Two Excursions in Four Dimensions

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C. Anastasiou, R. Haindl, G. Sterman, Z. Yang and M. Zeng

JHEP 04, 222 (2021) [arXiv:2008.12293]

C. Anastasiou & G.S., JHEP (05) 2023 242, arXiv 2212.12162

G.S.& Aniruddha Venkata arXiv 230x.xxx

C. Anastasiou, Julia Karlen, G.S., A. Venkata . . .

Two examples of sidestepping IR regularization through local cancellations in momentum space

- Initial state: NNLO process-dependence in complex EW annihilation amplitudes.

- Final state: Local finiteness of IR safe weighted cross sections in leptonic annihilation to hadrons.

Start with the initial state . . .

Initial state: NNLO process-dependence in complex EW annihilation amplitudes

I'll assume, on good authority, that NNLO QCD amplitudes for EW annihilation processes to 2, 3 or more EW bosons with fixed momenta, $q_i^2 = M_i^2 \gg \Lambda_{QCD}$:

$$q(p_1)\bar{q}(p_2) \to W^+(q_1)\, W^-(q_2)\, Z(q_3), \quad gg \to H(q_1)\, H(q_2)\, H(q_3) \dots$$

- **are important, but complicated,**
- **and it might be nice to be able to compute them numerically efficiently,**
- **which would require momentum space integrals that are infrared finite locally. (And UV convergent.)**
- **We'll work in Feynman gauge.**

- I'll describe an approach, based on the IR factorization of these amplitudes:

$$M_{a\bar{a}\to n}(p_1+p_2\to q_1+q_2+\ldots,\epsilon) \;= J^a(p_1,\epsilon)\,J_2^a(p_2,\epsilon)\; H_{a\bar{a}\to n}(p_1,p_2;q_1,q_2\ldots)$$

(We've absorbed the "soft function" into a definition of incoming jet subdiagrams J^a)

- with all dependence of the final state in

$$H_{a\bar{a}\to n}(p_1,p_2;q_1,q_2\ldots) \;=\; \frac{M_{a\bar{a}\to n}(p_1+p_2\to q_1+q_2+\ldots,\epsilon)}{J^a(p_1,\epsilon)\,J_2^a(p_2,\epsilon)}$$

- All true infrared singularities are absorbed into the jet functions.

- The "hard" function H is complex and complicated, and includes dynamics of intermediate states at momentum configurations that are not "soft" or "collinear". These "threshold" momentum configurations are amenable to numerical analysis on deformed momentum contours or by other means (see talk of Dario Kemanschah, shortly).

- For the process I'm talking about, this practical application is still in development.

- The essential point is that the singlet QCD form factor enjoys the same factorization with the same jet subdiagrams (we'll absorb H_1 into the Js):

$$F_{a\bar{a}\to 1}(p_1 + p_2 \to 1) \; = J^a(p_1,\epsilon)\, J^a(p_2,\epsilon) \;\; H_{a\bar{a}\to 1}(p_1,p_2)$$

- The idea is to use this knowledge to simplify a procedure for IR subtraction

$$H_{a\bar{a}\to n}(p_1,p_2;q_1,q_2\dots) \;\equiv\; \frac{M_{a\bar{a}\to n}(p_1+p_2\to q_1+q_2+\dots,\epsilon)}{F_{a\bar{a}\to 1}(p_1+p_2\to 1)}$$

- Just expand, each $L = (\alpha_s/\pi)^n L^{(n)}$, and then solve for $H^{(n)}$

$$H_{a\bar{a}\to n}(p_1,p_2;q_1,q_2\dots)\, F_{a\bar{a}\to 1}(p_1+p_2\to 1) \;=\; M_{a\bar{a}\to 1}(p_1+p_2\to q_1+q_2+\dots,\epsilon)$$

- or

$$H^{(1)} \;=\; M^{(1)} \;-\; F^{(1)}\, H^{(0)}$$

$$H^{(2)} \;=\; M^{(2)} \;-\; F^{(1)}\, H^{(1)} \;-\; F^{(2)}\, H^{(0)}$$

- **This construction for the hard-scattering is surely true for the full functions, but we want a result for the** *integrands*, $\mathcal{L} = \mathcal{M}$, $\mathcal{F}$, $\mathcal{H}$:

$$L_{a\bar{a}\to n}(p_1, p_2; q_1, q_2 \ldots) \;=\; \mathcal{L}^{(0)} + \int \frac{d^D k}{(2\pi)^D}\,\mathcal{L}_1(k) \;+ \int \frac{d^D k}{(2\pi)^D}\,\frac{d^D l}{(2\pi)^D}\,\mathcal{L}^{(2)}(k,l) \;+ \ldots$$

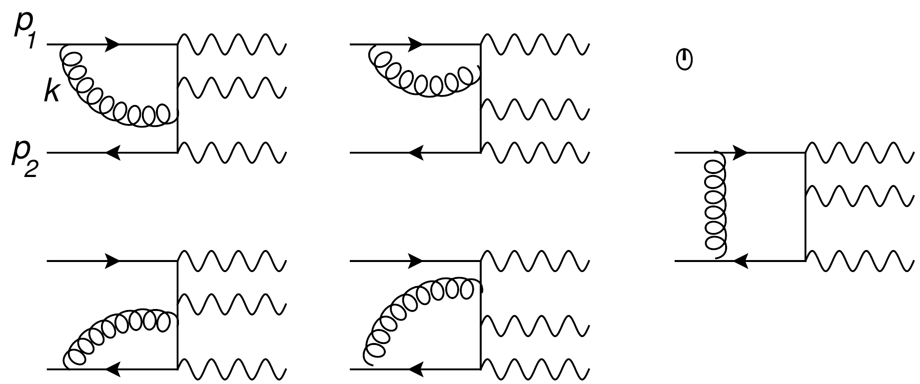
- **To be able to "give $\mathcal{H}$ to a computer" what we want is to show is:**

$$\begin{aligned} \mathcal{H}^{(1)} &= \mathcal{M}^{(1)} - \mathcal{F}^{(1)}\,\mathcal{H}^{(0)} \\ \mathcal{H}^{(2)} &= \mathcal{M}^{(2)} - \mathcal{F}^{(1)}\,\mathcal{H}^{(1)} - \mathcal{F}^{(2)}\,\mathcal{H}^{(0)} \end{aligned}$$

- **To get these local relations at two loops it will be necessary to modify the integrand by adding some IR "counterterms".**

• Let's first loop look at what happens at one loop: $\mathcal{H}^{(1)} = \mathcal{M}^{(1)} - \mathcal{F}^{(1)}\mathcal{H}^{(0)}$.

• IR singularities arise when $k \to 0$ and in the $k \propto p_{1,2}$ *collinear* limits:

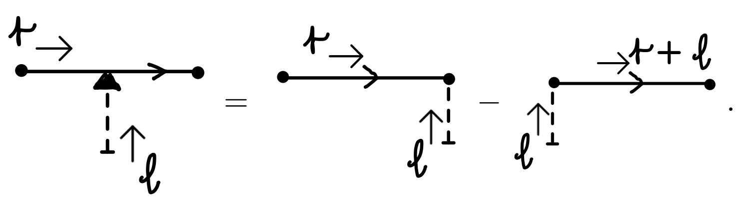


• When k gets collinear to p_1, singular behavior comes from

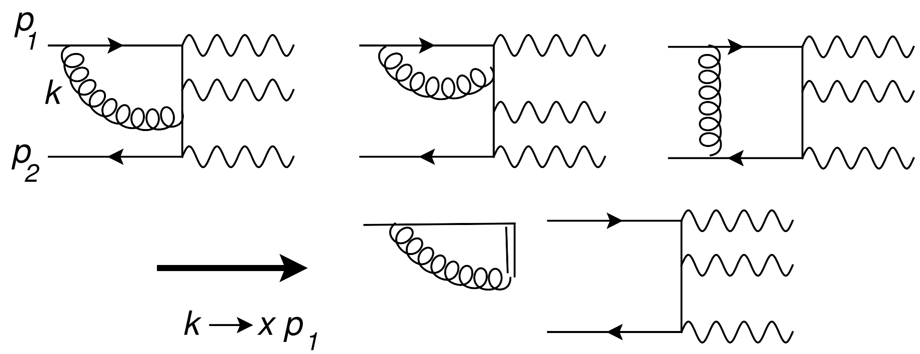
$$u(p_1)\gamma_\nu\,(\not p - \not k)\ \frac{-\eta^{\mu\nu}}{k^2+i\epsilon}\ \Rightarrow_{k\to xp_1}\ -u(p_1)\frac{(p_1-k)\cdot p_2}{p_2\cdot k}\,\frac{1}{k^2+i\epsilon}\,k^\nu$$

• Then in the collinear limit the gluon k is scalar-polarized and the "Feynman identity" applies, producing lots of pairwise cancellations,

$$\frac{i}{\not r + \not \ell}\,[-ig\not \ell]\,\frac{i}{\not r} = \frac{ig}{\not r} - \frac{ig}{\not r + \not \ell}$$



- And all k-dependence separates from the EW bosons . . .



- This is an algebraic relation, which is automatic when we combine the diagrams of the original amplitude. (The double line is $\sim\ p_{2\nu}/p_2\cdot k$)

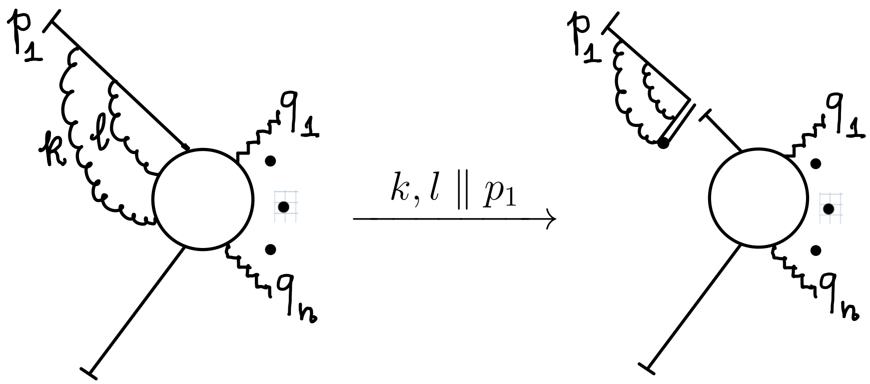
- The only k-dependent factor on the right equals the one-loop form factor in the k collinear to p_1 region, and $\mathcal{H}^{(1)} = \mathcal{M}^{(1)} - \mathcal{F}^{(1)}\,\mathcal{H}^{(0)}$ is confirmed locally. The same is true for the "soft" $k \to 0$ and collinear-p_2 limits.

- The single term $\mathcal{F}^{(1)}\,\mathcal{H}^{(0)}$ serves as a local IR subtraction for the full set of (5 for a VVV final state) diagrams of the original amplitude.

- The same holds for any EW final state of heavy bosons with this initial state, like $q\bar{q} \to W^+W^-W^+W^-$, or more.

- Could something like this work for two loops? Well, with a little help . . .

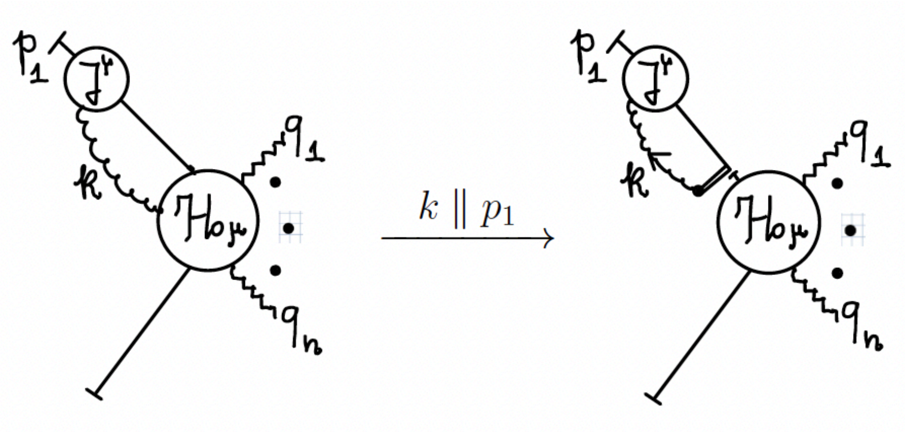
- Actually, when both gluons are collinear to either of the incoming quarks, or when one or both are soft, everything works just as at one loop. For example:



is algebraic and hence completely local.

- A reflection of gauge invariance through the Ward, Taylor-Slavnov (BRST) identities.

- Things get a little complicated when we try to see how a “single-collinear” gluon separates from the hard subdiagram at the *integrand* level,



- Compared to one loop, we encounter two qualitative complications, associated with an extra loop, either in the jet or hard part:

1. “loop polarizations” when $\mathcal{J}^\mu$ is a one-loop vertex or self energy.

2. “shift mismatches”, when $\mathcal{H}_\mu$ has the extra loop, and the Ward identity requires a shift in loop momentum.

- These complications are addressed by bespoke counterterms that integrate to zero, but reorganize the integrand. I’ll give basic examples that illustrate the detailed approach for our treatment of this region.

1. Collinear-singular loop polarizations occur in

$$\mathcal{J}^\mu(p_1,k,l) \;=\; \mathcal{J}_l(p_1,k)l^\mu + \mathcal{J}_k(p_1,l)k^\mu + \mathcal{J}_{p_1}(k,l)p_1^\mu \,.$$

Loop momentum l^μ (vertex or self-energy) may be in any direction, and gives a nonfactorizable collinear singularity.

- The l^μ part goes away after integration,

$$J^\mu(p_1,k,l) \;=\; J_k(p_1,l)k^\mu + J_{p_1}(k,l)p_1^\mu$$

so in the end it's all factorizable. But how to make this happen locally?

- Strategy is to identify an IR counterterm at the integrand level that integrates to zero:

$$\int \frac{d^Dl}{(2\pi)^D}\,\delta\mathcal{J}^\mu(p_1,k,l) \;=\; 0\,,$$

yet when added to $\mathcal{J}^\mu$ eliminates the unphysical loop polarizations:

$$\mathcal{J}^\mu(p_1,k,l) \;+\delta\mathcal{J}^\mu(p_1,k,l) \;=\; \tilde{\mathcal{J}}_k(p_1,k,l)k^\mu + \tilde{\mathcal{J}}_{p_1}(p_1,k,l)p_1^\mu \;+\text{IR finite}\,.$$

- And for a "quark jet" here it is:

$$\delta\mathcal{J}^\mu(k,l) = \frac{2(1-\epsilon)}{(p_1+k+l)^2}\left[\frac{2l^\mu+p_1^\mu+k^\mu}{l^2} - \frac{2(l+p_1)^\mu+k^\mu}{(l+p_1)^2}\right]\frac{\not\eta_1}{2p_1\cdot\eta_1}$$

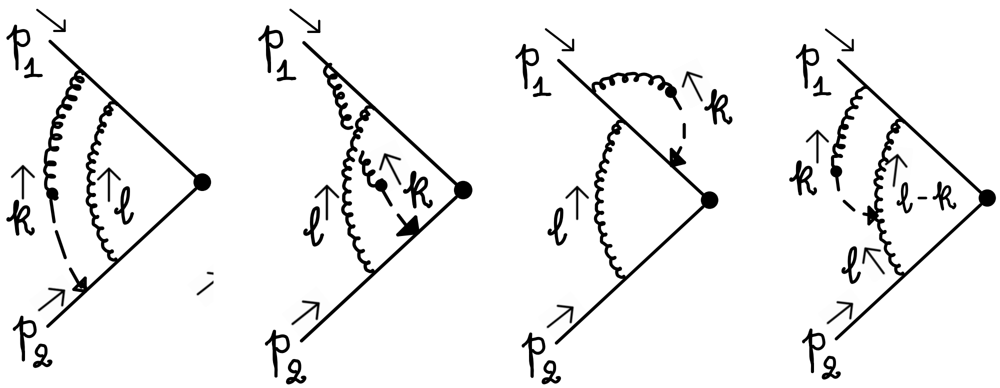
where we can take $\eta_1 = p_2$. We just add this to the integrand before integrating.

2. Shift mismatches and their counterterms.

- **For simplicity, illustrate with the form factor itself.**
  The coiled-dashed line with an arrow represents

$$\frac{(p_1 - k)\cdot p_2}{p_2 \cdot k}\,\frac{1}{k^2 + i\epsilon} k_\nu$$

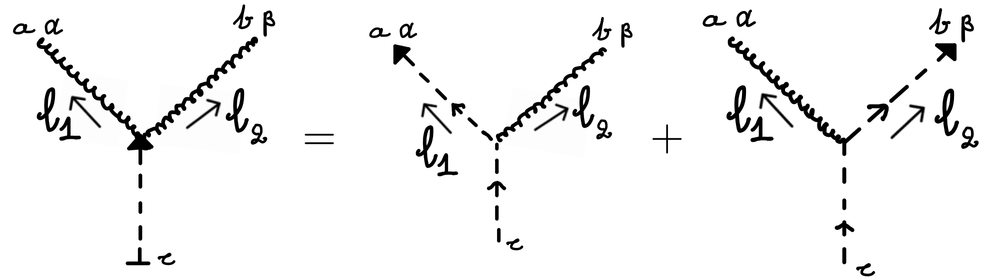
and in the region k^μ collinear to p_1, ℓ hard, the integrand behaves as



(a) C_F^2. (b) $C_F\left(C_F - \frac{C_A}{2}\right)$. (c) $C_F\left(C_F - \frac{C_A}{2}\right)$. (d) $\frac{C_F C_A}{2}$.

- **Again, the Ward identities are at the basis of factorization, but can we make it local?**

- What becomes of the "scalar-polarized" gluon – with the arrow? In QCD the Feynman identity analog involves ghosts:

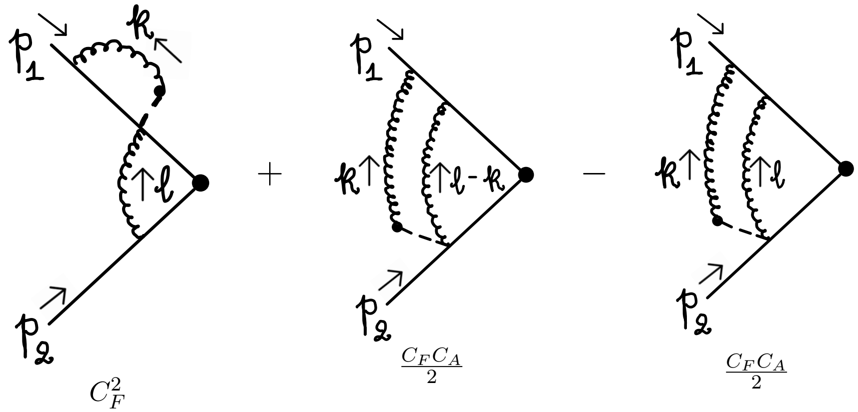


$$+ g_s f^{abc} \eta^{\alpha\beta} \left[\frac{1}{l_1^2} - \frac{1}{l_2^2} \right] .$$

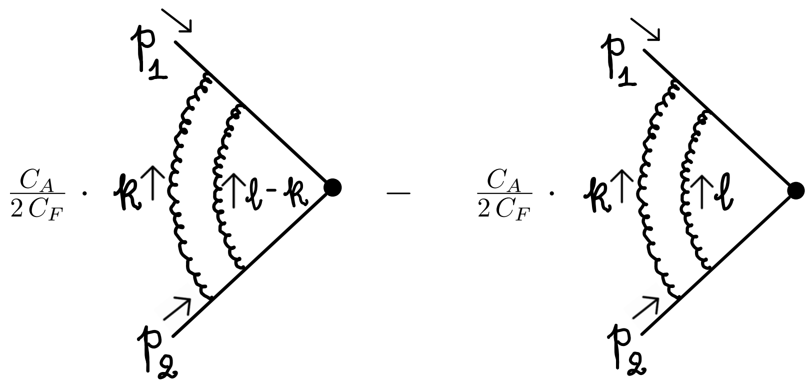
- Perhaps surprisingly, the ghost contributions factorize algebraically at two loops (in the form factor and general amplitudes). It's a little complicated, but all non-factoring singular behavior cancells "automatically" when diagrams are added. No counterterms required. (CA & GS, 2023)

- **Let's see what happens to the other two terms, the gluon analog of the Feynman identity in QED.**

- Here's what we get for the rest, an "almost factorized" form, but needs a shift of loop momentum. This is a reflection of the local nature of gauge invariance (QCD or QED).



- A counterterm that integrates to zero, but cancels the singularities of the unwanted terms locally – in both the k collinear to p_1 and p_2 regions



- The same "exotic-color planar" counterterms apply to arbitrary EW state.

Summary for “initial states in EW production”:

- There are more counterterms (including UV), but the ones we’ve seen illustrate the method.

- With a limited number of counterterms (roughly one per diagram) we can derive a hard function $\mathcal{H}^{(2)}$ that is free of IR divergences at NNLO, and can be computed numerically (the latter in progresss).

- In C. Anastasiou & GS (2023) this is demonstrated for quark-pair initiated processes.

- See talk by Julia Karlen at 2023 RadCor for glue-glue to multiple Higgs at two loops.

- Applications with color in the final state remain to be investigated,

- as well as the possibilities of $\mathrm{N^3LO}$ extensions. These may require further insight.

Final state: explicit IR finiteness in for leptonic annihilation to hadrons

- **In this case, a simple EW initial state to a complex (but IR safe) set of QCD final states.**
- **We'll find cancellations without regularization in the final state.**

- **The total cross section is the imaginary part of a current-current correlator by the optical theorem (unitarity):**

$$\sigma(Q) = \sigma_0 \, \mathrm{Im}\,\Pi(Q)$$
$$\Pi(Q) = \int d^4x \, e^{-iq\cdot x} \langle 0 | T[\mathcal{J}(0)\mathcal{J}(x)] | 0 \rangle \, .$$

(Neglecting indices on currents $\mathcal{J}$ . . . σ_0 is lowest-order cross section.)

- **Weighted cross sections:**

$$\Sigma[f,Q] \equiv \sum_N f(N) \, (2\pi)^4 \delta^4 \left(Q - P_N\right) \, \left|\langle 0 | \, \mathcal{J}(0) \, | N \rangle\right|^2 , \, ,$$

- **For infrared safety (perturbative finiteness) we require:**

$$f_C(\vec{q}_1, \ldots \vec{q}_i \ldots \vec{q}_{j-1}, \xi \vec{q}_i, \vec{q}_{j+1}, \ldots \vec{q}_{k_C}) = f_{C/j}(\vec{q}_1, \ldots (1+\xi)\vec{q}_i, \ldots \vec{q}_{j-1}, \vec{q}_{j+1}, \ldots \vec{q}_{k_C}) \, ,$$

- **It's usually a good thing to reduce the number of integrals.**

- **The advantages of doing energy integrals has been emphasized in the development of "Loop-tree duality"**
  - **S. Catani, T. Gleisberg, F. Krauss, G. Rodrigo and J.C. Winter, JHEP 09, 065 (2008) [arXiv:0804.3170] . . .**
  - **Z. Capatti, V. Hirschi, D. Kermanschah, A. Pelloni and B. Ruijl, [arXiv:2009.05509] and JHEP 04, 104 (2021) [arXiv:2010.01068 [hep-ph]].**

- **Here, let's see of what we can say in "old-fashioned" time-ordered perturbation theory (TOPT).**

- **TOPT – schematically (but with all the detail we need) for the correlator:**

$$\Pi(Q) = \sum_{G\in G_\Pi} \int dL_G \prod_{i=1}^{N_G} \frac{1}{2\omega_i} \sum_{\tau_G} \mathcal{N}[\tau_G]\, \pi_{\tau_G}(Q, L_G)\,,$$

$$\pi_{\tau_G}(Q, L_G) = \prod_{s=1}^{V_G-1} \frac{i}{Q\lambda_s - \sum_{j\in s}\omega_j + i\epsilon},$$

where

- dL_G **is measure of spatial loop momenta**
- **the** ω_j**s are on-shell energies of lines** $\sqrt{\vec{p}_j^{\,2} + m_j^2}$, **with** $\vec{p}_j$**s linear in loop momenta**
- τ_G **labels a time order of the** N_G **vertices – need all** $N_G!$ **of them.**
- $\mathcal{N}[\tau_G]$ **is a polynomial numerator factor.**
- **The** λ_s**s: Label** i **as (**$\mathcal{J}$**) vertex where momentum** q **flows in,** o**, where it flows out:**

$$\lambda_s = 1\,, \quad o > s > i\,,$$

$$\lambda_s = -1\,, \quad i > s > o$$

$$= 0\,, \quad \text{otherwise}\,.$$

- In a general cross section, states C are weighted by a function f_C,

$$\Sigma[f] \;=\; \sum_G \sum_C \sum_{\tau_G} \int dL_G \sum_{\tau_G} \mathcal{N}[\tau_G] \prod_{i=1}^{N_G} \frac{1}{2\omega_i}\, \sigma^{(C)}_{\tau_G}[f]\,, \tag{1}$$

where the "energy denominators" occur as

$$\sigma^{(C)}_{\tau_G}[f] \;=\; \prod_{s=C+1}^{V_G-1} \frac{i}{Q - \sum_{j\in s}\omega_j - i\epsilon}\, f_C(\vec{q}_1 \ldots \vec{q}_{k_C})$$
$$\times\, (2\pi)\,\delta\left(Q - \sum_{j\in C}\omega_j\right) \prod_{s=1}^{C-1} \frac{i}{Q - \sum_{j\in s}\omega_j + i\epsilon}\,,$$

- For $f_C = 1$ (total cross section), repeated use of

$$2\pi\delta(x) = \frac{i}{x+i\epsilon} - \frac{i}{x-i\epsilon}$$

gives the optical theorem: $\sigma(Q) \;=\; \sigma_0\, \mathrm{Im}\,\Pi(Q)$.

- For a general weighted cross section, we get (G.S., A. Venkata)

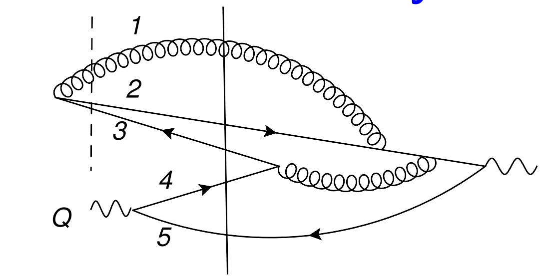
$$\sum_{C=1}^{n+1} \sigma_{\tau_G}^{(C)}[f] = \left(\prod_{s=1}^{n+1} \frac{i}{E_s - \sum_{j\in s} \omega_j + i\epsilon} f_{n+1} \right.$$
$$+ \sum_{C=1}^{n} \prod_{s=C+1}^{n+1} \frac{i}{E_s - \sum_{j\in s} \omega_j - i\epsilon} (f_C - f_{C+1}) \prod_{s=1}^{C} \frac{i}{E_s - \sum_{j\in s} \omega_j + i\epsilon}$$
$$\left. - \prod_{s=1}^{n+1} \frac{i}{E_s - \sum_{j\in s} \omega_j - i\epsilon} f_1 \right) .$$

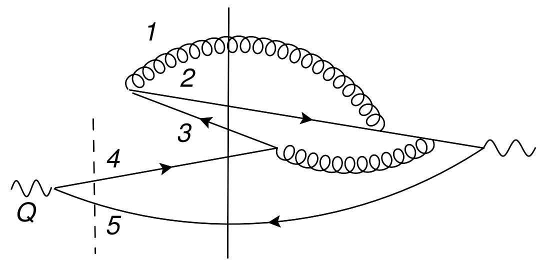
- First and final terms are free of pinches in momentum contours.

- For remaining terms, cancellations at all "pinch surfaces" from
  1. vanishing of $f_C - f_{C+1}$ term by term when states C and $C+1$ are pinched
  2. by summing over C when intermediate states are off-shell in "renormalization parts"

- When the terms are combined before integration, their sum becomes integrable, and amenable to numerical evaluation.

- **Important aside: As a practical matter, in its standard form, TOPT suffers from denominators with "pseudo-physical" states.**
- **An example shows how they occur, and how they cancel:**





- **Unphysical singularity at $Q = \sum_{i=1}^{5} \omega_i$, but . . .**

$$\left[\frac{1}{-\omega_1 - \omega_2 - \omega_3 + i\epsilon} + \frac{1}{Q - \omega_4 - \omega_5 + i\epsilon} \right] \delta\left(Q - \sum_{i=1}^{5} \omega_i \right) = 0$$

- **This cancellation is completely general. See W.J. Torres Bobadilla, JHEP 04, 183 (2021) [arXiv:2102.05048], Z. Capatti, Phys. Rev. D107 (2023) 5, L051902 (2211.09653) and Venkata & GS, to appear.**

Summary: final state cancellations

- **In TOPT, unitarity is realized locally in spatial momenta. Sums over final states are integrals over loop momenta.**

- **To eliminate unphysical TOPT singularities, reformulate simple "Time-ordered" (TOPT) to "Partial time-ordered" (PTOPT) using poset formalism (A. Venkata and GS, to appear). Up to orders of integrals, equivalent to results in Z. Capatti, Phys. Rev. D107.**

- **Also closely related to "flow ordering" introduced in M. Borinsky, Z. Capatti, E. Laenen and A. Salas-Bernárdez, JHEP 01, 172 (2023) [arXiv:2210.05532 [hep-th]].**

In conclusion

- **Local initial-state IR factorization & final state cancellations have the potential to add another tool for improving precision in Standard Model calculations, complementing analytic methods.**

- **Combining final state cancellations (unitarity) and initial-state IR factorization (causality) may open additional doors.**