Acta Physica Austriaca, Suppl. VII, 32—70 (1970)

INTRODUCTION TO THE USE OF NON-LINEAR TECHNIQUES IN S-MATRIX THEORY*

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CONTENTS

1. INTRODUCTION

I am going to explain to you how one can tackle certain problems in S-matrix theory that involve non-linear functional equations. A physicist's usual reaction to a non-linear equation of this kind would be either to try to get an approximate solution by iteration, or to introduce a linearization, perhaps in the neighbourhood

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*Lecture given at IX. Internationale Universitätswochen für Kernphysik, Schladming, February 23 - March 7,1970 of a known approximate solution. I will introduce some concepts of Banach space analysis [1], which will enable us to put these intuitive ideas on a rigorous basis. The advantage is that one can sometimes prove the existence of solutions of the exact equations, without any approximations. For almost all of these talks, I will limit myself to the Contraction Mapping Principle, which is perhaps the simplest technique available, and corresponds precisely to trying to find a function, say $\Phi(x,y,\ldots)$, that satisfies a non-linear functional equation,

$$\Phi(x,y,\ldots) = P[\Phi;x,y,\ldots] \tag{1.1}$$

by means of an iteration

$$\Phi_{n+1}(x,y,\ldots) = P[\Phi_n;x,y,\ldots] \quad . \tag{1.2}$$

I will develop the proof that, when certain "contraction" conditions are observed, then, not only does the iteration (1.2) converge in a well-defined sense, but the limit-function satisfies the exact equation (1.1) . Moreover, we will be able to estimate the error involved in stopping the iteration after N steps, which will clearly be very useful, since in practice one always has to truncate an iteration, if only because someone else wants to use the computer.

I will first of all apply the technique to a problem that has been considered by Martin [2], namely, under what circumstances does a measurement of the differential cross-section, in the elastic region, serve to determine the phase-shifts uniquely, via the elastic unitarity condition. It turns out that, if the modulus of the amplitude satisfies a certain explicit condition, then its phase is uniquely determined (except for an overall sign). I will give a slightly simplified version of part of Martin's proof.

In this application, one can use a Banach space of continuous functions; but when the non-linear equations also involve principal-value integrals, one needs to use a space of Hölder-continuous functions. I will introduce this space and apply it to the pion-pion equations in the Shirkov [3] approximation. My proof will be similar in some ways to Warnock's [4] treatment of the Low equation, except that I will consider the equation with no subtractions, since this simplifies matters. In particular, I will need no cut-off.

Then I will consider the exact Mandelstam [5] equations for pion-pion scattering; and I will explain the simplest version of the existence proof I [6] worked out two years ago. Again Hölder-continuous functions are used, but this time with respect to two variables.

Lastly, I propose to sketch the progress that has been made with generalizations of this last proof, in particular the introduction of subtractions [7] and CDD poles [8]. I will also mention the possible use of the Newton-Kantorovich method, which is the rigorous way to linearize in the vicinity of an approximate solution. I will finish by indicating a few outstanding problems, which may, or may not be tractable.

2. CONTRACTION MAPPING PRINCIPLE

A Banach space is a complete, normed, linear metric space. We will always be talking about Banach spaces in which the elements or "points" are functions, either of one or of two variables. That is, each of our spaces will consist in a set of functions that satisfy certain specific properties that are characteristic of the space in question. The norm of a function, $\Phi(x)$, is a number that is associated with $\Phi(x)$, and is written $\|\Phi\|$. This assignment of numbers (norms) to the functions cannot be done in a completely arbitrary way, but must be such that, for any Φ and Ψ belonging to the space, the following properties hold good:

$$\|\Phi\| \geq 0 \qquad (2.1)$$

$$\|\Phi\| = 0 \text{ if and only if } \Phi(x) \equiv 0 \qquad (2.2)$$

$$\|\Phi+\Psi\| \leq \|\Phi\| + \|\Psi\| \qquad (2.3)$$

The requirement of linearity means that, for any real (or complex) number, λ,

$$\|\lambda \Phi\| = |\lambda| \, \|\Phi\| \qquad (2.4)$$

To say that the normed space is complete is to assert that every Cauchy sequence of functions in the space converges to a function that belongs to the space. To say that $\{\Phi_n\}$ is a Cauchy sequence means that, for any $\varepsilon>0$, one can find an N such that

$$\|\Phi_{p+n} - \Phi_p\| < \varepsilon \qquad (2.5)$$

for any $p \geq N$, and $n=1,2,3,\ldots$. If the space is complete, then there necessarily exists a limit function, Φ, which belongs to the space. That is, there is a Φ such that

$$\|\Phi - \Phi_p\| < \varepsilon \qquad (2.6)$$

for any $p \geq N$. It is these two properties of having a linear norm structure, and completeness that make the use of Banach spaces indispensable in functional analysis.

The contraction mapping principle, specialized to a Banach space, can be stated as follows: Suppose that the non-linear operator, P, maps a complete set in the space into itself, and that Φ and Ψ are any two "points" belonging to this set, with

$$\Phi' = P(\Phi) \qquad (2.7)$$

$$\psi' = P(\psi) \qquad (2.8)$$

If

$$\|\Phi' - \Psi'\| \leq k\|\Phi - \Psi\| \tag{2.9}$$

where $k<1$, then the equation

$$\Phi = P(\Phi) \tag{2.10}$$

has a unique solution in the set in question, which may be obtained by iteration

$$\Phi_{n+1} = P(\Phi_n) \tag{2.11}$$

so long as Φ_o belongs to the set.

This principle, which I will prove in a moment, is just the common sense statement that an iteration converges if successive steps get smaller and smaller. I want to draw your attention to the fact, however, that it is crucial that the set be complete in the first place. The technique of proof is in fact to show that the sequence $\{\Phi_n\}$, defined by eq. (2.11), is Cauchy. For

$$\|\Phi_{n+p} - \Phi_p\| \leq \sum_{m=p}^{n+p-1} \|\Phi_{m+1} - \Phi_m\| \tag{2.12}$$

by eq. (2.3). Now from eq. (2.11) and (2.7) - (2.9),

$$\|\Phi_{m+1} - \Phi_m\| \leq k \|\Phi_m - \Phi_{m-1}\| \tag{2.13}$$

so that, by iteration,

$$\|\Phi_{m+1} - \Phi_m\| \leq k^m \|\Phi_1 - \Phi_o\| \ . \tag{2.14}$$

Hence, from eq. (2.12),

$$\|\Phi_{n+p} - \Phi_p\| \leq \|\Phi_1 - \Phi_o\| \sum_{m=p}^{n+p-1} k^m \leq \|\Phi_1 - \Phi_o\| \frac{k^p}{1-k} \ . \tag{2.15}$$

Since $k<1$, it follows that, for any pre-assigned $\varepsilon>0$, one can certainly choose p so that the right-hand side of eq. (2.15) is smaller than ε, so that $\{\Phi_n\}$ is a Cauchy sequence, and hence has a limit, say Φ. It is easy to see, by letting $n\to\infty$ in eq. (2.15), that

$$\|\Phi - \Phi_p\| \leq \|\Phi_1 - \Phi_o\| \frac{k^p}{1-k} \ . \tag{2.16}$$

This is a useful inequality, since it gives a bound on the error committed by stopping the iteration after p steps.

In order to finish the proof in the tidy way that mathematicians like, we should show (a) that the limit function, Φ, really satisfies eq. (2.10), and (b) that it is the only function (within the complete set in question) that does so. In view of eq. (2.16), given any $\varepsilon > 0$, we can certainly find a p such that

$$\|\Phi - \Phi_m\| < \varepsilon/2 \qquad (2.17)$$

for all $m \geq p$. Then

$$\|P(\Phi)-\Phi\| \leq \|P(\Phi)-P(\Phi_p)\| + \|P(\Phi_p)-\Phi\| \leq k\|\Phi-\Phi_p\| + $$
$$+ \|\Phi_{p+1} - \Phi\| < k\,\frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon \quad . \qquad (2.18)$$

Since ε can be as small as one likes, one must have

$$\|P(\Phi) - \Phi\| = 0 \qquad (2.19)$$

from which eq. (2.10) follows, by virtue of property (2.2).

Lastly, one can prove, by reductio ad absurdum, that Φ is locally unique. For suppose, on the contrary, that there were two different functions, Φ and Ψ, belonging to the complete set, such that

$$\Phi = P(\Phi) \qquad (2.20)$$

and

$$\Psi = P(\Psi) \quad . \qquad (2.21)$$

According to the contraction condition, eq. (2.9),

$$\|\Phi-\Psi\| \leq k\,\|\Phi-\Psi\| \quad . \qquad (2.22)$$

Since $\Phi-\Psi$ is not identically zero, it follows from eq. (2.1) and (2.2) that

$$\|\Phi - \Psi\| > 0 \quad . \qquad (2.23)$$

Hence eq. (2.22) implies

$$k \gtrsim 1 \quad . \tag{2.24}$$

This is absurd, since one knows that $k < 1$.

3. SPACE OF CONTINUOUS FUNCTIONS

A simple, and often very useful Banach space is the set of all continuous functions, $\Psi(x)$, $-1 \le x \le 1$, with the norm

$$\|\Psi\| = \sup_{-1 \le x \le 1} |\Psi(x)| \quad . \tag{3.1}$$

It is easy to check eq. (2.1) - (2.4). To show that the space is complete, let $\{\Psi_n(x)\}$ be a Cauchy sequence in the space. For a given, fixed x, one knows, by the Bolzano-Weierstrass theorem, that $\Psi_n(x)$ tends to a limit, that may be called $\Psi(x)$, as $n \to \infty$. It has to be shown that $\Psi(x)$ is continuous, and so belongs to the space. Now, for any n,

$$|\Psi(x_1) - \Psi(x_2)| \le |\Psi(x_1) - \Psi_n(x_1)| + |\Psi(x_2) - \Psi_n(x_2)| + $$

$$+ |\Psi_n(x_1) - \Psi_n(x_2)| \quad . \tag{3.2}$$

Given any $\varepsilon > o$, one can certainly choose n so large that the first two terms on the right-hand side of eq. (3.2) are each less than $\varepsilon/3$. One can find a δ so small that

$$|\Psi_n(x_1) - \Psi_n(x_2)| < \varepsilon/3 \tag{3.3}$$

for all $|x_1 - x_2| \le \delta$, since $\Psi_n(x)$ is continuous. Hence

$$|\Psi(x_1) - \Psi(x_2)| < \varepsilon \tag{3.4}$$

for all $|x_1 - x_2| \le \delta$, which means that $\Psi(x)$ is continuous.

This space has been used by Martin to tackle the following problem: Suppose that you know the modulus, B , of a two-particle elastic scattering amplitude, in the elastic region (for example from a measurement of the differential scattering cross-section). Under what circumstances does the elastic unitarity condition serve to define uniquely the phase, Φ , of the scattering amplitude, and hence to determine uniquely the phase -shifts? The unitarity condition can be written

$$B(z)\ \sin\Phi(z) = \tag{3.5}$$
$$= \frac{1}{4\pi}\int_{-1}^{1} dz_1 \int_{0}^{2\pi} d\phi_1\ B(z_1)B(z_2)\exp\{i[\Phi(z_1)-\Phi(z_2)]\}$$

where z is the cosine of the scattering angle, and where the dependence on the energy has been suppressed. In eq. (3.5), z_2 is to be considered as a function of z, z_1 and ϕ_1, according to

$$z_2 = z\ z_1 + (1-z^2)^{\frac{1}{2}}(1-z_1^2)^{\frac{1}{2}} \cos\phi_1 \quad . \tag{3.6}$$

Since B(z) is known, eq. (3.5) is to be regarded as an equation for the unknown $\Phi(z)$. Under what conditions on B is there a unique solution? We will apply the contraction mapping principle in the space of continuous functions on the domain $-1\leq z\leq 1$.

The equation (3.5) can be rewritten

$$\Phi(z) \equiv P[\Phi;z] = \tag{3.7}$$
$$= \sin^{-1}\{\int d\Omega_1\ H(z,z_1,\phi_1)\cos[\Phi(z_1)-\Phi(z_2)]\}$$

where

$$H(z,z_1,\phi_1) = \frac{B(z_1)B(z_2)}{4\pi\ B(z)} \tag{3.8}$$

and where the symmetry of the integral (3.5) has been used to eliminate the imaginary part. Suppose that H is such that

$$\int d\Omega_1\ H(z,z_1,\phi_1) \leq \sin\mu < 1 \tag{3.9}$$

where $0<\mu<\frac{\pi}{2}$. This imposes a restriction on B(z). From eq. (3.7), it follows that

$$|\Phi(z)| \le \sin^{-1}\{\int d\Omega\, H(z,z_1,\phi_1)\} \le \mu \ . \tag{3.10}$$

Let Φ_{max} and Φ_{min} be the maximum and minimum values of $\Phi(z)$. If we define $\Phi(o)$ to lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, then, since we are looking for a solution of eq. (3.7) that is continuous, it follows from (3.10) that

$$-\frac{\pi}{2} < \Phi_{min} \le \Phi_{max} < \frac{\pi}{2} \tag{3.11}$$

Hence

$$0 \le \Phi_{max} - \Phi_{min} < \pi \tag{3.12}$$

I will now show, following Martin, that in fact eq. (3.10) can be strengthened to

$$0 \le \Phi(z) \le \mu \ . \tag{3.13}$$

Consider two cases: either

$$0 \le \Phi_{max} - \Phi_{min} \le \frac{\pi}{2} \quad , \tag{3.14}$$

in which case $\cos[\Phi(z_1)-\Phi(z_2)]$ in eq. (3.7) can never be negative, so that $\Phi(z) \ge o$ for all z. On the other hand, if

$$\frac{\pi}{2} < \Phi_{max} - \Phi_{min} < \pi \quad , \tag{3.15}$$

then $\cos[\Phi(z_1)-\Phi(z_2)]$ could apparently be negative for some values of z_1 and z_2, but eq. (3.7) implies that

$$\begin{aligned} \sin \Phi_{min} &\ge \int d\Omega_1\, H(z,z_1,\phi_1)\cos[\Phi_{max} - \Phi_{min}] \ge \\ &\ge \sin\mu[\cos\Phi_{max}\cos\Phi_{min} + \sin\Phi_{max}\sin\Phi_{min}] \end{aligned} \tag{3.16}$$

since $\cos[\Phi_{max} - \Phi_{min}]$ is negative, by assumption (3.15). Hence

$$\sin\Phi_{min} \ge \frac{\sin\mu\cos\Phi_{max}\cos\Phi_{min}}{1 - \sin\mu\sin\Phi_{max}} \ge \frac{\sin\mu\cos^2\mu}{1-\sin^2\mu} = \sin\mu \tag{3.17}$$

This contradicts eq. (3.10), so in fact the apparent alternative (3.15) is disallowed.

This result can be rephrased as follows: Consider the mapping

$$\Phi'(z) = P[\Phi(z);z] \tag{3.18}$$

where P is defined in eq. (3.7). What has been shown is that if eq. (3.9) is satisfied, then the set in the Banach space of continuous functions that is defined by

$$0 \leq \Phi(z) \leq \mu \tag{3.19}$$

is mapped into itself by P . In technical terms, this set would be described as the intersection of the ball

$$||\Phi|| \leq \mu \tag{3.20}$$

with the norm of eq. (3.1), and the cone

$$\Phi(z) \geq 0 \quad . \tag{3.21}$$

A cone in a Banach space is a set such that if Φ belongs to it, then so does $c\Phi$, where c is any nonnegative real number.

If one were to iterate eq. (3.18), according to

$$\Phi_{n+1} = P[\Phi_n] \tag{3.22}$$

for $n = 0,1,2,...$, with $\Phi_o(z)$ satisfying (3.19), it is clear that, for any $\sin\mu < 1$, the infinite set of iterates all satisfy eq. (3.19). Martin has shown that these iterates have at least one limit-point in the set (3.19), but the proof involves the Schauder principle, which I do not intend to explain in this course. If $\sin\mu$ is substantially smaller than unity, one can use the contraction mapping principle to show that there is one, and only one limit-point in the space of continuous functions. I will explain this to you.

Suppose that $\Phi^a(z)$ and $\Phi^b(z)$ are any two continuous functions that satisfy the inequality (3.19). Then it follows from eq. (3.18), by a series of elementary trigonometric manipulations, that

$$\sin \tfrac{1}{2}[\phi'^a(z)-\phi'^b(z)] = 2 \sec \tfrac{1}{2}[\phi'^a(z)-\phi'^b(z)] \times$$

$$\times \int d\Omega_1 \; H(z,z_1,\phi_1) \sin \tfrac{1}{2}[\phi^a(z_1)-\phi^b(z_1)] \times$$

$$\times \cos \tfrac{1}{2}[\phi^a(z_2)-\phi^b(z_2)] \sin\tfrac{1}{2}[\phi^a(z_1) +$$

$$+ \phi^b(z_1)-\phi^a(z_2)-\phi^b(z_2)] \tag{3.23}$$

The inequality (3.19), which has been shown to hold also for ϕ', implies

$$|\sin\tfrac{1}{2}[\phi'^a(z)-\phi'^b(z)]| \le \frac{2\sin^2\mu}{\cos\mu} \sup_{-1\le z_1\le 1} [\sin\tfrac{1}{2} \phi^a(z_1)- -\phi^b(z_1)]| \tag{3.24}$$

Since

$$x \ge \sin x \ge \frac{2x}{\pi} \tag{3.25}$$

for $0 \le x \le \frac{\pi}{2}$, it follows from eq. (3.24) that

$$|\phi'^a(z)-\phi'^b(z)| \le \frac{\pi\sin^2\mu}{\cos\mu} \sup_{-1\le z\le 1} |\phi^a(z)-\phi^b(z)| \tag{3.26}$$

or

$$||\phi'^a-\phi'^b|| \le \frac{\pi\sin^2\mu}{\cos\mu} ||\phi^a - \phi^b|| \tag{3.27}$$

The condition for a contraction mapping is accordingly

$$\frac{\pi\sin^2\mu}{\cos\mu} < 1 \tag{3.28}$$

or

$$\sin\mu < [\frac{2}{1+(1+4\pi^2)^{1/2}}]^{1/2} \tag{3.29}$$

The bound (3.29) imposes a restriction on the magnitude of B(z), the modulus of the amplitude, for the applicability of the above contraction mapping proof of the existence and uniqueness of a solution of the equation (3.7). One can extend the domain of the proof by remarking that

$$||\phi||_1 \equiv \sup_{-1\le z\le 1} |\sin\tfrac{1}{2}\phi(z)| \tag{3.30}$$

can be used as an alternative norm. One has to check the triangle inequality, eq. (2.3), but this is easily done (exercise). With this norm, one has, directly from eq. (3.24),

$$\|\phi'^a - \phi'^b\|_1 \leq \frac{2\sin^2\mu}{\cos\mu} \|\phi^a - \phi^b\|_1 \tag{3.31}$$

and this leads to the requirement

$$\sin\mu < \left[\frac{2}{1 + (17)^{1/2}}\right]^{1/2} \approx 0.62 \tag{3.32}$$

which is an improvement.

By including in longer trigonometrical manipulations, Martin has managed to make the uniqueness proof work for

$$\sin\mu < 0.79 \quad . \tag{3.33}$$

I will not go into his proof, which would take us too far afield, without introducing any new point of principle. I can refer you to his paper if you are interested. Incidentally, there is an outstanding problem: there is some reason to expect that one should have uniqueness for any

$$\sin\mu < 1 \tag{3.34}$$

but no-one has been able to bridge the gap between 0.79 and 1.00 . I leave it as an exercise for the student to extend the proof to 1.00, or to find a counter-example. In either case, tell André Martin immediately!

4. PRINCIPAL-VALUE INTEGRALS

When one has to deal with mappings that involve principal-value integrals, one can no longer use the space of continuous functions, because the principal-value integral of a continuous function is not necessarily continuous. I will show you how to prove that the principal-value integral of a Hölder-continuous function is itself Hölder continuous. Then we will construct a Banach space of Hölder-continuous functions, in which we will use the contraction mapping theorem again.

The theorem I will prove is as follows: Suppose that

$$f(x) = \frac{P}{\pi}\int_0^1 \frac{dx'\ \sigma(x')}{x'-x}\ , \tag{4.1}$$

where $\sigma(x)$ satisfies

$$\sigma(o) = 0 = \sigma(1) \tag{4.2}$$

and

$$|\sigma(x_1)-\sigma(x_2)| \le \xi|x_1-x_2|^\mu \tag{4.3}$$

for any x_1, x_2 in the interval [0,1] where ξ is constant, and where μ satisfies $0<\mu<1$. Eq. (4.3) is the statement of Hölder continuity. Then we will prove that

$$|f(x_1)-f(x_2)| \le c\xi|x_1-x_2|^\mu \tag{4.4}$$

for any x_1, x_2 in [0,1], where c depends only on the Hölder index, μ.

One has to be a little bit careful about the end points of the integration range, in order to avoid logarithm singularities. It is for this reason that one needs eq. (4.2). One can work the proof most elegantly by extending formally the integration range in eq. (4.1) to

$$f(x) = \frac{P}{\pi}\int_{-2}^2 \frac{dx'\ \sigma(x')}{x'-x} \tag{4.5}$$

by defining $\sigma(x')=0$ for $-2\le x'\le 0$ and $1\le x'\le 2$. Because of eq. (4.2), one can extend the Hölder-continuity (4.3) over the whole range $-2\le x_1,x_2\le 2$. For example, suppose $0\le x_1\le 1$ and $-2\le x_2\le 0$. Then

$$\begin{aligned} |\sigma(x_1)-\sigma(x_2)| &= |\sigma(x_1)-0| = \\ &= |\sigma(x_1)-\sigma(o)| \le \xi|x_1-0|^\mu \le \xi|x_1-x_2|^\mu\ , \end{aligned} \tag{4.6}$$

and similarly for the other possibilities. Note the use of eq. (4.2) in the second line. Although x' now ranges over [-2,2] , x is still restricted to the range [0,1] in eq. (4.5).

One has

$$f(x) = \frac{1}{\pi}\int_{-2}^{2} dx' \frac{\sigma(x')-\sigma(x)}{x'-x} + \frac{\sigma(x)}{\pi} P\int_{-2}^{2} \frac{dx'}{x'-x} \qquad (4.7)$$

so that

$$|f(x_1)-f(x_2)| \le B_1 + B_2 \qquad (4.8)$$

where

$$B_1 = \frac{1}{\pi}\left|\int_{-2}^{2} dx'\left\{\frac{\sigma(x')-\sigma(x_1)}{x'-x_1} - \frac{\sigma(x')-\sigma(x_2)}{x'-x_2}\right\}\right| \qquad (4.9)$$

and

$$B_2 = \frac{1}{\pi}\left|\sigma(x_1)\log\frac{2-x_1}{2+x_1} - \sigma(x_2)\log\frac{2-x_2}{2+x_2}\right| . \qquad (4.10)$$

We have to tackle B_1 and B_2 in turn, and show that each is less than $|x_1-x_2|^{\mu}$, multiplied by a constant. Consider B_1 first, and suppose $x_2 \ge x_1$ for definiteness. Define $\theta = x_2-x_1$. We have a delicate piece of engineering to do. We will divide the integral (4.9) into the interval $x_1-2\theta \le x' \le x_1+2\theta$, which we might call Ω , and the rest, called $\bar{\Omega}$. I will leave you to check that $x_1+2\theta \ge x_2$ so that we have both "Cauchy points", $x'=x_1$ and $x'=x_2$, inside the interval Ω . Also you can check that $x_1-2\theta \ge -2$ and $x_1+2\theta \le 2$, which is why I extended the integration range as far as I did. Write

$$B_1 \le B_{11} + B_{12} + B_{13} \qquad (4.11)$$

where

$$B_{11} = \frac{1}{\pi}\int_{\Omega} dx'\left\{\left|\frac{\sigma(x')-\sigma(x_1)}{x'-x_1}\right| + \left|\frac{\sigma(x')-\sigma(x_2)}{x'-x_2}\right|\right\} \qquad (4.12)$$

$$B_{12} = \frac{1}{\pi}\left|\int_{\bar{\Omega}} dx'\{\sigma(x') - \sigma(x_1)\}\left\{\frac{1}{x'-x_1} - \frac{1}{x'-x_2}\right\}\right| \qquad (4.13)$$

and

$$B_{13} = \frac{1}{\pi}\left|\int_{\bar{\Omega}} \frac{dx'}{x'-x_2}\{[\sigma(x')-\sigma(x_1)]-[\sigma(x')-\sigma(x_2)]\}\right| . \qquad (4.14)$$

So far as B_{11} is concerned, we use the Hölder continuity directly to yield

$$B_{11} \leq \frac{\xi}{\pi} \int_{\Omega} dx' \{ |x'-x_1|^{-1+\mu} + |x'-x_2|^{-1+\mu} \} =$$

$$= \frac{\xi}{\pi\mu} [1+2^{1+\mu} + 3^{\mu}] |x_1-x_2|^{\mu} \quad , \tag{4.15}$$

so that piece has the right form. For B_{12} one can be a bit more brutal, because the integrand has no singularity:

$$B_{12} \leq \frac{1}{\pi} \int_{\bar{\Omega}} dx' |\sigma(x')-\sigma(x_1)| \left| \frac{x_1-x_2}{(x'-x_1)(x'-x_2)} \right| \leq$$

$$\leq \frac{|x_1-x_2|\xi}{\pi} \int_{\bar{\Omega}} dx' |x'-x_1|^{-1+\mu} |x'-x_2|^{-1} \leq$$

$$\leq \frac{2^{\mu}\,\xi}{\pi(1-\mu)} |x_1-x_2|^{\mu} \quad , \tag{4.16}$$

which is nice again. Finally, B_{13} is easy:

$$B_{13} \leq \frac{1}{\pi} |\sigma(x_1)-\sigma(x_2)| \left| \int_{\bar{\Omega}} \frac{dx'}{x'-x_2} \right| \leq \frac{\xi}{\pi} \log 3 |x_1-x_2|^{\mu} \quad , \tag{4.17}$$

thus completing the hat-trick (for those of you conversant with gaming, or cricket jargon).

We still have to perform upon B_2, but this presents no difficulty. From eq. (4.10),

$$B_2 \leq B_{21} + B_{22} \quad , \tag{4.18}$$

where

$$B_{21} = \frac{1}{\pi} |\sigma(x_1)-\sigma(x_2)| \log \frac{2+x_1}{2-x_1} \quad , \tag{4.19}$$

and

$$B_{22} = \frac{1}{\pi} |\sigma(x_2)| \{ \log(1+ \frac{x_2-x_1}{2+x_1}) + \log(1+ \frac{x_2-x_1}{2-x_2}) \} \quad . \tag{4.20}$$

Clearly

$$B_{21} \leq \frac{\xi}{\pi} \log 3 |x_1-x_2|^{\mu} \tag{4.21}$$

since log 3 is the largest value $\log(2+x_1/2-x_1)$ can have, for $0 \leq x_1 \leq 1$. I extended the range of x' up to two precisely to avoid the logarithmic divergence that we would have had otherwise. From eq. (4.2) and (4.3),

$$|\sigma(x_2)| = |\sigma(x_2)-\sigma(o)|$$
$$\leq \xi\, x_2^{\mu}$$
$$\leq \xi \quad ; \tag{4.22}$$

and since

$$\log(1 + A) \leq A \, , \tag{4.23}$$

for $A \geq 0$, it follows that

$$B_{22} \leq \frac{3\xi}{2\pi}|x_1-x_2|^{\mu} \tag{4.24}$$

Thus one has proved eq. (4.4), with the explicit estimate

$$c = \frac{1}{\pi}\left\{\frac{1}{\mu}(1+2^{1+\mu}+ 3^{\mu}) + \frac{2^{\mu}}{1-\mu} + 2 \log 3 + \frac{3}{2}\right\} \tag{4.25}$$

Notice that this explodes as $\mu \to o$ or $\mu \to 1$.

5. SPACE OF HÖLDER-CONTINUOUS FUNCTIONS

I have shown that the property of Hölder-continuity is transmitted through a principal-value integration, as it were. Now, I will now show how one can construct a complete space of Hölder-continuous functions; and then we will be ready to tackle a non-linear, singular integral equation.

Consider the space of all functions, $\sigma(x)$, defined for $0 \leq x \leq 1$, for which

$$\sigma(o) = 0 \quad , \tag{5.1}$$

and for which the condition of Hölder-continuity, eq. (4.3), is satisfied for some ξ. Consider the following norm

$$\|\sigma\| \equiv \sup_{0 \leq x_1, x_2 \leq 1} \frac{|\sigma(x_1)-\sigma(x_2)|}{|x_1 - x_2|^{\mu}} \quad . \tag{5.2}$$

One can easily check that the conditions (2.1) - (2.4) are satisfied, so that the set of all functions satisfying (5.1), and with a norm (5.2), constitute a linear, normed space.

I will now show that this space is complete, that is, it is a Banach space. The proof follows the lines of the corresponding proof for the space of continuous functions with the norm (3.1), but is a little more involved.

Let $\{\sigma_n\}$ be a uniformly bounded Cauchy sequence in the space, i.e.

$$\|\sigma_n\| \leq B \tag{5.3}$$

for all n; and, given any $\varepsilon > o$, there exists an N such that

$$\|\sigma_{N+p} - \sigma_N\| < \varepsilon , \tag{5.4}$$

for p=1,2,3,... . It has to be shown that

$$\sigma_n \to \sigma^* , \tag{5.5}$$

where the limit-function, σ^*, must belong to the space (i.e. it must be Hölder-continuous). It will be shown in fact that

$$\|\sigma^*\| \leq B . \tag{5.6}$$

First of all, it follows from the Bolzano-Weierstrass theorem that $\sigma^*(x)$ exists, if the limit (5.5) is understood in terms of the "usual" topology, i.e. for any $\bar{\varepsilon}>o$, there exists an n_o such that

$$|\sigma_n(x) - \sigma^*(x)| < \bar{\varepsilon} \tag{5.7}$$

for all $n>n_o$. For any N and p, it follows directly from the triangle inequality that

$$\frac{|\sigma^*(x_1)-\sigma^*(x_2)|}{|x_1 - x_2|^\mu} \leq \frac{|\sigma^*(x_1)-\sigma_{N+p}(x_1)|}{|x_1 - x_2|^\mu} + \frac{|\sigma_{N+p}(x_2)-\sigma^*(x_2)|}{|x_1 - x_2|^\mu}$$

$$+ \frac{|[\sigma_{N+p}(x_1)-\sigma_N(x_1)]-[\sigma_{N+p}(x_2)-\sigma_N(x_2)]|}{|x_1 - x_2|^\mu} + \frac{|\sigma_N(x_1)-\sigma_N(x_2)|}{|x_1 - x_2|^\mu} \tag{5.8}$$

The last term here is not greater than $\|\sigma_N\|$, whereas the penultimate term is bounded by $\|\sigma_{N+p} - \sigma_N\|$. Hence one can choose N such that the sum of these two terms is not greater than $B+\varepsilon$, while p is still completely free. Lastly, for any given x_1 and x_2, $x_1 \neq x_2$ one choose p to be so great that

$$|\sigma^*(x_1)-\sigma_{N+p}(x_1)| < \varepsilon|x_1-x_2|^\mu \tag{5.9}$$

and

$$|\sigma^*(x_2)-\sigma_{N+p}(x_2)| < \varepsilon|x_1-x_2|^\mu \tag{5.9}$$

This is certainly possible, according to eq. (5.7), if one sets $\bar{\varepsilon} = \varepsilon|x_1-x_2|^\mu$. Hence eq. (5.8) reduces to

$$\frac{|\sigma^*(x_1)-\sigma^*(x_2)|}{|x_1 - x_2|^\mu} < B + 3\varepsilon \quad . \tag{5.10}$$

Since ε may be as small as one likes, one may drop it from eq. (5.10), if $<$ is replaced by $\leq$. Moreover, for any $\varepsilon > o$,

$$|\sigma^*(o)| = |\sigma^*(o) - \sigma_N(o)| < \varepsilon \tag{5.11}$$

for N large enough. But since ε can be made indefinitely small, it follows that $\sigma^*(o)$ must vanish. Hence eq. (5.6) has been demonstrated, and with it the completeness of the space.

6. APPLICATION TO THE SHIRKOV EQUATIONS

I will first apply these results on Hölder-continuous functions to the Shirkov pion-pion equations, in the SP approximation.Let $F^I(s,t)$ be the total pion-pion scattering amplitude, the superscript, I=0,1,2, being the isospin. One can write a dispersion relation for the forward amplitude:

$$F(s,o) = \frac{1}{\pi}\int_4^\infty \frac{ds'}{s'-s}\,\mathrm{Im}\,F(s',o) + \frac{1}{\pi}\int_{-\infty}^{o} \frac{ds'}{s'-s}\,\mathrm{Im}\,F(s',o) \ . \tag{6.1}$$

Change the integration variable in the second integral from s' to 4-s', and use the crossing relation

$$F(s,o) = \eta\beta\eta\, F(4-s,o) \tag{6.2}$$

where the crossing matrices are

$$\beta = \frac{1}{6}\begin{pmatrix} 2 & 6 & 10 \\ 2 & 3 & -5 \\ 2 & -3 & 1 \end{pmatrix} \tag{6.3}$$

$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.4}$$

to obtain

$$F(s,o) = \frac{1}{\pi}\int_4^\infty ds'\left[\frac{1}{s'-s} - \frac{\eta\beta\eta}{s'+s-4}\right]\mathrm{Im}\, F(s',o) \quad . \tag{6.5}$$

Now introduce the approximation of retaining only S and P waves, so that

$$F^I(s,o) = f^I(s) \tag{6.6}$$

for I=0,2,, , where f^0 and f^2 are the S-wave amplitudes, and

$$F^1(s,o) = 3f^1(s) \tag{6.7}$$

where f^1 is the P-wave amplitude. The real part of eq.(6.5) becomes

$$\mathrm{Re}\, f(s) = \frac{P}{\pi}\int_4^\infty ds'\left[\frac{1}{s'-s} + \frac{\gamma}{s'+s-4}\right]\mathrm{Im}\, f(s') \tag{6.8}$$

where

$$\gamma = \begin{pmatrix} \frac{1}{3} & -3 & \frac{5}{3} \\ -\frac{1}{9} & \frac{1}{2} & \frac{5}{18} \\ \frac{1}{3} & \frac{3}{2} & \frac{1}{6} \end{pmatrix} \tag{6.9}$$

The unitarity relation connects the real and imaginary parts of f(s) according to

$$\mathrm{Im}\, f^I(s) = \left(\frac{s-4}{s}\right)^{1/2}\left\{[\mathrm{Re}\, f^I(s)]^2 + [\mathrm{Im}\, f^I(s)]^2\right\} + v^I(s) \tag{6.10}$$

where v(s) is the contribution from the inelastic channels. It must vanish below the inelastic threshold, s=16 . In terms of the usual elasticity function, $\eta^I(s)$, one has

$$v^I(s) = \frac{1}{4}\{1-[\eta^I(s)]^2\} \quad . \qquad (6.11)$$

It will be assumed that $\eta(s)$, or equivalently $v(s)$, is known, and the problem is to construct solutions, $f(s)$, of the non-linear singular system (6.8) and (6.10) .

The equation (6.10), with Re $f(s)$ regarded as being defined in terms of Im $f(s)$ by (6.8), is a non-linear expression for Im $f(s)$ in terms of itself, which may be summarized

$$\text{Im } f(s) = P[\text{Im } f;s] \quad . \qquad (6.12)$$

We will seek to find a solution in the space of Hölder-continuous functions, since eq. (6.8) involves a principal value integral.

A minor difficulty is that the integral in eq. (6.8) is over the infinite range $4 \le s' < \infty$, whereas, in Section 4, the proof was given for a finite domain, $0 \le x' \le 1$. It would be possible to transform eq. (6.8) according to $x = \frac{4}{s}$ and $x' = \frac{4}{s'}$, but it is neater simply to translate the theorem of Section 4 by the same transformation, read backwards. This, however, involves one significant change. Instead of eq. (4.1), consider

$$\bar{f}(x) = \frac{P}{\pi}\int_0^1 dx' \, \bar{\sigma}(x')\left[\frac{1}{x-x'} + \frac{1}{x'}\right] \quad ,(6.13)$$

since the above transformation takes this into

$$\bar{f}\left(\frac{4}{s}\right) = \frac{P}{\pi}\int_4^\infty \frac{ds'}{s'-s}\, \bar{\sigma}\left(\frac{4}{s'}\right) \quad . \qquad (6.14)$$

Suppose that $\bar{\sigma}(x')$ satisfies the same conditions (4.2) and (4.3) as did $\sigma(x')$. Then $\bar{f}(x)$ will satisfy the condition of Hölder-continuity, eq. (4.4), since the extra term $1/x'$ in eq. (6.13) has no effect on this proof. More than this,

$$\bar{f}(0) = 0 \qquad (6.15)$$

due to cancellation between the two terms in the square parentheses in eq. (6.13). I leave you to check this rigorously, given that $\bar{\sigma}(x')$ is Hölder-continuous. Set

$$\bar{f}(\tfrac{4}{s}) = F(s) \quad ; \quad \bar{\sigma}(\tfrac{4}{s}) = \Psi(s) \quad . \tag{6.16}$$

The theorem may be re-phrased as follows: If

$$F(s) = \frac{P}{\pi}\int_4^\infty \frac{ds'}{s'-s}\,\Psi(s') \quad , \tag{6.17}$$

where

$$\Psi(4) = 0 = \Psi(\infty) \ , \tag{6.18}$$

and

$$|\Psi(s_1) - \Psi(s_2)| \le b\left|\frac{s_1-s_2}{s_1 s_2}\right|^\mu \quad , \tag{6.19}$$

for $4 \le s_1, s_2 < \infty$, then

$$|F(s_1)-F(s_2)| \le Cb\left|\frac{s_1-s_2}{s_1 s_2}\right|^\mu \tag{6.20}$$

for $4 \le s_1, s_2 < \infty$, where C only depends on μ , and

$$F(\infty) = 0 \ , \tag{6.21}$$

this equation being the translation of eq. (6.15). Note that in general $F(4) \neq 0$.

A solution of the system (6.12) will be looked for in the Banach space of Hölder-continuous functions, which, in terms of the variable s, may be described as the set of all functions $\Psi(s)$, defined on $4 \le s \le \infty$, for which

$$\Psi(\infty) = 0 \ , \tag{6.22}$$

and which have a finite norm

$$\|\Psi\| = \sup_{4 \le s_1, s_2 < \infty} \frac{|\Psi(s_1)-\Psi(s_2)|}{\left|\frac{s_1-s_2}{s_1 s_2}\right|^\mu} \quad . \tag{6.23}$$

We have now set up the apparatus with which to probe eq. (6.12). Consider the mapping

$$\text{Im } f'(s) = P[\text{Im } f;s] \quad . \tag{6.24}$$

Suppose that Im f(s) belongs to the space defined by eqs. (6.22) and (6.23), and in fact satisfies

$$\|\text{Im } f(s)\| \leq b \ , \tag{6.25}$$

and also

$$\text{Im } f(4) = 0 \ . \tag{6.26}$$

Suppose that the known function, v(s), belongs to the ball

$$\|v(s)\| \leq B \ , \tag{6.27}$$

and to the cone defined by

$$v(s) = 0 \ , \tag{6.28}$$

for $4 \leq s \leq 16$, and

$$v(s) \geq 0 \tag{6.29}$$

for $s > 16$.

It follows from eq. (6.8), the properties (6.25) and (6.26), and the theorem embodied in eqs.(6.17) - (6.21), that

$$\|\text{Re } f\| \leq C_1 b \ , \tag{6.30}$$

where C_1 is a quantity that depends only on the Hölder index, μ . I leave you to show that there is no difficulty in handling the second piece of eq. (6.8), which only involves a vulgar non-singular integral.

From eq. (6.10), rewritten with a prime on the left-hand side, one sees that

$$\|\text{Im } f'(s)\| \leq [C_1^2+1]b^2+B \quad . \tag{6.31}$$

Moreover, because of the phase-space factor in eq. (6.10), one sees that

$$\text{Im } f'(4) = 0 \quad . \tag{6.32}$$

Hence, if one can find values of b and B such that

$$\Gamma\, b^2 + B \leq b \ , \tag{6.33}$$

where

$$\Gamma = C_1^2 + 1 \ , \tag{6.34}$$

then one will have shown that P has mapped the set (6.25), (6.26) into itself.

The inequality (6.33) is equivalent to

$$(b-b_+)(b-b_-) \leq 0 \quad , \tag{6.35}$$

where

$$b_\pm = \frac{1 \pm (1-4\Gamma B)^{1/2}}{2\Gamma} \quad , \tag{6.36}$$

so that if

$$B \leq (4\Gamma)^{-1} \tag{6.37}$$

then the roots $b_\pm$ are real, and then if b satisfies

$$b_- \leq b \leq b_+ \quad , \tag{6.38}$$

it follows that the inequality (6.35) is indeed observed, so that the set has been mapped into itself.

To complete the contraction mapping proof, one has to consider any two functions, Im $f^{(1)}(s)$ and Im $f^{(2)}(s)$, that belong to the set (6.25), (6.26). It follows immediately from eq. (6.8) that

$$\|\mathrm{Re}\, f^{(1)}(s)-\mathrm{Re}\, f^{(2)}(s)\| \leq C_1\|\mathrm{Im}\, f^{(1)}(s)-\mathrm{Im}\, f^{(2)}(s)\| \; . \tag{6.39}$$

From eq. (6.10), one can write

$$\mathrm{Im}\, f'^{I(1)}(s)-\mathrm{Im}\, f'^{I(2)}(s) = \tag{6.40}$$

$$= \left(\frac{s-4}{s}\right)^{1/2}\{[\mathrm{Re}\, f^{I(1)}(s)+\mathrm{Re}\, f^{I(2)}(s)][\mathrm{Re}\, f^{I(1)}(s)-\mathrm{Re}\, f^{I(2)}(s)]$$

$$+ \quad [\mathrm{Im}\, f^{I(1)}(s)+\mathrm{Im}\, f^{I(2)}(s)][\mathrm{Im}\, f^{I(1)}(s)-\mathrm{Im}\, f^{I(2)}(s)]\}$$

Hence

$$\|\mathrm{Im}\, f'^{(1)}-\mathrm{Im}\, f'^{(2)}\| \leq 2\Gamma b\|\mathrm{Im}\, f^{(1)}-\mathrm{Im}\, f^{(2)}\|, \tag{6.41}$$

so that the contraction condition is

$$b < (2\Gamma)^{-1} \; . \tag{6.42}$$

This condition is only consistent with (6.38) if (6.37) is weakened to

$$B < (4\Gamma)^{-1} \; , \tag{6.43}$$

for then $$b_{-} < (2\Gamma)^{-1} \quad , \tag{6.44}$$

as can be seen from eq. (6.36).

The conclusion is that if v(s) is such that (6.43) is satisfied, then one has a contraction mapping for any b that satisfies

$$b_{-} \leq b < (2\Gamma)^{-1} \ . \tag{6.45}$$

Hence it follows easily that the equations (6.8) and (6.10) have one, and only one solution in the ball

$$b \leq b_{-} \quad , \tag{6.46}$$

and no solutions in the annulus between this ball and the ball

$$b < (2\Gamma)^{-1} \quad . \tag{6.47}$$

Thus each allowed inelastic input, v(s), generates a locally unique Im f(s) . It will be shown that two different driving terms, $v_1(s)$ and $v_2(s)$, generate two different solutions, Im $f_1(s)$ and Im $f_2(s)$. For suppose the converse, namely that

$$\text{Im } f_1(s) \equiv \text{Im } f_2(s) \quad . \tag{6.48}$$

Then eq. (6.8) would mean that

$$\text{Re } f_1(s) \equiv \text{Re } f_2(s) \quad , \tag{6.49}$$

and so eq. (6.10) implies

$$v_1(s) \equiv v_2(s) \quad , \tag{6.50}$$

in contradiction to the supposition that v_1 and v_2 were different. This means that two different v's cannot generate the same Im f .

7. THE MANDELSTAM EQUATIONS

The Shirkov equations may be regarded as an approximation to the exact equations that were developed by Mandelstam in 1958 . We now turn to these equations, and we will develop a contraction mapping proof that involves no approximation, either of crossing symmetry, or of unitarity.

Part of the proof is closely parallel to that of the previous section. We will again look at unsubtracted equations, but now we have two variables, s and t, a decidedly non-trivial complication. The total pion-pion amplitude F(s,t), that is to be constructed, will have an unsubtracted Mandelstam representation,

$$F(s,t) = A(t,u)+\beta A(s,u)+\eta\beta\eta A(t,s) \quad , \qquad (7.1)$$

where

$$A(t,u) = \frac{\beta}{\pi^2}\int_4^\infty dt' \int_4^\infty du' \frac{\rho(t,u')}{(t'-t)(u'-u)} \quad , (7.2)$$

and the spectral-function is crossing-symmetric:

$$\rho(x,y) = \beta\rho(y,x) \quad . \qquad (7.3)$$

The isospin matrices have already been given in eqs.(6.3) and (6.4), and u=4-s-t .

As Mandelstam showed, the elastic unitarity relation, eq. (3.5), will be satisfied for $4\le s\le 16$ if

$$\rho(s,t) = \rho^{el}(s,t) + \beta\rho^{el}(t,s) \quad , \qquad (7.4)$$

where

$$\rho^{elI}(s,t) = \int_4^{g(s;t,4)} dt_1 \int_4^{g(s;t,t_1)} dt_2 \quad \times \qquad (7.5)$$

$$\times\; K(s;t,t_1,t_2)D^{I*}(s,t_1)D^{I}(s,t_2) \quad ,$$

with

$$D(s,t) = \frac{1}{\pi}\int_4^\infty ds'\left[\frac{1}{s'-s} + \frac{\eta\beta\eta}{s'-u}\right]\rho(s',t) \quad , \qquad (7.6)$$

$$K(s;t,t_1,t_2) = \qquad (7.7)$$

$$= \frac{4}{\pi}\,[s(s-4)]^{-1/2}\left[t^2+t_1^2+t_2^2-2t\;t_1-2t_1t_2-2t_2t-\frac{4tt_1t_2}{s-4}\right]^{-1/2}$$

$$g(s;t,t_1) = t+t_1+\frac{2tt_1}{s-4} - 2[tt_1(1+\frac{t}{s-4})(1+\frac{t_1}{s-4})]^{1/2} \qquad (7.8)$$

The form of eq. (7.4) guarantees that $\rho(s,t)$ satisfies the crossing symmetry, eq. (7.3). Moreover, one sees from eq. (7.5) that $\rho^{el}(s,t)$ vanishes when

$$g(s,t,4) \leq 4 \qquad (7.9)$$

that is, when

$$t \leq \frac{16s}{s-4} \qquad (7.10)$$

Hence, for any $s \geq 4$,

$$\rho^{el}(s,t) = 0 \qquad (7.11)$$

for $t \leq 16$. Hence eq. (7.4) means that

$$\rho(s,t) = \rho^{el}(s,t) \qquad (7.12)$$

for $s \leq 16$, i.e. elastic unitarity is exactly satisfied for $s \leq 16$, as should be the case. Above s=16, there is an inelastic contribution, $\beta\rho^{el}(t,s)$. In fact, one is free to add any crossing symmetric contribution that vanishes for $s \leq 16$, so as to preserve elastic unitarity. We will use this freedom to ensure that the inelastic unitarity constraints are not violated. In fact, in order to do this, it will prove necessary to re-cast the equations (7.4) - (7.6), and write them, not for $\rho^{el}(s,t)$ directly, but rather for

$$\bar{\rho}(s,t) \equiv \beta\rho^{el}(s,t) \quad . \qquad (7.13)$$

Instead of eq. (7.4) one has

$$\rho(s,t) = \beta[\bar{\rho}(s,t)+v(s,t)] +[\bar{\rho}(t,s)+v(t,s)] , \qquad (7.14)$$

where the identity $\beta^2 = 1$ has been used, and where $v(s,t)$ is an inelastic generating function, which is constrained to vanish for $s \leq 16$ and $t \leq 16$, and which will be chosen in such a way that the inelastic inequalities are observed for $s>16$. Eq. (7.5) will be rewritten

$$\bar{\rho}^{I}(s,t) = \sum_{J,M,N} \beta_{IJ}\beta_{JM}\beta_{JN}\iint dt_1 dt_2 K(s,t,t_1,t_2) d^{M*}(s,t_1) \times$$

$$\times\ d^N(s,t_2) \qquad (7.15)$$

where the summations are over the values 0,1,2, where the integration limits are as in eq. (7.5), and where

$$d(s,t) = \beta D(s,t) \quad . \qquad (7.16)$$

Since $\eta\beta\eta=\beta\eta\beta$, and $\rho(s,t)$ satisfies eq. (7.3), it follows from eq. (7.6) that

$$d(s,t) = \frac{1}{\pi}\int_4^\infty ds'\left[\frac{1}{s'-s} + \frac{\eta}{s'-u}\right] \rho(t,s') \qquad (7.17)$$

The reason that eqs.(7.14) - (7.17) are better than eqs. (7.4) - (7.6) will be explained later. One can regard eq. (7.15), with $d(s,t)$ defined by eq. (7.17), and $\rho(s,t)$ by eq. (7.14), as a non-linear equation for $\bar{\rho}(s,t)$ in terms of itself:

$$\bar{\rho}(s,t) = P[\bar{\rho};s,t] \quad . \qquad (7.18)$$

The idea will be to show that, for a suitable, given generating function, $v(s,t)$, the equation (7.18) defines a contraction mapping in a suitable space.

The real part of eq. (7.17) involves a principal value integral in the variable s, while eq. (7.14) contains an exchange of s and t. So, at the very least, our experience with the Shirkov equation would lead us to require double Hölder-continuity, with respect both to s and with respect to t . A new feature is that the behaviour with respect to t has to be preserved under the integration (7.15). As I will explain in a moment, it turns out that a simple power behaviour $t^{-\mu}$ is not so preserved, but the form $t^{-\mu}(\log t)^{-1-\varepsilon}$, $\varepsilon>0$, is preserved, if $0<\mu<\frac{1}{2}$.

Thus one is led to the following generalization of the space of eqs. (6.22) - (6.23) : The set of all functions $\bar{\rho}(s,t)$ defined for $4\leq s,\ t<\infty$, for which

$$\bar{\rho}(s,\infty) = 0 = \bar{\rho}(\infty,t) \quad , \qquad (7.19)$$

and for which there exists a norm

$$\|\bar{\rho}\| = \sup_{4\le s_1,s_2,t_1,t_2<\infty} \frac{|\rho(s_1,t_1)-\rho(s_2,t_2)|(\log\bar{s}\log\bar{t})^{1+\varepsilon}}{\left|\frac{s_1-s_2}{s_1s_2\bar{t}}\right|^{\mu}+\left|\frac{t_1-t_2}{t_1t_2\bar{s}}\right|^{\mu}} \tag{7.20}$$

where $\bar{s} = \min(s_1,s_2)$, $\bar{t} = \min(t_1,t_2)$, and the Hölder-index satisfies $0<\mu<\frac{1}{2}$. Thus, in particular

$$|\bar{\rho}(s_1,t)-\bar{\rho}(s_2,t)| \le \|\bar{\rho}\|\left|\frac{s_1-s_2}{s_1s_2}\right|^{\mu}(\log\bar{s})^{-1-\varepsilon}\ t^{-\mu}(\log t)^{-1-\varepsilon} \tag{7.21}$$

I will first indicate the outline of the existence proof, and then I will sketch in some of the difficult points in the algebra, which I do not have time to give in full detail.

Consider the mapping

$$\bar{\rho}'(s,t) = P[\bar{\rho};s,t] \quad . \tag{7.22}$$

Let $\bar{\rho}(s,t)$ belong to the ball

$$\|\bar{\rho}\| \le \bar{b} \quad , \tag{7.23}$$

and to the cone

$$\bar{\rho}(s,t) = 0 \tag{7.24}$$

for $t\le \frac{16s}{s-4}$. Let the known function belong to the ball

$$\|v\| \le B \tag{7.25}$$

and to the cone

$$v(s,t) = 0 \quad , \tag{7.26}$$

for $s\le 16$ and $t\le 16$.

First of all, it can be shown from eq. (7.17) and (7.14), much as in the one-dimensional case, that

$$|d(s_1,t)-d(s_2,t)| \le (\bar{b}+B)C_1\left|\frac{s_1-s_2}{s_1s_2\bar{t}}\right|^{\mu}(\log\bar{s}\ \log t)^{-1-\varepsilon} \tag{7.27}$$

where C_1 depends only on μ. One has to carry the extra factor $(\log s)^{-1-\varepsilon}$ through the proof, and there is the extra t-dependence in the denominator s'-u=s+s+t-4 in eq. (7.17) to worry about, but this is not too hard. Next, one can use (7.27) to show from eq. (7.15), with a prime on the left-hand side, that

$$\|\bar{\rho}'\| \leq \Gamma(\bar{b}+B)^2 \qquad (7.28)$$

where Γ only depends on the Hölder index, μ. I will sketch the algebra leading to this result in a moment. The condition that the ball (7.23) be mapped into itself is then that

$$\Gamma(\bar{b}+B)^2 \leq \bar{b} \qquad (7.29)$$

or, if one defines

$$b = \bar{b} + B \quad , \qquad (7.30)$$

the condition is

$$\Gamma b^2 + B \leq b \quad . \qquad (7.31)$$

This inequality is exactly the same as eq. (6.33), so the solution is the same, viz.

$$B \leq (4\Gamma)^{-1} \qquad (7.32)$$

and

$$b_- \leq b \leq b_+ \qquad (7.33)$$

with $b_\pm$ defined in eq. (6.36).

Consider now two functions, $\bar{\rho}^{(1)}(s,t)$ and $\bar{\rho}^{(2)}(s,t)$ each of which belongs to the set (7.23), (7.24). One proves that

$$\|\bar{\rho}'^{(1)} - \bar{\rho}'^{(2)}\| \leq 2\Gamma b\|\bar{\rho}^{(1)} - \bar{\rho}^{(2)}\| \quad , \qquad (7.34)$$

so that, as in eqs. (6.41) - (6.45), the conditions for a contraction mapping are

$$B < (4\Gamma)^{-1} \qquad (7.35)$$

$$b_- \leq b < (2\Gamma)^{-1} \quad . \qquad (7.36)$$

One has again that a locally unique solution, in this case $\bar{\rho}(s,t)$, is generated by each $v(s,t)$ for which eq. (7.35) holds; and that different generating functions, $v(s,t)$, give necessarily different solutions, $\rho(s,t)$.

I will now give some details of the derivation of eq. (7.28) from eq. (7.27), and we will incidentally see why the factor $(\log t)^{-1-\varepsilon}$ is needed. I will in fact explain in detail only the simpler problem of showing that

$$|\bar{\rho}'(s,t)| \le \Gamma b^2 (s\ t)^{-\mu} (\log s \log t)^{-1-\varepsilon}\ , \tag{7.37}$$

given

$$|d(s,t)| \le Cb(s\ t)^{-\mu}\ (\log s \log t)^{-1-\varepsilon}\ . \tag{7.38}$$

One has, from eq. (7.15), that

$$|\bar{\rho}'(s,t)| \le \frac{4}{\pi}\ C^2 b^2 (s-4)^{-1/2} s^{-1/2-2\mu} (\log s)^{-2-2\varepsilon} \times$$

$$\times \int_4^{g(s;t,4)} dt_1 t_1^{-\mu} (\log t_1)^{-1-\varepsilon} y(t,t_1) \tag{7.39}$$

where

$$y(t,t_1) = \int_4^{g(s;t,t_1)} dt_2 [h(s;t,t_1)-t_2]^{-1/2} [g(s;t,t_1)- -t_2]^{-1/2} t_2^{-\mu} (\log t_2)^{-1-\varepsilon} \tag{7.40}$$

with

$$h(s;t,t_1) = g(s;t,t_1) + 4t^{1/2} t_1^{1/2} (1+\frac{t}{s-4})^{1/2} (1+\frac{t_1}{s-4})^{1/2} \tag{7.41}$$

Now $[h(s;t,t_1)-t_2]^{-1/2}$ can be majorized in eq. (7.40) by

$$[h(s;t,t_1)-g(s;t,t_1)]^{-1/2} = \tag{7.42}$$

$$= \frac{1}{2}\ t^{-1/4} t_1^{-1/4} (1+\frac{t}{s-4})^{1/4} (1+\frac{t_1}{s-4})^{1/4} \le \frac{1}{2} (\frac{s-4}{t t_1})^{1/2}\ .$$

The factor $t_2^{-\mu} (\log t_2)^{-1-\varepsilon}$ can be written

$$t_2^{-1/2}\ t^{1/2-\mu} (\log t_2)^{-1-\varepsilon}$$

and it may be shown that

$$t_2^{1/2-\mu} (\log t_2)^{-1-\varepsilon}$$

is majorized by

$$C_2 [(s-4)\frac{t}{t_1}]^{1/2-\mu} (\frac{s}{s-4})^{\mu} [\log(\frac{4t}{t_1})]^{-1-\varepsilon} \tag{7.43}$$

where C_2 is constant. Hence

$$y(t,t_1) \le \frac{C_2}{2} s^{\mu}(s-4)^{1-2\mu} t^{-\mu} t_1^{-1+\mu} (\log\frac{4t}{t_1})^{-1-\varepsilon} \times$$

$$\times \int_4^g \frac{dt_2}{(g-t_2)^{1/2} t_2^{1/2}} \tag{7.44}$$

The integral that is left here may be shown to be less than 4, so that one finds

$$|\bar{\rho}'(s,t)| \le \frac{8}{\pi} C^2 b^2 C_2 s^{-1/2-\mu}(s-4)^{1/2-2\mu}(\log s)^{-2-2\varepsilon} t^{-\mu}$$

$$\int_4^t \frac{dt_1}{t_1}(\log t_1)^{-1-\varepsilon} (\log\frac{4t}{t_1})^{-1-\varepsilon} . \tag{7.45}$$

The integral here may be divided into two pieces, the first being

$$\int_4^{t^{1/2}} \frac{dt_1}{t_1}(\log t_1)^{-1-\varepsilon} (\log\frac{4t}{t_1})^{-1-\varepsilon} \le$$

$$\le [\log(4t^{1/2})]^{-1-\varepsilon} \int_4^{t^{1/2}} \frac{dt_1}{t_1}(\log t_1)^{-1-\varepsilon} \le$$

$$\le 2^{1+\varepsilon}(\log t)^{-1-\varepsilon}\left[\frac{(\log t_1)^{-\varepsilon}}{-\varepsilon}\right]_4^{t^{1/2}} \le$$

$$\le \frac{2}{\varepsilon}(\log 2)^{-\varepsilon} (\log t)^{-1-\varepsilon} . \tag{7.46}$$

Notice how crucial the power $(-1-\varepsilon)$ was in the second and third lines here. The other piece of the integral in eq. (7.45), from $t^{1/2}$ to t, may be shown to have a similar bound, with the help of the transformation $t_1 \to t/t_1$. On gathering together the pieces, one finds a majorant of the form (7.37).

One now has to show that $\bar{\rho}'(s,t)$ is Hölder-continuous with respect to s and with respect to t. These two pieces of the proof are best treated separately. The Hölder-continuity with respect to t can simply be derived from the bound (7.38), whereas one needs eq. (7.27) to show that $\bar{\rho}'(s,t)$ is Hölder-continuous with respect to s . One has to break up the differences of the double integrals, evaluated at different points, into lots of little pieces, and work very patiently. The work follows the lines of the above proof of eq. (7.37), but is more complicated. I will simply refer you to the original references.

8. INELASTIC UNITARITY

It has been shown that one can construct solutions, in fact an infinite number of solutions, of the non-linear equation (7.18). Each solution satisfies crossing-symmetry and, for $4 \le s \le 16$, exact elastic unitarity. In general, the inelastic inequalities would be violated for $s > 16$, but we will now show that, if some extra constraints are imposed on $v(s,t)$, then we can arrange that these inequalities are safe.

At the fixed-point of the mapping (7.22), we know that the amplitude, $F(s,t)$, has the unsubtracted Mandelstam representation, eqs.(7.1), (7.2). On combining this with the partial-wave projection

$$f_\ell(s) = \frac{1}{s-4} \int_{4-s}^{0} dt\ P_\ell\left(1+ \frac{2t}{s-4}\right) F(s,t), \quad (8.1)$$

we find the Froissart-Gribov form

$$f_\ell(s) = \frac{1}{s-4} \int_{4}^{\infty} dt\ Q_\ell\left(1+ \frac{2t}{s-4}\right) D(s,t) \quad , \quad (8.2)$$

where D was defined in eq. (7.6) and is related to d by eq. (7.16). The imaginary part of (8.2), for $s \ge 4$, is

$$\mathrm{Im}\ f_\ell(s) = \frac{1}{s-4} \int_{4}^{\infty} dt\ Q_\ell\left(1+ \frac{2t}{s-4}\right) \rho(s,t) \ . \quad (8.3)$$

Now $\rho(s,t)$ was written in eq. (7.14) as the sum of four parts, the first part being

$$\beta\bar{\rho}(s,t) = \rho^{el}(s,t) \quad . \tag{8.4}$$

This part must just yield the elastic contribution to $\mathrm{Im}\, f_\ell(s)$. That is, we must have

$$\frac{1}{s-4}\int_4^\infty dt\ Q_\ell(1+\frac{2t}{s-4})\rho^{el}(s,t) = (\frac{s-4}{s})^{1/2}|f_\ell(s)|^2 \tag{8.5}$$

This is in fact true, and it follows from the identity

$$(s-4)\int_0^\infty dt\ Q_\ell(1+\frac{2t}{s-4})K(s;t,t_1,t_2) = $$
$$= Q_\ell(1+\frac{2t_1}{s-4})Q_\ell(1+\frac{2t_2}{s-4}) \quad . \tag{8.6}$$

If we take the contribution (8.5) on to the left-hand side of eq. (8.3), what remains is

$$\mathrm{Im}\, f_\ell(s) - (\frac{s-4}{s})^{1/2}|f_\ell(s)|^2 = \tag{8.7}$$
$$= \frac{1}{s-4}\int_{16}^\infty dt\ Q_\ell(1+\frac{2t}{s-4})[\bar{\rho}(t,s)+\beta v(s,t)+v(t,s)]$$

The inelastic unitarity constraint is that the left-hand side of this equation should be non-negative for $s\geq 16$. I will show that one can arrange that

$$[\bar{\rho}(t,s) + \beta v(s,t) + v(t,s)] \geq 0 \quad , \tag{8.8}$$

everywhere. Now we can prove, simply by glancing at the Laplace representation

$$Q_\ell(z) = \int_0^\infty du[z+ \cosh u(z^2-1)^{1/2}]^{-\ell-1} \quad , \tag{8.9}$$

that

$$Q_\ell(1+\frac{2t}{s-4}) \geq 0 \quad , \tag{8.10}$$

for $s>4$, $t>0$. Hence eqs. (8.8) and (8.10) imply that

$$\mathrm{Im}\, f_\ell(s) - (\frac{s-4}{s})^{1/2}|f_\ell(s)|^2 \geq 0 \quad , \tag{8.11}$$

for $s\geq 16$, directly from eq. (8.7).

I have, then, to demonstrate eq. (8.8). It is convenient to divide the s-t plane into four pieces:

$$\begin{array}{llll} \text{I.} & 4\le s\le 20 & , & 4\le t\le 20 \\ \text{II.} & 4\le s\le 20 & , & t>20 \\ \text{III.} & s>20 & , & 4\le t\le 20 \\ \text{IV.} & s>20 & , & t>20 \end{array} \tag{8.12}$$

It will be supposed that, in addition to the requirements (7.25) and (7.26), v(s,t) also satisfies

$$\beta v(s,t)+v(t,s) \geq \gamma B\left[\frac{(s-20)(t-20)}{s^2t^2}\right]^{\mu} (\log s \log t)^{-1-\varepsilon} \tag{8.13}$$

for s and t in IV, and also

$$g(s,t) \geq \gamma B\left[\frac{t-20}{t^2}\right]^{\mu} (\log t)^{-1-\varepsilon} \quad , \tag{8.14}$$

and

$$\beta g(s,t) \geq \gamma B\left[\frac{t-20}{t^2}\right]^{\mu} (\log)^{-1-\varepsilon} \quad , \tag{8.15}$$

both for s and t in II, where

$$g(s,t) = P\int_{16}^{\infty} ds'\left[\frac{1}{s'-s} + \frac{\eta}{s'-u}\right]\left[v(s',t)+ +\beta v(t,s')\right] \tag{8.16}$$

In eqs. (8.13) - (8.15), the number γ is to satisfy

$$0 < \gamma < 1 \tag{8.17}$$

The basis of the proof is that while $\beta v(s,t)+v(t,s)$ in IV, and $g(s,t)$ and $\beta g(s,t)$ in II, are of the order B and <u>positive</u>, $\bar{\rho}(s,t)$ is only of order B^2. This can be seen from eqs. (7.28) and (7.30), which imply that, at the fixed-point,

$$\begin{aligned} \|\bar{\rho}\| &\leq \Gamma(b_-)^2 = \\ &= \Gamma\left[\frac{2B}{1+(1-4\Gamma B)^{1/2}}\right]^2 \\ &\leq 4\Gamma B^2 \quad . \end{aligned} \tag{8.18}$$

Hence, in IV the inequality can certainly be arranged, simply by choosing B so small that $\beta v(s,t)+v(t,s)$, which is positive, swamps $\bar{\rho}(t,s)$, which could be negative. The little tail of $\bar{\rho}(t,s)$ in $16<s<20$, $t>20$, likewise presents no difficulty, but what we have to do is to show that $\bar{\rho}(t,s)$ can be made non-negative in III, or, what is equivalent, that $\bar{\rho}(s,t)$ can be made non-negative in II.

This will be done by requiring that $\bar{\rho}(s,t)$ belongs to the cone

$$\bar{\rho}(s,t) \geq 0 \qquad (8.19)$$

for $4 \leq s \leq 20$, $t > 20$ and by showing that this restricted set is still mapped into itself. This will then complete the proof. The proof of eq. (8.19) is somewhat complicated by the fact that some of the elements of the crossing matrices, β and η, are negative.

I think it might be clearer where the proof is going if I give it backwards. In eq. (7.15) the symmetry between M and N, and between t_1 and t_2, may be exploited to replace $d^{M*}(s,t_1)d^N(s,t_2)$ by

$$\text{Re } d^M(s,t_1)\text{Re } d^N(s,t_2) + \rho^M(t_1,s)\rho^N(t_2,s) \; . \qquad (8.20)$$

It will be shown that Re $d(s,t)$ is non-negative in I and II, for all isospin states. Thus (8.20) can be made non-negative throughout I and II, because in I $\rho(t,s)$ vanishes, while in II it is of a higher order in B than is Re $d(s,t)$. This is enough to prove the positivity of $\bar{\rho}(s,t)$ in II, because the kernel, K, in eq. (7.15), is positive, and moreover

$$\sum_J \beta_{IJ} \beta_{JM} \beta_{JN} \geq 0 \qquad (8.21)$$

for all I, M. N, even though β_{IJ} itself has negative elements. I leave you to check this important result.

So, how do we prove that Re $d(s,t)$ is non-negative in I and II ? In II it follows from (8.14) and the "order of B" argument. In I it follows because we can prove that

$$\rho(t,s) = \beta\rho(s,t) \geq 0 \qquad (8.22)$$

for $s \geq 80 \frac{t+20}{t-4}$, and this positive contribution to $d(s,t)$ can be shown to more than compensate the contribution from $\frac{16t}{t-4} \leq s \leq 80 \frac{t+20}{t-4}$. Finally, eq. (8.22), or equivalently

$$\rho(s,t) \geq 0 \qquad (8.23)$$

for $t \geq 80 \frac{s+20}{s-4}$, is demonstrated by using eq. (8.16), and the fact that, for $t \geq 80 \frac{s+20}{s-4}$, the integrand in eq. (7.5) contains some positive contribution, which can be relied upon to force (8.23). If you would like to see this proof in full detail, I refer you to the original paper.

9. PROBLEMS AND PROSPECTS

The above existence proof has been generalized by the introduction of subtractions [7] and CDD poles [8]. The way that one does this is to subtract out from eq. (7.17) a finite number of partial waves, which must then be treated separately from the double spectral function equations. For example, if $\sigma(t)$ is the absorptive part of the S-wave in the t-channel, then eq. (7.17) can be rewritten

$$d(s,t) = \sigma(t) + \frac{1}{\pi}\int_4^\infty ds'\left[\frac{1}{s'-s} + \frac{\eta}{s'-u} - \frac{1+\eta}{t-4}\log\left(1+\frac{t-4}{s'}\right)\right]\times \rho(t',s) \qquad (9.1)$$

Now $\sigma(t)$ can be determined from an S-wave dispersion relation, in which the left-hand discontinuity is given exactly in terms of $d(s,t)$. A double mapping $(\bar{\rho},\sigma)\to(\bar{\rho}',\sigma')$ is involved, and consequently a double contraction mapping. One can have double-spectral functions that diverge now as $s\to\infty$, so that the Mandelstam representation needs subtractions. So far, it is only known how to construct elastic spectral functions that need one subtraction, although the inelastic generating function may need more subtractions [7]. The fiercest divergence that it has proved possible to allow so far [9] is

$$|F(s,t)| \lesssim \text{const } t(\log t)^{-2-\varepsilon} \qquad (9.2)$$

as $t\to\infty$, for $0\leq s\leq 16$. This would allow happily for the ρ-trajectory, but not for the Pomeranchuk. It is not yet known how to go beyond (9.2), without spoiling the in -

elastic unitarity bounds. It is possible to resolve the S-wave part of eq. (9.1) by N/D equations instead of straightforward dispersion relations, and then one can add CDD poles [8] thus further enlarging the set of crossing-symmetric, unitary functions.

To conclude, I want to mention the Newton-Kantorovich method, and show how it might be used as a computer algorithm for proceeding from small to large values of the coupling. Define the operator $\Phi \equiv 1-P$ by

$$\Phi[\bar{\rho};s,t] = \bar{\rho}(s,t) - \iint dt_1 dt_2\ K(s;t,t_1 t_2) d^*(s,t_1) d(s,t_2) \tag{9.3}$$

where it is understood that $d(s,t)$ is defined in terms of $\bar{\rho}(s,t)$ by (7.17) and (7.14). The first Fréchet derivative of $\Phi[\bar{\rho}]$ with respect to $\bar{\rho}$ may be defined to be the <u>linear operator</u>, $\Phi'[\bar{\rho}]$, if it exists, such that if $h(x,y)$ is any function belonging to the Banach space, then

$$\lim_{\lambda\to 0} \left\| \frac{\Phi[\bar{\rho}+\lambda h]-\Phi[\bar{\rho}]}{\lambda} - \Phi'[\bar{\rho}]h \right\| = 0\ , \tag{9.4}$$

where λ is a real number. We can get Φ' by differentiating eq. (9.3) with respect to $\bar{\rho}$ at fixed v. The result is

$$\Phi'[\bar{\rho}]h(s,t) =$$
$$= h(s,t) - 2\mathrm{Re}\iint dt_1 dt_2 K(s;t,t_1 t_2) d^*(s,t_1) g(s,t_2) \tag{9.5}$$

where

$$g(s,t) = \frac{1}{\pi}\int_4^\infty ds'\left[\frac{1}{s'-s} + \frac{n}{s'-u}\right][h(s',t)+\beta h(t,s')] . \tag{9.6}$$

The second Fréchet derivative is defined analogously, and it may be written

$$\Phi'' h_1 h_2(s,t) = -2\mathrm{Re}\iint dt_1 dt_2 K(s;t,t_1 t_2)\, g_1^*(s,t_1) g_2(s,t_2) \tag{9.7}$$

where the connection between g_1 and h_1, and between g_2 and h_2, is the same as that between g and h. Note that Φ'' is a constant operator, that is, it does not depend on

$\bar{\rho}$. This simplifies the application of the Newton-Kantorovich method, since one can immediately obtain a numerical bound on $\|\Phi''\|$.

Let us consider the so-called "modified Newton iteration", namely

$$\bar{\rho}_{n+1} = \bar{\rho}_n - [\Phi'(\bar{\rho}_o)]^{-1} \Phi(\bar{\rho}_n) \ . \qquad (9.8)$$

A sufficient condition for the convergence of this iteration is that $\Phi'(\bar{\rho}_o)$ have an inverse, and that $\bar{\rho}_1$ be so close to $\bar{\rho}_o$ that

$$\|\bar{\rho}_1 - \bar{\rho}_o\| \cdot \|[\Phi'(\bar{\rho}_o)]^{-1}\| \cdot \|\Phi''\| \leq \frac{1}{2} \ . \qquad (9.9)$$

One could use this technique to proceed from small to large values of $\|v\|$. For example, let us replace v(s,t) by $\lambda v(s,t)$, where λ is a number. Then we have already proved that there is a contraction mapping solution for λ small enough, say $\lambda<\lambda_c$. One could attempt to get a solution at a point outside the contraction circle, say at $\lambda=\lambda_c+\varepsilon$, by taking, as the starting point, $\bar{\rho}_o$, of the Newton-Kantorovich iteration, the known contraction-mapping solution at the point $\lambda=\lambda_c-\varepsilon$. Having obtained a new solution at $\lambda=\lambda_c+\varepsilon$, one could then use it as the starting point for an iteration at the point $\lambda=\lambda_c+2\varepsilon$, and so on.

The difficulty with this technique is that $\Phi'(\bar{\rho}_o)$ could fail to have an inverse outside the contraction circle. One could always ask the computer to work out $\|\Phi'(\rho_o)\|$, which in practice means the determinant of a matrix, as a preparatory step. If this is very small, it means that one is near a singular point of the iteration. One possible solution is to circumnavigate the bad point in the complex λ-plane. If, when one regains the real axis, the spectral-function is again real, and the inelastic inequalities are still safe (!), one can proceed to still larger values of λ, as if nothing has happened. However, so far no-one has proved that one definitely can escape from the contraction circle, but it would seem natural to expect that one could proceed at least some distance along the real λ-axis, before getting into trouble.

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