Homework

1. Evaluate the finite integral from lecture 1.
2. Prove IBP identities for the IBP vectors chosen to be loop momenta.
3. Derive the Baikov polynomial for the one-mass triangle integral from lecture 3. Evaluate it on maximal cut (all three propagators cut). What do you observe? Think about how is this related to the identity that we have derived.
4. Consider iterated integrals over one-forms $\{\omega_0, \omega_1\}$ evaluated on a path $\gamma : [0,1] \to M$, where M is the manifold where ω_i are holomorphic. Demonstrate that

$$\int_\gamma \omega_0 \int_\gamma \omega_1 = \int_\gamma \omega_0\omega_1 + \int_\gamma \omega_1\omega_0$$

   Hint: think of iterated integrations as a two-dimensional integral over a region in $[0,1]^2$

5. Find solutions of homogeneous DE at leading order in ϵ for each of the subtopologies for the tree-mass triangle integral from lecture 4. Compare these to the definitions of the basis that satisfies the canonical DE.
6. Derive the differential equations for the bubble integral families that we discussed in lecture 2. Use solutions of homogeneous DE to change the basis of integrals that leads to the canonical form of the DE.
7. Consider iterated integrals on $M = \mathbb{R}^2$, and a family of paths $\gamma(t) : [0,1] \to M$, $t \to (t^r, t^s)$ for any $r, s > 0$. Let $\omega_1 = \mathrm{d}x, \omega_2 = \mathrm{d}y$.

   Calculate explicitly the iterated integral $I_1 = \int_\gamma \omega_1\omega_2$ (the outermost integration is last). What can we say about the path dependence? Next find a one-form ω_x, such that $\omega_1 \wedge \omega_2 + \mathrm{d}\omega_x = 0$, and calculate a correction term $I_x = \int_\gamma \omega_x$. What do we see for $I_1 + I_x$?

   This example illustrates that iterated integrals of arbitrary one-forms on $M = \mathbb{R}^n$, $n \geq 2$, are not in general homotopy functionals, and the issue can be traced to non-vanishing $\omega_i \wedge \omega_j$.

8. Use you favorite computer algebra system for this exercise. Consider a rational function

$$f = \frac{7x^3 - 8x^2y - 2xy^2 + y^3}{x^2y - xy^2}$$

   Pretend you don't know what the function is and can only evaluate it numerically. Construct a dense ansatz of all monomials up to degree 3 for numerator and denominator. Use numerical evaluations (either over rationals or prime fields) to obtain a linear system of equations for the ansatz parameters and solve it.

   Now assume you know that this function is regular apart from simple poles at $x = 0, y = 0, x = y$, and it is invariant under rescaling $x \to xt, y \to yt$ (you still can only evaluate it numerically). Construct a refined ansatz using this information and compare the number of free parameters.