Fermilab

Alexander von Humboldt
Stiftung/Foundation

Alaric: A NLL accurate Parton Shower algorithm

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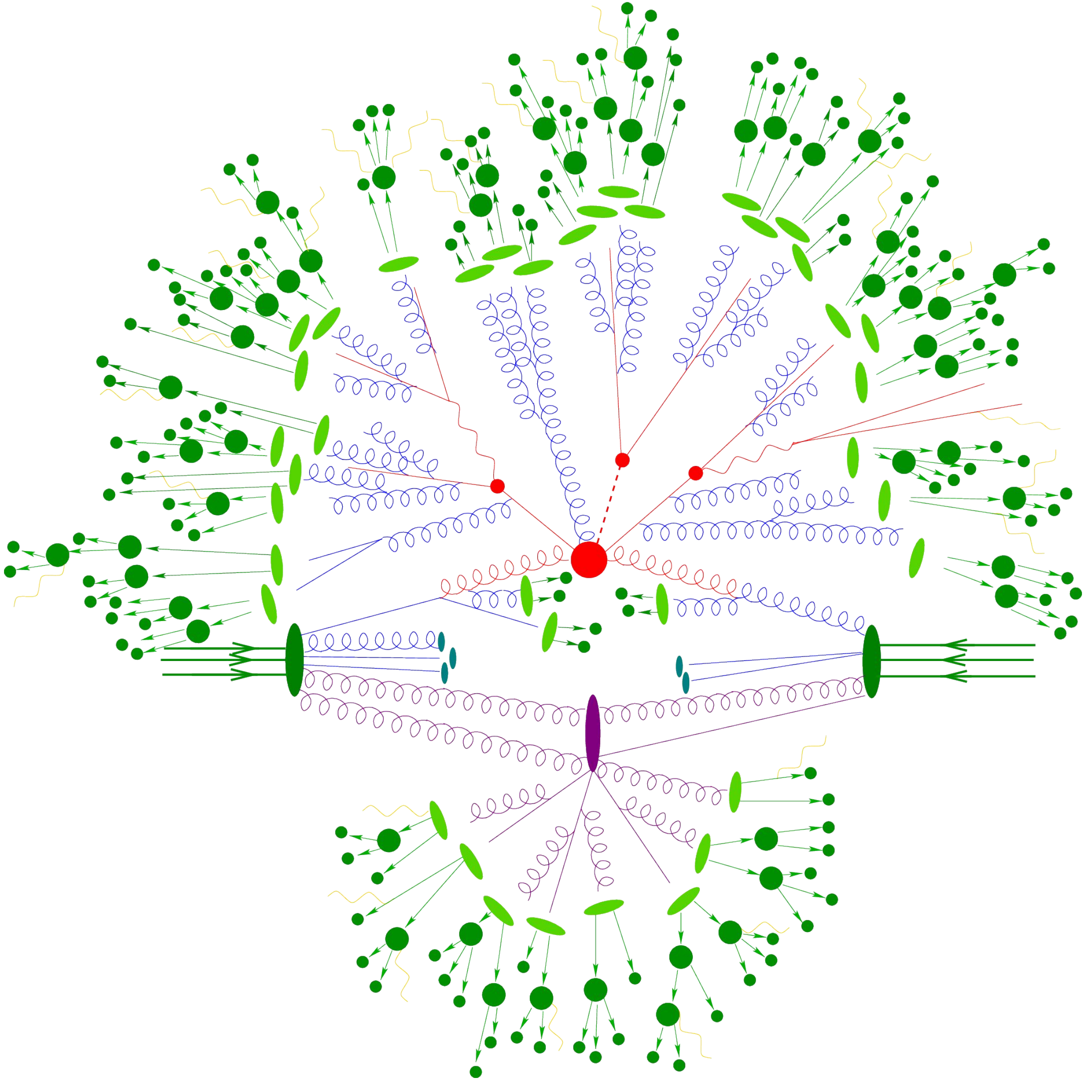
2208.06057, in collaboration with Stefan Höche, Frank Krauss, Daniel Reichelt & Marek Schönherr

Event Generators

Crucial for precision Collider Physics

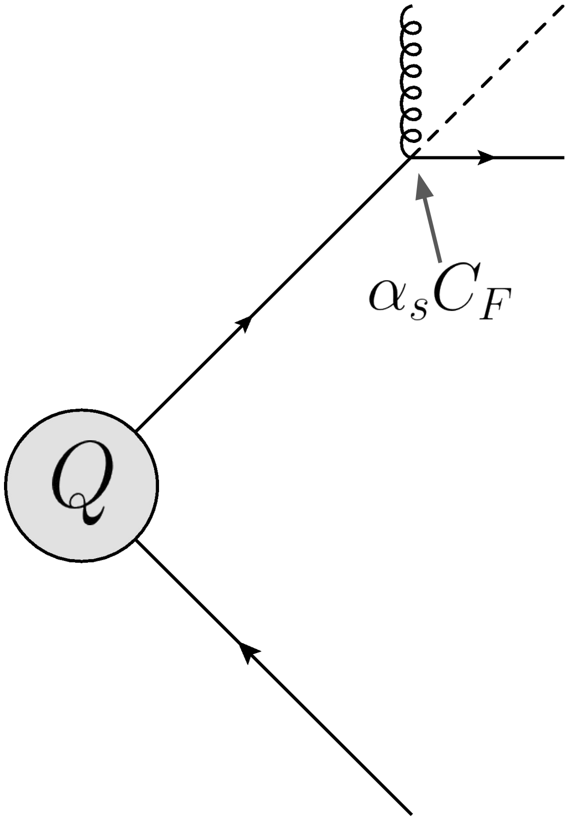
Combine different physics at different scales:

- Hard Process
- Parton Shower
- Underlying Interaction
- Hadronization
- QED FSR
- Hadron Decays



Parton Showers

Here: Leading colour



Gluon emission most likely in singular limits!

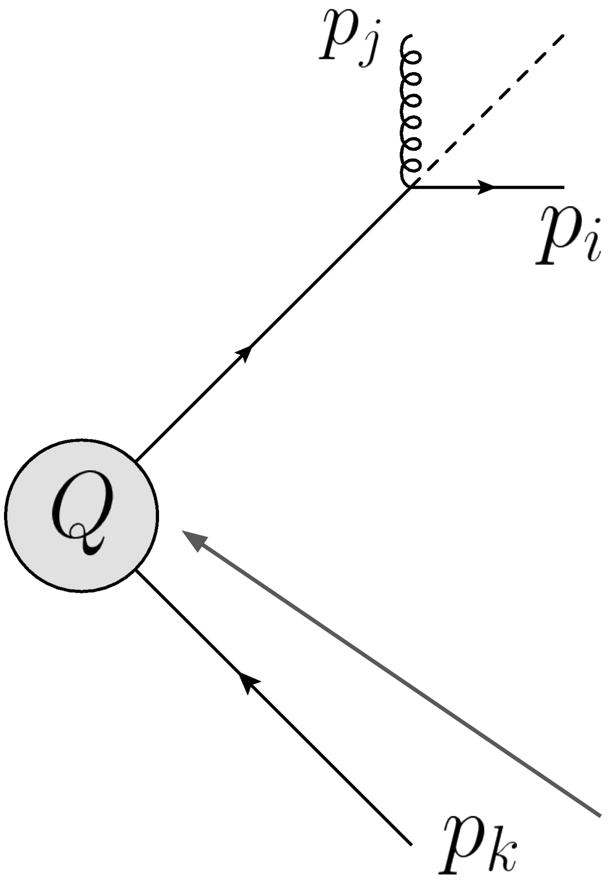
Depends on both dipole members

Gluon soft: $|\mathcal{M}_{qqg}|^2 \approx 2\frac{p_i p_k}{(p_i p_j)(p_j p_k)}|\mathcal{M}_{qq}|^2$

Gluon collinear: $|\mathcal{M}_{qqg}|^2 \approx \frac{P_{qq}(z)}{2(p_i p_j)}|\mathcal{M}_{qq}|^2$

Depends only on radiating parton

Parton Showers



Need momentum mapping between on-shell momenta:

$$\tilde{p}_i, \tilde{p}_k \Rightarrow p_i, p_k, p_j$$

Conditions:

$p_i \to z\tilde{p}_i$

$p_j \to (1-z)\tilde{p}_i$

in collinear limit, and

$Q^2 = \tilde{Q}^2$

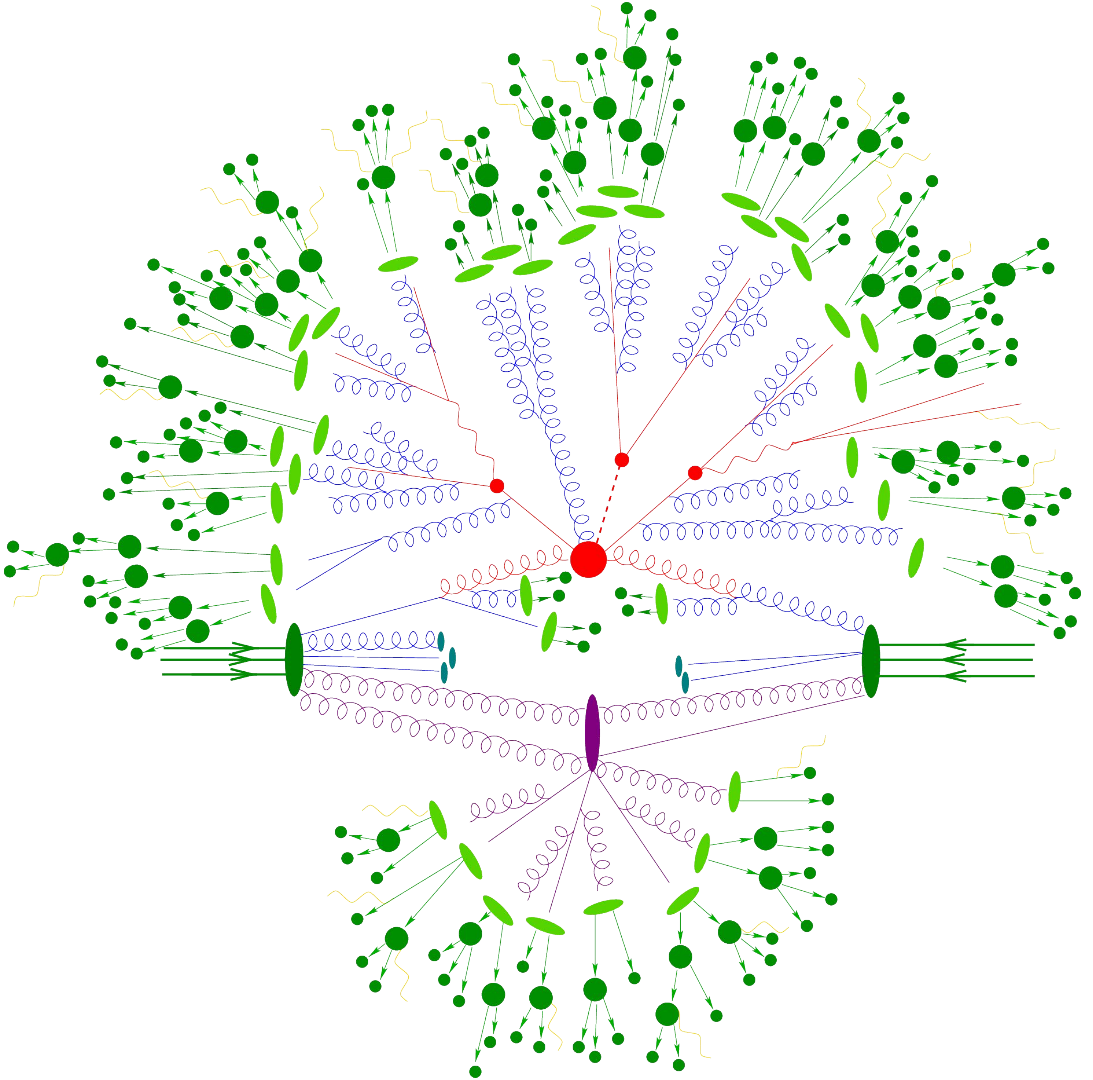
Also $\tilde{Q}$ may change

NLL Showers

Criteria for NLL accuracy:

- Generate correct square tree-level ME when one kinematic variable (e.g. $k_\perp$) for two emissions differ significantly and another one is similar (e.g. η)
- Reproduce NLL results for rIRC safe observables → Subsequent Emissions don't change previous ones significantly

[Dasgupta,Dreyer,Hamilton,Monni,Salam,Soyez] 2002.11114



Soft Radiation

Factorisation in the soft limit:

$$ {}_n\langle 1,\ldots,n|1,\ldots,n\rangle_n = -8\pi\alpha_s \sum_{i,k\neq j} {}_{n-1}\langle 1,\ldots,\not{j},\ldots,n\big|\mathbf{T}_i\mathbf{T}_k\, w_{ik,j}\big|1,\ldots,\not{j},\ldots,n\rangle_{n-1} $$

Eikonal factor:

$$ w_{ik,j} = \frac{p_i p_k}{(p_i p_j)(p_j p_k)} = \frac{1}{E_j^2}\frac{1-\cos\theta_{ik}}{(1-\cos\theta_{ij})(1-\cos\theta_{jk})} $$

Implementing the Eikonal in the collinear limit leads to double-counting of soft singularities

[Marchesini, Webber] *Nucl.Phys.B* 310 (1988) 461-526

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$W_{ik,j}$

Additive matching of singularities:

$$ W_{ik,j} = \tilde{W}^i_{ik,j} + \tilde{W}^k_{ki,j} $$

$$ \tilde{W}^i_{ik,j} = \frac{1}{2}\left(W_{ik,j} + \frac{1}{1-\cos\theta_{ij}} - \frac{1}{1-\cos\theta_{jk}}\right) $$

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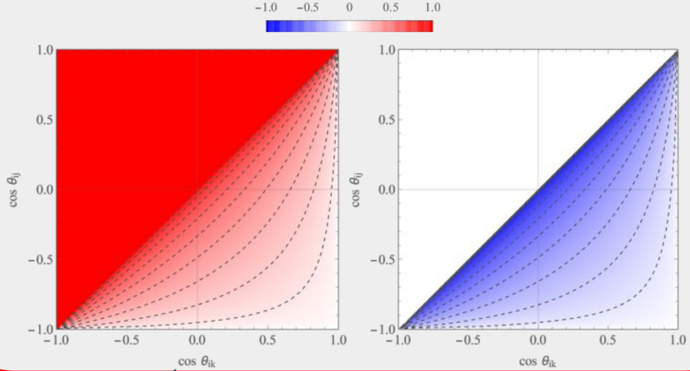
Option 2: Implement radiator differentially



Option 1:
Angular Ordering → Spoils NGLs

Azimuthal average:

$$ \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi^i_{jk}\tilde{W}^i_{ik,j} = \frac{\theta(\theta_{ik}-\theta_{ij})}{1-\cos\theta_{ij}} $$

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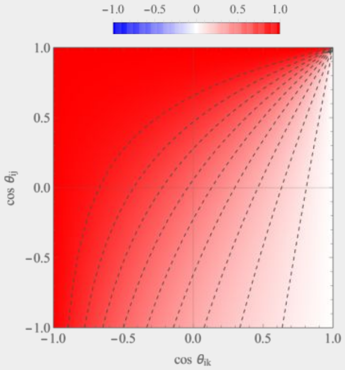
Multiplicative matching of singularities:

$$ W_{ik,j} = \bar{W}^i_{ik,j} + \bar{W}^k_{ki,j} $$

$$ \bar{W}^i_{ik,j} = W_{ik,j}\frac{1-\cos\theta_{jk}}{2-\cos\theta_{ij}-\cos\theta_{jk}} $$

[Catani, Seymour] hep-ph/9605323

Implement radiator differentially



Azimuthal average:

$$ \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi^i_{jk}\bar{W}^i_{ik,j} = \frac{1}{\sqrt{(A^i_{ik,j})^2-(B^i_{ik,j})^2}} $$

$$ A^i_{ij,k} = \frac{2-\cos\theta_{ij}(1+\cos\theta_{ik})}{1-\cos\theta_{ik}} $$

$$ B^i_{ij,k} = \frac{\sqrt{(1-\cos^2\theta_{ij})(1-\cos^2\theta_{ik})}}{1-\cos\theta_{ik}} $$

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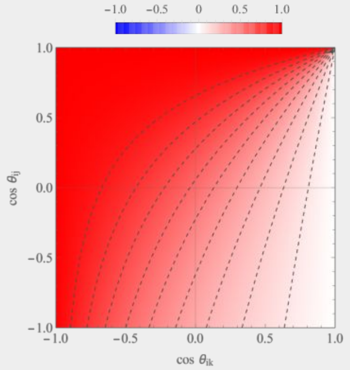


$$ \frac{1}{2p_ip_j}P_{(ij)i}(z) \to \frac{1}{2p_ip_j}P_{(ij)i}(z)+\delta_{(ij)i}\left[\frac{\bar{W}^i_{ik,j}}{E_j^2} - w^{(\mathrm{coll})}_{ik,j}(z)\right] $$

Splitting functions depend on azimuthal direction of spectator, strictly positive

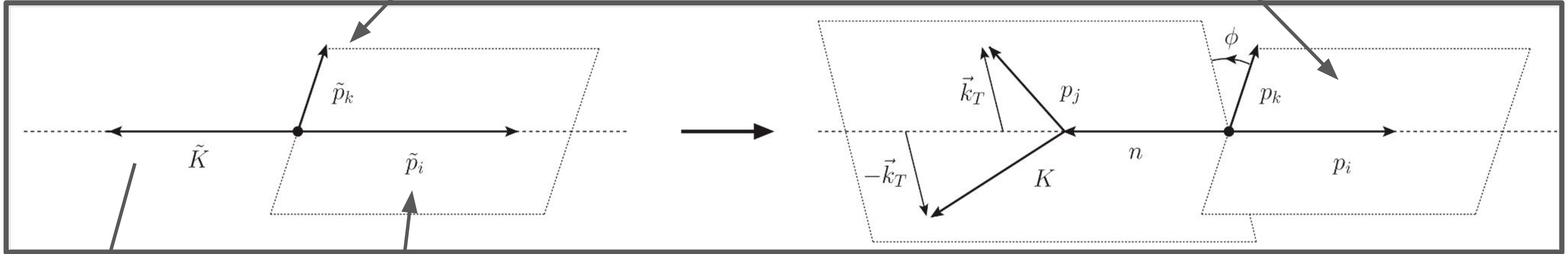
Momentum Mapping

Main Idea:
maintain directions of hard particles exactly

$$p_i = z\tilde{p}_i \qquad z = \frac{p_i n}{(p_i + p_j)n}$$

$$p_k = \tilde{p}_k$$

Colour Spectator



Colour neutral System

Emitter

Need to find K and p_j such that:

$$K^2 = \tilde{K}^2 \quad p_j \to (1-z)\tilde{p}_i$$

Shift:
$n = \tilde{K} + (1-z)\tilde{p}_i$

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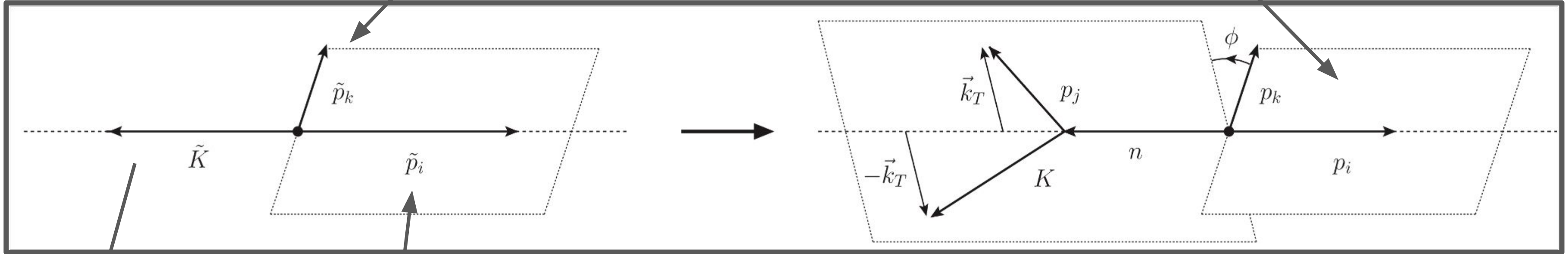

Colour neutral System

Emitter

$$v = \frac{p_i p_j}{p_i \tilde{K}} \qquad \kappa = \frac{\tilde{K}^2}{2\tilde{p}_i \tilde{K}}$$

$$p_j = (1-z)\tilde{p}_i + v\left(\tilde{K} - (1-z+2\kappa)\tilde{p}_i\right) + k_\perp$$
$$K = \tilde{K} - v\left(\tilde{K} - (1-z+2\kappa)\tilde{p}_i\right) - k_\perp$$

Recoil distributed to remaining momenta through Lorentz Transformation:

$$p_l^\mu \rightarrow \Lambda^\mu_\nu(K, \tilde{K}) p_l^\nu$$

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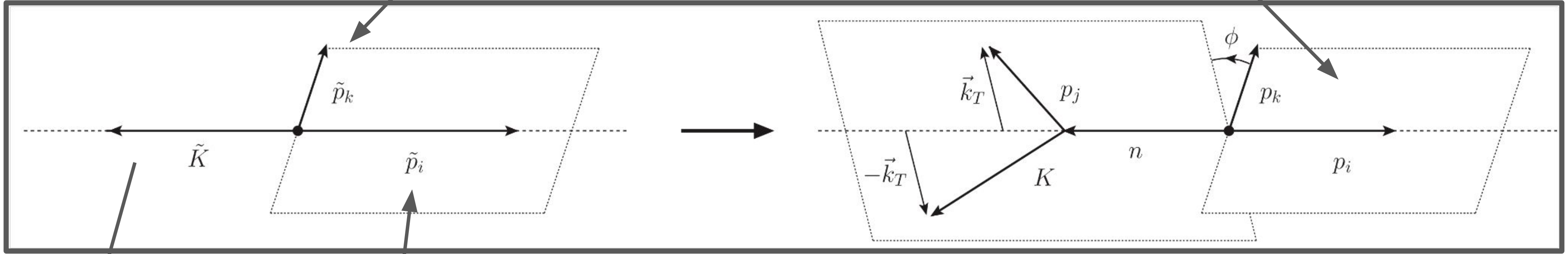

Colour neutral System

Emitter

$$v = \frac{p_i p_j}{p_i \tilde{K}} \qquad \kappa = \frac{\tilde{K}^2}{2\tilde{p}_i \tilde{K}}$$

Evolution variable:
$t = 2E_j^2(1 - \cos\theta_j^i) = v(1-z)2\tilde{p}_i\tilde{K}$
Corresponds to Lund plane $k_t^2 \rightarrow \beta_{\mathrm{PS}} = 0$

Recoil distributed to remaining momenta through Lorentz Transformation:

$$p_l^\mu \rightarrow \Lambda_\nu^\mu(K, \tilde{K}) p_l^\nu$$

Recoil

Recoil distributed to remaining momenta through Lorentz Transformation:

$$p_l^\mu \to \Lambda^\mu_\nu(K, \tilde{K}) p_l^\nu$$

Define

$$X^\mu = p_j^\mu - (1-z)\,\tilde{p}_i^\mu$$
$$= v\big(\tilde{K}^\mu - (1 - z + 2\kappa)\,\tilde{p}_i^\mu\big) + k_\perp^\mu$$

At most $\mathcal{O}(k_\perp)$ in logarithmically enhanced region

Recoil

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$$\Lambda^\mu{}_\nu(K, \tilde{K}) = g^\mu_\nu + \tilde{K}^\mu A_\nu + X^\mu B_\nu$$

$$A^\nu = 2\left[\frac{(\tilde{K}-X)^\nu}{(\tilde{K}-X)^2} - \frac{(\tilde{K}-X/2)^\nu}{(\tilde{K}-X/2)^2}\right] \qquad B^\nu = \frac{(\tilde{K}-X/2)^\nu}{(\tilde{K}-X/2)^2}$$

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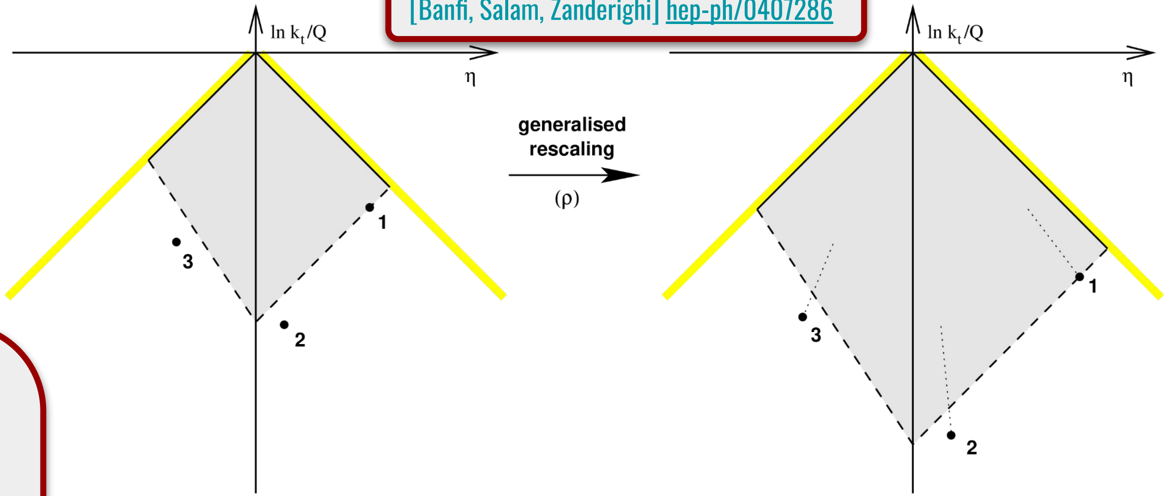
Suppressed by

$$\mathcal{O}(k_\perp / K)$$

$$\Lambda^\mu_\nu \approx g^\mu_\nu + \frac{K_\rho X_\sigma}{K^2} T^{\mu\rho\sigma}_\nu + \mathcal{O}(k_\perp^2)$$

Recoil

[Banfi, Salam, Zanderighi] hep-ph/0407286



For one emission, kinematic variables in the Lund plane scale like:

$$k_{t,l} \to k'_{t,l} = k_{t,l}\rho^{(1-\xi_l)/a+\xi_l/(a+b)}$$

$$\eta_l \to \eta'_l = \eta - \xi_l \frac{\ln \rho}{a+b}$$

$$\xi_l = \frac{\eta_l}{\eta_{l,\max}}$$

where a = 1 and b = 0 for Alaric

Working in the rest frame of the color dipole, the other momenta scale like:

$$\tilde{p}_l^0 \sim \rho^{1-\xi_l}$$

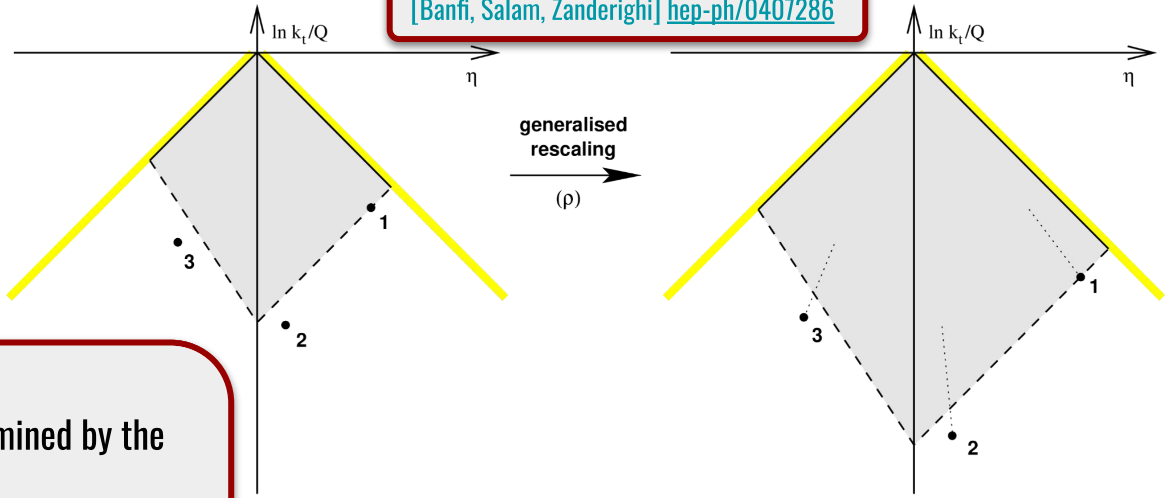
$$\tilde{p}_l^{1,2} \sim \rho$$

$$\tilde{p}_l^3 \sim \rho^{1-\xi_l}$$

for $\rho \to 0$

Recoil

[Banfi, Salam, Zanderighi] hep-ph/0407286



Scaling under an additional emission is determined by the Lorentz transformation in the limit $\rho \to 0$:

$$\Delta p_l^\mu = 2\,\frac{\tilde{K}X}{\tilde{K}^2}\frac{\tilde{p}_l\tilde{K}}{\tilde{K}^2}\,\tilde{K}^\mu - \frac{\tilde{p}_l X}{\tilde{K}^2}\,\tilde{K}^\mu + \frac{\tilde{p}_l\tilde{K}}{\tilde{K}^2}\,X^\mu$$

Scaling becomes:

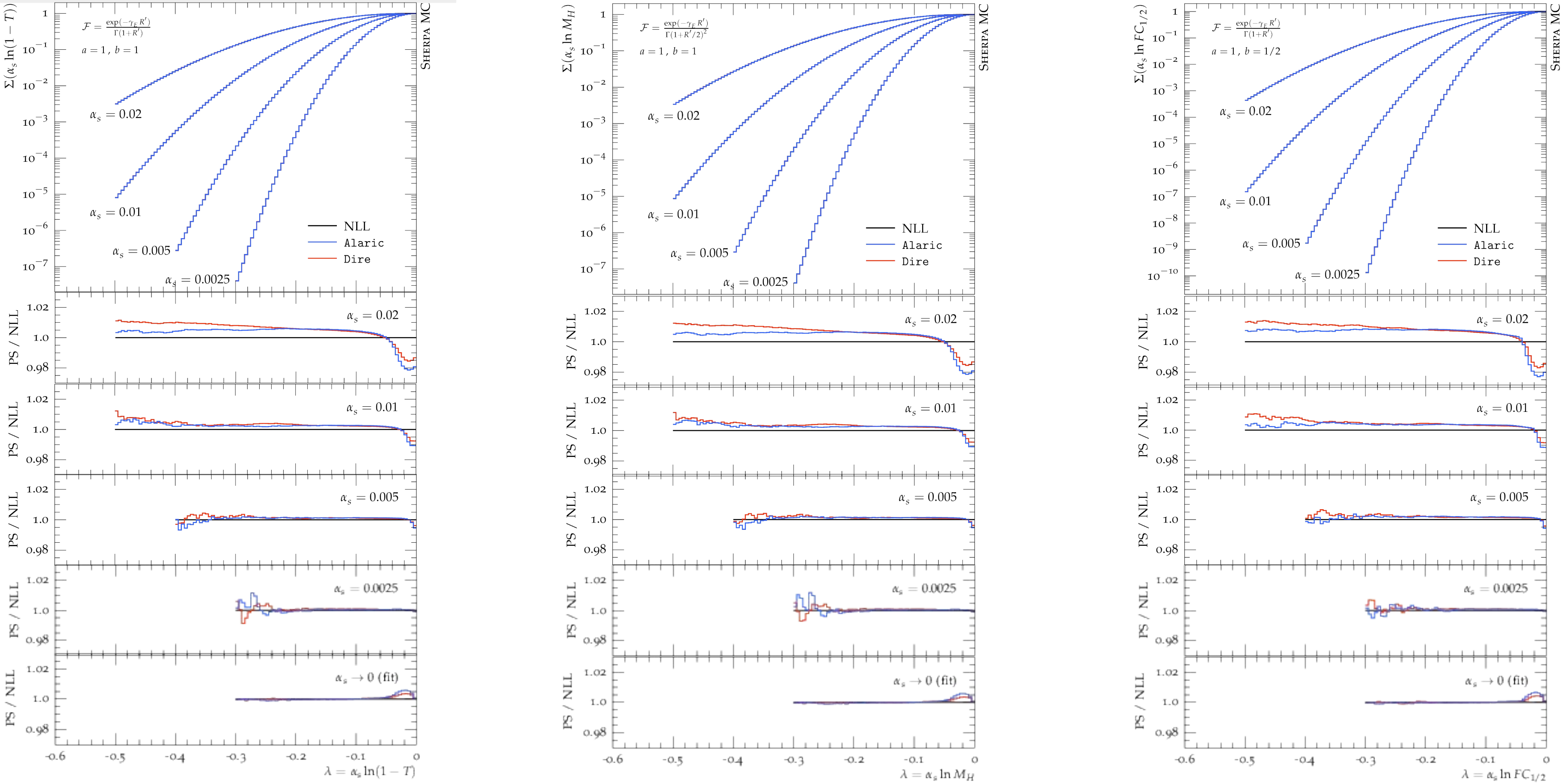
$$\Delta p_l^0 \sim \;\rho^{1-\xi_l}X^0 + \rho^{2-\xi_l-\max(\xi_i,\xi_j)}\tilde{K}^0 + \rho^{1-\xi_l}X^0 \;\sim \rho^{2-\xi_l-\max(\xi_i,\xi_j)} \qquad \tilde{p}_l^0 \sim \;\rho^{1-\xi_l}$$

$$\Delta p_l^{1,2} \sim \;\rho^{1-\xi_l}X^{1,2} \sim \rho^{2-\xi_l} \qquad \tilde{p}_l^{1,2} \sim \;\rho$$

$$\Delta p_l^3 \sim \;\rho^{1-\xi_l}X^3 \sim \rho^{2-\xi_l-\max(\xi_i,\xi_j)} \qquad \tilde{p}_l^3 \sim \;\rho^{1-\xi_l}$$

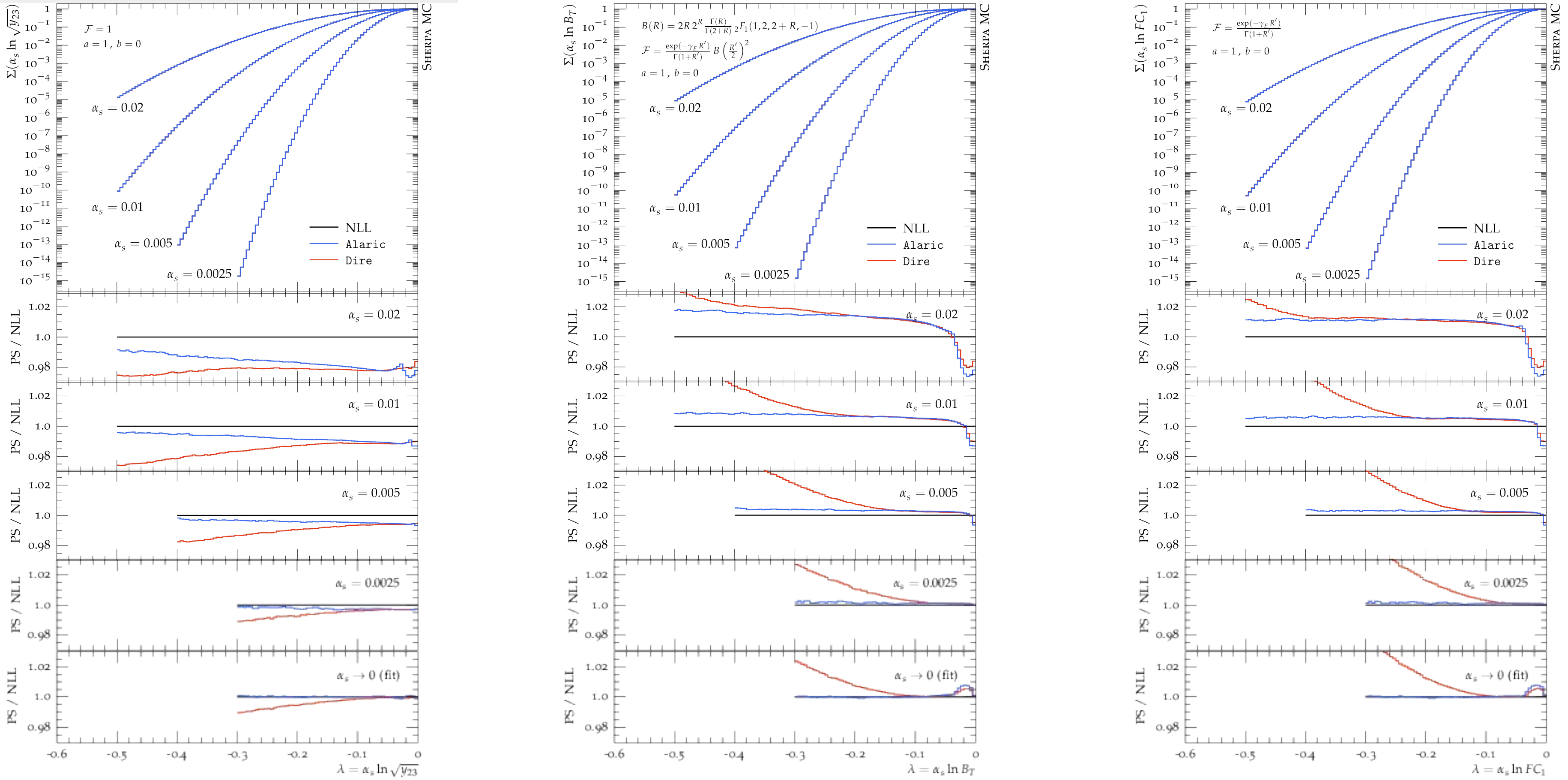
Numerical Tests

For Thrust, Heavy Jet mass and Fractional Energy Correlators with b = 1, Alaric is NLL and Dire is indistinguishable from NLL



Numerical Tests

For the Two-Jet rate, total Broadening and FC with b = 1
Alaric and Dire differ, here only Alaric is NLL accurate



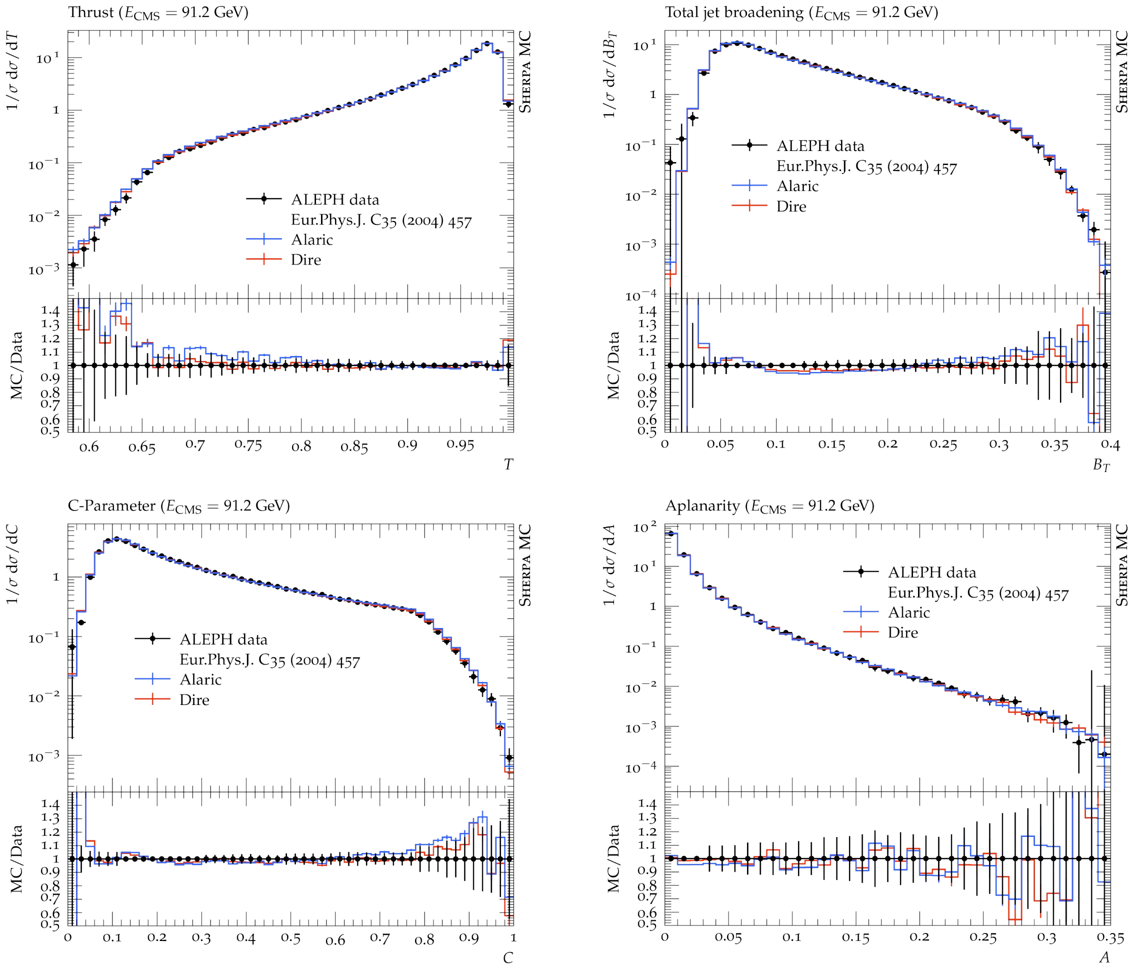
Let's look at Data

Details:

- CMW scheme
- Massless b- and c-quarks
- Flavour thresholds
- Hadronization through Lund string fragmentation

Comments:

- Perturbative region to the right, except for thrust
- Some deviations for Broadening and Aplanarity
- MECs and massive quarks will improve situation



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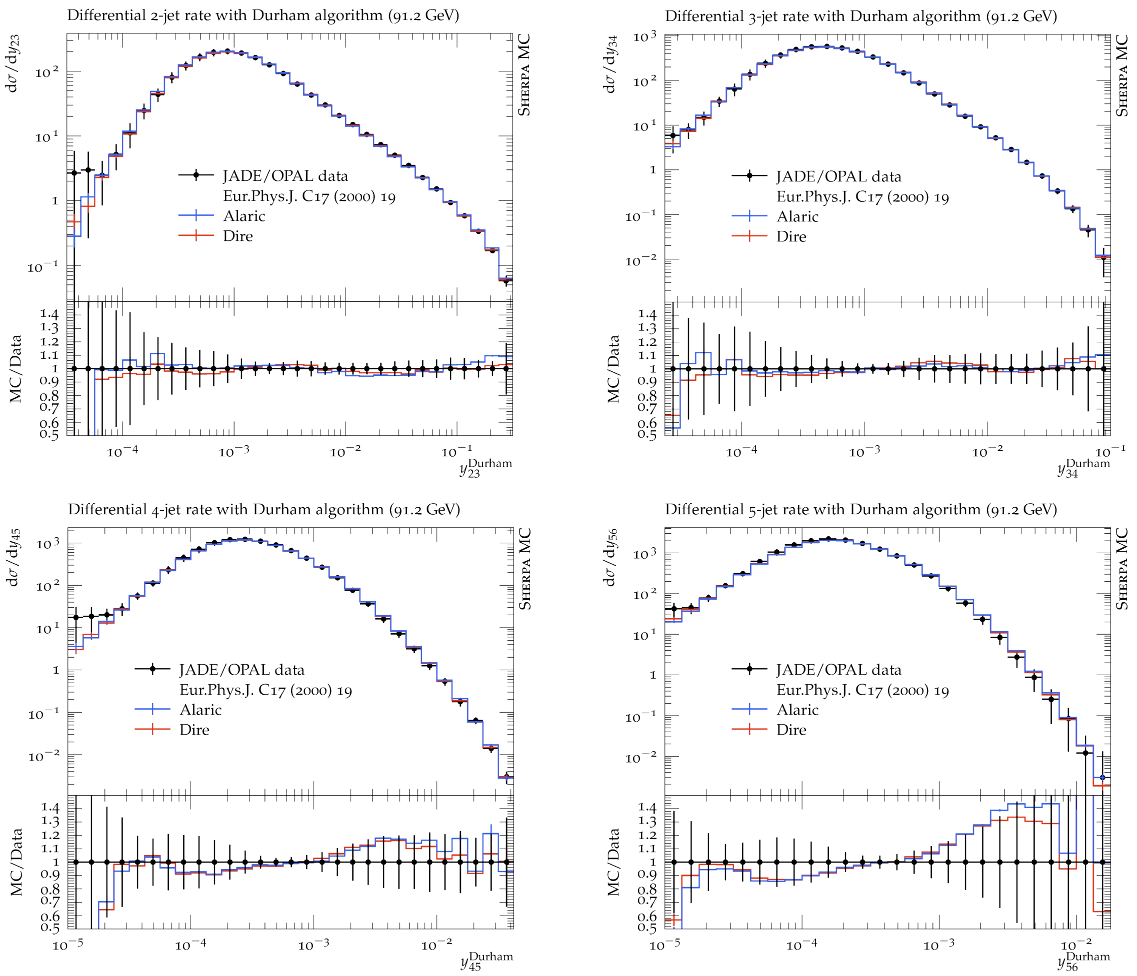
- b-quark mass corresponds to

$$y \approx 2.8 \times 10^{-3}$$



MC@NLO Matching

$$\sigma^{(\mathrm{NLO})} = \int \mathrm{d}\Phi_n \left[B + V + \int \mathrm{d}\Phi_{+1} S\right] + \int \mathrm{d}\Phi_{n+1} \left[R - S\right]$$

If we compare NLO calculation and PS expanded to first order in the strong coupling:

- NLO calculation: contains virtual corrections, one hard, soft or collinear emission
- PS: contains one soft or collinear emission virtual corrections approximated (unitarity)

$\longrightarrow$ Double counting of soft and collinear radiation!

In the MC@NLO scheme, the subtraction terms are chosen to be the PS evolution kernels $\longrightarrow$ Need to compute integrated terms with our momentum mapping, but the matching is fully differential!
[Frixione, Webber] hep-ph/0204244

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Evaluating the soft counterterm tells us something about efficient scale choices. In our case we obtain:

$$I_{\mathrm{soft}} \propto \left(\frac{\mu^2 (p_k n)}{(p_i p_k)(p_i p_n)}\right)^{\epsilon} \propto \left(\frac{\mu^2}{E_i^2 (1 - \cos\theta_k^i)}\right)^{\epsilon} \propto \left(\frac{\mu^2}{t}\right)^{\epsilon}$$

→ The logarithms resummed by the RG-evolution are large when the soft parton is emitted from a Dipole that originates from a soft or collinear splitting of k → k+i and correspond to the respective $k_\perp$

→ We can minimize the number of explicit higher-order corrections by choosing t as the renormalization scale

Conclusion

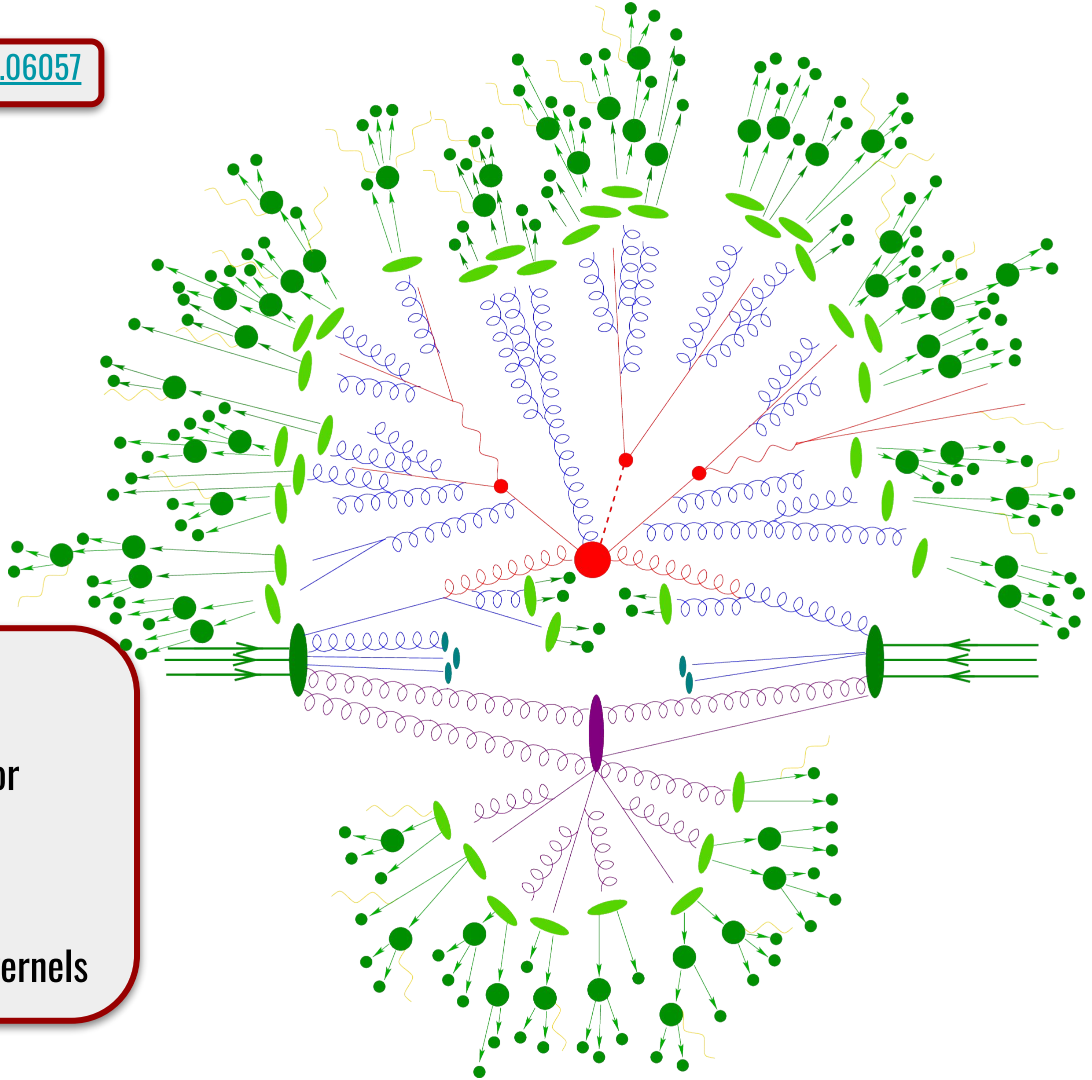
2208.06057

- New NLL accurate Parton Shower Algorithm: Alaric
- First dipole-like algorithm to disentangle colour and kinematics
- Strict positivity of evolution kernels
- Momentum mapping preserves directions of hard partons

Conceptually clear but need to be implemented:

- Initial state emitter and spectator
- Initial state emitter and final state spectator
- NLO matching

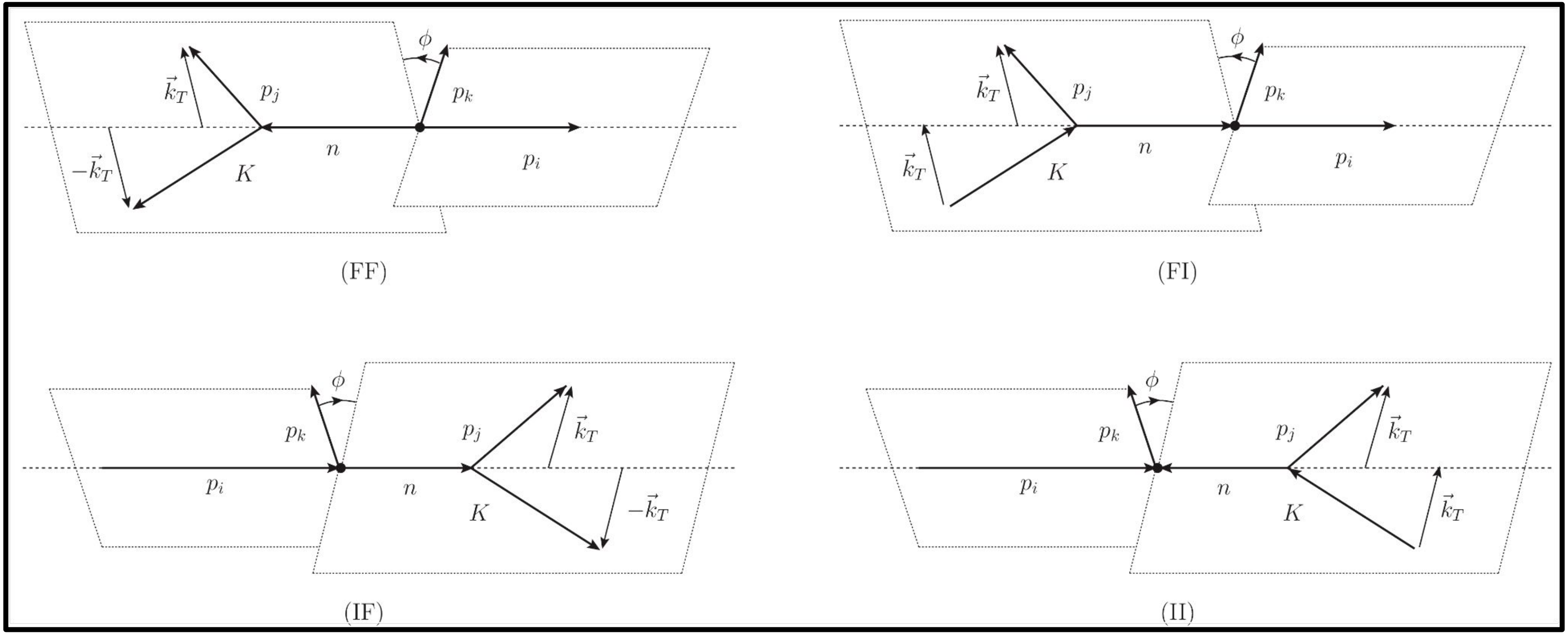
Next goals:
Massive quarks (see Benoit's talk), NLO splitting kernels



Backup

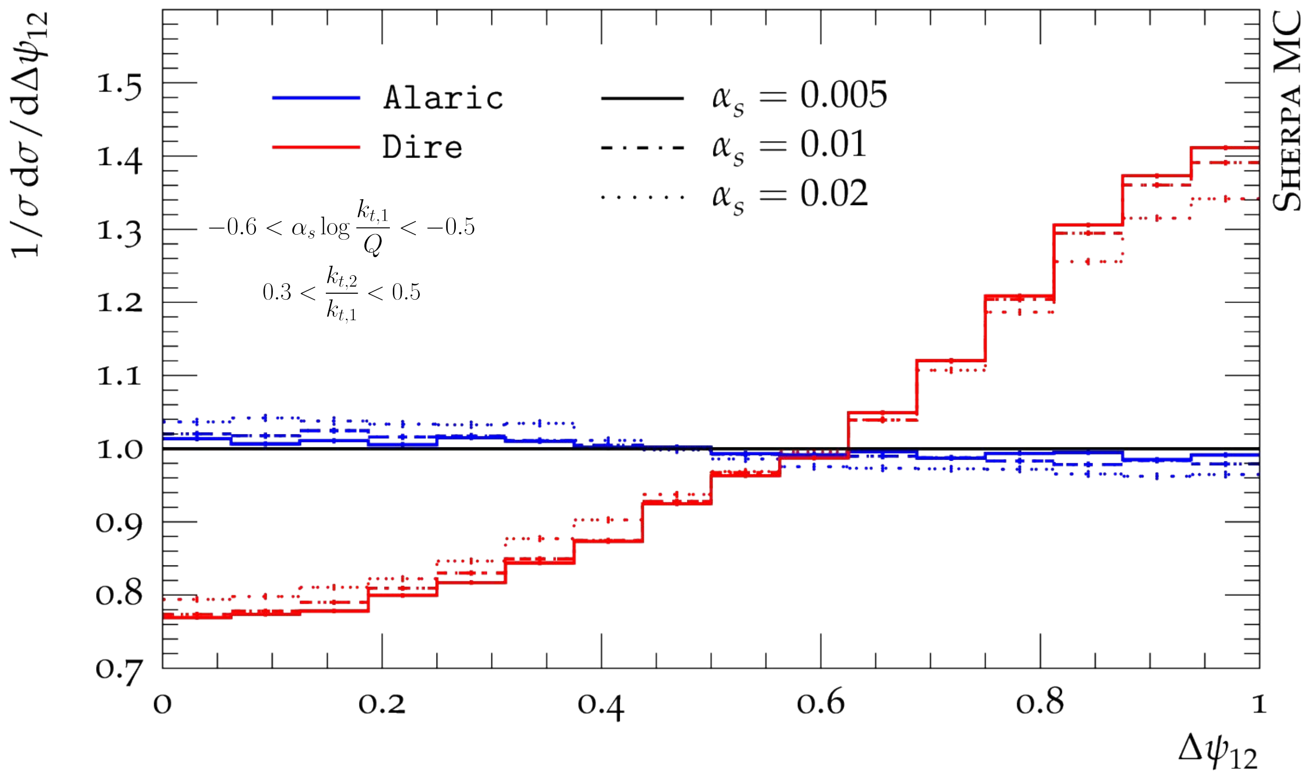
Recoil

Momentum mapping works for initial and final state emitters/spectator
→ e+ e-, pp, DIS, … all treated on same footing



Numerical Tests

Azimuthal angle between two Lund plane declusterings
Tests soft and rapidity separated emissions



NLO Matching

Alaric shares many similarities with
Catani-Seymour identified particle subtraction
$\rightarrow$ MC@NLO matching straightforward
Follow [Höche, Liebschner, Siegert] 1807.04348

Combined integrated subtraction term for identified parton production with a partonic fragmentation function:

$$\int_{m+1} \mathrm{d}\sigma^S + \int_m \mathrm{d}\sigma^C = \frac{1}{2} \sum_{i=g,q,\bar{q}} \sum_{\tilde{\imath}=1}^{m} \int_0^1 \frac{\mathrm{d}z}{z^{2-2\epsilon}} \int_m \mathrm{d}\sigma^B\left(p_1, \ldots, \frac{p_i}{z}, \ldots, p_m\right) \otimes \hat{\mathbf{I}}_{\tilde{\imath}i}^{(\mathrm{FS})}$$

Insertion operator:

$$\hat{\mathbf{I}}_{\tilde{\imath}i}^{(\mathrm{FS})} = \delta(1-z)\mathbf{I}_{\tilde{\imath}i} + \mathbf{P}_{\tilde{\imath}i} + \mathbf{H}_{\tilde{\imath}i}$$

$$\mathbf{I}_{\tilde{\imath}i}(p_1, \ldots, p_i, \ldots, p_m; \epsilon) = -\frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \sum_{k=1, k\neq\tilde{\imath}}^{m} \frac{\mathbf{T}_{\tilde{\imath}} \mathbf{T}_k}{\mathbf{T}_{\tilde{\imath}}^2} \left(\frac{4\pi\mu^2}{2p_i p_k}\right)^{\epsilon} \mathcal{V}_{\tilde{\imath}i}(\epsilon)$$

$$\mathbf{P}_{\tilde{\imath}i}\left(p_1, \ldots, \frac{p_i}{z}, \ldots, p_m; z; \mu_F\right) = \frac{\alpha_s}{2\pi} \sum_{k=1, k\neq\tilde{\imath}}^{m} \frac{\mathbf{T}_{\tilde{\imath}} \mathbf{T}_k}{\mathbf{T}_{\tilde{\imath}}^2} \ln \frac{z\mu_F^2}{2p_i p_k} \delta_{\tilde{\imath}i} P_{\tilde{\imath}i}(z)$$

$$\mathbf{H}_{\tilde{\imath}i}(p_1, \ldots, p_i, \ldots, p_m; n; z) = -\frac{\alpha_s}{2\pi} \sum_{k=1, k\neq\tilde{\imath}}^{m} \frac{\mathbf{T}_{\tilde{\imath}} \mathbf{T}_k}{\mathbf{T}_{\tilde{\imath}}^2} \left[\tilde{K}^{\tilde{\imath}i}(z) + \bar{K}^{\tilde{\imath}i}(z) + 2P_{\tilde{\imath}i}(z) \ln z + \mathcal{L}^{\tilde{\imath}i}(z; p_i, p_k, n)\right]$$

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$$\int_{m+1} \mathrm{d}\sigma^S + \int_m \mathrm{d}\sigma^C = \frac{1}{2} \sum_{i=g,q,\bar{q}} \sum_{\tilde{i}=1}^{m} \int_0^1 \frac{\mathrm{d}z}{z^{2-2\epsilon}} \int_m \mathrm{d}\sigma^B(p_1, \ldots, \frac{p_i}{z}, \ldots, p_m) \otimes \hat{\mathbf{I}}^{(\mathrm{FS})}_{\tilde{i}i}$$

Non-trivial integral:

$$\int_0^1 \mathrm{d}z\, \mathbf{H}_{\tilde{i}i}(p_1, \ldots, p_i, \ldots, p_m; n; z) = -\frac{\alpha_s}{2\pi} \sum_{k=1,k\neq\tilde{i}}^{m} \frac{\mathbf{T}_{\tilde{i}}\mathbf{T}_k}{\mathbf{T}^2_{\tilde{i}}} \left\{ \mathcal{K}^{\tilde{i}i} + \delta_{\tilde{i}i}\, \mathrm{Li}_2\left(1 - \frac{2\tilde{p}_i\tilde{p}_k\, \tilde{K}^2}{(\tilde{p}_i\tilde{K})(\tilde{p}_k\tilde{K})}\right) - \int_0^1 \mathrm{d}z\, P^{qq}_{\mathrm{reg}}(z) \ln \frac{n^2\tilde{p}_i\tilde{p}_k}{2z(\tilde{p}_i n)^2} \right\}$$

Evolution

Rewrite partial radiators:

$$\bar{w}^i_{ik,j} = \frac{\bar{W}^i_{ik,j}}{E_j^2} = \frac{\bar{W}_{ik,j}}{p_i p_j}$$

$$\bar{W}^i_{ik,j} = \frac{z}{1-z}(1-\cos\theta^i_j)\bar{W}_{ik,j}$$

Bounded from above by 2

$$\bar{w}^i_{ik,j} \leq 2 w^{(\mathrm{coll})}_{ik,j}$$

Evolution variable:

$$t = 2E_j^2(1-\cos\theta^i_j) = v(1-z)2\tilde{p}_i\tilde{K}$$

Corresponds to Lund plane $k_t^2 \;\rightarrow\; \beta_{\mathrm{PS}} = 0$

Differential splitting probabilities:

$$\mathrm{d}P^{i\,(\mathrm{soft})}_{ik,j}(t,z,\phi) = \mathrm{d}t\;\mathrm{d}z\;\frac{\mathrm{d}\phi}{2\pi}\frac{\alpha_s}{2\pi t}2C_i\bar{W}_{ik,j}$$

$$\mathrm{d}P^{i\,(\mathrm{coll})}_{ik,j}(t,z,\phi) = \mathrm{d}t\;\mathrm{d}z\;\frac{\mathrm{d}\phi}{2\pi}\frac{\alpha_s}{2\pi t}C_{\tilde{i}i}$$