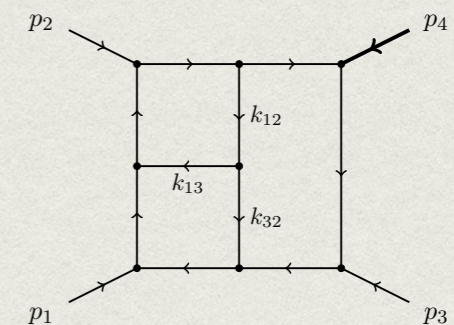
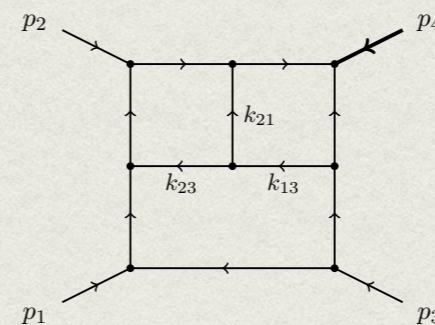
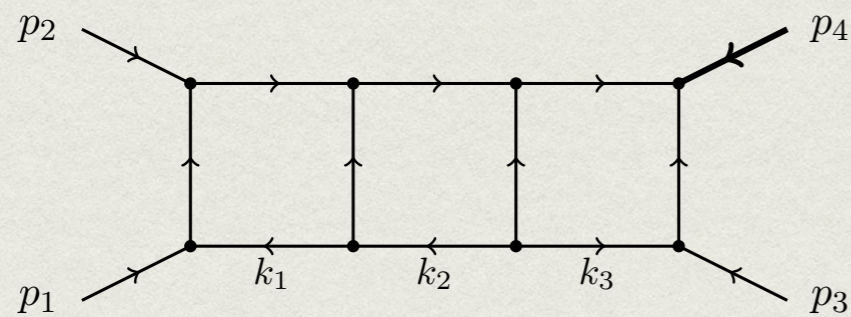
On the evaluation of three-loop non-planar Feynman diagrams for Higgs plus jet production

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In collaboration with Johannes Henn and Jungwon Lim

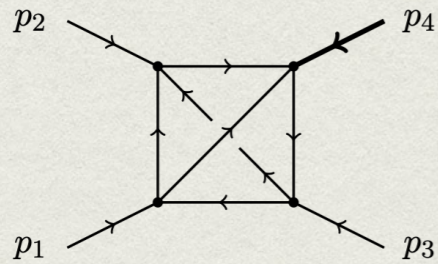
LoopFest XXI
June 26 — 28, 2023
SLAC National Accelerator Laboratory

This talk: we revisit the calculation of planar and summarise first results on non-planar three-loop families for three-loop integrals with one massive leg.

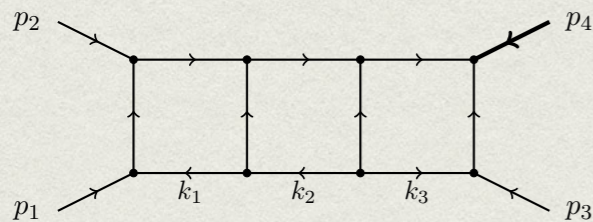


2302.12776 [hep-th]

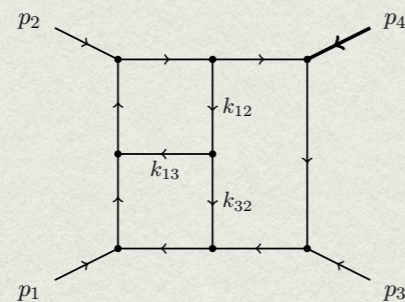
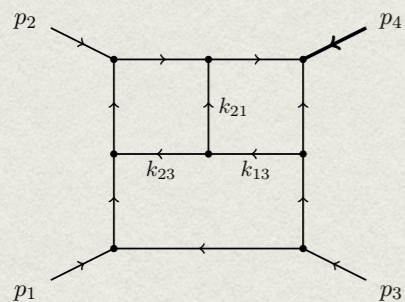
State-of-the-art at three-loop



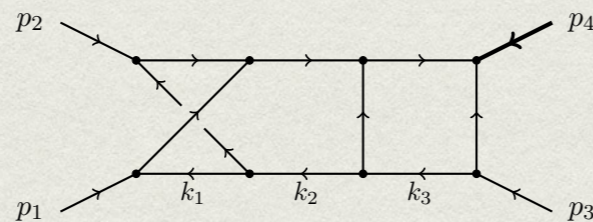
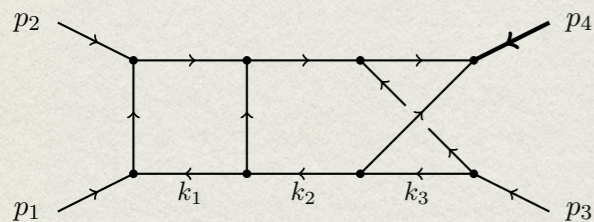
[Henn, Smirnov, Smirnov (2013)]



[di Vita, Mastrolia, Schubert, Yundin (2014)]



[Canko, Syrrakos (2021)]



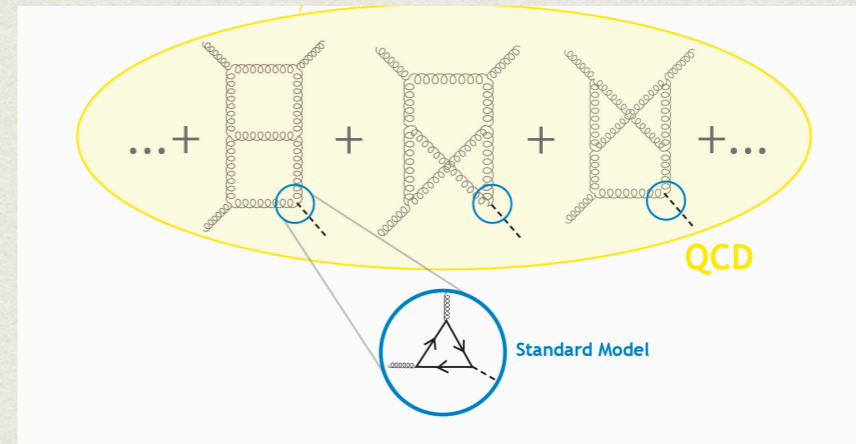
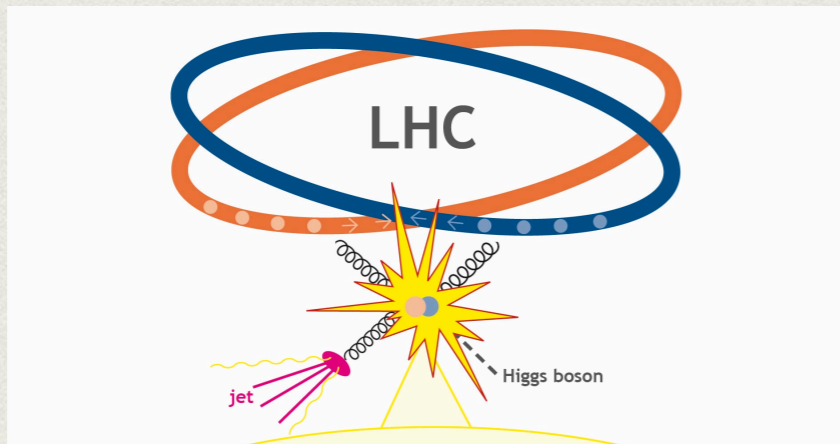
[Henn, Lim, WJT (2023)]

Outline

- Motivation
- Algorithms for computing Feynman integrals
- Observation on three-loop integrals
- Conclusions/Outlook

Motivation

- Phenomenology :: Higgs/vector + jet production \rightarrow N₃LO

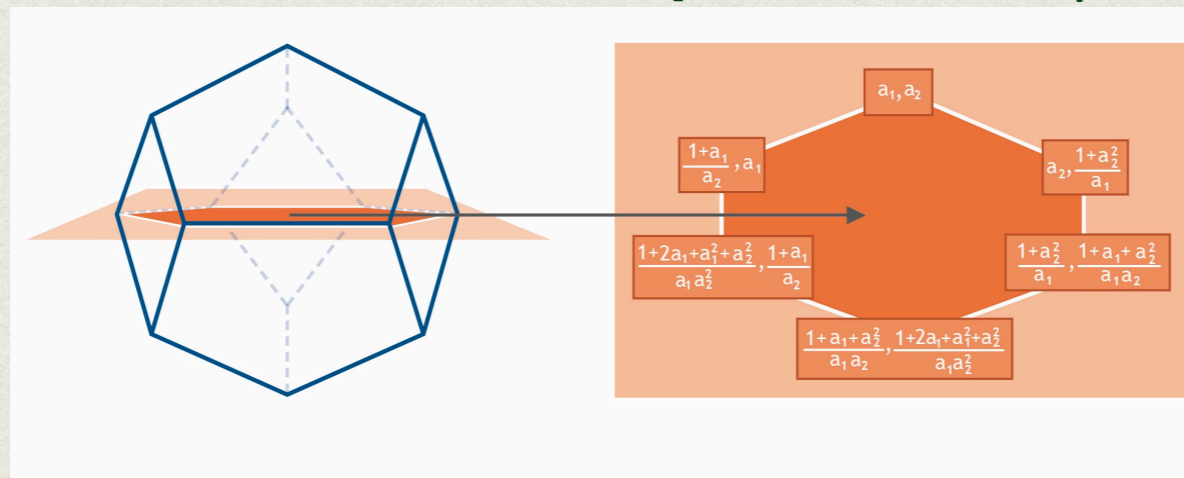


- Mathematically :: Bootstrapping approaches :: N=4 sYM form factors

[Dixon, Gurdogan, McLeod, Wilhelm (2020)]

C_2 Cluster algebras

[Chicherin, Henn, Papathanasiou (2020)]

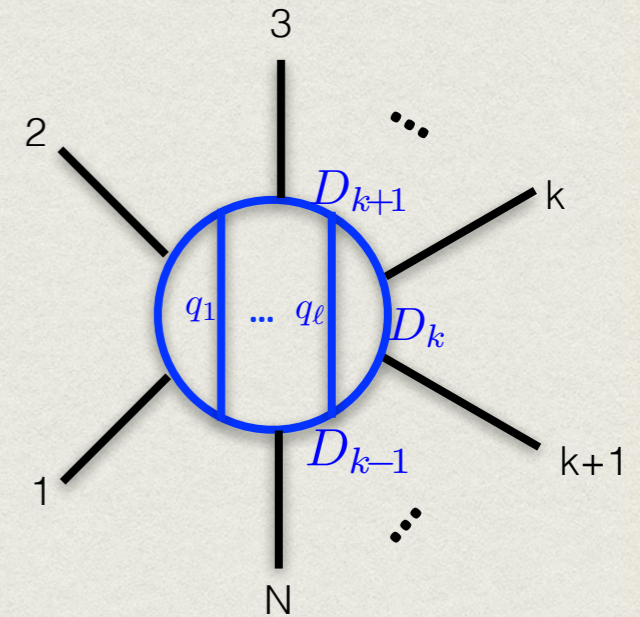


Algorithms for computing Feynman integrals

- In loop calculations, one finds

$$J_N^{(L),D}(1, \dots, n; n+1, \dots, m) = \int \prod_{i=1}^L \frac{d^D \ell_i}{i \pi^{D/2}} \frac{\prod_{k=n+1}^m D_k^{-\nu_k}}{\prod_{j=1}^n D_j^{\nu_j}}$$

$$D_i = q_i^2 - m_i^2 + i0$$



- DEQ :: Feynman integrals are not independent

$$\partial_x \vec{J}(x) = A_i(x, \epsilon) \vec{J}(x)$$

Canonical form

Conjecture: there exist a basis of uniform transcendental weight functions

[Henn (2013)]

$$\partial_x \vec{g}(x) = \epsilon B(x) \vec{g}(x) \longrightarrow d \vec{g}(x, \epsilon) = \epsilon (d\tilde{B}) \vec{g}(x; \epsilon)$$

$$\tilde{B} = \sum_k B_k \log \alpha_k(x)$$

Uniform weight function

Algorithms for computing Feynman integrals

Master Integrals

dlog / UT integrals

Canonical differential equations

Evaluation in terms of generalised polylogarithms

Numerical validation

• DlogBasis + UT integrals in subsectors.

[Wasser (2020)]

• IBP reductions w/ FIRE6.

[Smirnov, Chuharev (2019)]

• Differential equations:
in-house implementation+LiteRed.

[Lee (2012)]

• Analytic reconstruction w/ FiniteFlow.

[Peraro (2019)]

• Algebraic manipulation of GPLs with
PolylogTools

[Duhr, Dulat (2019)]

• Numerical evaluation w/ PySecDec
& FeynTrop

[Borowka et al (2017)]

[Borinsky, Munch, Tellander (2023)]

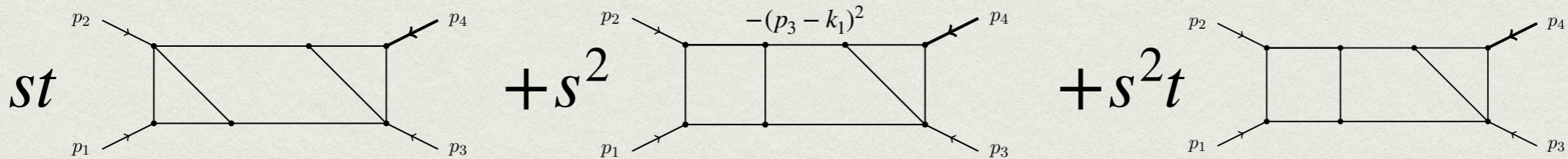
dlog/UT integrals

📌 Dlog basis :: set of integrals admitting a dlog representation

$$\mathcal{F}^{(L)} = \sum_{k=1}^{4L} c_k d \log g_1^{(k)} \wedge d \log g_2^{(k)} \wedge \dots \wedge d \log g_n^{(k)} \longrightarrow \int_{g_i=0} \mathcal{F} = 1$$

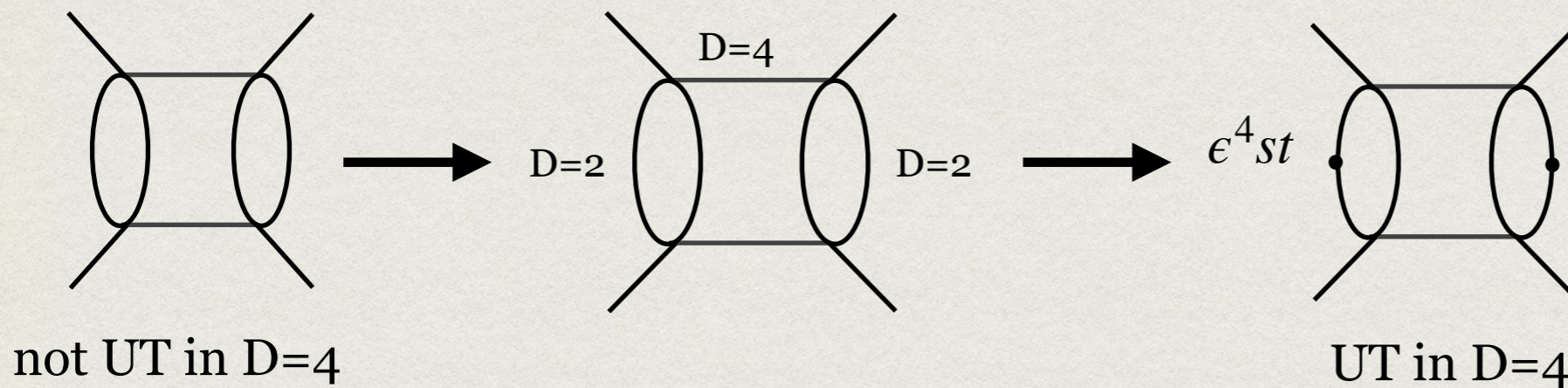
Automated by DlogBasis

[Wasser (2020)]



📌 Uniform transcendental basis :: dlog integrals in particular spacetime dimensions

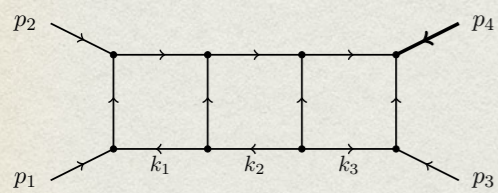
[Flieger, WJT (2022)]



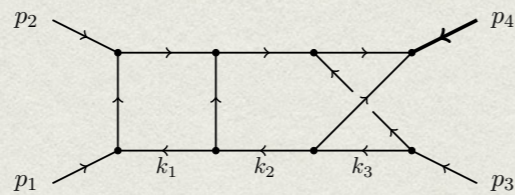
Leading Singularity $\left(\text{---} \left(\text{---} \text{---} \right) \text{---} \right) \sim \frac{1}{p^2}$

Differential equations in canonical form

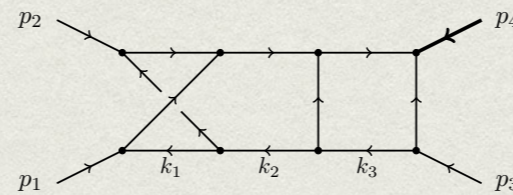
[Henn, Lim, WJT (2023)]



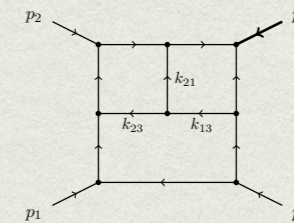
A



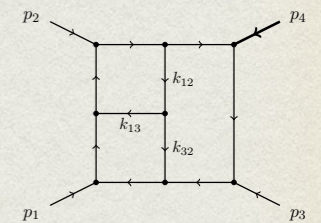
B1



B2



E1



E2

	A	B1	B2	E1	E2
# master integrals	83	150	114	166	117
# dlog integrals	75	124	106	151	116
# UT integrals	8	26	8	15	1
# Letters	6	8	6	6	6

$$d\vec{f}(\vec{x}, \epsilon) = \epsilon (d\tilde{A}) \vec{f}(\vec{x}; \epsilon),$$

$$\text{with } \tilde{A} = \left[\sum_k A_k \log \alpha_k(x) \right]$$

• Total derivative in terms of kinematic invariants
 $s = (p_1 + p_2)^2, t = (p_1 + p_3)^2, p_4^2 \neq 0.$

• Alphabet

$$\vec{\alpha} = \{\alpha_0, \dots, \alpha_8\} = \left\{ p_4^2, s, t, -p_4^2 + s + t, -p_4^2 + s, -p_4^2 + t, s + t, \right. \\ \left. - (p_4^2 - s)^2 + p_4^2 t, s^2 - p_4^2 (s - t) \right\}.$$

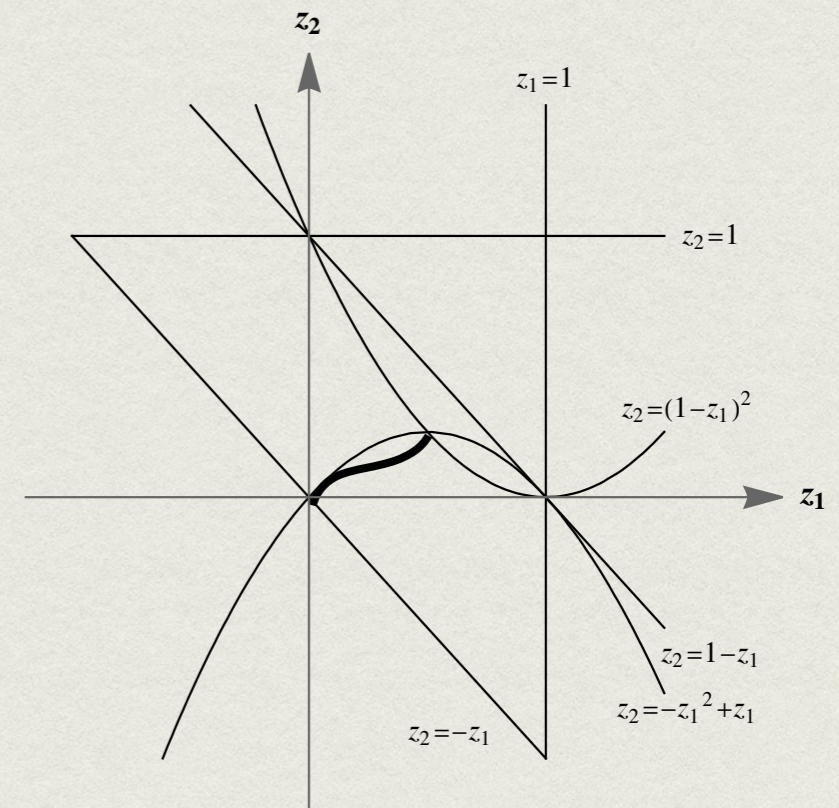
Evaluation of integrals in terms of GPLs

• Solve order-by-order in ϵ in terms of $z_1 = \frac{-s}{-p_4^2}$, $z_2 = \frac{-t}{-p_4^2}$.

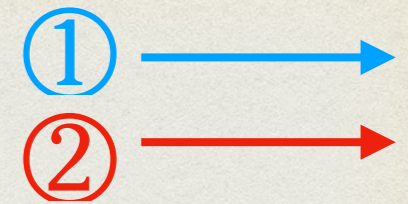
$$d\vec{f}_X = \epsilon \sum_{i=0}^8 \tilde{A}_{X;i} d \log \alpha_i \vec{f}_X$$

up-to transcendental weight six

$$\vec{f}(z_1, z_2; \epsilon) = \mathbb{P} \exp \left(\epsilon \int_{\gamma} d\tilde{A} \right) \vec{f}_0(\epsilon)$$



Evaluation of integrals in terms of GPLs

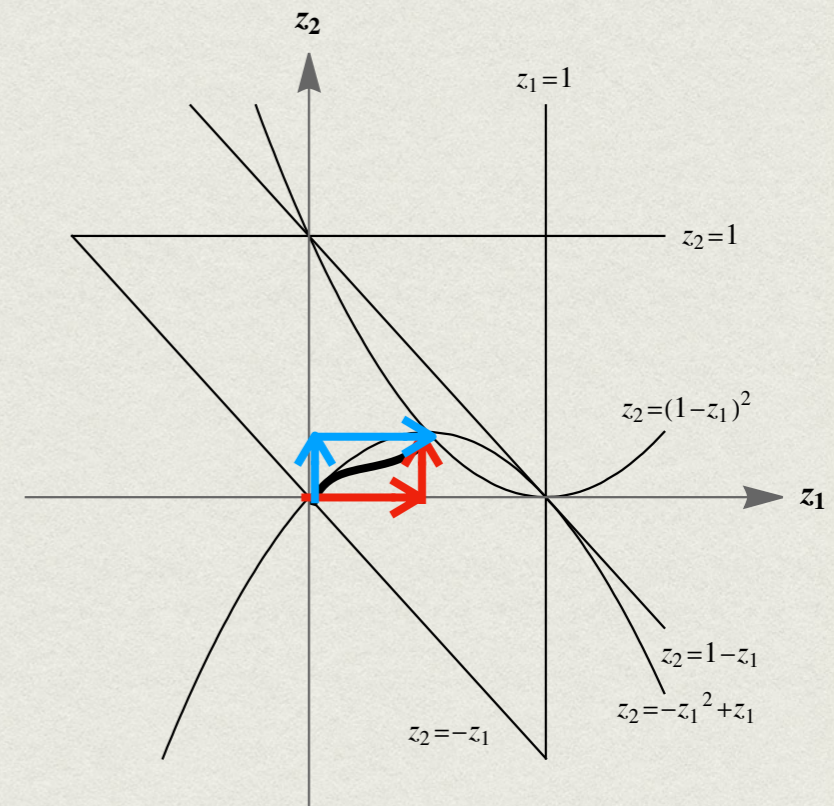


• Solve order-by-order in ϵ in terms of $z_1 = \frac{-s}{-p_4^2}$, $z_2 = \frac{-t}{-p_4^2}$.

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up-to transcendental weight six

$$\vec{f}(z_1, z_2; \epsilon) = \mathbb{P} \exp \left(\epsilon \int_{\gamma} d\tilde{A} \right) \vec{f}_0(\epsilon)$$



• Integration path γ made of two segments:

$$\vec{g}^{(n)}(z_2) = \vec{f}_0^{(n)} + \int_0^{z_2} d\bar{z}_2 \left[A_{z_2}(z_1, \bar{z}_2) \vec{g}^{(n-1)}(\bar{z}_2) - \partial_{\bar{z}_2} \int_0^{z_1} d\bar{z}_1 A_{z_1}(\bar{z}_1, \bar{z}_2) \vec{f}^{(n-1)}(\bar{z}_1, \bar{z}_2) \right]$$

$$\vec{g}^{(n)}(z_1) = \vec{f}_0^{(n)} + \int_0^{z_1} d\bar{z}_1 \left[A_{z_1}(\bar{z}_1, z_2) \vec{g}^{(n-1)}(\bar{z}_1) - \partial_{\bar{z}_1} \int_0^{z_2} d\bar{z}_2 A_{z_2}(\bar{z}_1, \bar{z}_2) \vec{f}^{(n-1)}(\bar{z}_1, \bar{z}_2) \right]$$

Evaluation of integrals in terms of GPLs

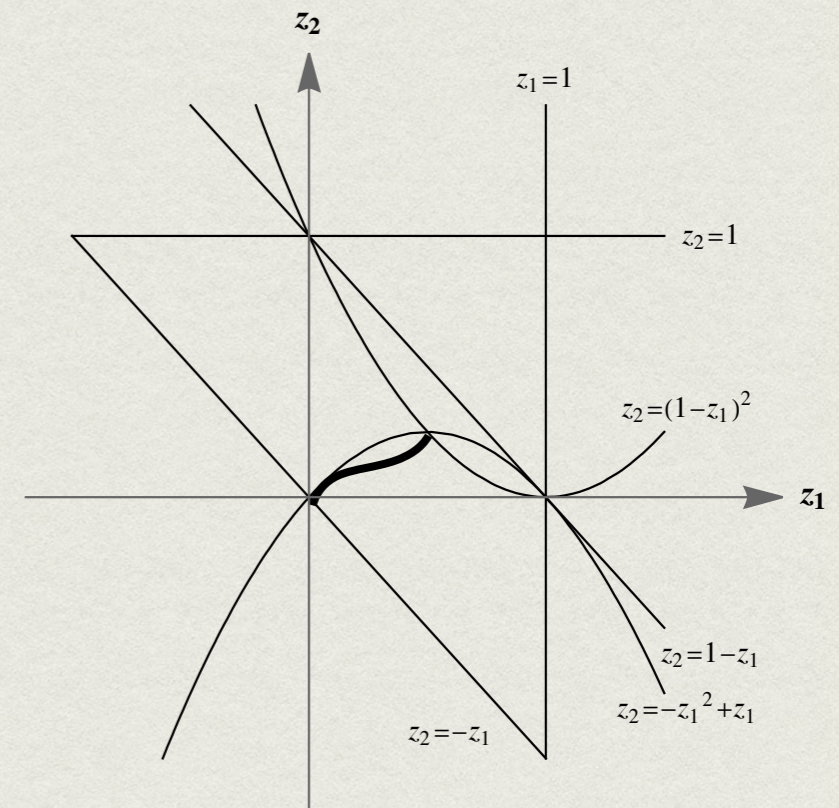
- Solve order-by-order in ϵ in terms of $z_1 = \frac{-s}{-p_4^2}$, $z_2 = \frac{-t}{-p_4^2}$.

$$d\vec{f}_X = \epsilon \sum_{i=0}^8 \tilde{A}_{X;i} d \log \alpha_i \vec{f}_X$$

up-to transcendental weight six

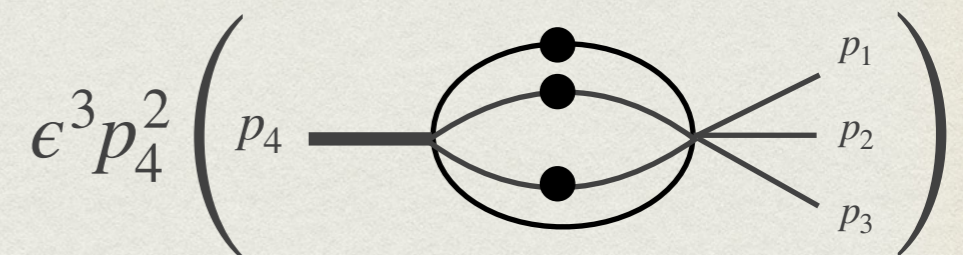
$$\vec{f}(z_1, z_2; \epsilon) = \mathbb{P} \exp \left(\epsilon \int_{\gamma} d\tilde{A} \right) \vec{f}_0(\epsilon)$$

$\vec{f}_0(\epsilon)$ is a boundary vector



- Boundary conditions :: fixed by studying physical and unphysical thresholds.

$$\lim_{\alpha_i \rightarrow 0} \vec{f} = \alpha_i^{\epsilon \tilde{A}_i} \vec{f}(\alpha_i = 0)$$



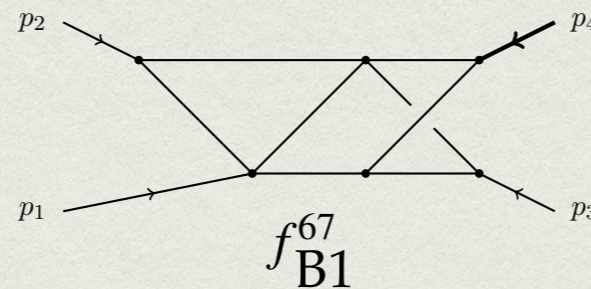
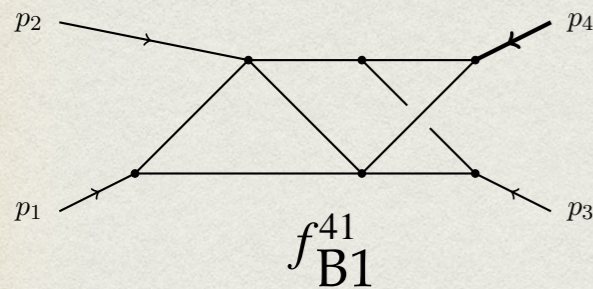
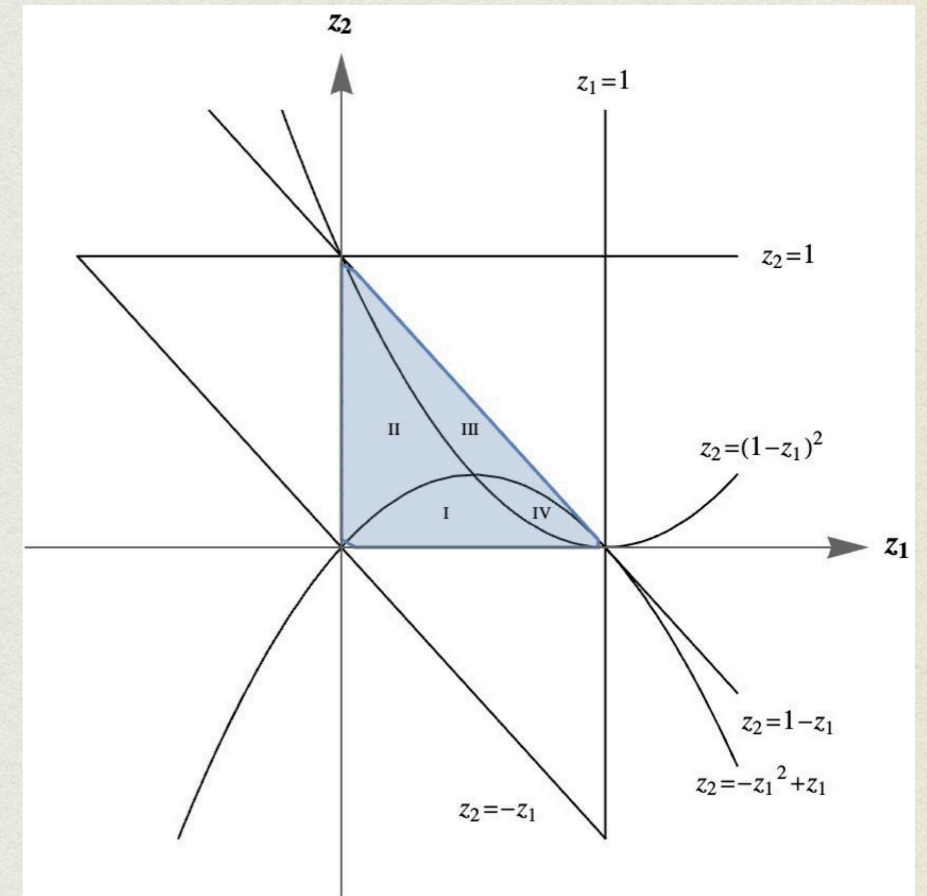
Only one integral needed up-to ϵ^6
(or transcendental weight six)

Numerical validation

• Analytic expression for master integrals evaluated numerically for sample points in the Euclidean (shaded region) region using GiNac through PolyLogTools.

• Perfect agreement w/ PySecDec and FeynTrop.

• Integrals of family B1 are manifestly real-valued in region I.



Integral	Evaluation point	ϵ^3		ϵ^4		ϵ^5		ϵ^6	
		Analytic	PYSECDEC	Analytic	PYSECDEC	Analytic	PYSECDEC	Analytic	PYSECDEC
f_{B1}^{41}	Point 1	0.3768713705	0.37687137(8)	0.2595847621	0.259585(2)	-24.1653497052	-24.1653(2)	-255.4746048147	-255.474(2)
	Point 2	0.0882252953	0.08822531(6)	0.1851070156	0.185107(1)	-3.5650885140	-3.56509(1)	-45.4350139041	-45.4350(2)
f_{B1}^{67}	Point 1	-6.1800769944	-6.1800771(7)	-37.5823284468	-37.58232(7)	-38.4079844011	-38.4080(4)	897.7904682990	897.790(7)
	Point 2	0.3592309958	0.35923099(3)	-1.1083670295	-1.108367(1)	-38.2406764190	-38.2407(1)	-367.9705607540	-367.970(1)

Table 3. Numerical check of integrals f_{B1}^{41} and f_{B1}^{67} against PYSECDEC at the kinematic points: point 1: $\{s, t, p_4^2\} = \{-0.11, -0.73, -1.00\}$, and point 2: $\{s, t, p_4^2\} = \{-0.18, -0.013, -0.25\}$.

New letters in the alphabet

[Henn, Lim, WJT (2023)]

$$\vec{\alpha} = \{\alpha_0, \dots, \alpha_8\} = \left\{ p_4^2, s, t, -p_4^2 + s + t, -p_4^2 + s, -p_4^2 + t, s + t, -\left(p_4^2 - s\right)^2 + p_4^2 t, s^2 - p_4^2 (s - t) \right\}.$$

$$\epsilon^6 \left((p_4^2 - s)^2 - p_4^2 t \right) \left(\begin{array}{c} p_2 \\ \text{---} \\ p_1 \end{array} \right) \longleftrightarrow \epsilon^6 \left((p_4^2 - s)^2 - p_4^2 u \right) \left(\begin{array}{c} p_2 \\ \text{---} \\ p_1 \end{array} \right)$$

$p_1 \leftrightarrow p_2$ symmetry

Start appearing at weight 4

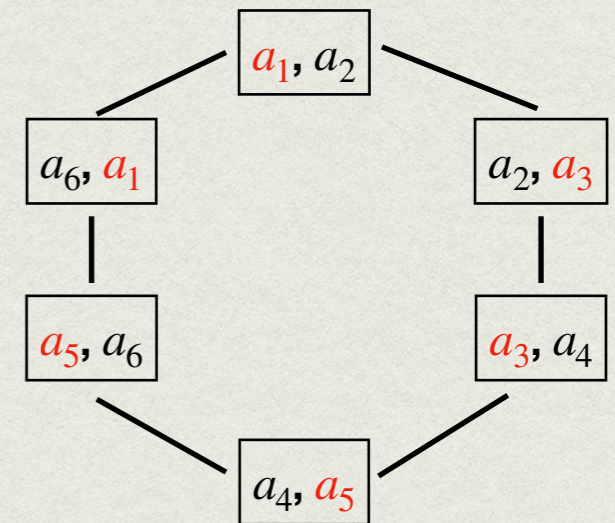
$$\mathcal{S} \left(f_{B1}^{41} \right) \Big|_{\epsilon^4} = 6 \left[\alpha_1 \otimes \alpha_1 \otimes \frac{\alpha_2}{\alpha_4} \otimes \alpha_7 - \alpha_1 \otimes \alpha_1 \otimes \alpha_4 \otimes \alpha_7 + \alpha_1 \otimes \frac{\alpha_4}{\alpha_2} \otimes \frac{\alpha_3}{\alpha_1 \alpha_4} \otimes \alpha_7 \right. \\ \left. + \alpha_2 \otimes \alpha_1 \otimes \frac{\alpha_1 \alpha_4}{\alpha_3} \otimes \alpha_7 + \alpha_2 \otimes \alpha_5 \otimes \frac{\alpha_3}{\alpha_1} - \frac{1}{2} \alpha_2 \otimes \alpha_5 \otimes \alpha_2 \otimes \alpha_7 + \dots \right]$$

Adjacency conditions

[Chicherin, Henn, Papathanasiou (2020)]

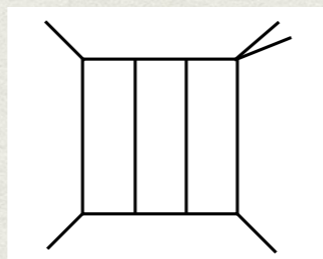
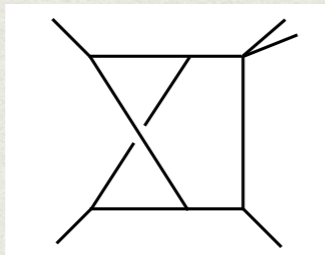
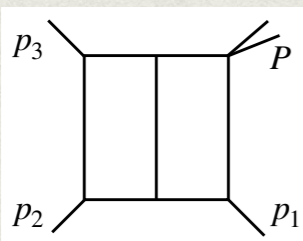
C_2 cluster algebra

$$\Phi_{C_2} = \{a_1, \dots, a_6\} = \left\{ a_1, a_2, \frac{1+a_2^2}{a_1}, \frac{1+a_1+a_1^2}{a_1 a_2}, \frac{1+2a_1+a_1^2+a_2^2}{a_1 a_2^2}, \frac{1+a_1}{a_2} \right\}$$



Connection to loop integrals w/ one massive leg

$$z_1 = -\frac{a_2^2}{1+a_1}, z_2 = -\frac{1+a_1+a_2^2}{a_1(1+a_1)} \quad \longrightarrow \quad \vec{\alpha} = \{z_1, z_2, 1-z_1-z_2, 1-z_1, 1-z_2, z_1+z_2\}$$

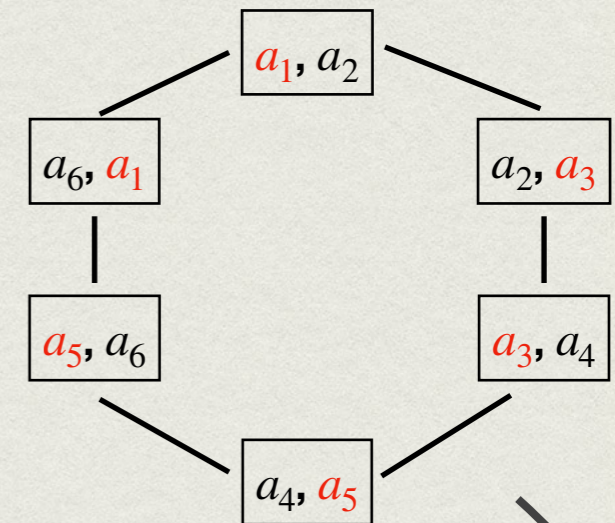


Adjacency conditions

[Chicherin, Henn, Papathanasiou (2020)]

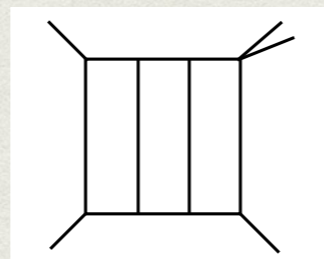
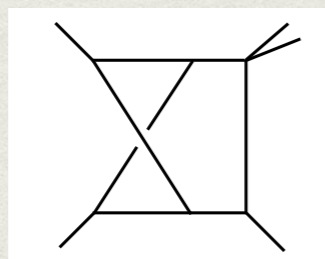
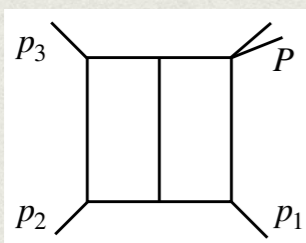
C_2 cluster algebra

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Connection to loop integrals w/ one massive leg

$$z_1 = -\frac{a_2^2}{1+a_1}, z_2 = -\frac{1+a_1+a_2^2}{a_1(1+a_1)} \longrightarrow \vec{\alpha} = \{z_1, z_2, 1-z_1-z_2, 1-z_1, 1-z_2, z_1+z_2\}$$



$$\tilde{A} = \sum_i \tilde{A}_i \log \alpha_i(\vec{z})$$

$$\tilde{A}_i \cdot \tilde{A}_j = 0 \implies \dots \otimes \alpha_i \otimes \alpha_j \otimes \dots$$

for $i, j \in \{4, 5, 6\}$ with $i \neq j$

Partially checked for tennis-court like diagrams (E1 & E2).

Adjacency conditions

[Henn, Lim, WJT (2023)]

$$d\vec{f}(\vec{z}; \epsilon) = \epsilon \left[\sum_i A_i d \log \alpha_i(\vec{z}) \right] \vec{f}_0(\epsilon)$$

$$\vec{\alpha} = \{\alpha_0, \dots, \alpha_8\} = \left\{ p_4^2, s, t, -p_4^2 + s + t, -p_4^2 + s, -p_4^2 + t, s + t, -(p_4^2 - s)^2 + p_4^2 t, s^2 - p_4^2 (s - t) \right\}.$$

From the C_2 cluster algebra one expects

$$A_i \cdot A_j = 0 \implies \dots \otimes \alpha_i \otimes \alpha_j \otimes \dots \quad \text{for } i, j \in \{4, 5, 6\} \text{ with } i \neq j$$

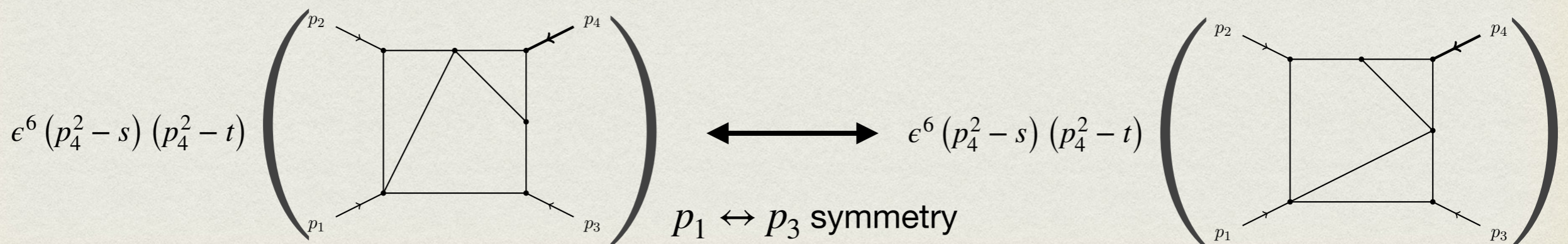
Satisfied for families A, B1, B2, E2.

Family E1

$$\tilde{A}_4 \cdot \tilde{A}_6 = \tilde{A}_6 \cdot \tilde{A}_4 = \tilde{A}_5 \cdot \tilde{A}_6 = \tilde{A}_6 \cdot \tilde{A}_5 = 0$$

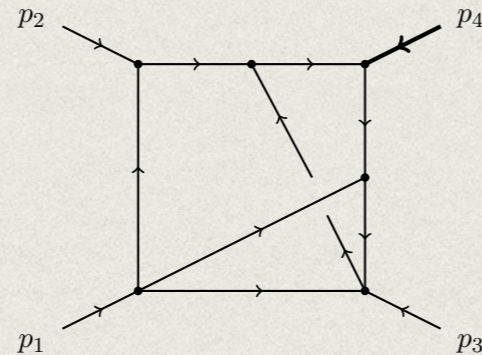
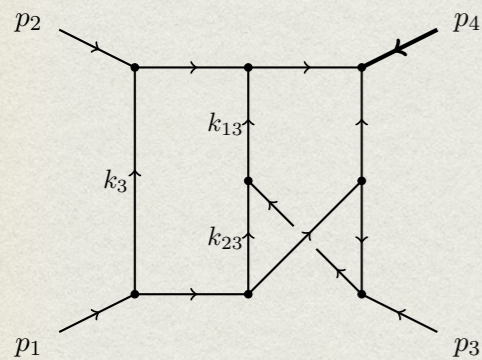
$$\tilde{A}_4 \cdot \tilde{A}_5 \neq 0 \text{ and } \tilde{A}_5 \cdot \tilde{A}_4 \neq 0 \implies \dots \otimes \alpha_4 \otimes \alpha_5 \otimes \dots$$

Start appearing at weight 5.



Preliminary observations

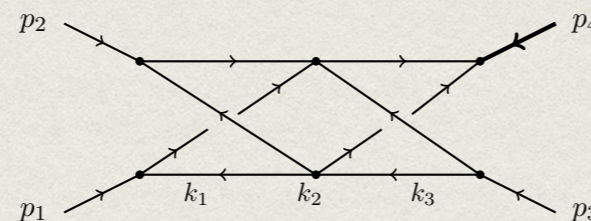
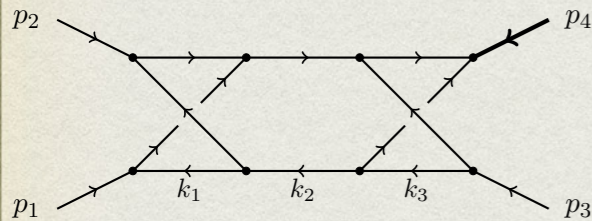
Appearance of more letters in subsectors



[Henn, Lim, WJT (work in progress)]

$$\{-(p_4^2)^2 + (p_4^2 + s)t\}$$

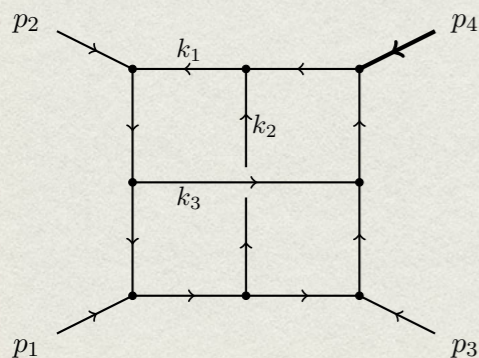
canonical DEQ :: Done



$$\left\{ \frac{-p_4^2 t - \sqrt{-p_4^2 s t (p_4^2 - s - t)}}{-p_4^2 t + \sqrt{-p_4^2 s t (p_4^2 - s - t)}}, \frac{st - \sqrt{-p_4^2 s t (p_4^2 - s - t)}}{st + \sqrt{-p_4^2 s t (p_4^2 - s - t)}} \right\}$$

Interesting families

[Henn, Lim, WJT + Canko, Syrrakos, Tancredi (work in progress)]



- * 371 MIs (FIRE6 & FiniteFlow).
- * 19 MIs in top sector.
- * DEQ on Maximal cut :: easy.
- * Canonical DEQ ?

Conclusions

• We have reached:

- Computed first non-planar families and revisited planar families for three-loop integrals with one massive leg.
- found new letters in the process of computing non-planar families.
- found counterexample of adjacency condition for tennis-court type integral.

• Open questions & future directions

- Complete three-loop non-planar integral families; unravel function space.
- Compute three-loop scattering amplitudes for Higgs/Vector boson plus jet production.
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Non-planar three-loop integrals

