Exercises for Feynman integrals*

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1 Schwinger parameters and graph polynomials

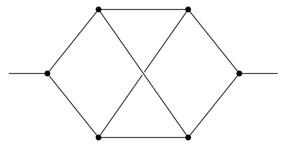
1. Starting from the Schwinger parameter representation

$$I(D,n,z) = \left(\prod_{e=1}^{N} \int_0^\infty \frac{x_e^{n_e-1}\mathrm{d}x_e}{\Gamma(n_e)}\right) \frac{e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{D/2}},$$

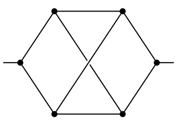
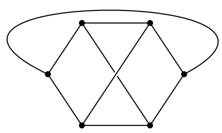
prove the projective and the Lee-Pomeransky representations.

Hint: Multiply with $1 = \int_0^\infty \delta(\rho - h(x))\mathrm{d}\rho$ *and change variables* $x_e \to \rho x_e$.

2. Compute the number of spanning trees of the following graph: *Hint: Use* $\mathcal{U} = \det A$.



3. Consider a graph G with two external legs, external momentum $p^2 = -1$, and vanishing internal masses $m_e = 0$. Let V denote the "vacuum" graph obtained by gluing the external legs into a new edge "0", for example

$G =$  $\Rightarrow$ $V =$ 

Show that:

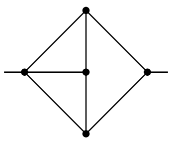
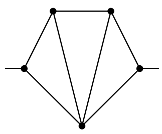
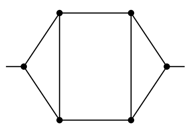
a) $\omega(V) = \omega(G) + n_0 - D/2$,

b) $\mathcal{U}_V = x_0\mathcal{U}_G + \mathcal{F}_G$, *Hint: trees and 2-forests.*

c) $I_G = \Gamma(D/2) \cdot P_V$ where

$$P_V := \operatorname*{Res}_{\omega(V)=0} I_V = \left(\prod_{e=0}^{N} \int_0^\infty \frac{x_e^{n_e-1}\mathrm{d}x_e}{\Gamma(n_e)}\right) \frac{\delta(1-h(x))}{\mathcal{U}_V^{D/2}}.$$

d) Conclude that in $D = 4$ dimensions with indices $n_e = 1$, the Feynman integrals of the following graphs all coincide:

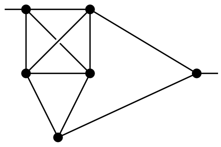
  

Remark. This is called the "glue-and-cut" symmetry.

[0]complementing lectures at the Amplitudes Summer School 2023. Adapted from MITP Amplitudes Games 2021.

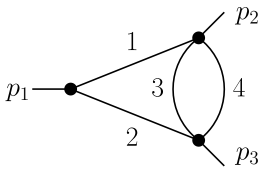
2 Power counting and factorization

1. Determine the leading order in the ε-expansion ($D = 4 - 2\varepsilon$) of the $n_e = 1$ integral



Hint: There are three nested divergences.

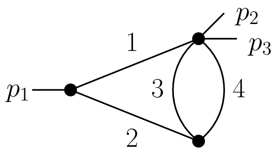
2. Consider the Feynman integral $I_G(D, n, z)$ of the graph

$G =$ 

   a) Compute the graph polynomials $\mathcal{U}$ and $\mathcal{F}$.
   b) Determine the two singular hyperplanes that contain the point $(D, n) = (4, 1, 1, 1, 1)$.
   c) Show that $\mathcal{U}$ and $\mathcal{F}$ factorize to leading order on the subdivergence, and conclude that the leading order of the ε-expansion is

$$I_G(4 - 2\varepsilon, 1, 1, 1, 1, z) = \frac{1}{2\varepsilon^2} + \mathcal{O}\left(\varepsilon^{-1}\right).$$

   d) Show that $I_G - I_{G'}$ is finite at $(D, n) = (4, 1, 1, 1, 1)$, where G' is the graph

$G' =$ 

   Hint: Compute both residues.

   Remark. Such 'rerouting' of momentum flow to compute divergent parts is used in the $\mathcal{R}^*$ operation and infrared rearrangement [1].

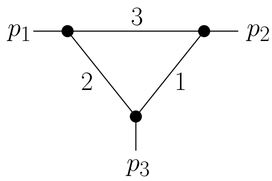
   e) For internal masses $m_e = 0$, obtain the subleading order ($\propto 1/\varepsilon$) of I_G.

   Hint: Compute $I_{G'}$ with the formula for the massless bubble integral in terms of Γ-functions.

   Remark. Such finite linear combinations are used in [2] to renormalize ϕ^4 at 6 loops.

3 Analytic continuation

Consider the following graph with $m_1^2 = m_2^2 = p_1^2 = p_2^2 = m^2$ and $m_3 = 0$:

$G =$ 

1. Show that $\mathcal{F}_{\{1,2\}} = 0$ for the tree subgraph with edges $\{1, 2\} = G - \{3\}$. Deduce, via the infrared factorization formula, that $\mathcal{F}_G$ must be independent of x_3.

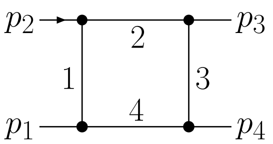
2. Confirm by computing $\mathcal{F}_G$ explicitly.
3. Draw the Newton polytope of $\mathcal{U}+\mathcal{F}$. *Hint: It has 5 facets.*
4. Describe the convergence domain in (D, n_1, n_2, n_3) by inequalities, and find all finite integrals in $D=6$ dimensions with integer n_e.
5. Set $D = 4-2\varepsilon$ and all $n_e = 1$. In the Lee-Pomeransky representation, insert $1 = \int_0^\infty \delta(\rho - x_1^{-1}) \mathrm{d}\rho$, rescale $x_e \to \rho^{\sigma_e} x_e$ for $\sigma = (-1,-1,-2)$, and factor out the lowest powers of ρ to make the infrared divergence explicit.
6. Integrate by parts in ρ to obtain the integral representation

$$I = -\frac{\Gamma(3-\varepsilon)}{2\varepsilon\Gamma(1-2\varepsilon)} \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3 \frac{x_1+x_2}{(\mathcal{U}+\mathcal{F})^{3-\varepsilon}}.$$

   and thus give a convergent integral formula for each coefficient in the ε-expansion.
7. Show that the leading order (coefficient of $1/\varepsilon$) is proportional to a bubble integral.
8. Explain where the divergence comes from in momentum space.

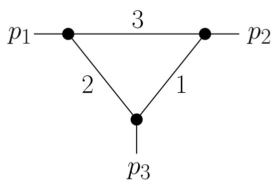
4 Polynomial reduction

1. Show that the Landau variety of the massless box integral ($m_e^2 = p_i^2 = 0$)



is $L = \{s, t, u\}$ where $\begin{cases} s = (p_1+p_2)^2 \\ t = (p_1+p_3)^2 \\ u = (p_1+p_4)^2 \end{cases}$

2. Consider the triangle integral for generic momenta p_1^2, p_2^2, p_3^2 as in the lecture, but with an internal mass $m_3 \neq 0$ (still $m_1 = m_2 = 0$):



   Show that with $\Delta = p_1^4 + p_2^4 + p_3^4 - 2p_1^2p_2^2 - 2p_1^2p_3^2 - 2p_2^2p_3^2$, its Landau variety is

$$L = \left\{ p_1^2, p_2^2, p_3^2, \Delta, m_3^2, m_3^2 - p_1^2, m_3^2 - p_2^2, (m_3^2 - p_1^2)(m_3^2 - p_2^2) + m_3^2 p_3^2 \right\}.$$

References

[1] K. G. Chetyrkin, *Combinatorics of R-, R^{-1}-, and R^*-operations and asymptotic expansions of Feynman integrals in the limit of large momenta and masses*, tech. rep., Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, Munich, Germany, Mar., 1991, arXiv:1701.08627 [hep-th].

[2] M. V. Kompaniets and E. Panzer, *Minimally subtracted six loop renormalization of O(n)-symmetric ϕ^4 theory and critical exponents*, *Phys. Rev. D* **96** (Aug., 2017) p. 036016, arXiv:1705.06483 [hep-th].