Exercises for Feynman integrals*

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1 Schwinger parameters and graph polynomials

1. Starting from the Schwinger parameter representation

$$I(D,n,z) = \left(\prod_{e=1}^{N} \int_0^\infty \frac{x_e^{n_e-1}\mathrm{d}x_e}{\Gamma(n_e)}\right) \frac{e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{D/2}},$$

prove the projective and the Lee-Pomeransky representations.

Hint: Multiply with $1 = \int_0^\infty \delta(\rho - h(x))\mathrm{d}\rho$ *and change variables* $x_e \to \rho x_e$.

Solution: Introducing ρ and rescaling x_e as suggested, the polynomials $\mathcal{U}$ and $\mathcal{F}$ get multiplied by ρ^L and ρ^{L+1}, respectively. From $x_e^{n_e-1}\mathrm{d}x_e$ we also get a factor ρ^{n_e} for every edge. Since $h(x)$ is homoeneous, the delta distribution becomes $\delta(\rho - \rho h(x)) = \delta(1-h(x))/\rho$ after the rescaling $x_e \to \rho x_e$. The integral over ρ is a gamma function,

$$I(D,n,z) = \left(\prod_{e=1}^{N} \int_0^\infty \frac{x_e^{n_e-1}\mathrm{d}x_e}{\Gamma(n_e)}\right) \frac{\delta(1-h(x))}{\mathcal{U}^{D/2}} \underbrace{\int_0^\infty \rho^{\omega-1} e^{-\rho\mathcal{F}/\mathcal{U}}\mathrm{d}\rho}_{\Gamma(\omega)(\mathcal{U}/\mathcal{F})^\omega}$$

which proves the projective representation of the Feynman integral. Applying the same procedure to the Lee-Pomeransky integral, we get

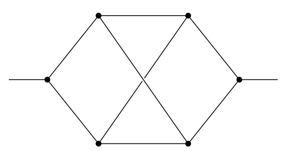
$$\left(\prod_{e=1}^{N} \int_0^\infty \frac{x_e^{n_e-1}\mathrm{d}x_e}{\Gamma(n_e)}\right)(\mathcal{U}+\mathcal{F})^{-D/2} = \left(\prod_{e=1}^{N} \int_0^\infty \frac{x_e^{n_e-1}\mathrm{d}x_e}{\Gamma(n_e)}\right)\delta(1-h(x))\int_0^\infty \rho^{\omega-1}(\mathcal{U}+\rho\mathcal{F})^{-D/2}\mathrm{d}\rho$$

The substitution $\rho \to \rho\cdot\mathcal{U}/\mathcal{F}$ gives

$$\int_0^\infty \rho^{\omega-1}(\mathcal{U}+\rho\mathcal{F})^{-D/2}\mathrm{d}\rho = \left(\frac{\mathcal{U}}{\mathcal{F}}\right)^\omega \mathcal{U}^{-D/2}\int_0^\infty \frac{\rho^{\omega-1}}{(1+\rho)^{D/2}}\mathrm{d}\rho = \frac{\beta(\omega, D/2-\omega)}{\mathcal{U}^{D/2-\omega}\mathcal{F}^\omega},$$

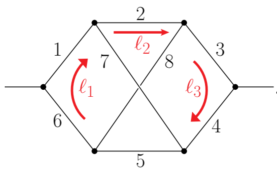
Euler's beta function. In terms of gamma functions, $\beta(\omega, D/2-\omega) = \Gamma(\omega)\Gamma(D/2-\omega)/\Gamma(D/2)$, hence with the prefactor $\Gamma(D/2)/\Gamma(D/2-\omega)$ of the Lee-Pomeransky representation, only $\Gamma(\omega)$ remains. We have thus arrived again at the projective representation of the Feynman integral.

2. Compute the number of spanning trees of the following graph: *Hint: Use* $\mathcal{U} = \det A$.



[0]complementing lectures at the Amplitudes Summer School 2023. Adapted from MITP Amplitudes Games 2021.

Solution: Consider the loop momentum flow and edge labels as indicated in



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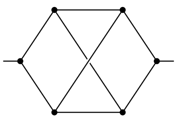
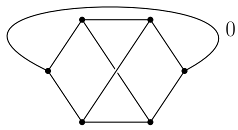
The loop momentum through edges 7,8,5 of X is then $\ell_1 - \ell_2$, $\ell_2 - \ell_3$ and $\ell_1 - \ell_2 + \ell_3$, respectively. This gives the matrix

$$A = \begin{pmatrix} x_1 + x_5 + x_6 + x_7 & -x_5 - x_7 & x_5 \\ -x_5 - x_7 & x_2 + x_5 + x_7 + x_8 & -x_5 - x_8 \\ x_5 & -x_5 - x_8 & x_3 + x_4 + x_5 + x_8 \end{pmatrix}.$$

The number of spanning trees is equal to $\mathcal{U} = \det A$ evaluated at $x_e = 1$ for every edge e,

$$\mathcal{U}|_{x_e=1} = \det \begin{pmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{pmatrix} = 36.$$

3. Consider a graph G with two external legs, external momentum $p^2 = -1$, and vanishing internal masses $m_e = 0$. Let V denote the "vacuum" graph obtained by gluing the external legs into a new edge "0", for example

$G =$  $\Rightarrow$ $V =$ 

Show that:

a) $\omega(V) = \omega(G) + n_0 - D/2$,

Solution: V has one additional edge ($\Rightarrow +n_0$) and one additional loop ($\Rightarrow -D/2$).

b) $\mathcal{U}_V = x_0 \mathcal{U}_G + \mathcal{F}_G$, *Hint: trees and 2-forests.*

Solution: Every spanning tree T of V either:

- does *not* contain 0, in which case T is a spanning tree of V; or
- *does* contain 0, in which case $F := T \setminus \{0\} = T_1 \sqcup T_2$ is a 2-forest of G with one external leg in T_1 and the other external leg in T_2, hence a 2-forest that contributes to $\mathcal{F}_G$.

c) $I_G = \Gamma(D/2) \cdot P_V$ where

$$P_V := \operatorname*{Res}_{\omega(V)=0} I_V = \left(\prod_{e=0}^{N} \int_0^\infty \frac{x_e^{n_e - 1} \mathrm{d}x_e}{\Gamma(n_e)} \right) \frac{\delta(1 - h(x))}{\mathcal{U}_V^{D/2}}.$$

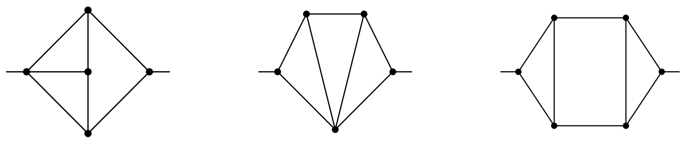
Solution: The integral over x_0 is (compare with the Lee-Pomeransky integral in part 1.)

$$\frac{1}{\Gamma(n_0)} \int_0^\infty \frac{x_0^{n_0 - 1} \mathrm{d}x_0}{(x_0 \mathcal{U}_G + \mathcal{F}_G)^{D/2}} = \frac{\beta(n_0, D/2 - n_0)}{\Gamma(n_0) \mathcal{F}_G^{D/2}} \left(\frac{\mathcal{F}_G}{\mathcal{U}_G} \right)^{n_0} = \frac{\Gamma(D/2 - n_0)}{\Gamma(D/2)} \frac{1}{\mathcal{U}_G^{n_0} \mathcal{F}_G^{D/2 - n_0}}.$$

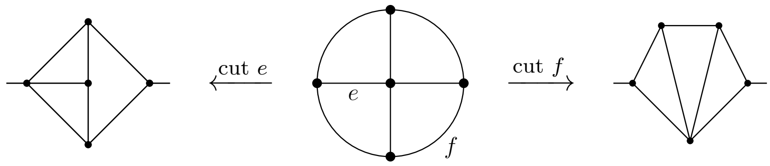
In the integral for P_V, we have $\omega(V) = 0$, hence $D/2 - n_0 = \omega(G)$. The above thus becomes

$$P_V = \frac{\Gamma(\omega(G))}{\Gamma(D/2)} \left(\prod_{e=0}^{N} \int_0^\infty \frac{x_e^{n_e - 1} \mathrm{d}x_e}{\Gamma(n_e)} \right) \frac{\delta(1 - h(x))}{\mathcal{U}_G^{D/2 - \omega(G)} \mathcal{F}_G^{\omega(G)}} = \frac{1}{\Gamma(D/2)} \cdot I_G.$$

d) Conclude that in $D = 4$ dimensions with indices $n_e = 1$, the Feynman integrals of the following graphs all coincide:

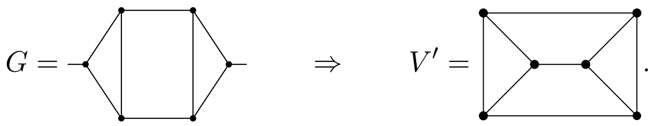


Solution: The left and centre graphs glue to the same graph V, the wheel with 4 spokes:



Also the indices agree, because $n_0 = 1$ in both cases. Hence both of these propagator integrals are equal to the period of V with unit indices everywhere. (FYI: it is $P_V = 20\zeta(5)$.)

The third propagator glues into a different graph:

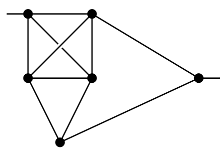


But the central, glued edge has index $n_0 = D/2 - \omega(G) = 0$ so that it can be contracted, therefore $P_{V'}|_{n_0=0} = P_V$ because $V'/0 \cong V$.

Remark. This is called the "glue-and-cut" symmetry.

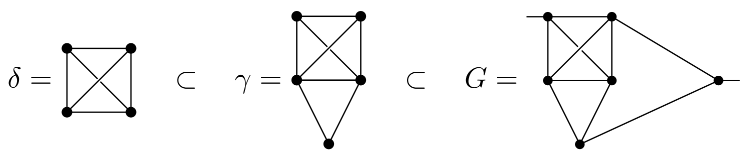
2 Power counting and factorization

1. Determine the leading order in the ε-expansion ($D = 4 - 2\varepsilon$) of the $n_e = 1$ integral



Hint: There are three nested divergences.

Solution: There is a logarithmic overall divergence $\omega|_{n_e=1} = 5\varepsilon$ and two nested, logarithmic subdivergences



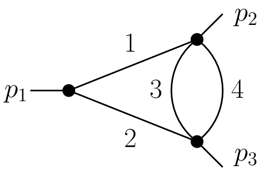
with $\omega(\delta)|_{n_e=1} = 3\varepsilon$ and $\omega(\gamma)|_{n_e=1} = 4\varepsilon$. Hence the leading order is a triple pole $1/[\omega \cdot \omega(\gamma) \cdot \omega(\delta)]|_{n_e=1} = 1/(60\varepsilon^3)$ with coefficient

$$\operatorname*{Res}_{\omega(\delta)=0} \operatorname*{Res}_{\omega(\gamma)=0} \operatorname*{Res}_{\omega=0} I_G = P_\delta \cdot P_{\gamma/\delta} \cdot P_{G/\gamma} = 6\zeta(3) \cdot 1 \cdot 1$$

where δ is the wheel with 3 spokes mentioned in the lecture, and $\gamma/\delta \cong G/\gamma$ are both isomorphic to the 1-loop bubble graph. In conclusion,

$$I_G(4 - 2\varepsilon, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, z) = \frac{\zeta(3)}{10\varepsilon^3} + \mathcal{O}\left(\varepsilon^{-2}\right).$$

2. Consider the Feynman integral $I_G(D, n, z)$ of the graph

$G =$ 

a) Compute the graph polynomials $\mathcal{U}$ and $\mathcal{F}$. **Solution:**

$$\mathcal{U} = x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$$
$$\mathcal{F} = -p_1^2x_1x_2(x_3 + x_4) - p_2^2x_1x_3x_4 - p_3^2x_2x_3x_4 + \mathcal{U}(m_1^2x_1 + m_2^2x_2 + m_3^2x_3 + m_4^2x_4)$$

b) Determine the two singular hyperplanes that contain the point $(D, n) = (4, 1, 1, 1, 1)$.

Solution:

- overall divergence: $\omega = 0$ where $\omega = n_1 + n_2 + n_3 + n_4 - D$
- subdivergence $\{3, 4\}$: $\omega(\{3, 4\}) = 0$ where $\omega(\{3, 4\}) = n_3 + n_4 - D/2$

c) Show that $\mathcal{U}$ and $\mathcal{F}$ factorize to leading order on the subdivergence, and conclude that the leading order of the ε-expansion is

$$I_G(4 - 2\varepsilon, 1, 1, 1, 1, z) = \frac{1}{2\varepsilon^2} + \mathcal{O}\left(\varepsilon^{-1}\right).$$

Solution: With $x_3 \to x_3\rho$ and $x_4 \to x_4$, the leading orders as $\rho \to 0$ are

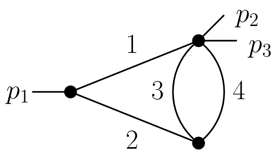
- $\mathcal{U} \to \rho(x_3 + x_4)(x_1 + x_2) + \mathcal{O}(\rho^2)$
- $\mathcal{F} \to \rho(x_3 + x_4)\left[-p_1^2x_1x_2 + (x_1 + x_2)(m_1^2x_1 + m_2^2x_2)\right] + \mathcal{O}(\rho^2)$

By factorization, the coefficient of the double pole $1/[\omega \cdot \omega(\{3, 4\})]$ is

$$\operatorname*{Res}_{\omega(\{3,4\})=0}\operatorname*{Res}_{\omega=0} I_G = \operatorname*{Res}_{\omega(\{3,4\})=0} P_G = P_{\{3,4\}} \cdot P_{G/\{3,4\}} = 1 \cdot 1 = 1$$

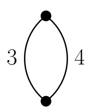
is the square of the period of the bubble (note both the subgraph $\{3, 4\}$, and the quotient graph $G/\{3, 4\}$, are bubble graphs). Hence the double pole is $1/[\omega \cdot \omega(\{3, 4\})] = 1/[2\varepsilon \cdot \varepsilon]$.

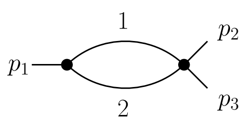
d) Show that $I_G - I_{G'}$ is finite at $(D, n) = (4, 1, 1, 1, 1)$, where G' is the graph

$G' =$ 

Hint: Compute both residues.

Solution: The graph G' has the same overall power counting, and also the same subdivergence $\omega(\{3, 4\}) = 0$. The corresponding sub- and quotient graphs



$\subset G, G'$, $\quad G/\{3, 4\} =$ 

$= G'/\{3, 4\}$

are the same for G as they are for G', hence they yield the same residue

$$\operatorname*{Res}_{\omega(\{3,4\})=0} I_G = P_{\{3,4\}} \cdot I_{G/\{3,4\}} = P_{\{3,4\}} \cdot I_{G'/\{3,4\}} = \operatorname*{Res}_{\omega(\{3,4\})=0} I_{G'}.$$

The residue $P_G = \operatorname{Res}_{\omega=0} I_G$ of the overall divergence depends only on $\mathcal{U}$, but not $\mathcal{F}$. Since G and G' differ only in the attachment of the external legs, they have the same $\mathcal{U}_G = \mathcal{U}_{G'}$, and therefore $P_G = P_{G'}$. Hence, the difference $I_G - I_{G'}$ has neither a pole on $\omega = 0$ nor at $\omega(\{3, 4\}) = 0$.

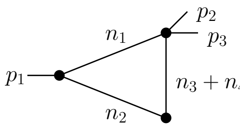
Remark. Such 'rerouting' of momentum flow to compute divergent parts is used in the $\mathcal{R}^*$ operation and infrared rearrangement [1].

e) For internal masses $m_e = 0$, obtain the subleading order ($\propto 1/\varepsilon$) of I_G.

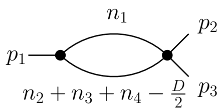
Hint: Compute $I_{G'}$ with the formula for the massless bubble integral in terms of Γ-functions.

Solution: Let $B(n_1, n_2) = \Gamma(D/2 - n_1)\Gamma(D/2 - n_2)\Gamma(n_1 + n_2 - D/2)/[\Gamma(n_1)\Gamma(n_2)\Gamma(D - n_1 - n_2)]$ so that the bubble integral is $(-p^2)^{-\omega} B(n_1, n_2)$. Integrating out the subloop yields

$$I_{G'} = B(n_3, n_4) \times$$



$$= B(n_3, n_4) \times$$



$$= B(n_3, n_4) B(n_1, n_2 + n_3 + n_4 - D/2)(-p_1^2)^{-\omega}$$

Plugging in $n_e = 1$ and $D = 4 - 2\varepsilon$, this gives the ε-expansion

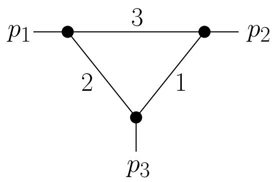
$$\begin{aligned} I_G(4 - 2\varepsilon, 1, 1, 1, 1, z) = I_{G'} + \mathcal{O}\left(\varepsilon^0\right) &= (-p_1^2)^{-2\varepsilon} \frac{\Gamma(1-\varepsilon)^3 \Gamma(1-2\varepsilon)\Gamma(\varepsilon)\Gamma(2\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(2-2\varepsilon)\Gamma(2-3\varepsilon)} + \mathcal{O}\left(\varepsilon^0\right) \\ &= \frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{5}{2} - \gamma_E - \log(-p_1^2)\right) + \mathcal{O}\left(\varepsilon^0\right) \end{aligned}$$

Remark. Such finite linear combinations are used in [2] to renormalize ϕ^4 at 6 loops.

3 Analytic continuation

Consider the following graph with $m_1^2 = m_2^2 = p_1^2 = p_2^2 = m^2$ and $m_3 = 0$:

$$G =$$



1. Show that $\mathcal{F}_{\{1,2\}} = 0$ for the tree subgraph with edges $\{1, 2\} = G - \{3\}$. Deduce, via the infrared factorization formula, that $\mathcal{F}_G$ must be independent of x_3.

   Solution: The graph polynomials for the tree are $\mathcal{U}_{\{1,2\}} = 1$ and

   $$\mathcal{F}_{\{1,2\}} = (m^2 x_1 + m^2 x_2)\mathcal{U}_{\{1,2\}} - p_1^2 x_2 - p_2^2 x_1 = x_1(m^2 - p_2^2) + x_2(m^2 - p_1^2) = 0.$$

   Under the scaling $(x_1, x_2) \to (\rho x_1, \rho x_2)$ of the tree edges, the IR-factorization formula gives

   $$\mathcal{F}_G = \rho^1 \mathcal{U}_{G/\{1,2\}} \mathcal{F}_{\{1,2\}} + \mathcal{O}\left(\rho^2\right) = \mathcal{O}\left(\rho^2\right)$$

   hence every term in $\mathcal{F}_G$ is of degree ≥ 2 in the variables (x_1, x_2). But we know that $\mathcal{F}_G$ is homogeneous of degree 2 in all variables (because G has 1 loop); hence $\mathcal{F}_G$ cannot have any x_3.

2. Confirm by computing $\mathcal{F}_G$ explicitly.

   Solution:

   $$\begin{aligned} \mathcal{F}_G &= m^2(x_1 + x_2)(x_1 + x_2 + x_3) - p_1^2 x_2 x_3 - p_2^2 x_1 x_3 - p_3^2 x_1 x_2 \\ &= m^2(x_1 + x_2)^2 - p_3^2 x_1 x_2 + x_3 \underbrace{(m^2 x_1 + m^2 x_2 - p_1^2 x_2 - p_2^2 x_1)}_{=\mathcal{F}_{\{1,2\}}=0} \\ &= m^2(x_1 + x_2)^2 - p_3^2 x_1 x_2. \end{aligned}$$

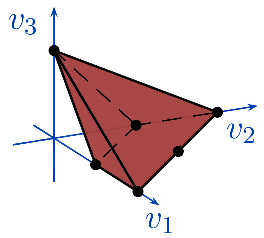
3. Draw the Newton polytope of $\mathcal{U}+\mathcal{F}$. *Hint: It has 5 facets.*

Solution: The Lee-Pomeransky polynomial

$$\mathcal{U}_G+\mathcal{F}_G = x_1+x_2+x_3+x_1^2m^2+x_2^2m^2+x_1x_2\left(2m^2-p_3^2\right)$$

has 6 different monomials. We read off the Newton polytope

$$\mathrm{NP} = \mathrm{conv}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\} =$$



4. Describe the convergence domain in (D,n_1,n_2,n_3) by inequalities, and find all finite integrals in $D=6$ dimensions with integer n_e.

Solution: The Newton polytope is a pyramid with apex $(0,0,1)$ over a quadrilateral in the (x_1,x_2)-plane. This pyramid has 5 supporting hyperplanes (facets) that we can read off easily:

$$v_1\geq 0,\qquad v_2\geq 0,\qquad v_3\geq 0,$$
$$v_1+v_2+v_3\geq 1,\qquad v_1+v_2+2v_3\leq 2.$$

For example, $v_3=0$ is the plane containing the base quadrilateral; and $v_1+v_2+2v_3=2$ is the supporting hyperplane that contains the three vertices $(2,0,0)$, $(0,2,0)$ and $(0,0,1)$.

The convergence region is $\mathrm{Re}(n)\in\mathrm{interior}(D/2\cdot\mathrm{NP})$, hence the conditions are

$$\mathrm{Re}(n_1)>0,\qquad \mathrm{Re}(n_2)>0,\qquad \mathrm{Re}(n_3)>0,$$
$$\mathrm{Re}(n_1+n_2+n_3)>\mathrm{Re}(D/2),\qquad \mathrm{Re}(n_1+n_2+2n_3)<\mathrm{Re}(D).$$

For $D=6$, the only integer solutions are $n=(2,1,1)$ and $n=(1,2,1)$.

5. Set $D=4-2\varepsilon$ and all $n_e=1$. In the Lee-Pomeransky representation, insert $1=\int_0^\infty\delta(\rho-x_1^{-1})\mathrm{d}\rho$, rescale $x_e\to\rho^{\sigma_e}x_e$ for $\sigma=(-1,-1,-2)$, and factor out the lowest powers of ρ to make the infrared divergence explicit.

Solution: Under this scaling, we have

$$\mathrm{d}x_1\to\rho^{-1}\mathrm{d}x_1,\qquad \mathrm{d}x_2\to\rho^{-1}\mathrm{d}x_2,\qquad \mathrm{d}x_3\to\rho^{-2}\mathrm{d}x_3,$$
$$\mathcal{U}\to\rho^{-2}(x_3+\rho x_1+\rho x_2),\qquad \mathcal{F}\to\rho^{-2}\mathcal{F},\qquad \delta(\rho-x_1^{-1})\to\rho^{-1}\delta(1-x_1).$$

With $D/2=2-\varepsilon$ and $\omega=1+\varepsilon$, the Lee-Pomeransky integral thus becomes

$$I=\frac{\Gamma(2-\varepsilon)}{\Gamma(1-2\varepsilon)}\int_0^\infty\mathrm{d}x_1\int_0^\infty\mathrm{d}x_2\int_0^\infty\mathrm{d}x_3\,\delta(1-x_1)\int_0^\infty\frac{\rho^{-2\varepsilon-1}\mathrm{d}\rho}{(x_3+\mathcal{F}+\rho x_1+\rho x_2)^{2-\varepsilon}}.$$

The integral over ρ is divergent at the lower boundary, unless $\omega(\sigma)=-2\varepsilon>0$.

6. Integrate by parts in ρ to obtain the integral representation

$$I=-\frac{\Gamma(3-\varepsilon)}{2\varepsilon\Gamma(1-2\varepsilon)}\int_0^\infty\mathrm{d}x_1\int_0^\infty\mathrm{d}x_2\int_0^\infty\mathrm{d}x_3\frac{x_1+x_2}{(\mathcal{U}+\mathcal{F})^{3-\varepsilon}}.$$

and thus give a convergent integral formula for each coefficient in the ε-expansion.

Solution: The integration by parts in ρ gives

$$\int_0^\infty \rho^{-2\varepsilon-1}\mathrm{d}\rho\,(x_3+\mathcal{F}+\rho x_1+\rho x_2)^{\varepsilon-2} = \frac{\varepsilon-2}{2\varepsilon}\int_0^\infty \frac{(x_1+x_2)\rho^{-2\varepsilon}\mathrm{d}\rho}{(x_3+\mathcal{F}+\rho x_1+\rho x_2)^{3-\varepsilon}}$$

Inverting the scaling, $x_1 \to x_1\rho$, $x_2 \to x_2\rho$ and $x_3 \to x_3\rho^2$ to return to the original Schwinger parameters and to get rid of ρ, the resulting integral representation is as stated. The divergence at $\varepsilon = 0$ is explicit in the prefactor, but (in contrast to the original Lee-Pomeransky representation) the integral over the Schwinger parameters is holomorphic also at $\varepsilon = 0$. Hence we can expand under the integral:

$$I = -\frac{\Gamma(3-\varepsilon)}{2\Gamma(1-2\varepsilon)}\sum_{k\geq 0}\frac{\varepsilon^{k-1}}{k!}\int_0^\infty \mathrm{d}x_1\int_0^\infty \mathrm{d}x_2\int_0^\infty \mathrm{d}x_3 \frac{x_1+x_2}{(\mathcal{U}+\mathcal{F})^3}\log^k(\mathcal{U}+\mathcal{F}).$$

7. Show that the leading order (coefficient of $1/\varepsilon$) is proportional to a bubble integral.

   Solution: For the leading order, the ρ-integral in 6. simplifies to

$$\frac{1}{2\varepsilon}\int_0^\infty \rho^{-2\varepsilon}\mathrm{d}\rho\,\frac{\partial}{\partial\rho}\frac{1}{(x_3+\mathcal{F}+\rho x_1+\rho x_2)^{2-\varepsilon}} = -\frac{1}{2\varepsilon}\frac{1}{(x_3+\mathcal{F})^2}+\mathcal{O}\left(\varepsilon^0\right)$$

   and the subsequent x_3-integral is straightforward, leaving:

$$I = -\frac{1}{2\varepsilon}\int_0^\infty \mathrm{d}x_1\int_0^\infty \mathrm{d}x_2\frac{\delta(1-x_1)}{\mathcal{F}}+\mathcal{O}\left(\varepsilon^0\right).$$

   Up to the prefactor, this is identical to the bubble integral of $G/3$ in $D=2$ dimensions.

8. Explain where the divergence comes from in momentum space.

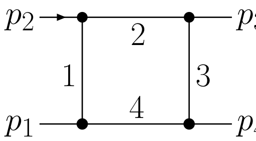
   Solution: Let ℓ denote the momentum flowing through edge 3. Then

$$\begin{aligned} I(D,1,1,1,z) &= \int_{\mathbb{R}^{1,D-1}}\frac{\mathrm{d}^D\ell}{i\pi^{D/2}}\frac{1}{-\ell^2}\frac{1}{m^2-(\ell+p_1)^2}\frac{1}{m^2-(\ell-p_2)^2} \\ &= \int_{\mathbb{R}^{1,D-1}}\frac{\mathrm{d}^D\ell}{i\pi^{D/2}}\frac{1}{-\ell^2}\frac{1}{\ell^2+2\ell p_1}\frac{1}{\ell^2-2\ell p_2} \end{aligned}$$

   For $\ell \to 0$, the integrand grows with $\|\ell\|^{-4}$, while the volume element in $D=4$ scales as $\|\ell\|^3\,\mathrm{d}\,\|\ell\|$. This shows a logarithmic divergence at $\ell \to 0$.

4 Polynomial reduction

1. Show that the Landau variety of the massless box integral ($m_e^2 = p_i^2 = 0$)



   is $L = \{s,t,u\}$ where $\begin{cases} s = (p_1+p_2)^2 \\ t = (p_1+p_3)^2 \\ u = (p_1+p_4)^2 \end{cases}$

   Solution: Start with the singularities of the integrand, in the projective representation with $\delta(1-x_4)$: Set $s = -(p_1+p_2)^2$ and $t = -(p_1+p_4)^2$, then

$$S = \{\mathcal{U}|_{x_4=1}, \mathcal{F}|_{x_4=1}\} = \{1+x_1+x_2+x_3, sx_2+tx_1x_3\}.$$

   Reduction of x_1:

$$S_1 = \{\underbrace{1+x_2+x_3, s, x_2}_{x_1\to 0}, \underbrace{t, x_3}_{x_1\to\infty}, \underbrace{sx_2-tx_3(1+x_2+x_3)}_{\text{resultant}}\}$$

Reduction of x_2:

$$S_{1,2} = \{s, t, x_3, 1 + x_3, s - tx_3\}$$

Reduction of x_3:

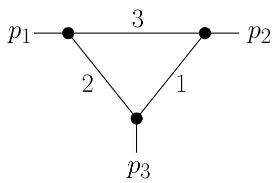
$$S_{1,2,3} = \{s, t, s + t\}$$

Hence the Landau variety $L \subseteq S_{1,2,3}$ has at most 3 components. Clearly $\{s = 0\}$ and $\{t = 0\}$ are necessary, since the special cases $n = (0, 1, 0, 1)$ and $n = (1, 0, 1, 0)$ correspond to bubble integrals $\propto s^{-\varepsilon}$ and $\propto t^{-\varepsilon}$, respectively, which have singularities at $s = 0$ and $t = 0$. To see that singularities at $s + t = 0$ also appear, consider for example the finite box integral in $D = 6$:

$$\begin{aligned}
I(6,1,1,1,1,s,t) &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3}{(sx_2 + tx_1x_3)(1 + x_1 + x_2 + x_3)^2}, \quad \text{set } x_2 \to x_2x_3 \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3}{(sx_2 + tx_1)(1 + x_1 + x_3(1 + x_2))^2} \\
&= \int_0^\infty \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2}{(sx_2 + tx_1)(1 + x_1)(1 + x_2)} = \int_0^\infty \frac{\mathrm{d}x_1}{(1 + x_1)(tx_1 - s)} \log \frac{tx_1}{s} \\
&= \frac{\pi^2 + \log^2(s/t)}{2(s + t)}
\end{aligned}$$

Taking two variations around $s = 0$, this becomes $(2i\pi)^2/(s + t)$, which clearly has a pole at $s + t = 0$. In conclusion, $L = S_{1,2,3} = \{s, t, s + t\}$.

2. Consider the triangle integral for generic momenta p_1^2, p_2^2, p_3^2 as in the lecture, but with an internal mass $m_3 \neq 0$ (still $m_1 = m_2 = 0$):



Show that with $\Delta = p_1^4 + p_2^4 + p_3^4 - 2p_1^2p_2^2 - 2p_1^2p_3^2 - 2p_2^2p_3^2$, its Landau variety is

$$L = \left\{p_1^2, p_2^2, p_3^2, \Delta, m_3^2, m_3^2 - p_1^2, m_3^2 - p_2^2, (m_3^2 - p_1^2)(m_3^2 - p_2^2) + m_3^2p_3^2\right\}.$$

Solution: In the projective representation with $x_3 = 1$, the singularities of the integrand are

$$S = \Big\{ \underbrace{1 + x_1 + x_2}_{\mathcal{U}|_{x_3=1}}, \underbrace{-p_1^2x_2 - p_2^2x_1 - p_3^2x_1x_2 + m_3^2(1 + x_1 + x_2)}_{\mathcal{F}|_{x_3=1}} \Big\}$$

Reduction of x_1:

$$S_1 = \Big\{ \underbrace{1 + x_2, m_3^2(1 + x_2) - p_1^2x_2}_{x_1 \to 0}, \underbrace{m_3^2 - p_2^2 - p_3^2x_2}_{x_1 \to \infty}, \underbrace{(1 + x_2)(p_2^2 + p_3^2x_2) - p_1^2x_2}_{\text{resultant } [\mathcal{U}, \mathcal{F}]} \Big\}$$

Reduction of x_2:

$$S_{1,2} = \Big\{ \underbrace{m_3^2, m_3^2 - p_2^2, p_2^2}_{x_2 \to 0}, \underbrace{m_3^2 - p_1^2, p_3^2}_{x_2 \to \infty}, \underbrace{p_1^2, m_3^2 + p_3^2 - p_2^2, (m_3^2 - p_1^2)(m_3^2 - p_2^2) - m_3^2p_3^2, \Delta}_{\text{resultants}} \Big\}$$

where $\Delta = p_1^4 + p_2^4 + p_3^4 - 2p_1^2p_2^2 - 2p_1^2p_3^2 - 2p_2^2p_3^2$ denotes the discriminant that also appeared in the massless case. By symmetry (flipping $x_1 \leftrightarrow x_2$ and $p_1 \leftrightarrow p_2$), note

$$S_{2,1} = S_{1,2}\big|_{p_1 \leftrightarrow p_2} = \Big(S_{1,2} \setminus \{m_3^2 + p_3^2 - p_2^2\}\Big) \cup \{m_3^2 + p_3^2 - p_1^2\}.$$

Hence the component $m_3^2 + p_3^2 - p_2^2 \in S_{1,2}$ is spurious, and we get the upper bound

$$L \subseteq S_{1,2} \cap S_{2,1} = \left\{ p_1^2, p_2^2, p_3^2, \Delta, m_3^2, m_3^2 - p_1^2, m_3^2 - p_2^2, (m_3^2 - p_1^2)(m_3^2 - p_2^2) + m_3^2 p_3^2 \right\}.$$

In fact, all these singularities indeed appear, hence $L = S_{1,2} \cap S_{2,1}$.

For example, setting $n_1 = 0$, the contracted graph is a bubble with one massless and one massive propagator. In $D = 4$ with the massive propagator squared ($n_3 = 2$), this bubble integral is

$$I(4,0,1,2,z) = \int_0^\infty \frac{x_3 \mathrm{d}x_3}{(1+x_3)(m_3^2 x_3(1+x_3) - p_1^2 x_3)} = \frac{1}{-p_1^2} \log \frac{m_3^2 - p_1^2}{m_3^2}.$$

This exhibits singularities at $p_1^2 = 0$, $m_3^2 = 0$ and $m_3^2 - p_1^2 = 0$. By symmetry, the bubble with $n = (1,0,2)$ also gives singularities at $p_2^2 = 0$ and $m_3^2 - p_2^2 = 0$. The massless bubble $n = (1,1,0)$ is proportional to a power of p_3^2. In summary, the bubble quotients imply the lower bound

$$L \supseteq \left\{ p_1^2, p_2^2, p_3^2, m_3^2, m_3^2 - p_1^2, m_3^2 - p_2^2 \right\}.$$

The component $(m_3^2 - p_1^2)(m_3^2 - p_2^2) + m_3^2 p_3^2 = 0$ is the leading Landau singularity of the triangle. For example, it appears in

$$I(4,1,1,2,z) = \int_0^\infty \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2}{\mathcal{F}^2|_{x_3=1}} = \frac{1}{(m_3^2 - p_1^2)(m_3^2 - p_2^2) + m_3^2 p_3^2} \log \frac{(m_3^2 - p_1^2)(m_3^2 - p_2^2)}{-p_3^2 m_3^2}.$$

Finally, the fact that $\Delta \in L$ is in some sense the most complicated. It is an example of a "singularity of the second type" [**?**]. As the calculation above shows, Δ arises as the discriminant of the resultant $[\mathcal{U}, \mathcal{F}]$. In particular, it is absent in the integral $I(4,1,1,2,z)$ above because that only depends on $\mathcal{F}$. One can see Δ explicitly for example as the denominator of

$$I(4,1,1,1,z) = \int_0^\infty \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2}{\mathcal{U}\mathcal{F}|_{x_3=1}} = \frac{1}{\sqrt{\Delta}} \times \{\text{sum of several dilogarithms}\}.$$

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