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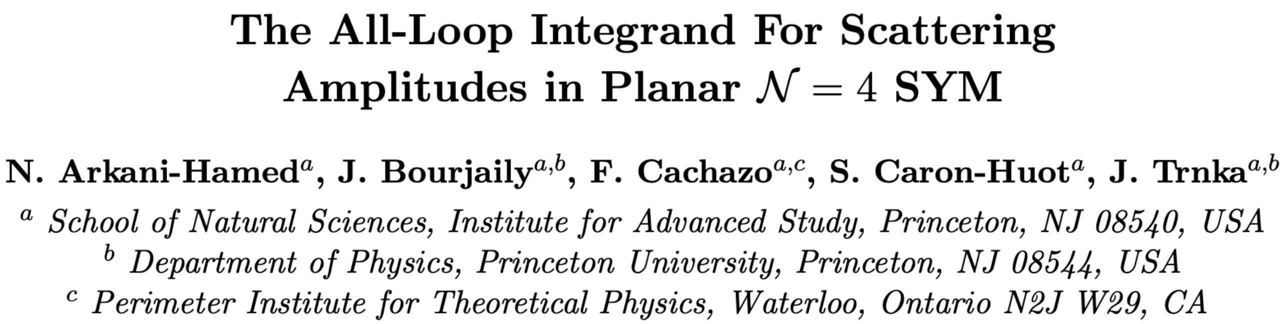
OXFORD

QCD Scattering Amplitudes

Beyond the Planar Limit

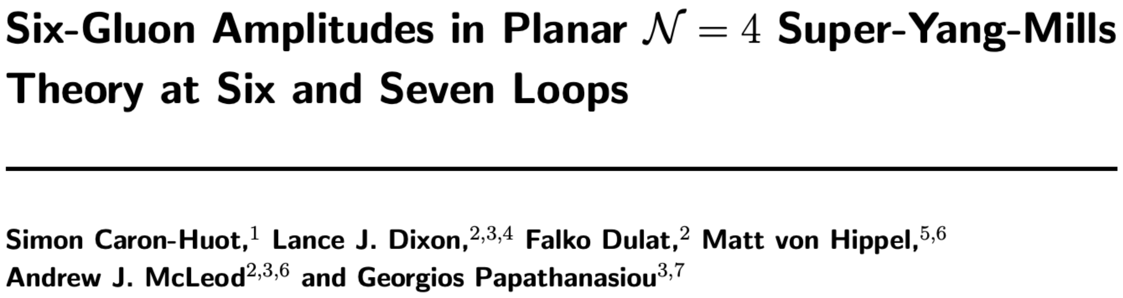
Giulio Gambuti - 06/08/2023 - Amplitudes, CERN

The Planar Limit



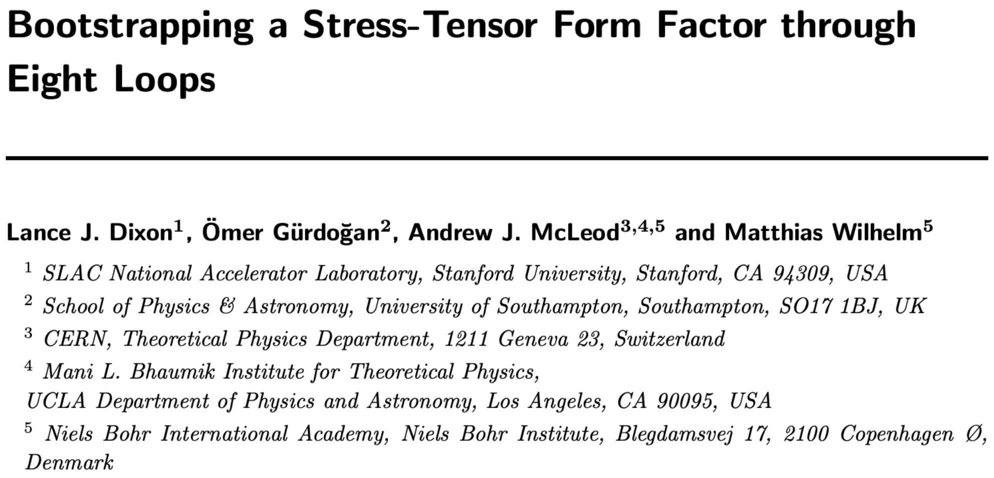
The All-Loop Integrand For Scattering Amplitudes in Planar $\mathcal{N} = 4$ SYM

N. Arkani-Hamed[a], **J. Bourjaily**[a,b], **F. Cachazo**[a,c], **S. Caron-Huot**[a], **J. Trnka**[a,b]
[a] *School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA*
[b] *Department of Physics, Princeton University, Princeton, NJ 08544, USA*
[c] *Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2J W29, CA*



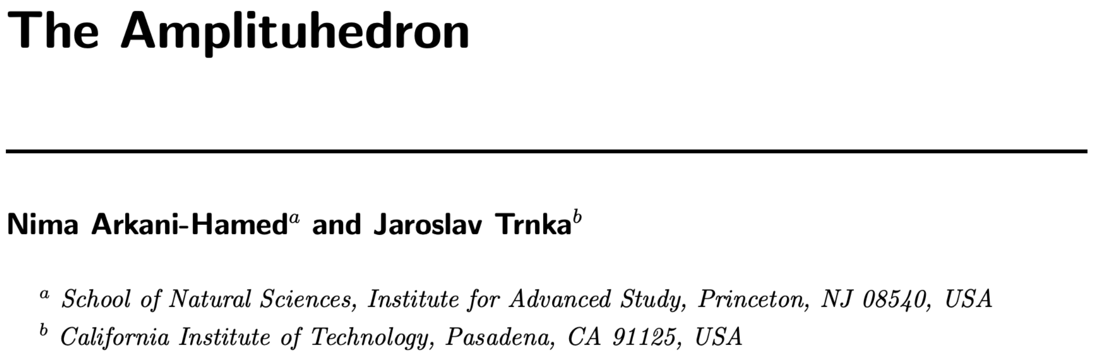
Six-Gluon Amplitudes in Planar $\mathcal{N} = 4$ Super-Yang-Mills Theory at Six and Seven Loops

Simon Caron-Huot,[1] **Lance J. Dixon,**[2,3,4] **Falko Dulat,**[2] **Matt von Hippel,**[5,6]
Andrew J. McLeod[2,3,6] **and Georgios Papathanasiou**[3,7]



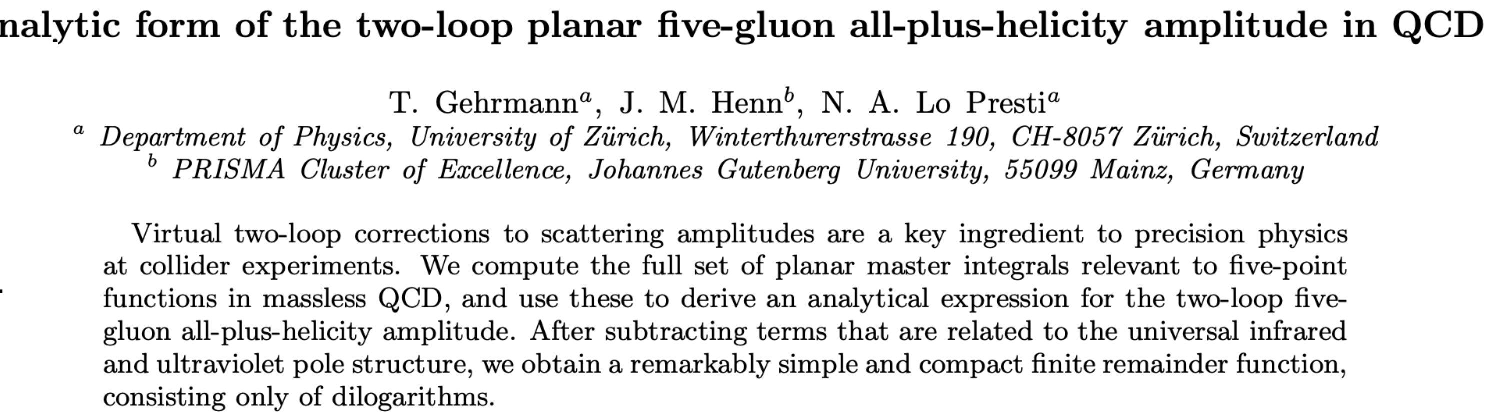
Bootstrapping a Stress-Tensor Form Factor through Eight Loops

Lance J. Dixon[1], **Ömer Gürdoğan**[2], **Andrew J. McLeod**[3,4,5] **and Matthias Wilhelm**[5]
[1] *SLAC National Accelerator Laboratory, Stanford University, Stanford, CA 94309, USA*
[2] *School of Physics & Astronomy, University of Southampton, Southampton, SO17 1BJ, UK*
[3] *CERN, Theoretical Physics Department, 1211 Geneva 23, Switzerland*
[4] *Mani L. Bhaumik Institute for Theoretical Physics,*
UCLA Department of Physics and Astronomy, Los Angeles, CA 90095, USA
[5] *Niels Bohr International Academy, Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark*



The Amplituhedron

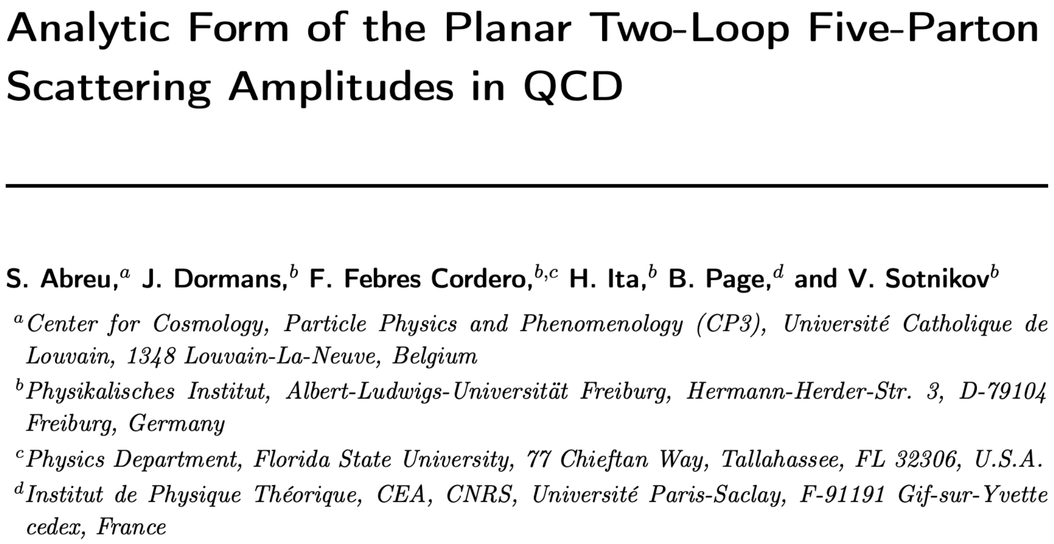
Nima Arkani-Hamed[a] **and Jaroslav Trnka**[b]
[a] *School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA*
[b] *California Institute of Technology, Pasadena, CA 91125, USA*



Analytic form of the two-loop planar five-gluon all-plus-helicity amplitude in QCD

T. Gehrmann[a], J. M. Henn[b], N. A. Lo Presti[a]
[a] *Department of Physics, University of Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland*
[b] *PRISMA Cluster of Excellence, Johannes Gutenberg University, 55099 Mainz, Germany*

Virtual two-loop corrections to scattering amplitudes are a key ingredient to precision physics at collider experiments. We compute the full set of planar master integrals relevant to five-point functions in massless QCD, and use these to derive an analytical expression for the two-loop five-gluon all-plus-helicity amplitude. After subtracting terms that are related to the universal infrared and ultraviolet pole structure, we obtain a remarkably simple and compact finite remainder function, consisting only of dilogarithms.



Analytic Form of the Planar Two-Loop Five-Parton Scattering Amplitudes in QCD

S. Abreu,[a] **J. Dormans,**[b] **F. Febres Cordero,**[b,c] **H. Ita,**[b] **B. Page,**[d] **and V. Sotnikov**[b]
[a] *Center for Cosmology, Particle Physics and Phenomenology (CP3), Université Catholique de Louvain, 1348 Louvain-La-Neuve, Belgium*
[b] *Physikalisches Institut, Albert-Ludwigs-Universität Freiburg, Hermann-Herder-Str. 3, D-79104 Freiburg, Germany*
[c] *Physics Department, Florida State University, 77 Chieftan Way, Tallahassee, FL 32306, U.S.A.*
[d] *Institut de Physique Théorique, CEA, CNRS, Université Paris-Saclay, F-91191 Gif-sur-Yvette cedex, France*

… and many more

Outline

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- Computation of non-planar QCD scattering amplitudes

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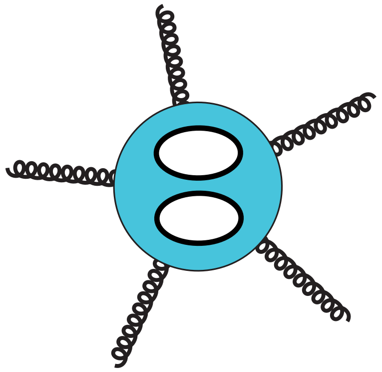
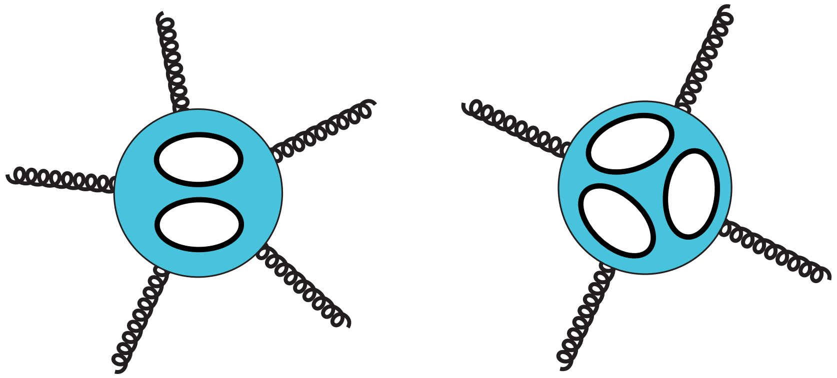
    - 2-loop 5-point



Outline

- Computation of non-planar QCD scattering amplitudes
  - 2-loop   5-point
  - 3-loop   4-point



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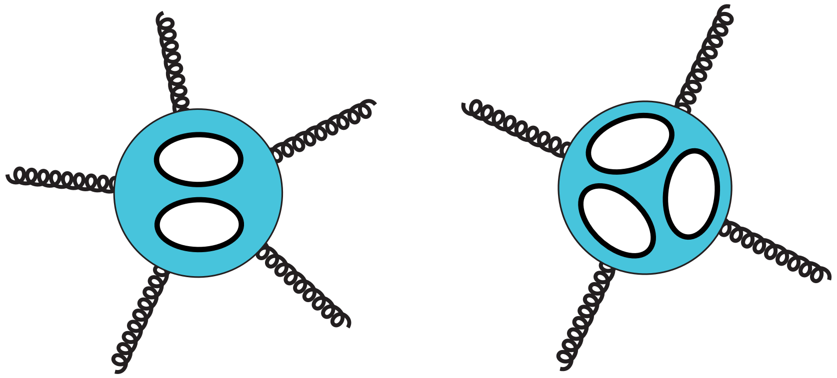

- A step towards new master integrals

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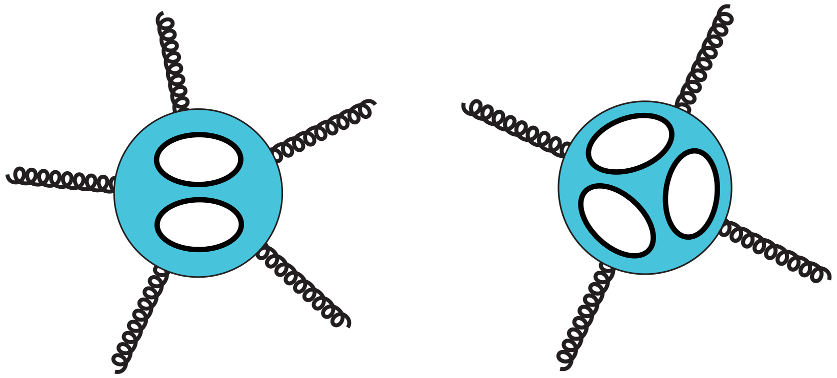

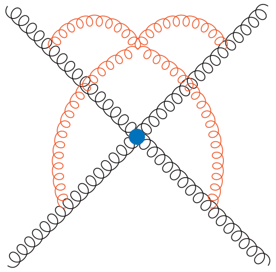

- A step towards new master integrals
- Their infrared structure…

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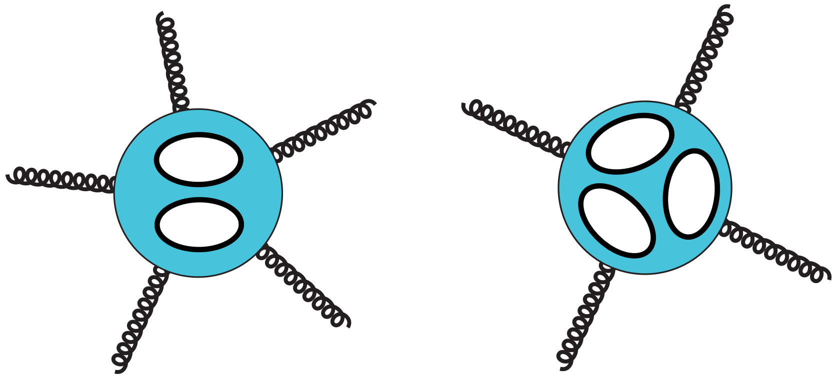

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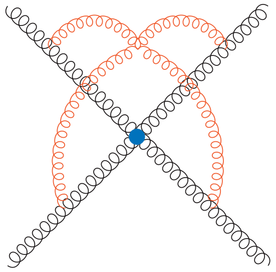
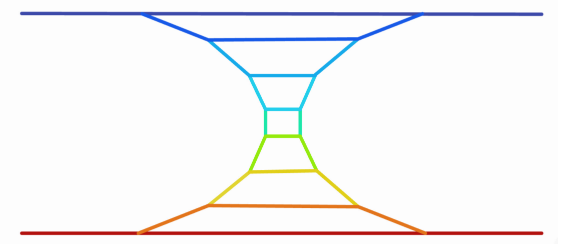


- Their infrared structure…
- …and Regge limit



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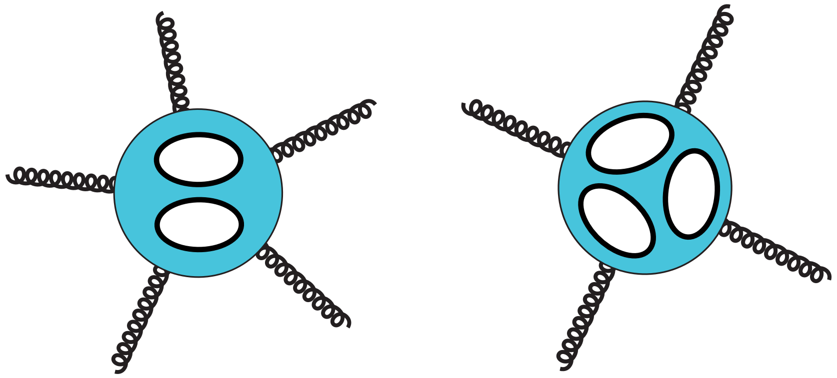

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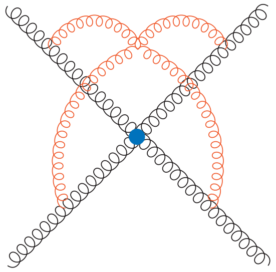

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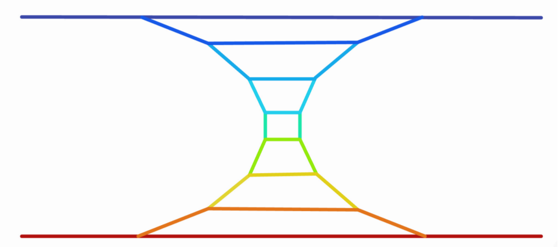
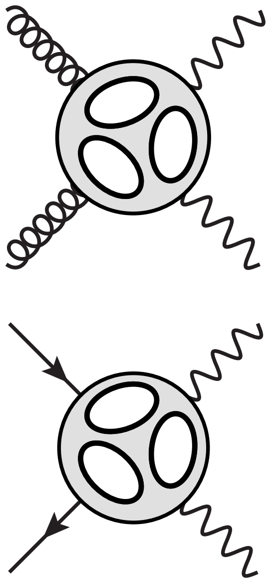
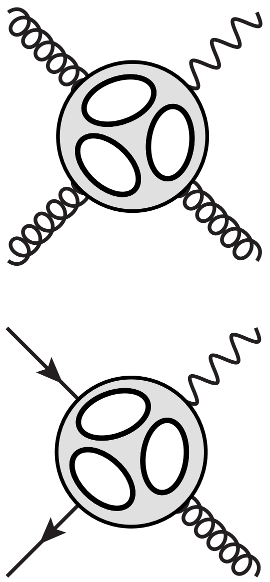


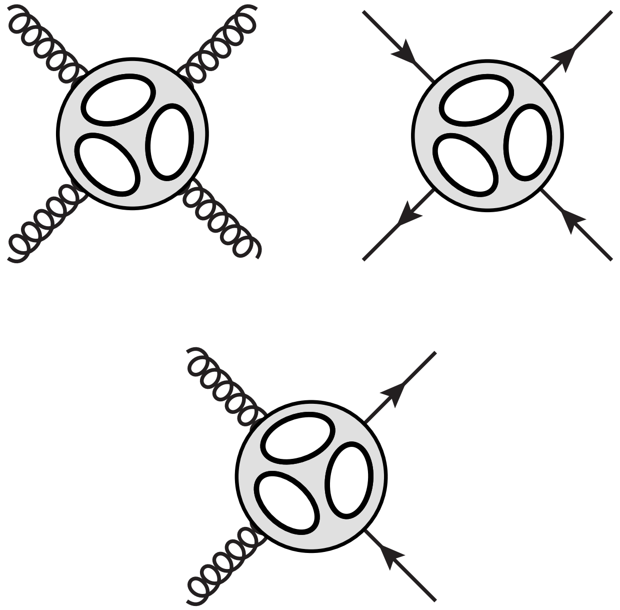


Caola, von Manteuffel,
Tancredi:
2011.13946(PRL)

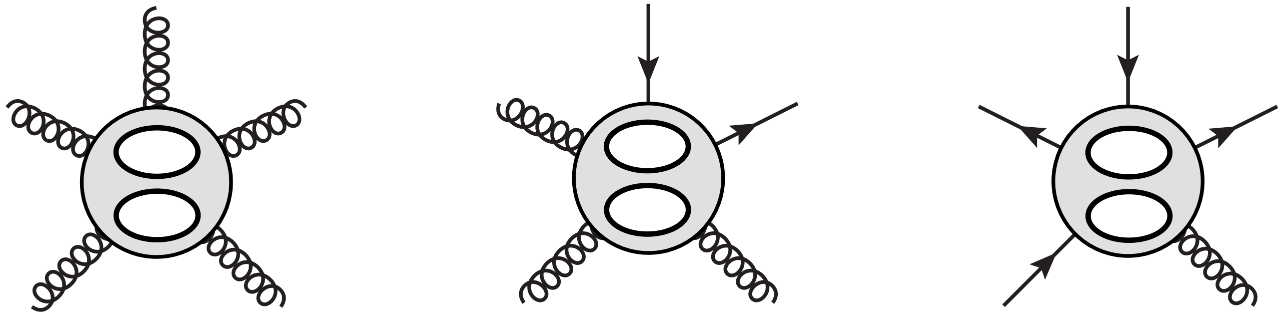
Bargiela, Caola, von Manteuffel,
Tancredi:
2111.13595(JHEP)



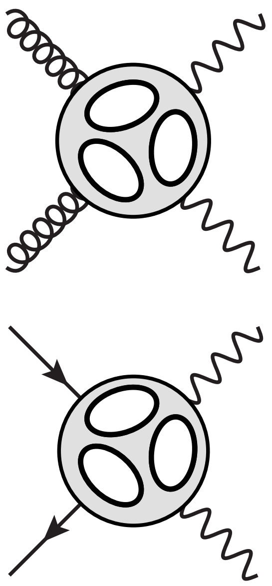
Bargiela, Chakraborty, GG:
2212.14069(PRD)



Chakraborty, Caola, GG,
Tancredi, von Manteuffel:
2108.00055(JHEP),
2207.03503(JHEP),
2112.11097(PRL)



Agarwal, Buccioni, Caola, Devoto, GG, von Manteuffel, Tancredi: soon!



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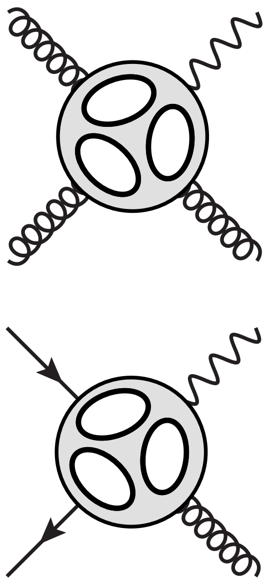

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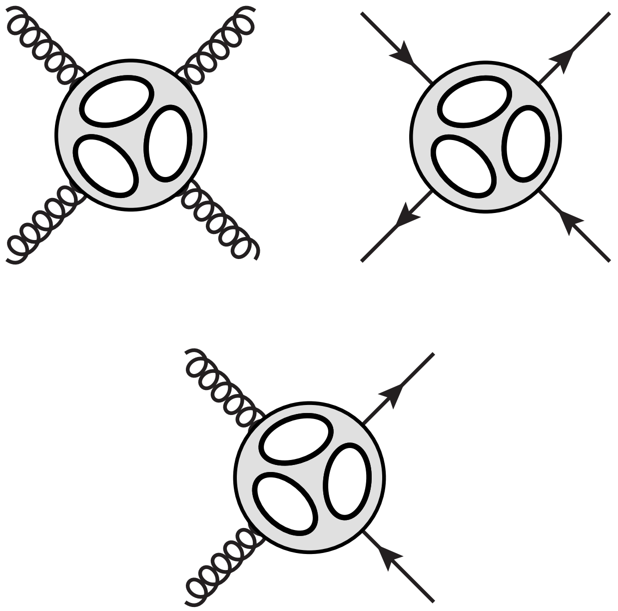

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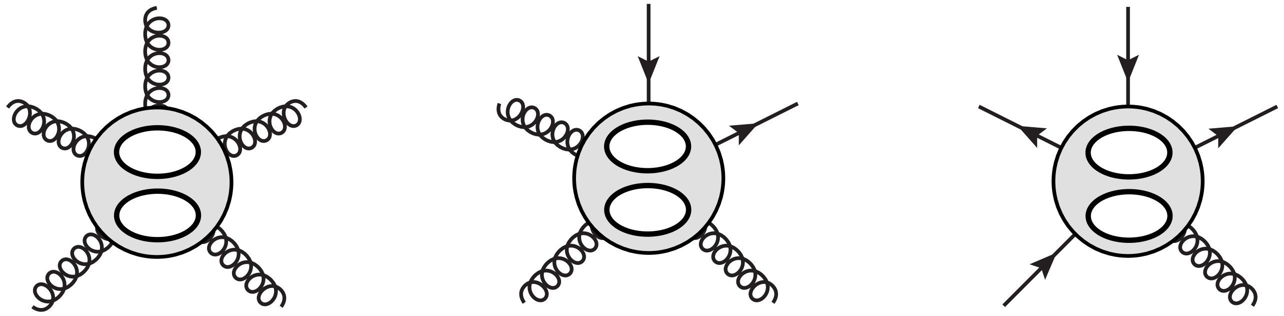
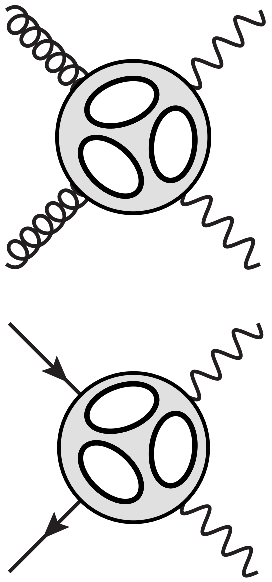


Agarwal, Buccioni, Caola, Devoto, GG, von Manteuffel, Tancredi: soon!

simplest QCD amplitudes
non-trivial beyond the planar limit



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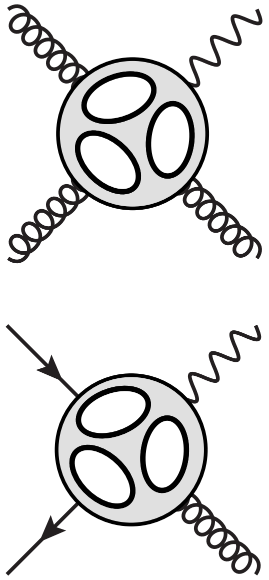

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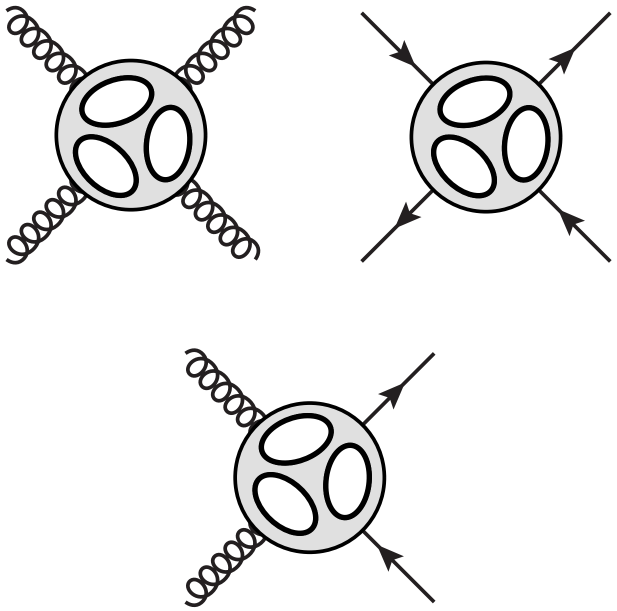

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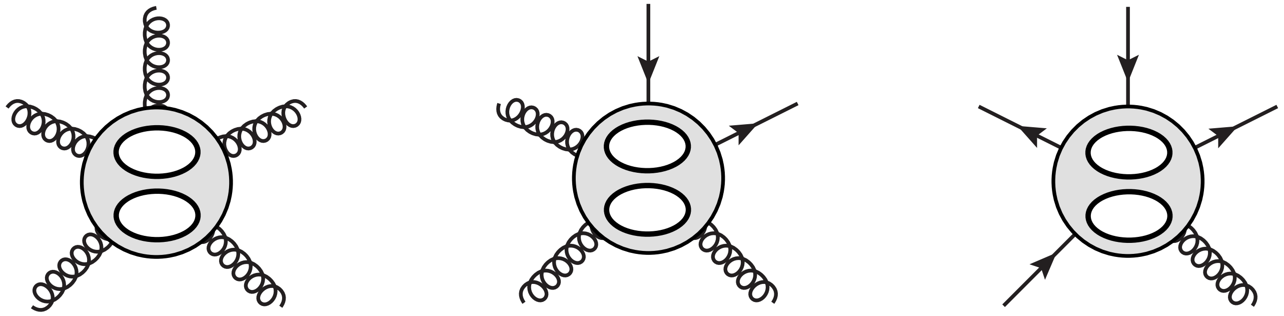

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simplest QCD amplitudes
non-trivial beyond the planar limit

& relevant for phenomenology

$$\mathcal{H} = \sum_{m,c} R^{mc}\, \mathcal{M}_m\, \mathcal{C}_c$$

feynman
diagrams

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feynman
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> 100k

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feynman diagrams → helicity projection

> 100k

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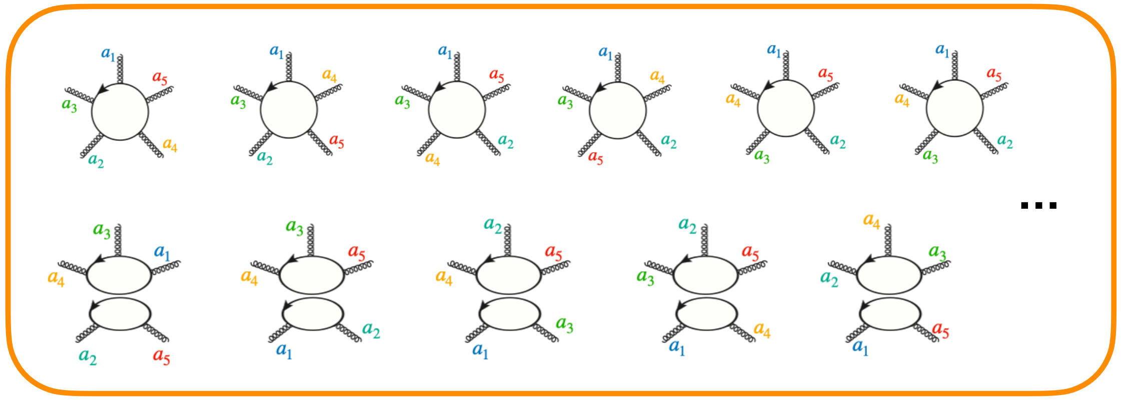
$$\mathcal{A} = \sum \mathcal{H}_h \overline{\mathcal{T}}_h$$

Tancredi, Peraro:
1906.03298, 2012.00820

feynman diagrams → helicity projection → colour decomposition

> 100k



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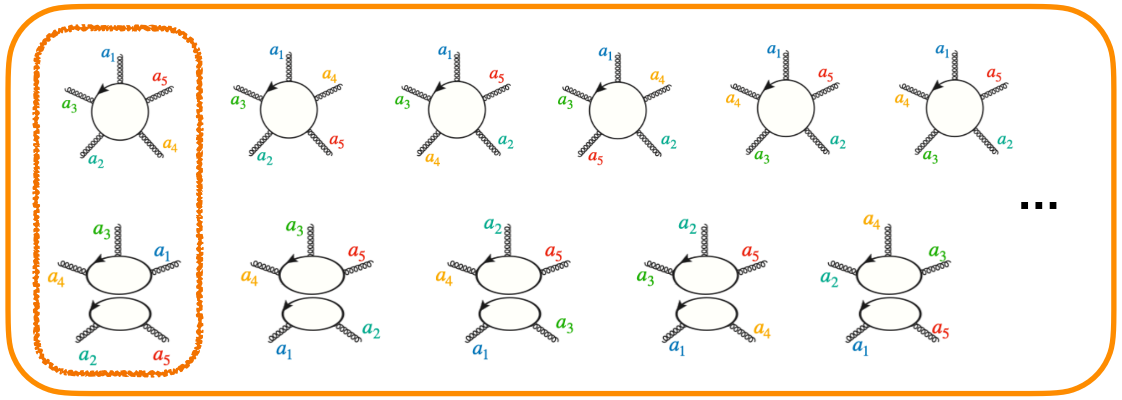

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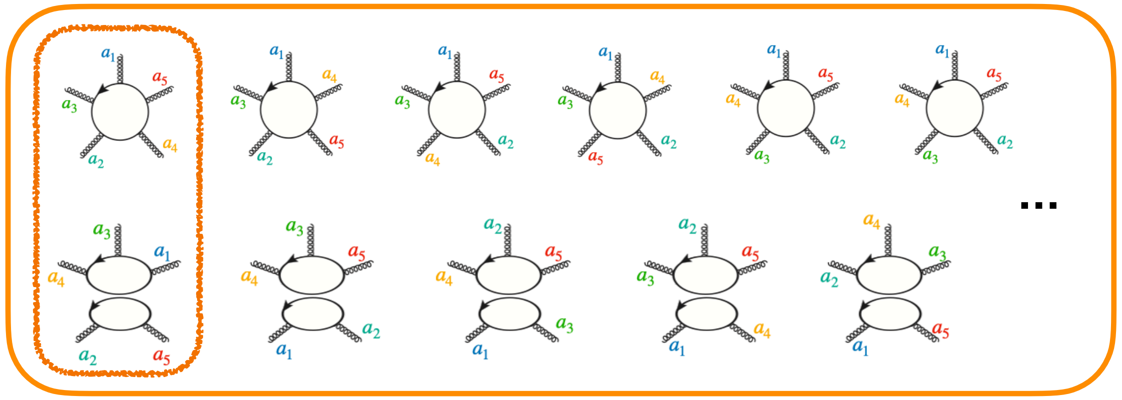

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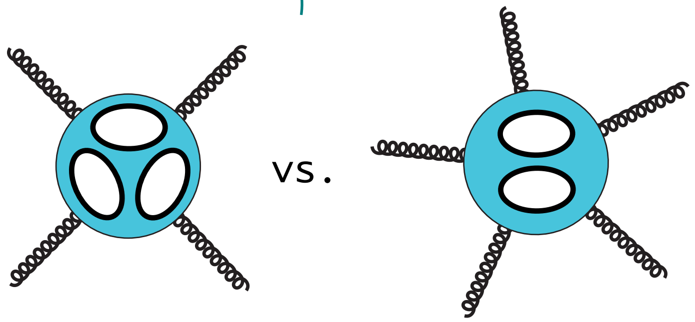


combinatorics kinematics

finite fields reconstruction

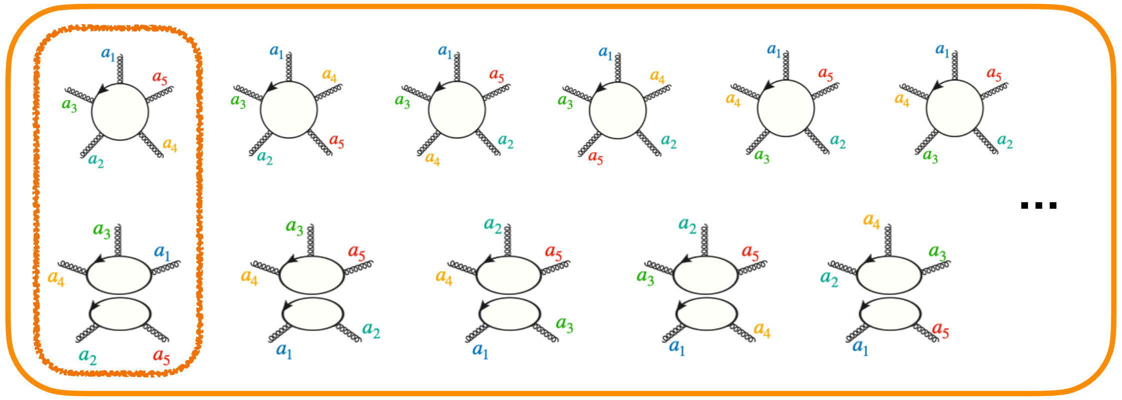
syzygy techniques

FinRed
(von Manteuffel)

feynman diagrams → helicity projection → colour decomposition → integration by parts

\> 100k

$$\frac{s_{12}s_{13}+(d-4)s_{23}s_{14}}{s_{24}s_{35}}$$



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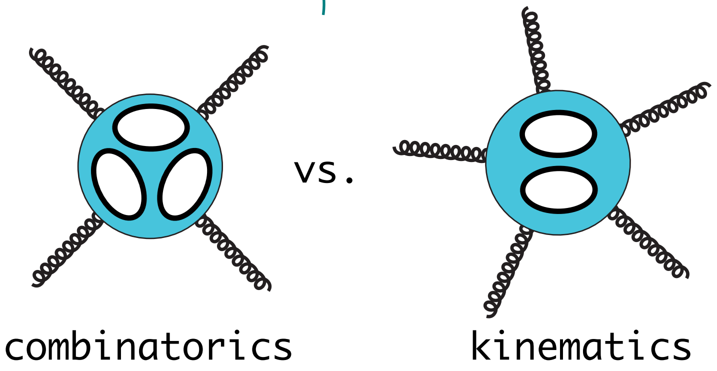

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syzygy techniques

FinRed
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feynman diagrams → helicity projection → colour decomposition → integration by parts

\> 100k

What are the right variables?

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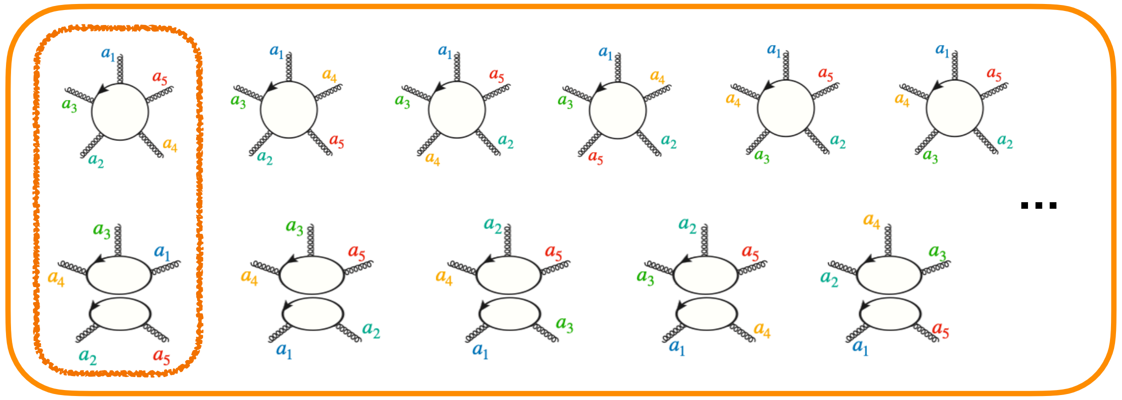

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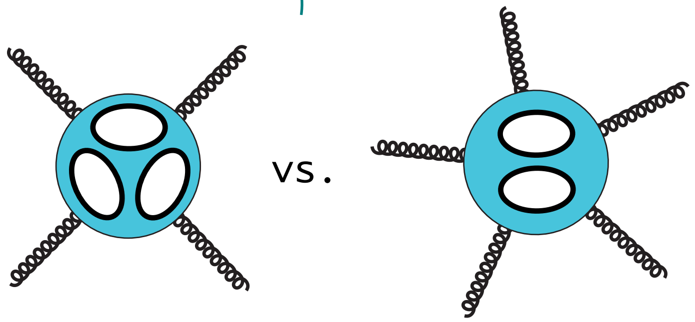
Tancredi, Peraro:
1906.03298, 2012.00820



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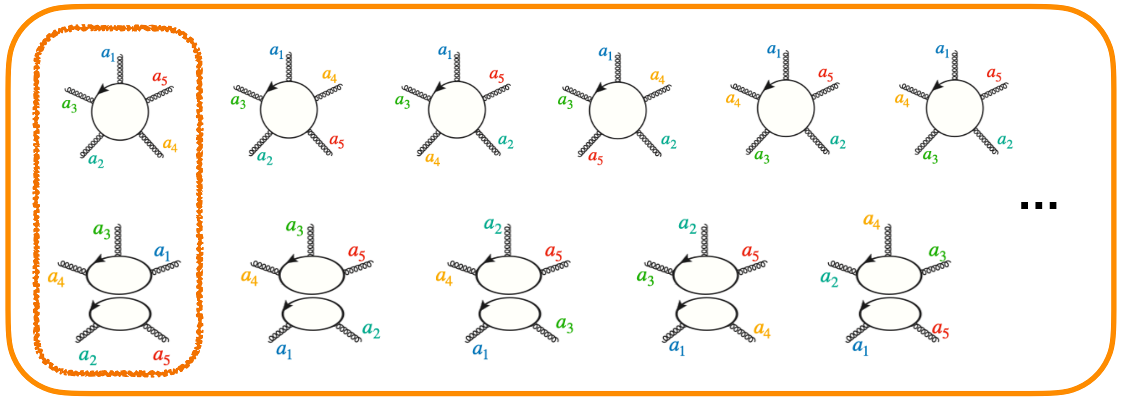

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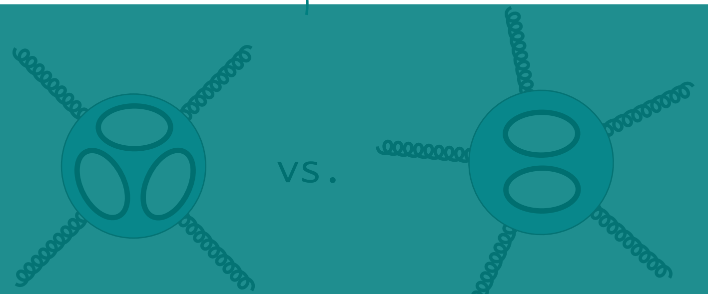

combinatorics

kinematics

What are the right masters?

finite fields reconstruction

syzygy techniques

FinRed
(von Manteuffel)

Which are the Right Masters ?

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with David Kosower, Pavel Novichkov, and Lorenzo Tancredi

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$$\int [\mathrm{d}^d \ell] \, \frac{\mathcal{N}}{\mathcal{D}_1 \ldots \mathcal{D}_e}$$

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with David Kosower, Pavel Novichkov, and Lorenzo Tancredi

$$\mathcal{A} = \mathcal{A}_{IR} + \mathcal{A}_{hard}$$

$$\int [\mathrm{d}^d \ell] \, \frac{\mathcal{N}}{\mathcal{D}_1 \ldots \mathcal{D}_e} = \int [\mathrm{d}^d \ell] \frac{c_0 + c_1(k_1 \cdot \ell_1) + c_2(k_1 \cdot \ell_2) + \ldots}{\mathcal{D}_1 \ldots \mathcal{D}_e}$$

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- UV finite ← Weinberg's Theorem

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- UV finite ← Weinberg’s Theorem
- IR finite ← Landau equations +
  power counting (Anastasiou,Sterman: 1812.03753)

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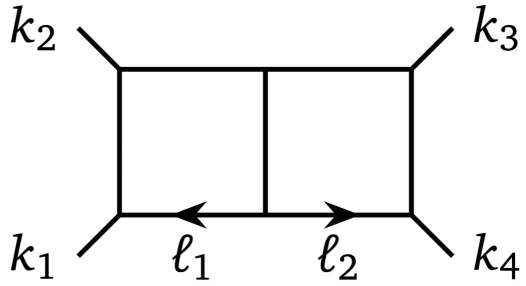
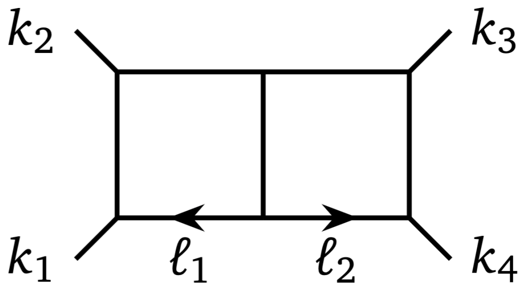
$$\int [\mathrm{d}^d\ell]\, \frac{\mathcal{N}}{\mathcal{D}_1 \ldots \mathcal{D}_e} = \int [\mathrm{d}^d\ell] \frac{c_0 + c_1(k_1 \cdot \ell_1) + c_2(k_1 \cdot \ell_2) + \ldots}{\mathcal{D}_1 \ldots \mathcal{D}_e} = \mathcal{O}(\epsilon^0)$$

- UV finite ← Weinberg's Theorem
- IR finite ← Landau equations +
  power counting (Anastasiou,Sterman: 1812.03753)

automated GG, Kosower, Novichkov, Tancredi: to appear



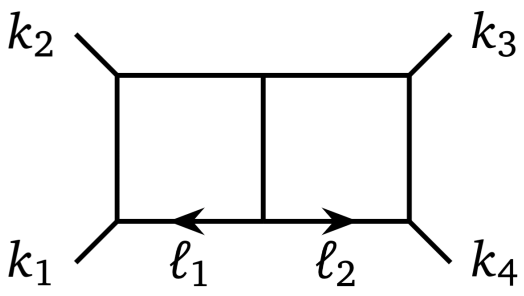
$$\boxed{\mathcal{N}_1 = G\begin{pmatrix} \ell_1 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix} G\begin{pmatrix} \ell_2 & 3 & 4 \\ 1 & 2 & 4 \end{pmatrix}, \quad \mathcal{N}_2 = G\begin{pmatrix} \ell_1 & 1 & 2 \\ \ell_2 & 3 & 4 \end{pmatrix}}$$



Gram determinant

$$G\begin{pmatrix} p_1 & \dots & p_n \\ q_1 & \dots & q_n \end{pmatrix} = \det(2p_i \cdot q_j)$$

$$\boxed{\mathcal{N}_1 = G\begin{pmatrix} \ell_1 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix} G\begin{pmatrix} \ell_2 & 3 & 4 \\ 1 & 2 & 4 \end{pmatrix}, \quad \mathcal{N}_2 = G\begin{pmatrix} \ell_1 & 1 & 2 \\ \ell_2 & 3 & 4 \end{pmatrix}}$$



Gram determinant

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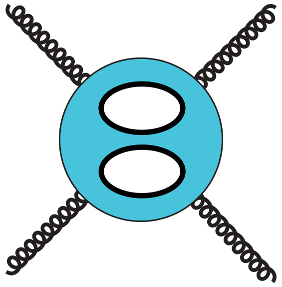

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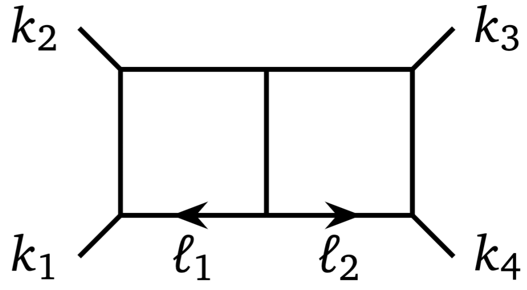

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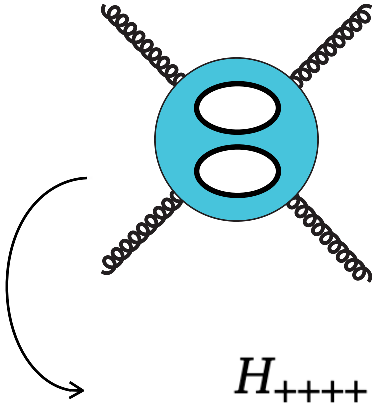
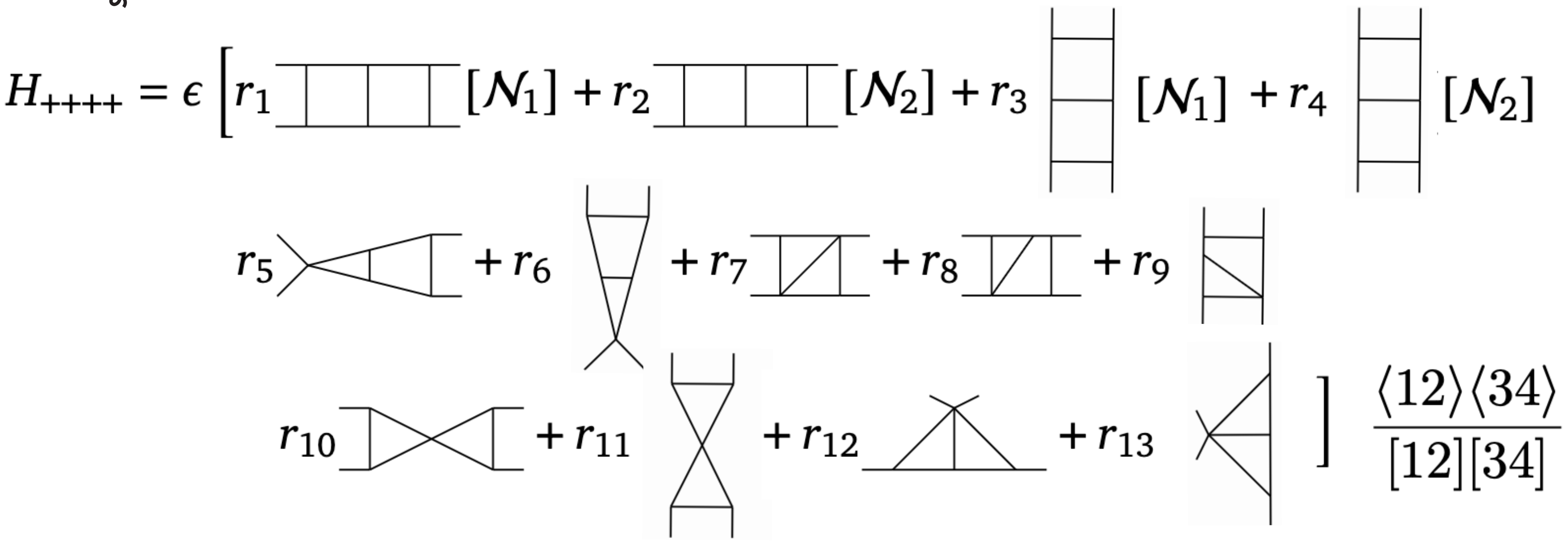
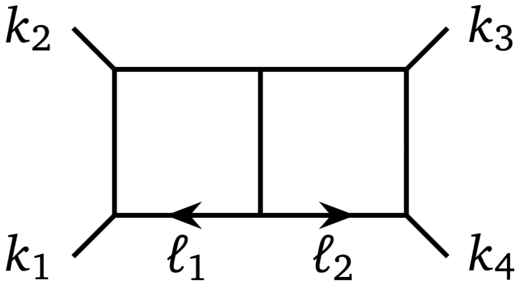




$$H_{++++} = \epsilon \Big[r_1 [\mathcal{N}_1] + r_2 [\mathcal{N}_2] + r_3 [\mathcal{N}_1] + r_4 [\mathcal{N}_2] + r_5 + r_6 + r_7 + r_8 + r_9 + r_{10} + r_{11} + r_{12} + r_{13} \Big] \frac{\langle 12\rangle\langle 34\rangle}{[12][34]}$$



Gram determinant

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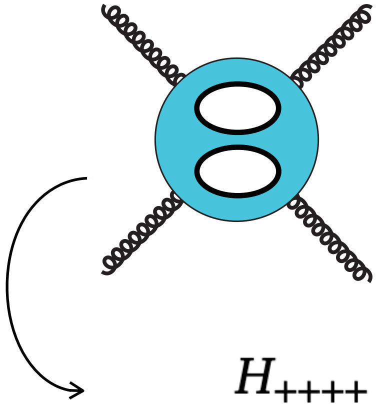
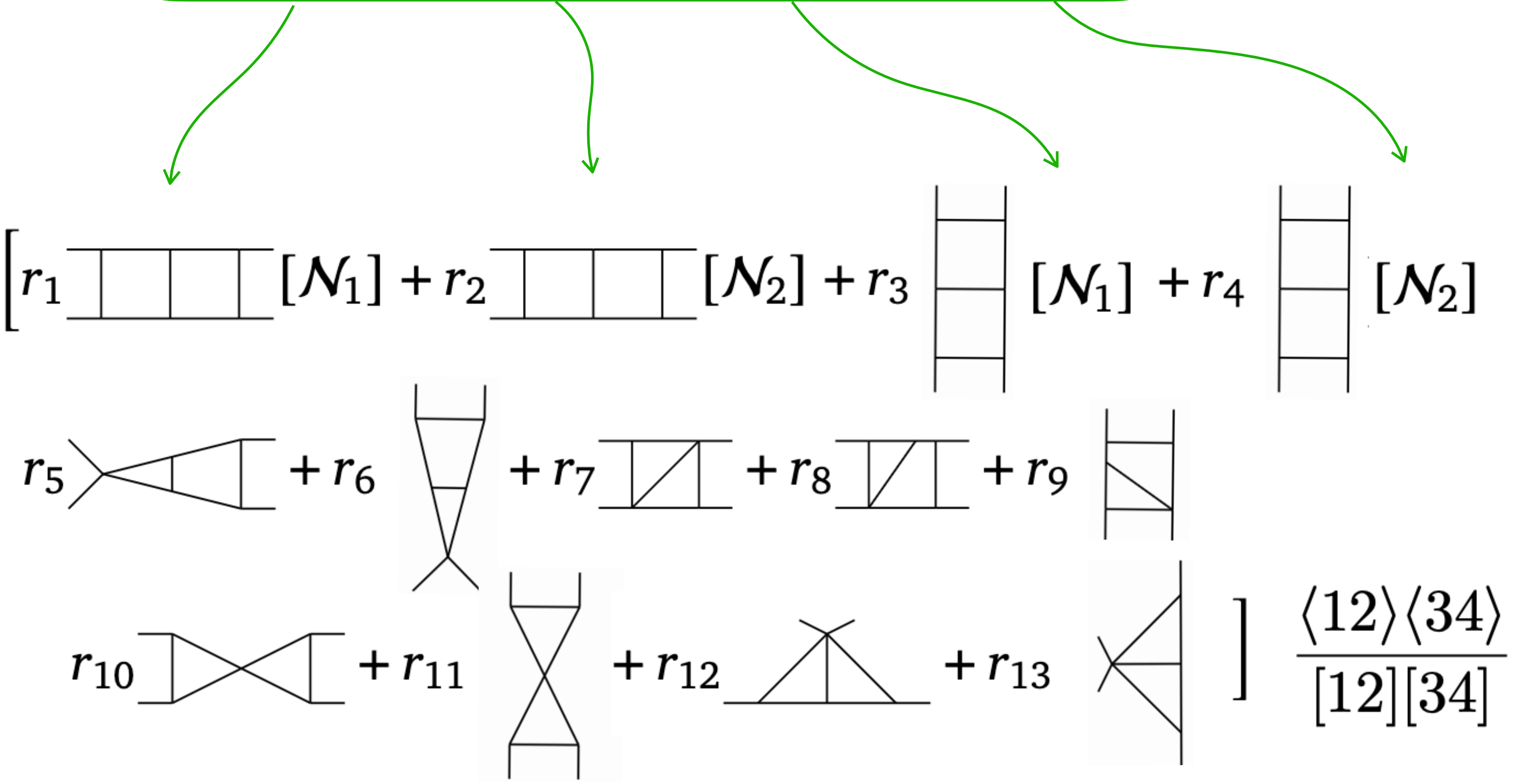
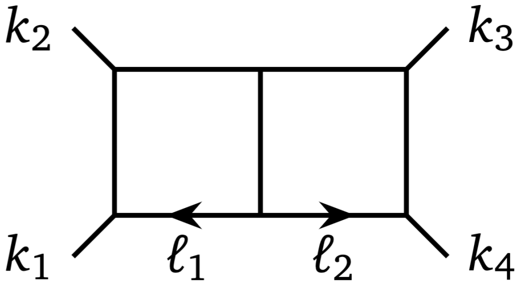




$$H_{++++} = \epsilon \Big[r_1 \, [\mathcal{N}_1] + r_2 \, [\mathcal{N}_2] + r_3 \, [\mathcal{N}_1] + r_4 \, [\mathcal{N}_2] \\ r_5 + r_6 + r_7 + r_8 + r_9 \\ r_{10} + r_{11} + r_{12} + r_{13} \Big] \frac{\langle 12\rangle\langle 34\rangle}{[12][34]}$$



Gram determinant

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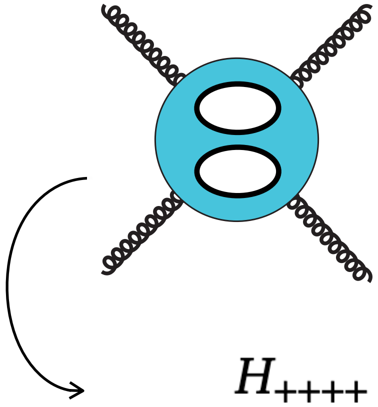
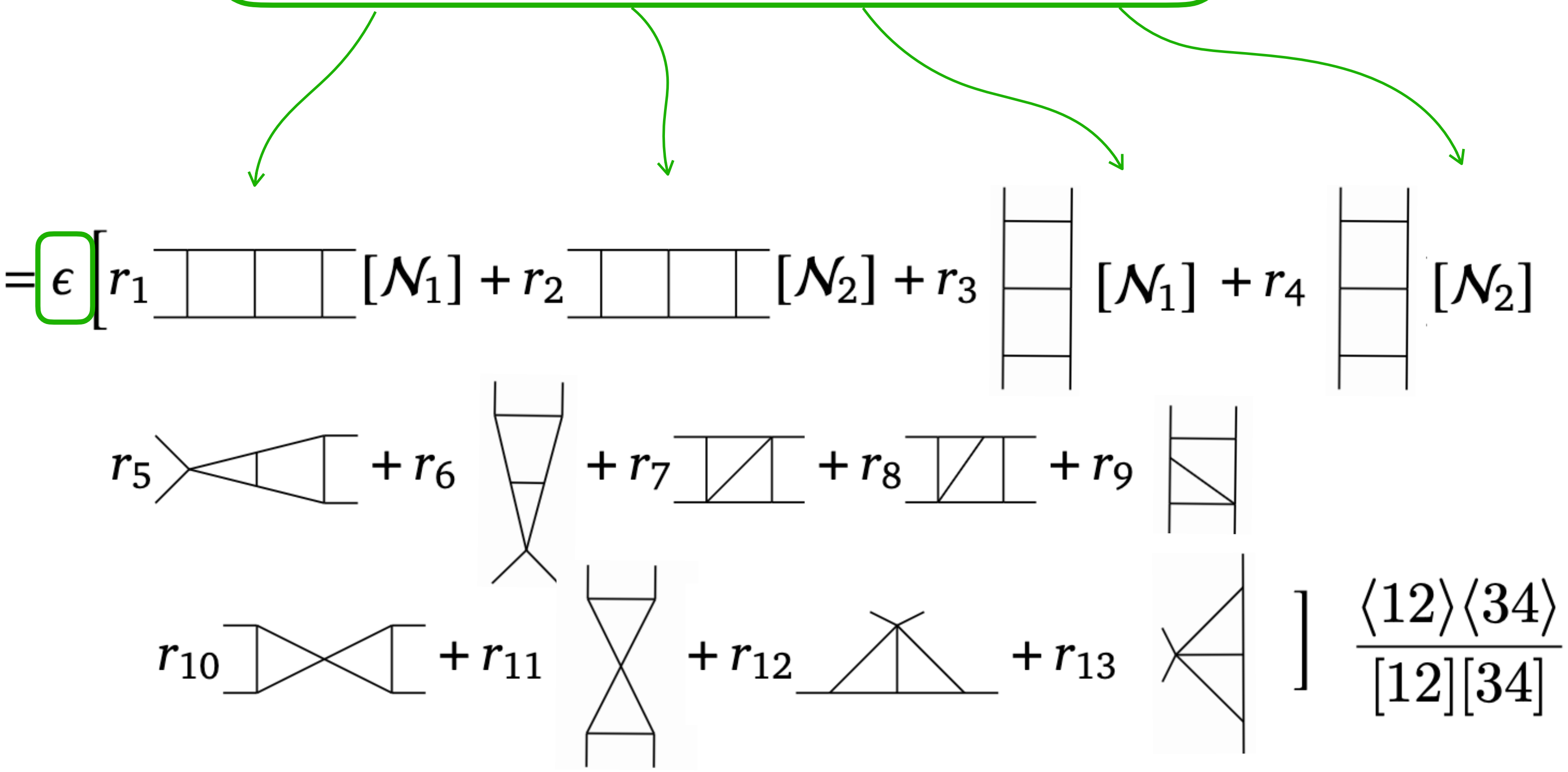
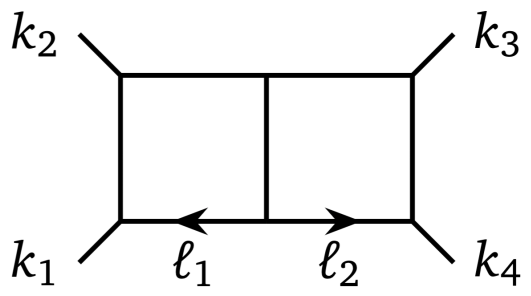




$$H_{++++} = \boxed{\epsilon}\Big[r_1 \, [\mathcal{N}_1] + r_2 \, [\mathcal{N}_2] + r_3 \, [\mathcal{N}_1] + r_4 \, [\mathcal{N}_2] \; r_5 + r_6 + r_7 + r_8 + r_9 \; r_{10} + r_{11} + r_{12} + r_{13} \Big] \frac{\langle 12\rangle\langle 34\rangle}{[12][34]}$$



Gram determinant

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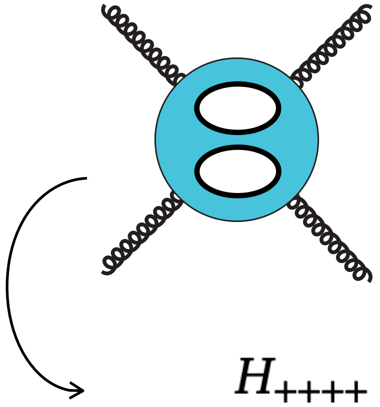
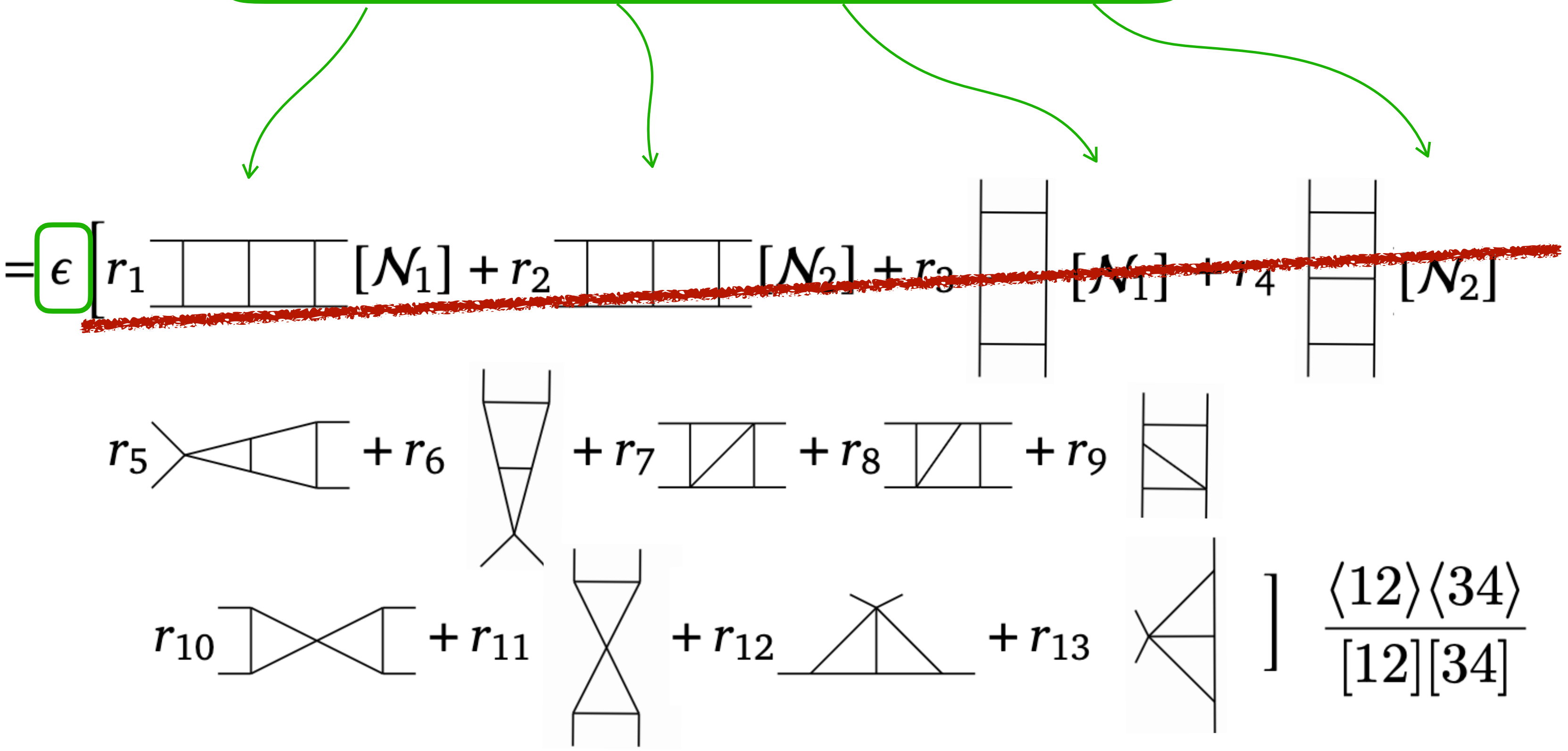
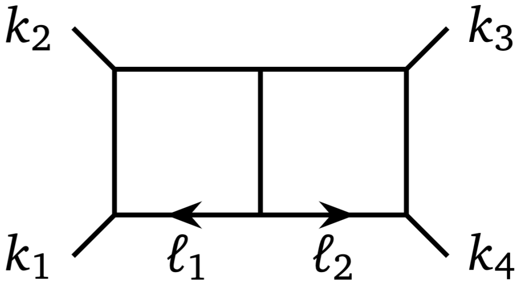




$$H_{++++} = \boxed{\epsilon}\Big[\cancel{r_1 [\mathcal{N}_1] + r_2 [\mathcal{N}_2] + r_3 [\mathcal{N}_1] + r_4 [\mathcal{N}_2]} + r_5 + r_6 + r_7 + r_8 + r_9 + r_{10} + r_{11} + r_{12} + r_{13}\Big]\frac{\langle 12\rangle\langle 34\rangle}{[12][34]}$$



Gram determinant

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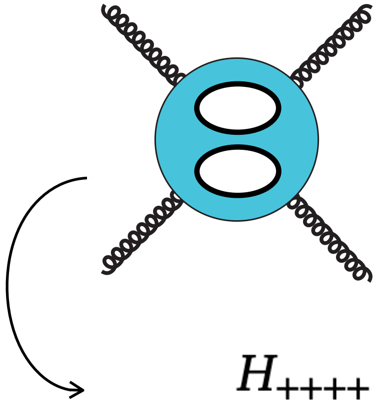
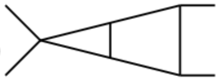
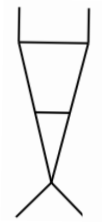
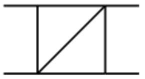
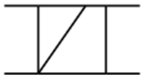
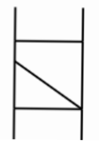
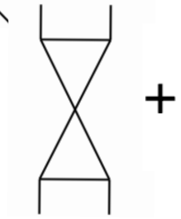
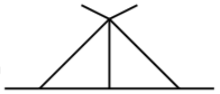
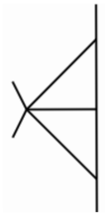
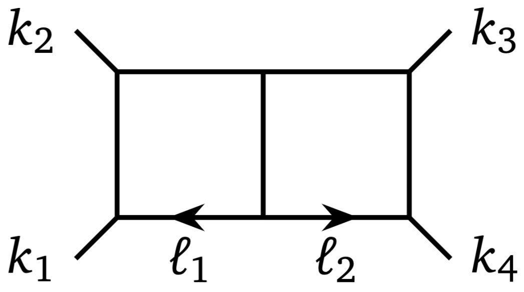


$H_{++++} = \boxed{\epsilon}\Big[r_5$  $+ r_6$  $+ r_7$  $+ r_8$  $+ r_9$ 

r_{10}  $+ r_{11}$  $+ r_{12}$  $+ r_{13}$  $\Big] \dfrac{\langle 12\rangle\langle 34\rangle}{[12][34]}$



Gram determinant

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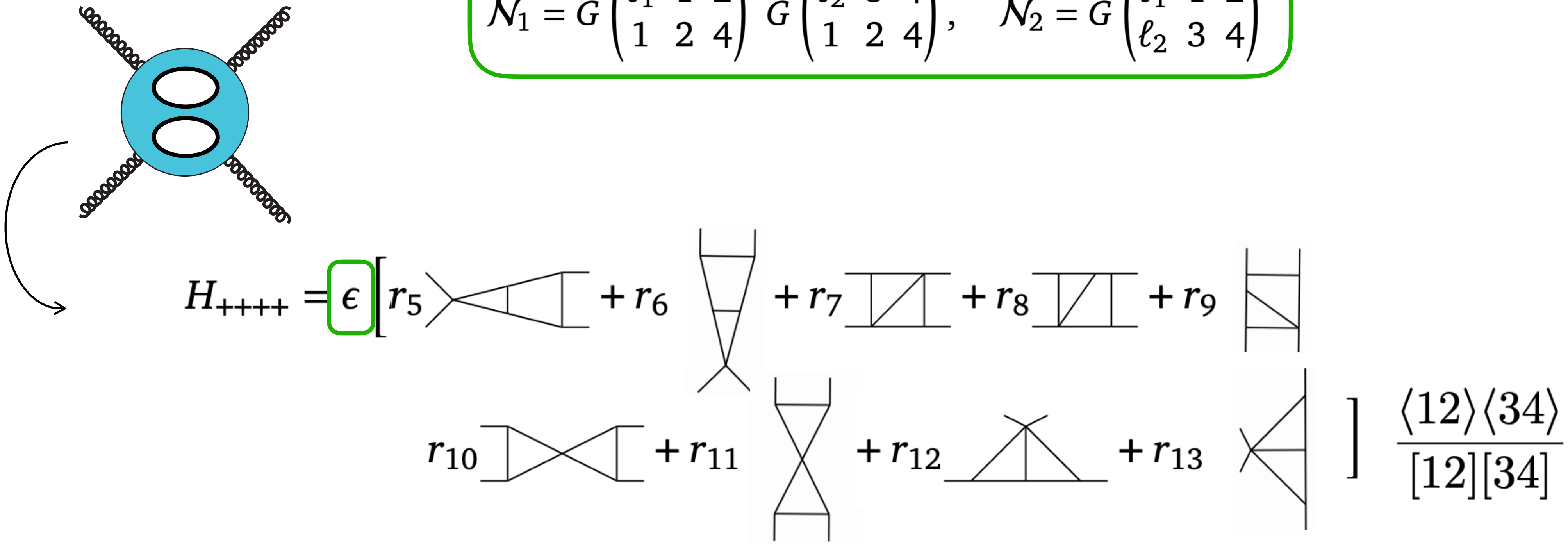
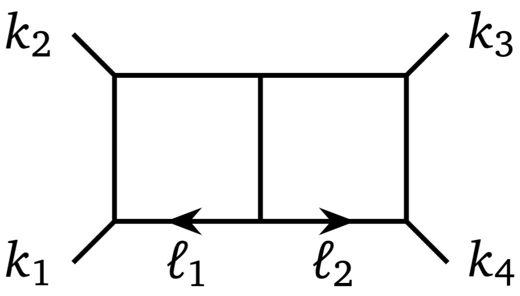


$$H_{++++} = \boxed{\epsilon}\Big[r_5 + r_6 + r_7 + r_8 + r_9 \quad r_{10} + r_{11} + r_{12} + r_{13} \Big] \frac{\langle 12\rangle\langle 34\rangle}{[12][34]}$$

–Largest Coefficients–



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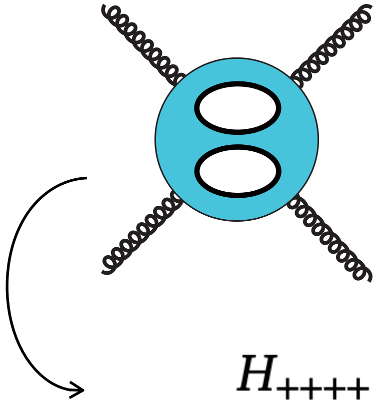
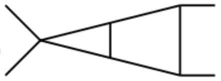
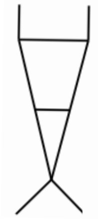
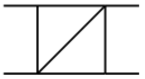
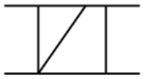
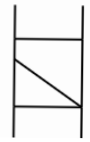
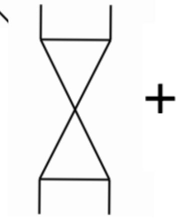
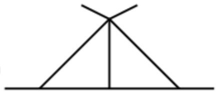
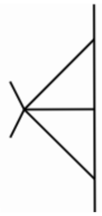
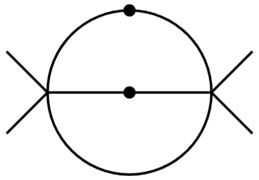


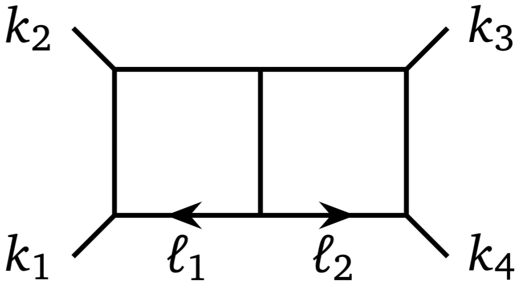
$H_{++++} = \boxed{\epsilon}\Big[r_5$  $+ r_6$  $+ r_7$  $+ r_8$  $+ r_9$ 

r_{10}  $+ r_{11}$  $+ r_{12}$  $+ r_{13}$  $\Big] \dfrac{\langle 12 \rangle \langle 34 \rangle}{[12][34]}$

–Largest Coefficients–

Canonical basis





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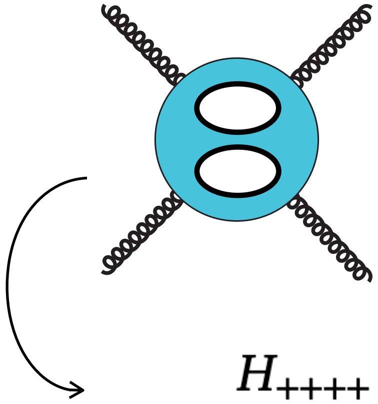
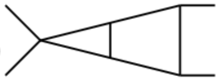
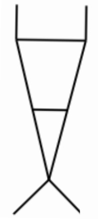
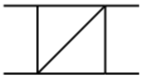
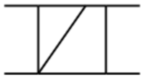
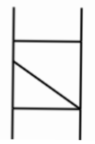
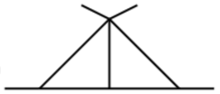
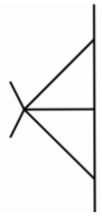
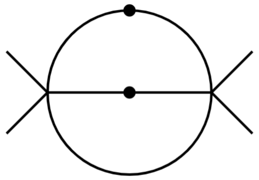


$H_{++++} = \boxed{\epsilon}\Big[r_5$  $+ r_6$  $+ r_7$  $+ r_8$  $+ r_9$ 

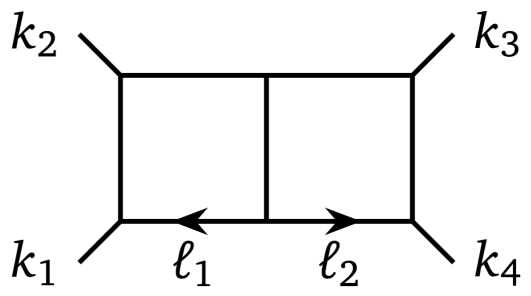
r_{10}  $+ r_{11}$  $+ r_{12}$  $+ r_{13}$  $\Big] \dfrac{\langle 12\rangle\langle 34\rangle}{[12][34]}$

–Largest Coefficients–

Canonical basis



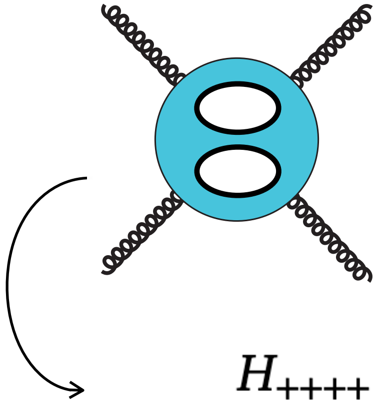
$$\begin{aligned}\longrightarrow\ &\big(30\,(-5+15\,x-36\,x^2+58\,x^3-27\,x^4-27\,x^5+18\,x^6)-3\,(-397+1453\,x-2942\,x^2+2082\,x^3+2501\,x^4-5381\,x^5+2196\,x^6)\,\epsilon\,+\\ &\quad 3\,(-1207+4242\,x-2487\,x^2-18\,110\,x^3+39\,899\,x^4-35\,856\,x^5+11\,061\,x^6)\,\epsilon^2-\\ &\quad 2\,(-2655+6948\,x+31\,603\,x^2-161\,272\,x^3+239\,397\,x^4-168\,768\,x^5+44\,793\,x^6)\,\epsilon^3\big)\,/\\ &\big(2\,(-1+x)^2\,x^2\,\epsilon\,(-3+2\,\epsilon)^2\,(-1+2\,\epsilon)^3\,(-2+3\,\epsilon)\,(-1+3\,\epsilon)\big)\end{aligned}$$

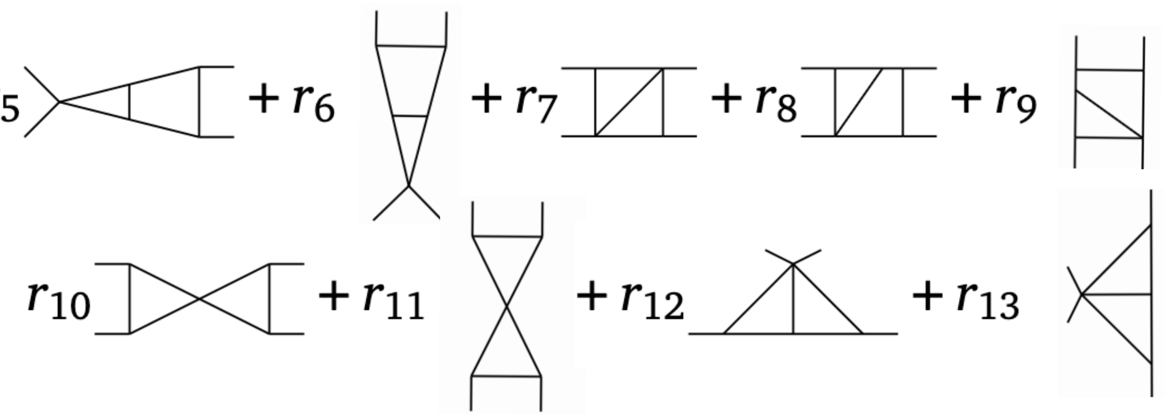


Gram determinant

$$G\begin{pmatrix} p_1 & \dots & p_n \\ q_1 & \dots & q_n \end{pmatrix} = \det(2p_i \cdot q_j)$$

$$\boxed{\mathcal{N}_1 = G\begin{pmatrix} \ell_1 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix} G\begin{pmatrix} \ell_2 & 3 & 4 \\ 1 & 2 & 4 \end{pmatrix}, \quad \mathcal{N}_2 = G\begin{pmatrix} \ell_1 & 1 & 2 \\ \ell_2 & 3 & 4 \end{pmatrix}}$$

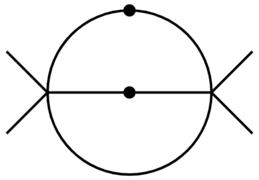




$$H_{++++} = \boxed{\epsilon}\Big[r_5 + r_6 + r_7 + r_8 + r_9 + r_{10} + r_{11} + r_{12} + r_{13}\Big]\frac{\langle 12\rangle\langle 34\rangle}{[12][34]}$$

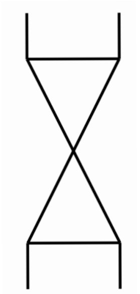
–Largest Coefficients–

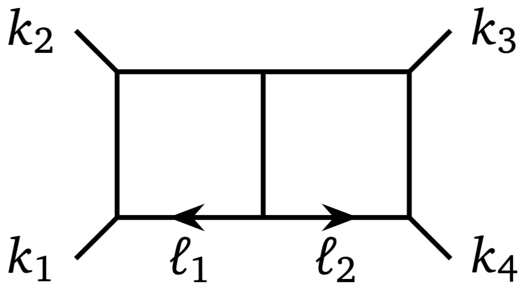
Canonical basis



$$\longrightarrow \Big(30\left(-5+15x-36x^2+58x^3-27x^4-27x^5+18x^6\right)-3\left(-397+1453x-2942x^2+2082x^3+2501x^4-5381x^5+2196x^6\right)\epsilon+3\left(-1207+4242x-2487x^2-18\,110x^3+39\,899x^4-35\,856x^5+11\,061x^6\right)\epsilon^2-2\left(-2655+6948x+31\,603x^2-161\,272x^3+239\,397x^4-168\,768x^5+44\,793x^6\right)\epsilon^3\Big)\Big/\left(2\,(-1+x)^2x^2\epsilon\,(-3+2\epsilon)^2(-1+2\epsilon)^3(-2+3\epsilon)(-1+3\epsilon)\right)$$

Modified basis

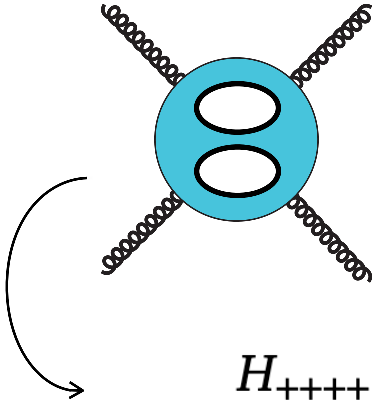


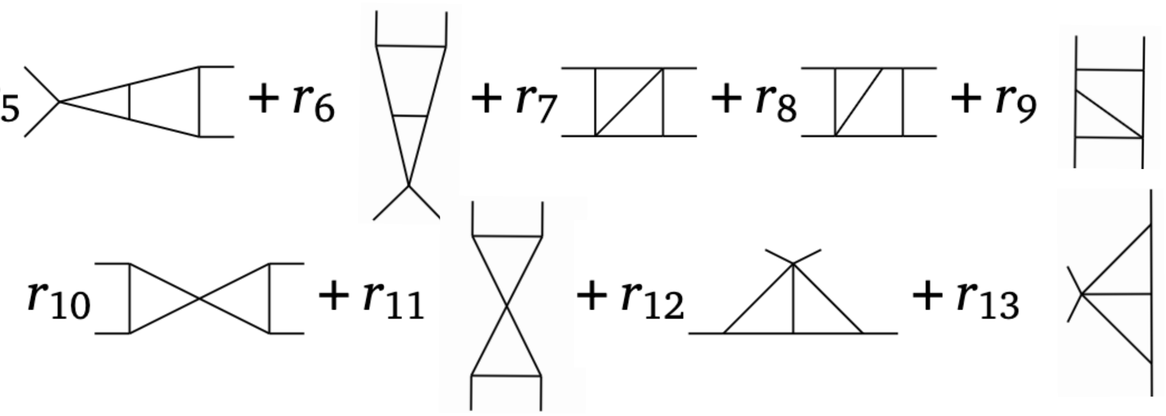


Gram determinant

$$G\begin{pmatrix} p_1 & \dots & p_n \\ q_1 & \dots & q_n \end{pmatrix} = \det(2p_i \cdot q_j)$$

$$\boxed{\mathcal{N}_1 = G\begin{pmatrix} \ell_1 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix} G\begin{pmatrix} \ell_2 & 3 & 4 \\ 1 & 2 & 4 \end{pmatrix}, \quad \mathcal{N}_2 = G\begin{pmatrix} \ell_1 & 1 & 2 \\ \ell_2 & 3 & 4 \end{pmatrix}}$$

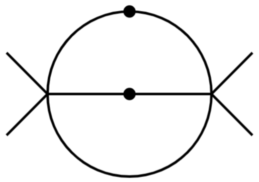




$$H_{++++} = \boxed{\epsilon}\Big[r_5 + r_6 + r_7 + r_8 + r_9 \\ r_{10} + r_{11} + r_{12} + r_{13} \Big] \frac{\langle 12\rangle\langle 34\rangle}{[12][34]}$$

–Largest Coefficients–

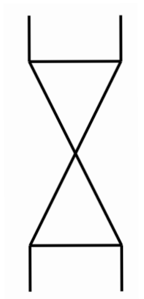
Canonical basis



→

$$\Big(30\left(-5+15x-36x^2+58x^3-27x^4-27x^5+18x^6\right)-3\left(-397+1453x-2942x^2+2082x^3+2501x^4-5381x^5+2196x^6\right)\epsilon+ \\ 3\left(-1207+4242x-2487x^2-18\,110x^3+39\,899x^4-35\,856x^5+11\,061x^6\right)\epsilon^2- \\ 2\left(-2655+6948x+31\,603x^2-161\,272x^3+239\,397x^4-168\,768x^5+44\,793x^6\right)\epsilon^3\Big)\Big/ \\ \left(2(-1+x)^2x^2\epsilon(-3+2\epsilon)^2(-1+2\epsilon)^3(-2+3\epsilon)(-1+3\epsilon)\right)$$

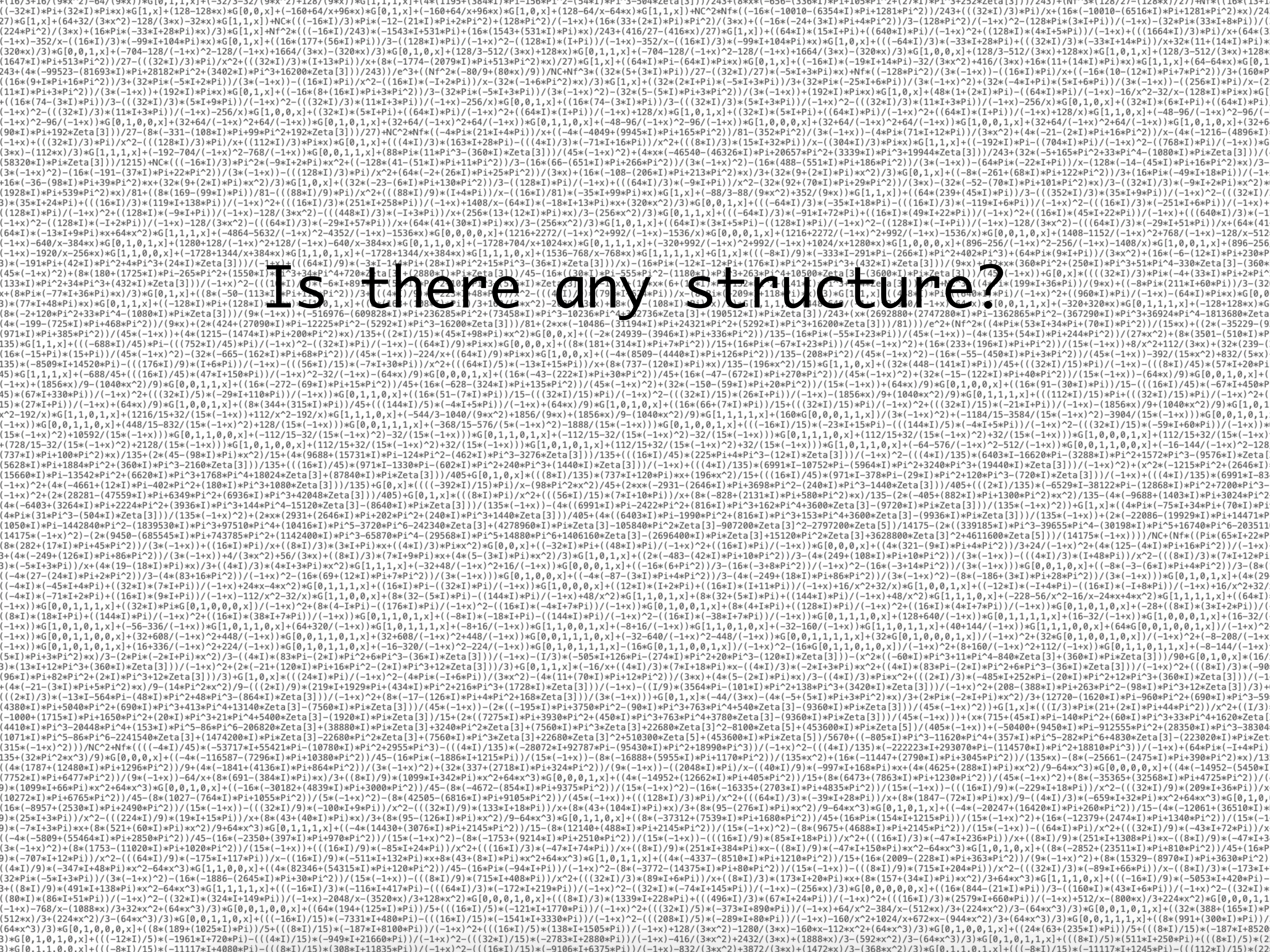
Modified basis



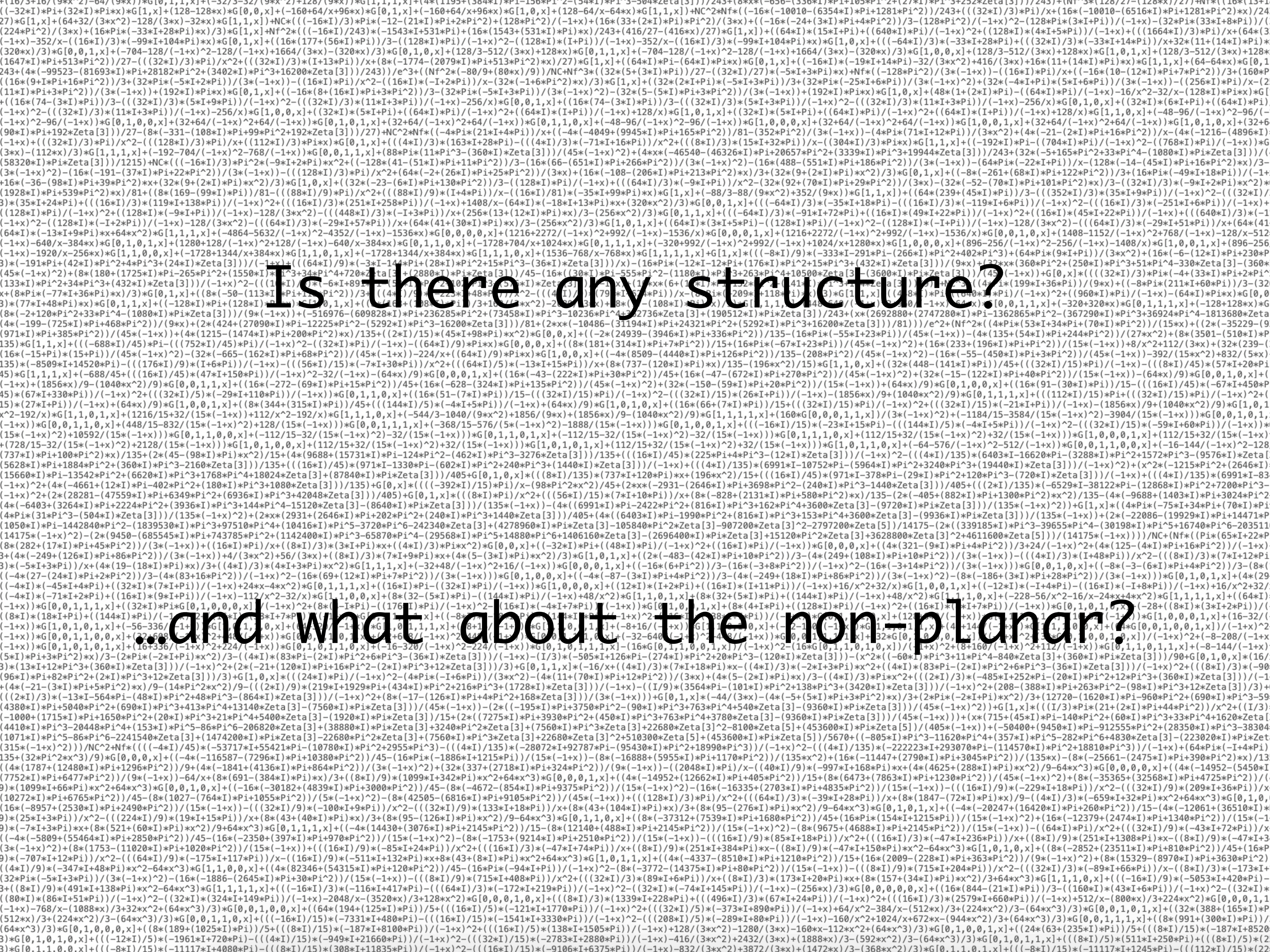
→

$$\epsilon\Big(48(-1+x)^3+8(-1+x)^2\left(67+71x+78x^2+30x^3\right)\epsilon-8(-1+x)^2\left(294+234x+605x^2+126x^3\right)\epsilon^2- \\ 4(-1+x)\left(1340-1456x+3091x^2-3398x^3+429x^4\right)\epsilon^3\Big)\Big/\left(3x^2(-3+2\epsilon)^2(-1+2\epsilon)^3(-2+3\epsilon)(-1+3\epsilon)\right)$$

```
+(16/3+16/(9*x^2)-64/(9*x))*G[0,1,1,x]+(-32/3-32/(9*x^2)+128/(9*x))*G[1,1,1,x]+(4*(1195+(384*I)*Pi-156*Pi^2-(54*I)*Pi^3-504*Zeta[3]))/243+(8*x*(-656-(336*I)*Pi+105*Pi^2+(27*I)*Pi^3+252*Zeta[3]))/243+(Nf^3*(128/27-(128*x)/27)+Nf*((16*(13+I
((-32*I)*Pi+(32*I)*Pi*x)*G[1,x]+(128-128*x)*G[0,0,x]+(-160+64/x+96*x)*G[0,1,x]+(-160+64/x+96*x)*G[1,0,x]+(128-64/x-64*x)*G[1,1,x])+NC^2*Nf*((-16*(-10010-(6354*I)*Pi+1281*Pi^2))/243+(((32*I)/3)*Pi)/x+(16*(-10010-(6516*I)*Pi+1281*Pi^2)*x)/24
27)*G[1,x]+(64+32/(3*x^2)-128/(3*x)-32*x)*G[1,1,x])+NC*(((-16*I)/3)*Pi*(-12-(21*I)*Pi+2*Pi^2)+(128*Pi^2)/(-1+x)+(16*(33+(2*I)*Pi)*Pi^2)/(3*x)+((-16*(-24+(3*I)*Pi+4*Pi^2))/3-(128*Pi^2)/(-1+x)^2-(128*Pi*(3*I+Pi))/(-1+x)-(32*Pi*(33*I+8*Pi))/(
(224*Pi^2)/(3*x)+(16*Pi*(-33*I+28*Pi)*x)/3)*G[1,x]+Nf^2*(((-16*I)/243)*(-1543*I+531*Pi)+(16*(1543+(531*I)*Pi)*x)/243+(416/27-(416*x)/27)*G[1,x])+((64*I)*(15*I+Pi)+((640*I)*Pi)/(-1+x)^2+((128*I)*(4*I+5*Pi))/(-1+x)+(((1664*I)/3)*Pi)/x+(64*(3
(-1+x)-352/x-((16*I)/3)*(-99*I+104*Pi)*x)*G[0,1,x]+((16*(177+(56*I)*Pi))/3-((128*I)*Pi)/(-1+x)^2-((128*I)*(I+Pi))/(-1+x)-352/x-((16*I)/3)*(-99*I+104*Pi)*x)*G[1,0,x]+(((-64*I)/3)*(-33*I+28*Pi)+(((32*I)/3)*(-33*I+14*Pi))/x+32*(11+(14*I)*Pi)*
(320*x)/3)*G[0,0,1,x]+(-704-128/(-1+x)^2-128/(-1+x)+1664/(3*x)-(320*x)/3)*G[0,1,0,x]+(128/3-512/(3*x)+128*x)*G[0,1,1,x]+(-704-128/(-1+x)^2-128/(-1+x)+1664/(3*x)-(320*x)/3)*G[1,0,0,x]+(128/3-512/(3*x)+128*x)*G[1,0,1,x]+(128/3-512/(3*x)+128*
(1647*I)*Pi+513*Pi^2))/27-(((32*I)/3)*Pi)/x^2+(((32*I)/3)*(I+13*Pi))/x+(8*(-1774-(2079*I)*Pi+513*Pi^2)*x)/27)*G[1,x]+((64*I)*Pi-(64*I)*Pi*x)*G[0,1,x]+((-16*I)*(-19*I+14*Pi)-32/(3*x^2)+416/(3*x)+16*(11+(14*I)*Pi)*x)*G[1,1,x]+(64-64*x)*G[0,1
243+(4*(-99523-(81693*I)*Pi+28182*Pi^2+(3402*I)*Pi^3+16200*Zeta[3]))/243))/e^3+((Nf^2*(-80/9+(80*x)/9))/NC+Nf^3*((32*(5+(3*I)*Pi))/27-((32*I)/27)*(-5*I+3*Pi)*x)+Nf*((-128*Pi^2)/(3*(-1+x))-((16*I)*Pi)/x+((-16*(10-(12*I)*Pi+7*Pi^2))/3+(160*P
((16*(9+I*Pi+16*Pi^2))/3+(32*Pi*(-5*I+2*Pi))/(3*(-1+x))-((16*I)*Pi)/x^2-((16*I)*(-I+2*Pi))/x-(32*(-1+6*Pi^2)*x)/3)*G[1,x]+((32*(2*I+Pi)*(-5*I+3*Pi))/3+(32*Pi*(-25*I+6*Pi))/(3*(-1+x)^2)+(32*(-4*I+Pi)*(5*I+6*Pi))/(3*(-1+x))-((256*I)*Pi)/x-(2
(11*I)*Pi+3*Pi^2))/(3*(-1+x))+(192*I)*Pi*x)*G[0,1,x]+((-16*(8+(16*I)*Pi+3*Pi^2))/3-(32*Pi*(-5*I+3*Pi))/(3*(-1+x)^2)-(32*(5-(5*I)*Pi+3*Pi^2))/(3*(-1+x))+(192*I)*Pi*x)*G[1,0,x]+(48*(1+(2*I)*Pi)-((64*I)*Pi)/(-1+x)-16/x^2-32/x-(128*I)*Pi*x)*G[
+((16*(74-(3*I)*Pi))/3-(((32*I)/3)*(5*I+9*Pi))/(-1+x)^2-(((32*I)/3)*(11*I+3*Pi))/(-1+x)-256/x)*G[0,0,1,x]+((16*(74-(3*I)*Pi))/3-(((32*I)/3)*(5*I+3*Pi))/(-1+x)^2-(((32*I)/3)*(11*I+3*Pi))/(-1+x)-256/x)*G[0,1,0,x]+((32*I)*(6*I+Pi)+((64*I)*Pi)/
(-1+x)^2-(((32*I)/3)*(11*I+3*Pi))/(-1+x)-256/x)*G[1,0,0,x]+((32*I)*(5*I+Pi)+((64*I)*Pi)/(-1+x)^2+((64*I)*(I+Pi))/(-1+x)+128/x)*G[1,0,1,x]+((32*I)*(5*I+Pi)+((64*I)*Pi)/(-1+x)^2+((64*I)*(I+Pi))/(-1+x)+128/x)*G[1,1,0,x]+(-48-96/(-1+x)^2-96/(-
(-1+x)^2-96/(-1+x))*G[0,1,0,0,x]+(32+64/(-1+x)^2+64/(-1+x))*G[0,1,0,1,x]+(32+64/(-1+x)^2+64/(-1+x))*G[0,1,1,0,x]+(-48-96/(-1+x)^2-96/(-1+x))*G[1,0,0,0,x]+(32+64/(-1+x)^2+64/(-1+x))*G[1,0,0,1,x]+(32+64/(-1+x)^2+64/(-1+x))*G[1,0,1,0,x]+(32+6
(90*I)*Pi+192*Zeta[3]))/27-(8*(-331-(108*I)*Pi+99*Pi^2+192*Zeta[3]))/27)+NC^2*Nf*((-4*Pi*(21*I+4*Pi))/x+((-4*(-4049+(9945*I)*Pi+165*Pi^2))/81-(352*Pi^2)/(3*(-1+x))-(4*Pi*(71*I+12*Pi))/(3*x^2)+(4*(-21-(2*I)*Pi+16*Pi^2))/x-(4*(-1216-(4896*I)
(-1+x)+(((32*I)/3)*Pi)/x^2-(((128*I)/3)*Pi)/x+((112*I)/3)*Pi*x)*G[0,1,x]+(((4*I)/3)*(163*I+28*Pi)-(((4*I)/3)*(-71*I+16*Pi))/x^2+(((8*I)/3)*(15*I+32*Pi))/x-((304*I)/3)*Pi*x)*G[1,1,x]+((-192*I)*Pi-((704*I)*Pi)/(-1+x)^2-((768*I)*Pi)/(-1+x))*G
(3*x)-(112*x)/3)*G[1,1,1,x]+(-192-704/(-1+x)^2-768/(-1+x))*G[0,0,1,1,x]+(88*Pi*(11*Pi^3-(360*I)*Zeta[3]))/(45*(-1+x)^2)+(4*x*(-46540-(46326*I)*Pi+20657*Pi^2+(3339*I)*Pi^3+19944*Zeta[3]))/243+(32*(-5+165*Pi^2+33*Pi^4-(1080*I)*Pi*Zeta[3]))/(
(58320*I)*Pi*Zeta[3]))/1215)+NC*(((-16*I)/3)*Pi^2*(-9*I+2*Pi)*x^2+((-128*(41-(51*I)*Pi+11*Pi^2))/3-(16*(66-(651*I)*Pi+266*Pi^2))/(3*(-1+x)^2)-(16*(488-(551*I)*Pi+186*Pi^2))/(3*(-1+x))-(64*Pi*(-22*I+Pi))/x-(128*(-14-(45*I)*Pi+16*Pi^2)*x)/3-
(3*(-1+x)^2)-(16*(-191-(37*I)*Pi+22*Pi^2))/(3*(-1+x))-(((128*I)/3)*Pi)/x^2+(64*(-2+(26*I)*Pi+25*Pi^2))/(3*x)+(16*(-108-(206*I)*Pi+213*Pi^2)*x)/3+(32*(9+(2*I)*Pi)*x^2)/3)*G[0,1,x]+((-8*(-261+(68*I)*Pi+122*Pi^2))/3+(16*Pi*(-49*I+18*Pi))/(-1+
+16*(-36-(98*I)*Pi+39*Pi^2)*x+(32*(9+(2*I)*Pi)*x^2)/3)*G[1,0,x]+((32*(-23-(6*I)*Pi+130*Pi^2))/3-((128*I)*Pi)/(-1+x)+(((64*I)/3)*(-9*I+Pi))/x^2-(32*(92+(70*I)*Pi+29*Pi^2))/(3*x)-(32*(-52-(70*I)*Pi+101*Pi^2)*x)/3-((32*I)/3)*(-9*I+2*Pi)*x^2)*
(1928*I)*Pi+539*Pi^2)*x)/81+((8*(169-(99*I)*Pi))/81-(((88*I)/9)*Pi)/x^2+(((88*I)/9)*(I+4*Pi))/x-((16*I)/81)*(-35*I+99*Pi)*x)*G[1,x]+(-88/3-88/(9*x^2)+352/(9*x))*G[1,1,x])+((64*(239+(45*I)*Pi))/3-(((352*I)/3)*(35*I+9*Pi))/(-1+x)^2-(((32*I)/
3)*(35*I+24*Pi)+(((16*I)/3)*(119*I+138*Pi))/(-1+x)^2+(((16*I)/3)*(251*I+258*Pi))/(-1+x)+1408/x-(64*I)*(-18*I+13*Pi)*x+(320*x^2)/3)*G[0,0,1,x]+(((-64*I)/3)*(-35*I+18*Pi)-(((16*I)/3)*(-119*I+6*Pi))/(-1+x)^2-(((16*I)/3)*(-251*I+6*Pi))/(-1+x)+
((128*I)*Pi)/(-1+x)^2+((128*I)*(-9*I+Pi))/(-1+x)-128/(3*x^2)-(((448*I)/3)*(-I+3*Pi))/x+(256*(13+(12*I)*Pi)*x)/3-(256*x^2)/3)*G[0,1,1,x]+(((-64*I)/3)*(-91*I+72*Pi)+((16*I)*(49*I+22*Pi))/(-1+x)^2+((16*I)*(45*I+22*Pi))/(-1+x)+(((640*I)/3)*(-1
(-1+x)^2-((128*I)*(-I+2*Pi))/(-1+x)-128/(3*x^2)-(((64*I)/3)*(-29*I+57*Pi))/x+(64*(41+(30*I)*Pi)*x)/3-(256*x^2)/3)*G[1,0,1,x]+((64*I)*(3*I+5*Pi)-((128*I)*Pi)/(-1+x)^2-((128*I)*(-I+Pi))/(-1+x)-128/(3*x^2)-(((64*I)/3)*(-29*I+51*Pi))/x+(64*(41
(64*I)*(-13*I+9*Pi)*x+64*x^2)*G[1,1,1,x]+(-4864-5632/(-1+x)^2-4352/(-1+x)-1536*x)*G[0,0,0,0,x]+(1216+2272/(-1+x)^2+992/(-1+x)-1536/x)*G[0,0,0,1,x]+(1216+2272/(-1+x)^2+992/(-1+x)-1536/x)*G[0,0,1,0,x]+(1408-1152/(-1+x)^2+768/(-1+x)-128/x-512
(-1+x)-640/x-384*x)*G[0,1,0,1,x]+(1280+128/(-1+x)^2+128/(-1+x)-640/x-384*x)*G[0,1,1,0,x]+(-1728+704/x+1024*x)*G[0,1,1,1,x]+(-320+992/(-1+x)^2+992/(-1+x)+1024/x+1280*x)*G[1,0,0,0,x]+(896-256/(-1+x)^2-256/(-1+x)-1408/x)*G[1,0,0,1,x]+(896-256
(-1+x)-1920/x-256*x)*G[1,1,0,0,x]+(-1728+1344/x+384*x)*G[1,1,0,1,x]+(-1728+1344/x+384*x)*G[1,1,1,0,x]+(1536-768/x-768*x)*G[1,1,1,1,x]+G[1,x]*(((-8*I)/9)*(-333*I-291*Pi-(266*I)*Pi^2+402*Pi^3)+(64*Pi*(9*I+Pi))/(3*x^2)+(16*(-6-(12*I)*Pi+230*P
3)*(-191*Pi+(42*I)*Pi^2+4*Pi^3+(24*I)*Zeta[3]))/(-1+x)+(((64*I)/9)*(-3*I-144*Pi+(28*I)*Pi^2+15*Pi^3-(36*I)*Zeta[3]))/x)-(16*Pi*(-12*I-12*Pi+(176*I)*Pi^2+15*Pi^3+(432*I)*Zeta[3]))/(9*x)+(32*x*(360*Pi^2+(250*I)*Pi^3+51*Pi^4-330*Zeta[3]-(360*
(45*(-1+x)^2)+(8*(180+(1725*I)*Pi-265*Pi^2+(1550*I)*Pi^3+34*Pi^4+720*Zeta[3]+(2880*I)*Pi*Zeta[3]))/45-(16*((30*I)*Pi-555*Pi^2-(1180*I)*Pi^3+263*Pi^4+10500*Zeta[3]-(3600*I)*Pi*Zeta[3]))/(45*(-1+x))+G[0,x]*((((32*I)/3)*Pi*(-4+(33*I)*Pi+2*Pi^
(133*I)*Pi^2+34*Pi^3+(432*I)*Zeta[3]))/(-1+x)^2-(((16*I)/9)*(-6*I+891*Pi-(59*I)*Pi^2+102*Pi^3+(576*I)*Zeta[3]))/(-1+x)+(16*x*(6+(12*I)*Pi-152*Pi^2-(141*I)*Pi^3+144*Zeta[3]))/9))+NC^3*((4*Pi*(199*I+36*Pi))/(9*x)+((-8*Pi*(211*I+60*Pi))/3-(32
x+(8*Pi*(-77*I+36*Pi)*x)/3)*G[0,1,x]+((8*(-50-(113*I)*Pi+118*Pi^2))/3+(((4*I)/9)*(-223*I+48*Pi))/x^2-(((4*I)/9)*(797*I+624*Pi))/x-(8*Pi*(-209*I+118*Pi)*x)/3)*G[1,1,x]+((448*I)*Pi+((640*I)*Pi)/(-1+x)^2+((960*I)*Pi)/(-1+x)-(64*I)*Pi*x)*G[0,0
3)*(77*I+48*Pi)*x)*G[0,1,1,x]+((-128*I)*Pi+(128*I)*Pi*x)*G[1,0,1,x]+((8*(163+(108*I)*Pi))/3+160/(3*x^2)-2080/(3*x)+(8*(77-(108*I)*Pi)*x)/3)*G[1,1,1,x]+(448+640/(-1+x)^2+960/(-1+x)-64*x)*G[0,0,1,1,x]+(-320+320*x)*G[0,1,1,1,x]+(-128+128*x)*G
(8*(-2+120*Pi^2+33*Pi^4-(1080*I)*Pi*Zeta[3]))/(9*(-1+x))+(-516976-(609828*I)*Pi+236285*Pi^2+(73458*I)*Pi^3-10236*Pi^4+362736*Zeta[3]+(190512*I)*Pi*Zeta[3])/243+(x*(2692880+(2747280*I)*Pi-1362865*Pi^2-(367290*I)*Pi^3+36924*Pi^4-1813680*Zeta
(4*(-199-(725*I)*Pi+468*Pi^2))/(9*x)+(2*(424+(27090*I)*Pi-12225*Pi^2-(5292*I)*Pi^3-16200*Zeta[3]))/81+(2*x*(-10486-(31194*I)*Pi+24321*Pi^2+(5292*I)*Pi^3+16200*Zeta[3]))/81)))/e^2+(Nf^2*((4*Pi*(53*I+34*Pi+(70*I)*Pi^2))/(15*x)+((2*(-35229-(9
(971*I)*Pi+385*Pi^2))/(45*(-1+x))+(4*(1215-(1474*I)*Pi+200*Pi^2)*x)/135+((2*I)/15)*(45*I+98*Pi)*x^2)*G[0,0,x]+((-2*(24939-(3946*I)*Pi+336*Pi^2))/135-(16*Pi*(-55*I+23*Pi))/(45*(-1+x))-(4*(135+(54*I)*Pi+244*Pi^2))/(27*x^2)+(8*(3501-(510*I)*P
135)*G[1,1,x]+(((-688*I)/45)*Pi-(((752*I)/45)*Pi)/(-1+x)^2-((32*I)*Pi)/(-1+x)-((64*I)/9)*Pi*x)*G[0,0,0,x]+((8*(181+(314*I)*Pi+7*Pi^2))/15+(16*Pi*(-67*I+23*Pi))/(45*(-1+x)^2)+(16*(233+(196*I)*Pi+Pi^2))/(15*(-1+x))+8/x^2+112/(3*x)+(32*(239-(
(16*(-15+Pi)*(15+Pi))/(45*(-1+x)^2)-(32*(-665-(162*I)*Pi+68*Pi^2))/(45*(-1+x))-224/x+((64*I)/9)*Pi*x)*G[1,0,0,x]+((-4*(8509-(4440*I)*Pi+126*Pi^2))/135-(208*Pi^2)/(45*(-1+x)^2)-(16*(-55-(450*I)*Pi+3*Pi^2))/(45*(-1+x))-392/(15*x^2)+832/(5*x)
135)*(-8509*I+14520*Pi)-(((176*I)/9)*(I+6*Pi))/(-1+x)-(((56*I)/15)*(-7*I+30*Pi))/x^2+(((64*I)/5)*(-13*I+15*Pi))/x+(8*(737-(120*I)*Pi)*x)/135-(196*x^2)/15)*G[1,1,0,x]+((32*(448-(141*I)*Pi))/45+(((32*I)/15)*Pi)/(-1+x)-(((8*I)/45)*(57*I+20*Pi
45)*G[1,1,1,x]+(-688/45+(((16*I)/45)*(47*I+150*Pi))/(-1+x)^2-32/(-1+x)-(64*x)/9)*G[0,0,0,1,x]+((16*(-43-(222*I)*Pi+30*Pi^2))/45+(16*(-47-(672*I)*Pi+270*Pi^2))/(45*(-1+x)^2)+(32*(-15-(122*I)*Pi+40*Pi^2))/(15*(-1+x))-(64*x)/9)*G[0,0,1,0,x]+(
(-1+x)+(1856*x)/9-(1040*x^2)/9)*G[0,0,1,1,x]+((16*(-272-(69*I)*Pi+15*Pi^2))/45+(16*(-628-(324*I)*Pi+135*Pi^2))/(45*(-1+x)^2)+(32*(-150-(59*I)*Pi+20*Pi^2))/(15*(-1+x))+(64*x)/9)*G[0,1,0,0,x]+((16*(91-(30*I)*Pi))/15-(((16*I)/45)*(-67*I+450*P
45)*(67*I+330*Pi))/(-1+x)^2+(((32*I)/5)*(-29*I+110*Pi))/(-1+x))*G[0,1,1,0,x]+((16*(51-(7*I)*Pi))/15-(((32*I)/15)*Pi)/(-1+x)^2-(((32*I)/15)*(26*I+Pi))/(-1+x)-(1856*x)/9+(1040*x^2)/9)*G[0,1,1,1,x]+(((112*I)/15)*Pi+(((32*I)/15)*Pi)/(-1+x)^2+(
15)*(27*I+Pi))/(-1+x)+(64*x)/9)*G[1,0,0,1,x]+((8*(344+(315*I)*Pi))/45+(((144*I)/5)*(-4*I+5*Pi))/(-1+x)+(64*x)/9)*G[1,0,1,0,x]+((16*(66+(7*I)*Pi))/15+(((32*I)/15)*Pi)/(-1+x)^2+(((32*I)/15)*(-21*I+Pi))/(-1+x)-(1856*x)/9+(1040*x^2)/9)*G[1,0,1
x^2-192/x)*G[1,1,0,1,x]+(1216/15+32/(15*(-1+x))+112/x^2-192/x)*G[1,1,1,0,x]+(-544/3-1040/(9*x^2)+1856/(9*x)+(1856*x)/9-(1040*x^2)/9)*G[1,1,1,1,x]+(160*G[0,0,0,1,1,x])/(3*(-1+x)^2)+(-1184/15-3584/(15*(-1+x)^2)-3904/(15*(-1+x)))*G[0,0,1,0,1,
(-1+x))*G[0,0,1,1,0,x]+(448/15-832/(15*(-1+x)^2)+128/(15*(-1+x)))*G[0,0,1,1,1,x]+(-368/15-576/(5*(-1+x)^2)-1888/(15*(-1+x)))*G[0,1,0,0,1,x]+(((-16*I)/15)*(-23*I+15*Pi)-(((144*I)/5)*(-4*I+5*Pi))/(-1+x)^2-(((32*I)/15)*(-59*I+60*Pi))/(-1+x))*
(15*(-1+x)^2)+10592/(15*(-1+x)))*G[0,1,1,0,0,x]+(-112/15-32/(15*(-1+x)^2)-32/(15*(-1+x)))*G[0,1,1,0,1,x]+(-112/15-32/(15*(-1+x)^2)-32/(15*(-1+x)))*G[0,1,1,1,0,x]+(112/15+32/(15*(-1+x)^2)+32/(15*(-1+x)))*G[1,0,0,0,1,x]+(112/15+32/(15*(-1+x)
+(728/15-32/(15*(-1+x)^2)+2128/(15*(-1+x)))*G[1,0,1,0,0,x]+(112/15+32/(15*(-1+x)^2)+32/(15*(-1+x)))*G[1,0,1,0,1,x]+(112/15+32/(15*(-1+x)^2)+32/(15*(-1+x)))*G[1,0,1,1,0,x]+(-64-576/(-1+x)^2-512/(-1+x))*G[0,0,1,1,0,0,x]+(-16-144/(-1+x)^2-128
(737*I)*Pi+100*Pi^2)*x)/135+(2*(45-(98*I)*Pi)*x^2)/15+(4*(9688+(15731*I)*Pi-124*Pi^2-(462*I)*Pi^3-3276*Zeta[3]))/135+(((16*I)/45)*(225*Pi+4*Pi^3-(12*I)*Zeta[3]))/(-1+x)^2-(((4*I)/135)*(6403*I-16620*Pi-(3288*I)*Pi^2+1572*Pi^3-(9576*I)*Zeta[
(5628*I)*Pi+1884*Pi^2+(360*I)*Pi^3-2160*Zeta[3]))/135+(((16*I)/45)*(971*I-1330*Pi-(602*I)*Pi^2+240*Pi^3+(1440*I)*Zeta[3]))/(-1+x)+(((4*I)/135)*(6991*I-10752*Pi-(5964*I)*Pi^2+3240*Pi^3+(19440*I)*Zeta[3]))/(-1+x)^2)+(x^2*(-1215*Pi^2+(2646*I)*
(15660*I)*Pi-13542*Pi^2+(6620*I)*Pi^3+1768*Pi^4+18024*Zeta[3]+(87840*I)*Pi*Zeta[3]))/405+G[0,1,0,x]*(((8*I)/135)*(737*I+120*Pi)*x+(196*x^2)/15+(((16*I)/45)*(971*I-378*Pi-(29*I)*Pi^2+120*Pi^3-(720*I)*Zeta[3]))/(-1+x)+(((4*I)/135)*(6991*I-83
(-1+x)^2+(4*(-4661+(12*I)*Pi-402*Pi^2+(180*I)*Pi^3+1080*Zeta[3]))/135)+G[0,x]*((((-392*I)/15)*Pi)/x-(98*Pi^2*x^2)/45+(2*x*(-2931-(2646*I)*Pi+3698*Pi^2-(240*I)*Pi^3-1440*Zeta[3]))/405+(((2*I)/135)*(-6529*I-38122*Pi-(12868*I)*Pi^2+7200*Pi^3-
(-1+x)^2+(2*(28281-(47559*I)*Pi+6349*Pi^2+(6936*I)*Pi^3+42048*Zeta[3]))/405)+G[0,1,x]*(((8*I)*Pi)/x^2+(((56*I)/15)*(7*I+10*Pi))/x+(8*(-828+(2131*I)*Pi+580*Pi^2)*x)/135-(2*(-405+(882*I)*Pi+1300*Pi^2)*x^2)/135-(4*(-9688+(1403*I)*Pi+3024*Pi^2
(4*(-6403+(3264*I)*Pi+2224*Pi^2+(3936*I)*Pi^3+144*Pi^4-15120*Zeta[3]-(8640*I)*Pi*Zeta[3]))/(135*(-1+x))-(4*((6991*I)*Pi-2422*Pi^2+(816*I)*Pi^3+162*Pi^4+3600*Zeta[3]-(9720*I)*Pi*Zeta[3]))/(135*(-1+x)^2))+G[1,x]*((4*Pi*(-75*I+34*Pi+(70*I)*Pi
(4*Pi*(31*Pi^3-(504*I)*Zeta[3]))/(135*(-1+x)^2)+(2*x*(2931+(2646*I)*Pi+202*Pi^2+(240*I)*Pi^3+1440*Zeta[3]))/405+(4*((6403*I)*Pi-1990*Pi^2+(816*I)*Pi^3+153*Pi^4+3600*Zeta[3]-(9936*I)*Pi*Zeta[3]))/(135*(-1+x))+(2*(-22086-(19929*I)*Pi+14471*P
(1050*I)*Pi-1442840*Pi^2-(1839530*I)*Pi^3+97510*Pi^4+(10416*I)*Pi^5-3720*Pi^6-242340*Zeta[3]+(4278960*I)*Pi*Zeta[3]-105840*Pi^2*Zeta[3]-907200*Zeta[3]^2-2797200*Zeta[5])/14175-(2*((339185*I)*Pi^3-39655*Pi^4-(30198*I)*Pi^5+16740*Pi^6-203511
(14175*(-1+x)^2)-(2*(9450-(685545*I)*Pi+743785*Pi^2+(1142400*I)*Pi^3-65870*Pi^4-(29568*I)*Pi^5+14880*Pi^6+1406160*Zeta[3]-(2696400*I)*Pi*Zeta[3]+15120*Pi^2*Zeta[3]+3628800*Zeta[3]^2+4611600*Zeta[5]))/(14175*(-1+x))))/NC+(Nf*((Pi*(65*I+22*P
(8*(282+(17*I)*Pi+45*Pi^2))/(3*(-1+x))+((16*I)*Pi)/x+((8*I)/3)*(3*I+Pi)*x+((4*I)/3)*Pi*x^2)*G[0,0,x]+((-32*I)*Pi+((48*I)*Pi)/(-1+x)^2+((16*I)*Pi)/(-1+x))*G[0,0,0,x]+((4*(321-(9*I)*Pi+4*Pi^2))/3+24/(-1+x)^2+(4*(125-(4*I)*Pi+16*Pi^2))/(-1+x)
3+(4*(-249+(126*I)*Pi+86*Pi^2))/(3*(-1+x))+4/(3*x^2)+56/(3*x)+((8*I)/3)*(7*I+9*Pi)*x+(4*(5-(3*I)*Pi)*x^2)/3)*G[1,0,1,x]+((2*(-483-(42*I)*Pi+10*Pi^2))/3-(4*(249+(108*I)*Pi+10*Pi^2))/(3*(-1+x))-(((4*I)/3)*(I+48*Pi))/x^2-(((8*I)/3)*(7*I+12*Pi
3)*(-5*I+3*Pi))/x+(4*(19-(18*I)*Pi)*x)/3+((4*I)/3)*(4*I+3*Pi)*x^2)*G[1,1,1,x]+(-32+48/(-1+x)^2+16/(-1+x))*G[0,0,0,1,x]+((-16*(6+Pi^2))/3-(16*(-3+8*Pi^2))/(-1+x)^2-(16*(-3+14*Pi^2))/(3*(-1+x)))*G[0,0,1,0,x]+((-8*(-3-(6*I)*Pi+4*Pi^2))/3-(8*(
((-4*(27-(24*I)*Pi+2*Pi^2))/3-(4*(83+16*Pi^2))/(-1+x)^2-(16*(69+(12*I)*Pi+7*Pi^2))/(3*(-1+x)))*G[0,1,0,0,x]+((-4*(-87-(3*I)*Pi+4*Pi^2))/3-(4*(-249+(18*I)*Pi+86*Pi^2))/(3*(-1+x)^2)-(8*(-186+(3*I)*Pi+28*Pi^2))/(3*(-1+x)))*G[0,1,0,1,x]+(4*(29
((-4*I)*(-45*I+4*Pi)+((32*I)*(7*I+Pi))/(-1+x)+24*x-4*x^2)*G[0,1,1,1,x]+((16*I)*Pi-((32*I)*Pi)/(-1+x))*G[1,0,0,0,x]+((12*I)*(I+2*Pi)+((16*I)*(I+11*Pi))/(-1+x)+16/x^2+32/x)*G[1,0,0,1,x]+((-12*I)*(-I+4*Pi)-((16*I)*(-I+8*Pi))/(-1+x)+16/x^2+32/
((-4*I)*(-71*I+2*Pi)+((16*I)*(9*I+Pi))/(-1+x)-112/x^2-32/x)*G[1,1,0,0,x]+(8*(32-(5*I)*Pi)-((144*I)*Pi)/(-1+x)+48/x^2)*G[1,1,0,1,x]+(8*(32+(5*I)*Pi)+((144*I)*Pi)/(-1+x)+48/x^2)*G[1,1,1,0,x]+(-228-56/x^2-16/x-24*x+4*x^2)*G[1,1,1,1,x]+((64*I)
(-1+x))*G[0,0,1,1,1,x]+((32*I)*Pi*G[0,1,0,0,0,x])/(-1+x)^2+(8*(4-I*Pi)-((176*I)*Pi)/(-1+x)^2-((16*I)*(-4*I+7*Pi))/(-1+x))*G[0,1,0,0,1,x]+(8*(4+I*Pi)+((128*I)*Pi)/(-1+x)^2+((16*I)*(4*I+7*Pi))/(-1+x))*G[0,1,0,1,0,x]+(-28+((8*I)*(3*I+2*Pi))/(
((8*I)*(18*I+Pi)+((144*I)*Pi)/(-1+x)^2+((16*I)*(38*I+7*Pi))/(-1+x))*G[0,1,0,1,1,x]+((-8*I)*(-18*I+Pi)-((144*I)*Pi)/(-1+x)^2-((16*I)*(-38*I+7*Pi))/(-1+x))*G[0,1,1,1,0,x]+(128+640/(-1+x))*G[0,1,1,1,1,x]+(16-32/(-1+x))*G[1,0,0,0,1,x]+(16-32/(
(-1+x))*G[1,0,1,0,1,x]+(-56-336/(-1+x))*G[1,0,1,1,0,x]+(64+320/(-1+x))*G[1,0,1,1,1,x]+(-8+16/(-1+x))*G[1,1,0,0,1,x]+(-8+16/(-1+x))*G[1,1,0,1,0,x]+(-32-160/(-1+x))*G[1,1,0,1,1,x]+(40+144/(-1+x))*G[1,1,1,0,0,x]+(64*G[0,0,1,0,0,1,x])/(-1+x)^2
(-1+x))*G[0,0,1,1,0,0,x]+(32+608/(-1+x)^2+448/(-1+x))*G[0,0,1,1,0,1,x]+(32+608/(-1+x)^2+448/(-1+x))*G[0,0,1,1,1,0,x]+(-32-640/(-1+x)^2-448/(-1+x))*G[0,0,1,1,1,1,x]+(32*G[0,1,0,0,0,1,x])/(-1+x)^2+(32*G[0,1,0,0,1,0,x])/(-1+x)^2+(-8-208/(-1+x
(-1+x))*G[0,1,0,1,0,1,x]+(16+336/(-1+x)^2+224/(-1+x))*G[0,1,0,1,1,0,x]+(-16-320/(-1+x)^2-224/(-1+x))*G[0,1,0,1,1,1,x]-(16*G[0,1,1,0,0,1,x])/(-1+x)^2-(16*G[0,1,1,0,1,0,x])/(-1+x)^2+(8+160/(-1+x)^2+112/(-1+x))*G[0,1,1,0,1,1,x]+(-8-144/(-1+x)
(5*I)*Pi+3*Pi^2)*x)/3-(2*Pi*(-2*I+Pi)*x^2)/3-((4*I)*(83*Pi-(2*I)*Pi^2+6*Pi^3-(36*I)*Zeta[3]))/(-1+x)-(I/3)*(-505*I+126*Pi-(274*I)*Pi^2+20*Pi^3-(120*I)*Zeta[3]))-(x^2*((-60*I)*Pi^3+11*Pi^4-840*Zeta[3]+(360*I)*Pi*Zeta[3]))/90+G[0,1,0,x]*(16/
3)*(13*I+12*Pi^3+(360*I)*Zeta[3]))/(-1+x)^2+(2*(-21+(120*I)*Pi+16*Pi^2-(2*I)*Pi^3+12*Zeta[3]))/3)+G[0,1,1,x]*(-16/x+((4*I)/3)*(7*I+18*Pi)*x-((4*I)/3)*(-2*I+3*Pi)*x^2+((4*I)*(83*Pi-(2*I)*Pi^2+6*Pi^3-(36*I)*Zeta[3]))/(-1+x)^2+(((8*I)/3)*(-90
(96*I)*Pi+82*Pi^2+(2*I)*Pi^3+12*Zeta[3]))/3)+G[1,0,x]*(((24*I)*Pi)/(-1+x)^2-(4*Pi*(-I+6*Pi))/(3*x^2)-(4*(11+(70*I)*Pi+12*Pi^2))/(3*x)+(4*(5-(2*I)*Pi)*x)/3-((4*I)/3)*Pi*x^2+(((2*I)/3)*(-485*I+252*Pi-(20*I)*Pi^2+12*Pi^3+(360*I)*Zeta[3]))/(-1
+(4*(-21-(3*I)*Pi+5*Pi^2)*x)/9-(14*Pi^2*x^2)/9-(((2*I)/9)*(219*I+1929*Pi+(434*I)*Pi^2+216*Pi^3+(1728*I)*Zeta[3]))/(-1+x)-((I/9)*(3564*Pi-(101*I)*Pi^2+138*Pi^3+(3420*I)*Zeta[3]))/(-1+x)^2+(208-(388*I)*Pi+263*Pi^2-(98*I)*Pi^3+12*Zeta[3])/3)+
(((2*I)/3)*(-13*I-564*Pi-(48*I)*Pi^2+48*Pi^3-(864*I)*Zeta[3]))/(-1+x)^2+(8*(-17-(126*I)*Pi+4*Pi^2+168*Zeta[3]))/(3*(-1+x)))+G[0,1,x]*(-44/(3*x)-(4*(-5+(5*I)*Pi+3*Pi^2)*x)/3+(2*Pi*(-2*I+Pi)*x^2)/3+(12720-(1620*I)*Pi-960*Pi^2+(690*I)*Pi^3-59
(4380*I)*Pi+5040*Pi^2+(690*I)*Pi^3+413*Pi^4+13140*Zeta[3]-(7560*I)*Pi*Zeta[3]))/(45*(-1+x))-(2*((-195*I)*Pi+3750*Pi^2-(90*I)*Pi^3+763*Pi^4+540*Zeta[3]-(9360*I)*Pi*Zeta[3]))/(45*(-1+x)^2))+G[1,x]*(((I/3)*Pi*(21+(2*I)*Pi+44*Pi^2))/x^2+((I/3)
(-1000+(1715*I)*Pi+1650*Pi^2+(20*I)*Pi^3+21*Pi^4+5400*Zeta[3]-(1920*I)*Pi*Zeta[3])/15+(2*((7275*I)*Pi+3930*Pi^2+(450*I)*Pi^3+763*Pi^4+3780*Zeta[3]-(9360*I)*Pi*Zeta[3]))/(45*(-1+x)))+(x*(715+(45*I)*Pi-140*Pi^2+(60*I)*Pi^3+33*Pi^4+1620*Zeta[
(4410*I)*Pi^3-20448*Pi^4+(153*I)*Pi^5-86*Pi^6-206820*Zeta[3]+(38880*I)*Pi*Zeta[3]+3240*Pi^2*Zeta[3]+(7560*I)*Pi^3*Zeta[3]+22680*Zeta[3]^2-8100*Zeta[5]+(453600*I)*Pi*Zeta[5])/(405*(-1+x))+(-50400+(9450*I)*Pi-912555*Pi^2+(28350*I)*Pi^3-38304
(1071*I)*Pi^5-86*Pi^6-2241540*Zeta[3]+(1474200*I)*Pi*Zeta[3]-22680*Pi^2*Zeta[3]+(7560*I)*Pi^3*Zeta[3]+22680*Zeta[3]^2+510300*Zeta[5]+(453600*I)*Pi*Zeta[5])/5670+((-805*I)*Pi^3-11620*Pi^4+(357*I)*Pi^5-282*Pi^6+4830*Zeta[3]-(223020*I)*Pi*Zeta
(315*(-1+x)^2)))/NC^2+Nf*((((-4*I)/45)*(-53717*I+55421*Pi-(10780*I)*Pi^2+2955*Pi^3)-(((4*I)/135)*(-28072*I+92787*Pi-(95430*I)*Pi^2+18990*Pi^3))/(-1+x)^2-(((4*I)/135)*(-222223*I+293070*Pi-(114570*I)*Pi^2+18810*Pi^3))/(-1+x)+(64*Pi*(-I+4*Pi)
135+(32*Pi^2*x^3)/9)*G[0,0,x]+((-4*(-116587-(7296*I)*Pi+10380*Pi^2))/45-(16*Pi*(-1886*I+1215*Pi))/(15*(-1+x))-(8*(-16888+(5955*I)*Pi+1170*Pi^2))/(135*x^2)+(16*(-11447+(2790*I)*Pi+3045*Pi^2))/(135*x)-(8*(-25661-(2475*I)*Pi+390*Pi^2)*x)/13
((4*(1787+(12480*I)*Pi+1296*Pi^2))/9+(4*(-1841+(4136*I)*Pi+864*Pi^2))/(3*(-1+x)^2)+(32*(337+(2718*I)*Pi+324*Pi^2))/(9*(-1+x))-((2048*I)*Pi)/x-((40*I)/9)*(-997*I+168*Pi)*x+(4*(4625+(288*I)*Pi)*x^2)/9-64*x^3)*G[0,0,0,0,x]+((4*(-14952-(5450*I
(7752*I)*Pi+6477*Pi^2))/(9*(-1+x))-64/x+(8*(691-(384*I)*Pi)*x)/3+((8*I)/9)*(1099*I+342*Pi)*x^2+64*x^3)*G[0,0,0,1,x]+((4*(-14952+(12662*I)*Pi+405*Pi^2))/15+(8*(6473+(7863*I)*Pi+1230*Pi^2))/(45*(-1+x)^2)+(8*(-35365+(32568*I)*Pi+4725*Pi^2))/(
9)*(1099*I+66*Pi)*x^2+64*x^3)*G[0,0,1,0,x]+((-16*(-30182+(4839*I)*Pi+3000*Pi^2))/45-(8*(-4672-(854*I)*Pi+9375*Pi^2))/(15*(-1+x)^2)-(16*(-16335+(2703*I)*Pi+4835*Pi^2))/(15*(-1+x))-(((16*I)/9)*(-229*I+18*Pi))/x^2-(((32*I)/9)*(209*I+36*Pi))/x
(10272*I)*Pi+6765*Pi^2))/45-(8*(1027-(764*I)*Pi+1055*Pi^2))/(5*(-1+x)^2)-(8*(42505-(6816*I)*Pi+9105*Pi^2))/(45*(-1+x))+(((128*I)/3)*Pi)/x^2+(((64*I)/3)*(-39*I+28*Pi))/x+(8*(1847-(72*I)*Pi)*x)/9-((4*I)/3)*(-659*I+32*Pi)*x^2+64*x^3)*G[0,1,0,
(16*(-8957+(2530*I)*Pi+2490*Pi^2))/(15*(-1+x))-(((32*I)/9)*(-100*I+9*Pi))/x^2-(((32*I)/9)*(133*I+18*Pi))/x+(8*(43+(104*I)*Pi)*x)/3+(8*(95-(276*I)*Pi)*x^2)/9-64*x^3)*G[0,1,0,1,x]+((-4*(-20247+(16420*I)*Pi+260*Pi^2))/15-(4*(-12061+(36510*I)*
9)*(25*I+3*Pi))/x^2-(((224*I)/9)*(19*I+15*Pi))/x+(8*(43+(40*I)*Pi)*x)/3+(8*(95-(126*I)*Pi)*x^2)/9-64*x^3)*G[0,1,1,0,x]+((8*(-37312+(7539*I)*Pi+1680*Pi^2))/45+(16*Pi*(154*I+1215*Pi))/(15*(-1+x)^2)+(16*(-12379+(2474*I)*Pi+1340*Pi^2))/(15*(-1
9)*(-7*I+3*Pi)*x+(8*(521+(60*I)*Pi)*x^2)/9+64*x^3)*G[0,1,1,1,x]+((-4*(14430+(3076*I)*Pi+2145*Pi^2))/15-(8*(12140+(488*I)*Pi+2145*Pi^2))/(15*(-1+x)^2)-(8*(9675+(4688*I)*Pi+2145*Pi^2))/(15*(-1+x))-((64*I)*Pi)/x^2+(((32*I)/9)*(-43*I+72*Pi))/x
((-4*(-5809+(55464*I)*Pi+2850*Pi^2))/45-(16*(-2350+(397*I)*Pi+970*Pi^2))/(15*(-1+x)^2)-(8*(-1753+(9214*I)*Pi+2510*Pi^2))/(15*(-1+x))-(((16*I)/9)*(85*I+18*Pi))/x^2+(((16*I)/3)*(-47*I+236*Pi))/x+((8*I)/9)*(251*I+1308*Pi)*x-((8*I)/9)*(-47*I+3
(3*(-1+x)^2)+(8*(1753-(11020*I)*Pi+1020*Pi^2))/(15*(-1+x))+(((16*I)/9)*(-85*I+24*Pi))/x^2+(((16*I)/3)*(-47*I+74*Pi))/x+((8*I)/9)*(251*I+384*Pi)*x-((8*I)/9)*(-47*I+150*Pi)*x^2-64*x^3)*G[1,0,1,0,x]+((8*(-2852+(23511*I)*Pi+810*Pi^2))/45+(16*P
9)*(-707*I+12*Pi))/x^2-(((64*I)/9)*(-175*I+117*Pi))/x-((16*I)/9)*(-511*I+132*Pi)*x+8*(43+(8*I)*Pi)*x^2+64*x^3)*G[1,0,1,1,x]+((4*(-4337-(8510*I)*Pi+1210*Pi^2))/15+(16*(2009-(228*I)*Pi+363*Pi^2))/(9*(-1+x)^2)+(8*(15329-(8970*I)*Pi+3630*Pi^2)
((4*I)/9)*(-347*I+48*Pi)*x^2-64*x^3)*G[1,1,0,0,x]+((4*(82346+(54315*I)*Pi+120*Pi^2))/45-(16*Pi*(-94*I+Pi))/(-1+x)^2-(8*(-3772-(14375*I)*Pi+80*Pi^2))/(15*(-1+x))-(((8*I)/9)*(715*I+204*Pi))/x^2-(((32*I)/3)*(-89*I+66*Pi))/x-((8*I)/3)*(-173*I+
(32*Pi*(-5*I+3*Pi))/(3*(-1+x)^2)-(16*(-1886-(2645*I)*Pi+30*Pi^2))/(15*(-1+x))-(((8*I)/9)*(715*I+408*Pi))/x^2+(((32*I)/3)*(89*I+6*Pi))/x+((8*I)/3)*(173*I+20*Pi)*x+(8*(157+(34*I)*Pi)*x^2)/3+64*x^3)*G[1,1,1,0,x]+(((-16*I)/9)*(-5053*I+420*Pi)-
3+((8*I)/9)*(491*I+138*Pi)*x^2-64*x^3)*G[1,1,1,1,x]+(((-16*I)/3)*(-116*I+417*Pi)-(((64*I)/3)*(-172*I+219*Pi))/(-1+x)^2-((32*I)*(-74*I+145*Pi))/(-1+x)-(256*x)/3)*G[0,0,0,0,0,x]+((16*(844-(21*I)*Pi))/3-((160*I)*(43*I+6*Pi))/(-1+x)^2-((32*I)*
((80*I)*(86*I+51*Pi))/(-1+x)^2-((32*I)*(324*I+149*Pi))/(-1+x)-2048/x-(3520*x)/3+128*x^2)*G[0,0,0,1,0,x]+(((8*I)/3)*(1339*I+228*Pi)+(((496*I)/3)*(67*I+24*Pi))/(-1+x)^2+(((16*I)/3)*(2579*I+660*Pi))/(-1+x)+512/x-(800*x)/3+224*x^2)*G[0,0,0,1,1
(-1+x)-768/x-(1088*x)/3+32*x^2+(64*x^3)/3)*G[0,0,1,0,0,x]+((64*(194+(125*I)*Pi))/5+(((16*I)/5)*(-121*I+1770*Pi))/(-1+x)^2+(((32*I)/5)*(-373*I+890*Pi))/(-1+x)+64/x^2-384/x-(512*x)/3+(224*x^2)/3-(64*x^3)/3)*G[0,0,1,0,1,x]+((32*(388+(165*I)*P
(512*x)/3+(224*x^2)/3-(64*x^3)/3)*G[0,0,1,1,0,x]+(((-16*I)/15)*(-7331*I+480*Pi)-(((16*I)/15)*(-1541*I+3330*Pi))/(-1+x)^2-(((208*I)/5)*(-289*I+80*Pi))/(-1+x)-160/x^2+1024/x+672*x-(944*x^2)/3+(64*x^3)/3)*G[0,0,1,1,1,x]+((8*(991+(300*I)*Pi))/
(64*x^3)/3)*G[0,1,0,0,0,x]+((8*(189+(1025*I)*Pi))/5+(((8*I)/15)*(-187*I+8100*Pi))/(-1+x)^2+(((16*I)/5)*(138*I+1505*Pi))/(-1+x)+128/(3*x^2)-1280/(3*x)-160*x-112*x^2+(64*x^3)/3)*G[0,1,0,0,1,x]+((24*(63+(235*I)*Pi))/5+(((8*I)/15)*(-187*I+8520
3)*G[0,1,0,1,0,x]+(((-12*I)/5)*(-1961*I+720*Pi)-(((4*I)/15)*(-949*I+21660*Pi))/(-1+x)^2-(((32*I)/15)*(-2783*I+2880*Pi))/(-1+x)-416/(3*x^2)+2432/(3*x)+(1888*x)/3-(592*x^2)/3-(64*x^3)/3)*G[0,1,0,1,1,x]+(((8*I)/5)*(511*I+250*Pi)+(((8*I)/5)*(2
3)*G[0,1,1,0,0,x]+(((-8*I)/15)*(-11117*I+4080*Pi)-(((8*I)/15)*(308*I+11835*Pi))/(-1+x)^2-(((16*I)/15)*(-9106*I+6375*Pi))/(-1+x)-832/(3*x^2)+3872/(3*x)+(1472*x)/3-(368*x^2)/3)*G[0,1,1,0,1,x]+(((-8*I)/15)*(-11117*I+1245*Pi)-(((112*I)/15)*(2
```



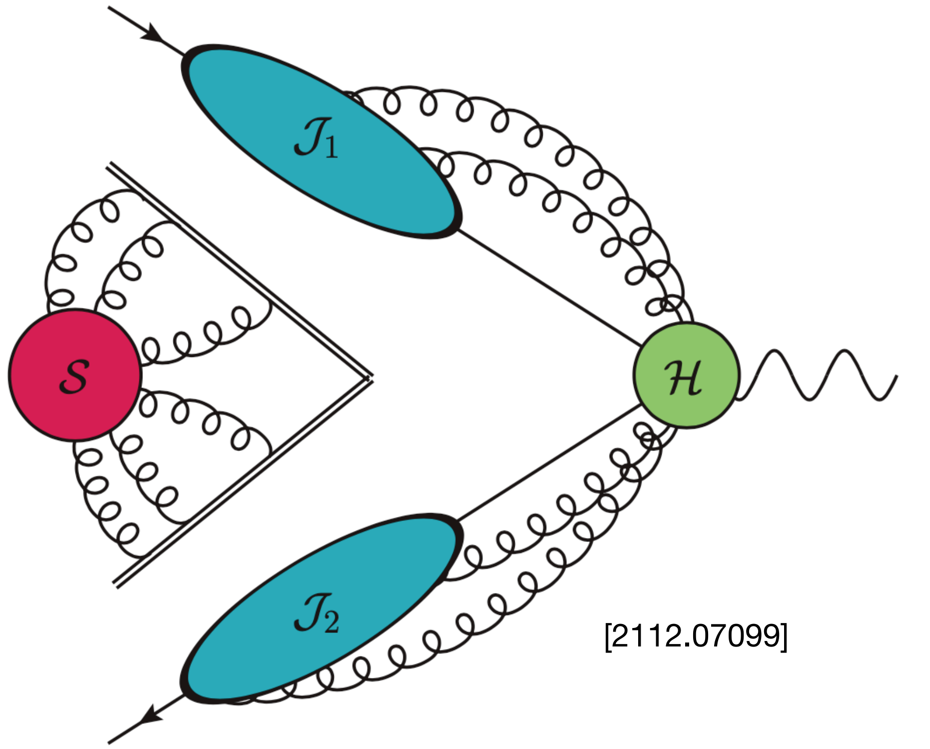
Is there any structure?



Is there any structure?

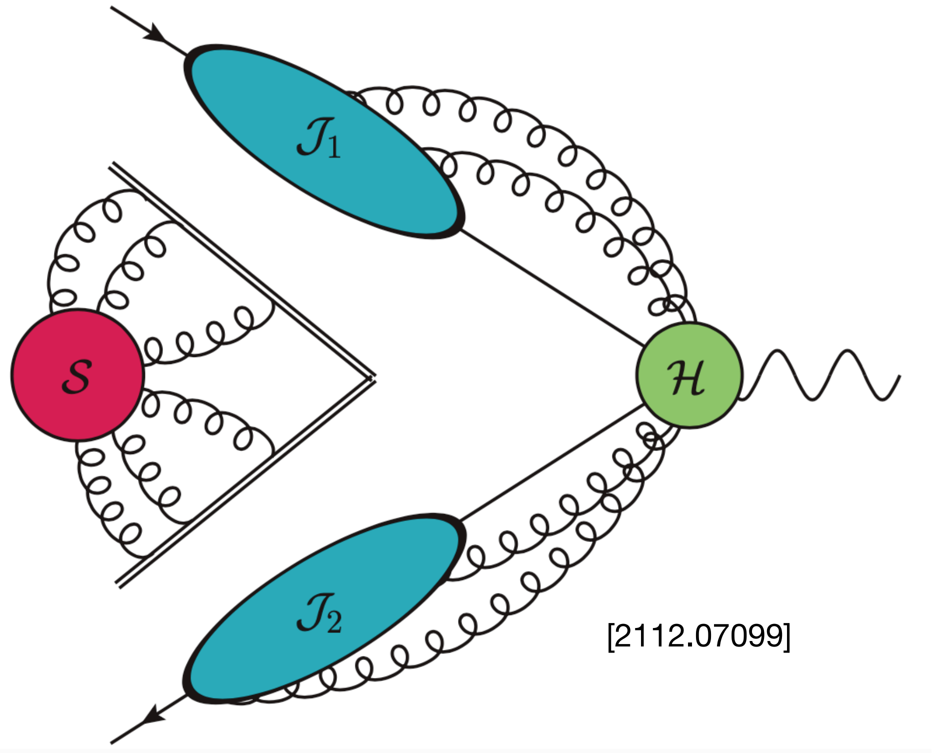
…and what about the non-planar?

Infrared



[2112.07099]

Infrared

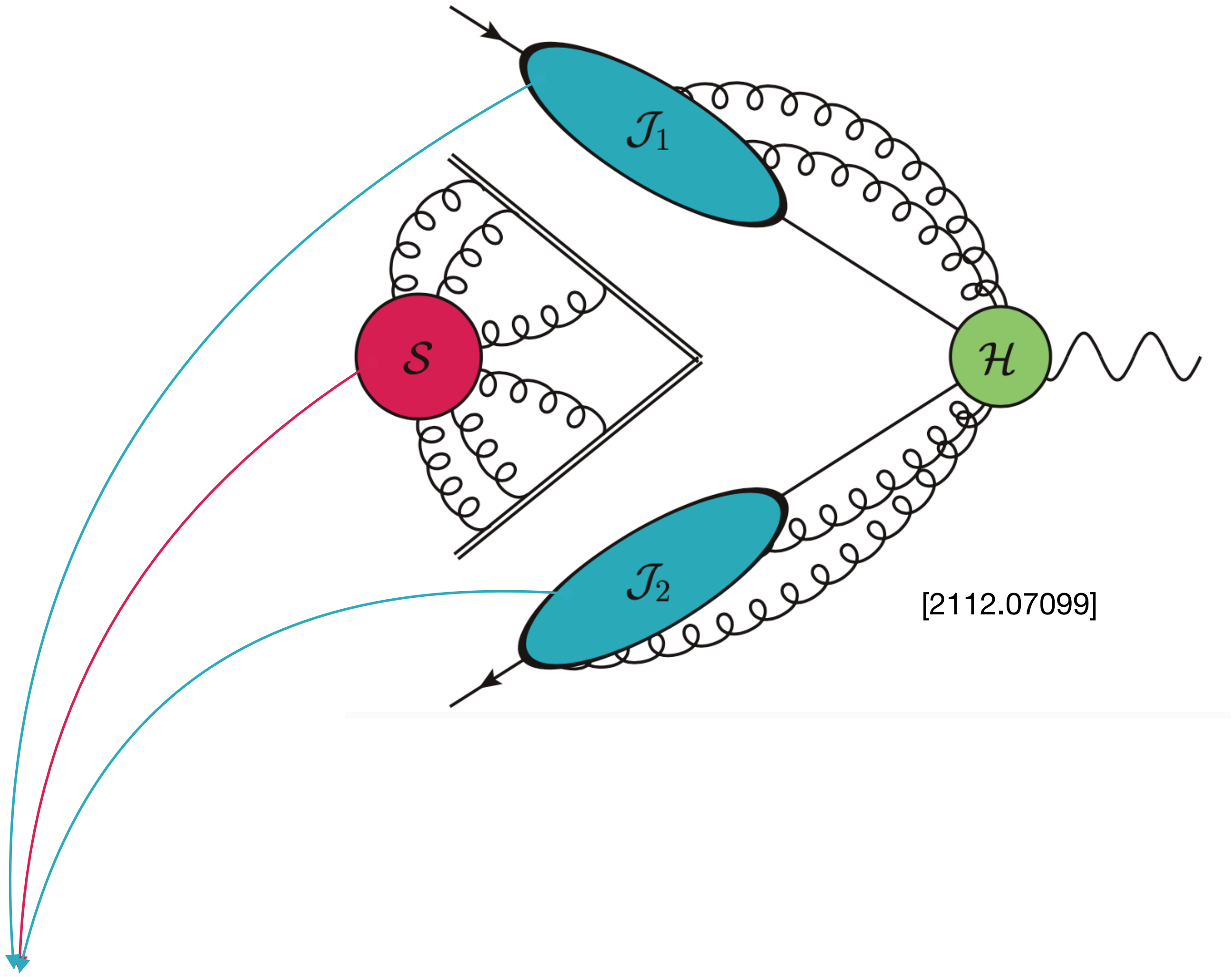


[2112.07099]

$$\mathcal{H}_{ren} = \mathcal{Z}_{IR}(\mu) \cdot \mathcal{H}_{hard}(\mu)$$

Catani: 9802439; Gardi, Magnea: 0901.1091, 0908.3273; Becher, Neubert: 0903.1126

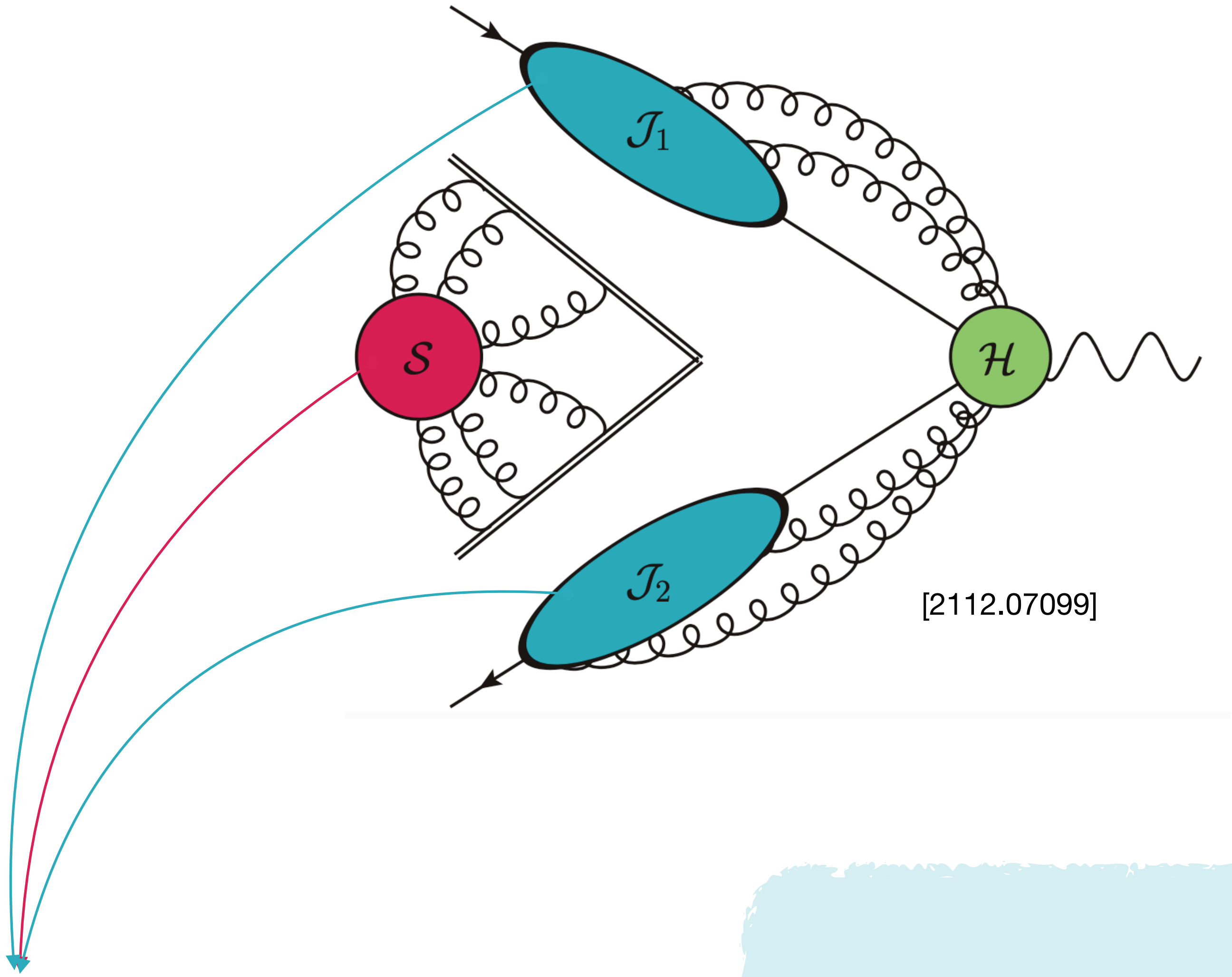
Infrared



$$\mathcal{H}_{ren} = \mathcal{Z}_{IR}(\mu) \cdot \mathcal{H}_{hard}(\mu)$$

Catani: 9802439; Gardi, Magnea: 0901.1091, 0908.3273; Becher, Neubert: 0903.1126

Infrared



[2112.07099]

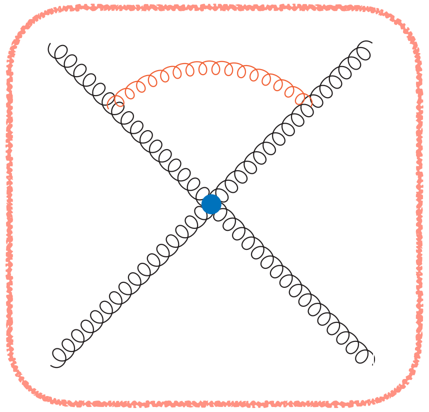
$$\mathcal{H}_{ren} = \mathcal{Z}_{IR}(\mu) \cdot \mathcal{H}_{hard}(\mu)$$

$$\mathcal{Z}_{IR}(\mu) = \mathbb{P}\exp\left[\int_{\mu}^{\infty} \frac{\mathrm{d}\mu'}{\mu'}\Gamma(\mu')\right]$$

Catani: 9802439; Gardi, Magnea: 0901.1091, 0908.3273; Becher, Neubert: 0903.1126

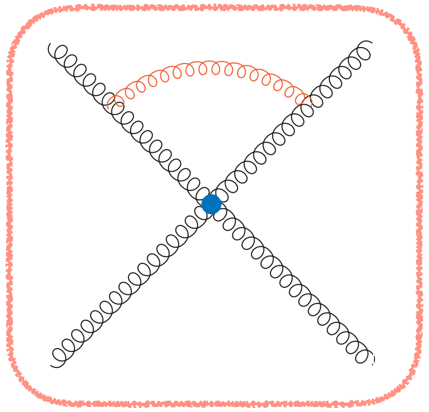
$$\Gamma(\mu) = \Gamma_{dipole}(\mu)$$

$$\Gamma(\mu) = \Gamma_{\underline{dipole}}(\mu)$$



$$\boxed{\mathbf{\Gamma}_{\text{dipole}}(\{p\},\mu) = \sum_{1\leq i<j\leq 4} \mathbf{T}_i^a\,\mathbf{T}_j^a\,\gamma^{\text{cusp}}(\alpha_{\text{s}})\ \log\left(\frac{\mu^2}{-s_{ij}-i\delta}\right) + \sum_{i=1}^{4}\gamma^i(\alpha_{\text{s}})}$$

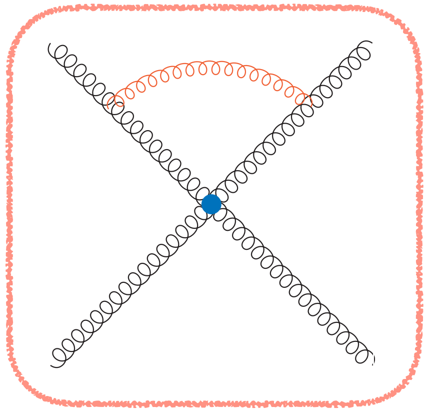
$$\Gamma(\mu) = \Gamma_{\underline{dipole}}(\mu)$$



$$\boxed{\mathbf{\Gamma}_{\text{dipole}}(\{p\},\mu) = \sum_{1\leq i<j\leq 4} \mathbf{T}_i^a\,\mathbf{T}_j^a\,\gamma^{\text{cusp}}(\alpha_s)\,\log\left(\frac{\mu^2}{-s_{ij}-i\delta}\right) + \sum_{i=1}^{4}\gamma^i(\alpha_s)}$$

1. conformal invariant cross ratios $\dfrac{s_{ij}s_{hk}}{s_{ih}s_{jk}}$

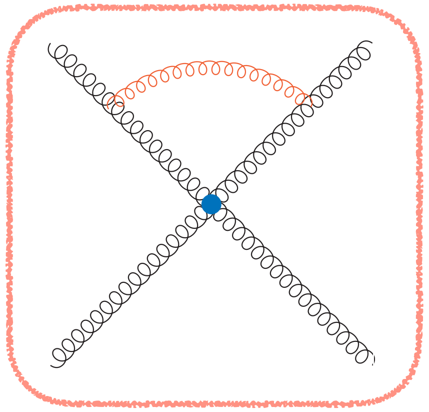
$$\Gamma(\mu) = \Gamma_{\underline{dipole}}(\mu)$$



$$\boxed{\boldsymbol{\Gamma}_{\rm dipole}(\{p\},\mu) = \sum_{1\leq i<j\leq 4} \mathbf{T}_i^a\,\mathbf{T}_j^a\,\gamma^{\rm cusp}(\alpha_{\rm s})\,\log\left(\frac{\mu^2}{-s_{ij}-i\delta}\right) + \sum_{i=1}^{4}\gamma^i(\alpha_{\rm s})}$$

1. conformal invariant cross ratios $\dfrac{s_{ij}s_{hk}}{s_{ih}s_{jk}}$
2. constant

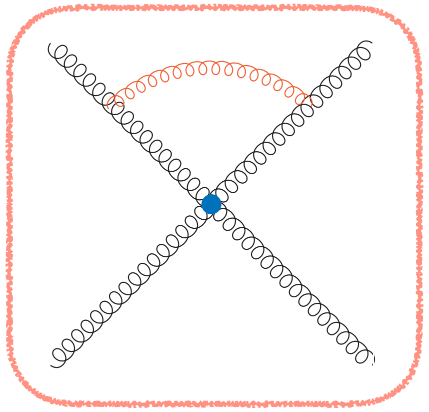
$$\Gamma(\mu) = \Gamma_{\underline{dipole}}(\mu)$$



$$\boldsymbol{\Gamma}_{\text{dipole}}(\{p\},\mu) = \sum_{1\leq i<j\leq 4} \mathbf{T}_i^a\,\mathbf{T}_j^a\,\gamma^{\text{cusp}}(\alpha_s)\,\log\left(\frac{\mu^2}{-s_{ij}-i\delta}\right) + \sum_{i=1}^{4}\gamma^i(\alpha_s)$$

		2 loop
1. conformal invariant cross ratios	$\frac{s_{ij}s_{hk}}{s_{ih}s_{jk}}$	✗
2. constant		✗

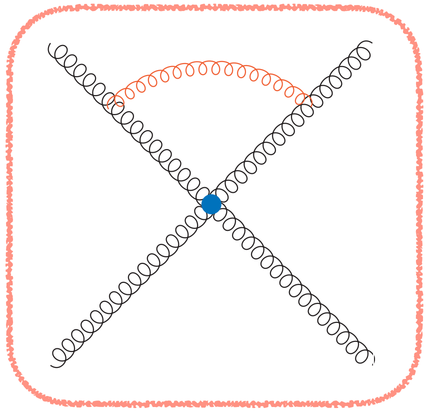
$$\Gamma(\mu) = \underline{\Gamma_{dipole}}(\mu)$$



$$\boxed{\mathbf{\Gamma}_{\rm dipole}(\{p\},\mu) = \sum_{1\leq i<j\leq 4} \mathbf{T}_i^a\,\mathbf{T}_j^a\,\gamma^{\rm cusp}(\alpha_{\rm s})\,\log\left(\frac{\mu^2}{-s_{ij}-i\delta}\right) + \sum_{i=1}^{4}\gamma^i(\alpha_{\rm s})}$$

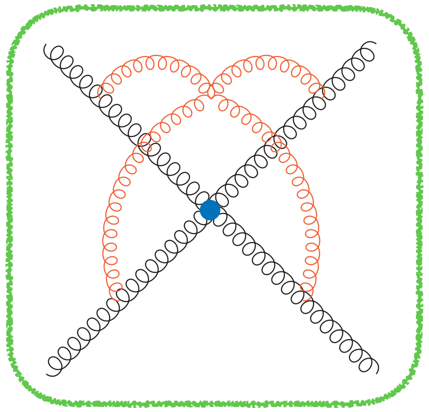
		2 loop	3 loop
1. conformal invariant cross ratios	$\frac{s_{ij}s_{hk}}{s_{ih}s_{jk}}$	✗	✓
2. constant		✗	✓

$$\Gamma(\mu) = \underline{\Gamma_{dipole}}(\mu) + \underline{\Delta_4}$$

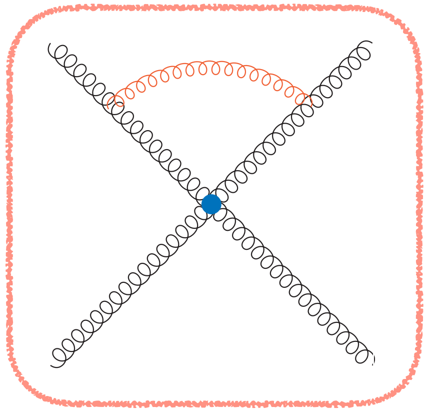


$$\boldsymbol{\Gamma}_{\text{dipole}}(\{p\},\mu) = \sum_{1\le i<j\le 4} \mathbf{T}_i^a\,\mathbf{T}_j^a\,\gamma^{\text{cusp}}(\alpha_{\text{s}})\,\log\left(\frac{\mu^2}{-s_{ij}-i\delta}\right) + \sum_{i=1}^{4}\gamma^i(\alpha_{\text{s}})$$

		2 loop	3 loop
1. conformal invariant cross ratios	$\frac{s_{ij}s_{hk}}{s_{ih}s_{jk}}$	✗	✓
2. constant		✗	✓



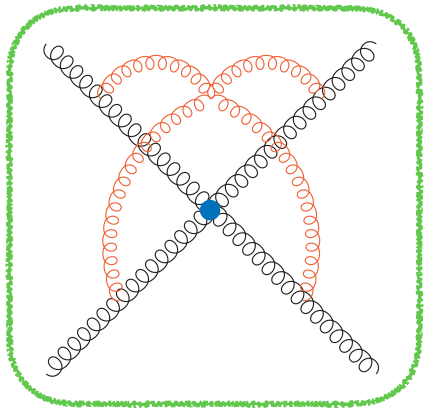
$$\Gamma(\mu) = \underline{\Gamma_{dipole}(\mu)} + \underline{\Delta_4}$$



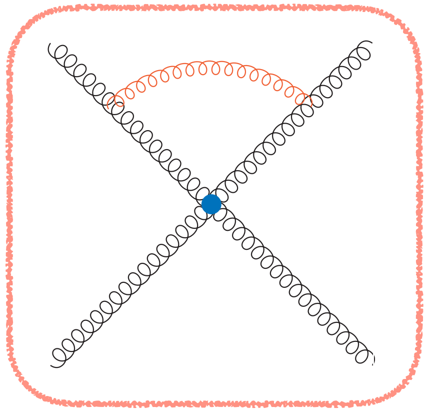
$$\boxed{\mathbf{\Gamma}_{\rm dipole}(\{p\},\mu) = \sum_{1\le i<j\le 4} \mathbf{T}_i^a\,\mathbf{T}_j^a\,\gamma^{\rm cusp}(\alpha_{\rm s})\,\log\left(\frac{\mu^2}{-s_{ij}-i\delta}\right) + \sum_{i=1}^{4}\gamma^i(\alpha_{\rm s})}$$

		2 loop	3 loop
1. conformal invariant cross ratios	$\frac{s_{ij}s_{hk}}{s_{ih}s_{jk}}$	✗	✓
2. constant		✗	✓

Non-planar



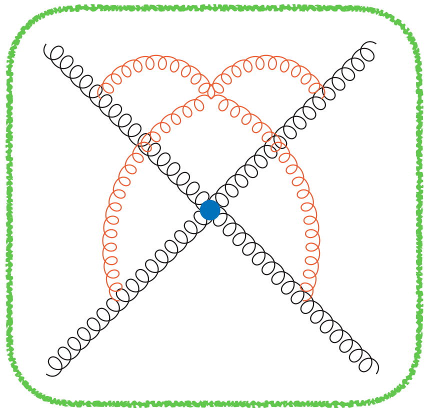
$$\Gamma(\mu) = \underline{\Gamma_{dipole}(\mu)} + \underline{\Delta_4}$$



$$\boxed{\mathbf{\Gamma}_{\rm dipole}(\{p\},\mu) = \sum_{1\le i<j\le 4} \mathbf{T}_i^a\,\mathbf{T}_j^a\,\gamma^{\rm cusp}(\alpha_{\rm s})\,\log\left(\frac{\mu^2}{-s_{ij}-i\delta}\right) + \sum_{i=1}^{4}\gamma^i(\alpha_{\rm s})}$$

		2 loop	3 loop
1. conformal invariant cross ratios	$\frac{s_{ij}s_{hk}}{s_{ih}s_{jk}}$	✗	✓
2. constant		✗	✓

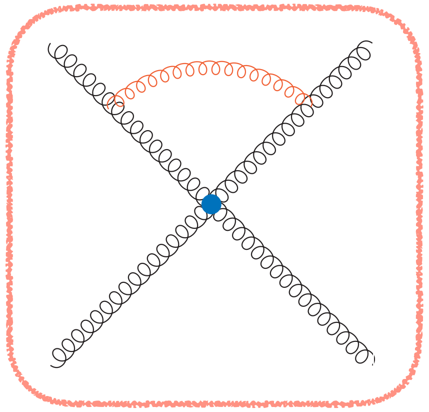
Non-planar



Pure gauge

Dixon: 0901.3414

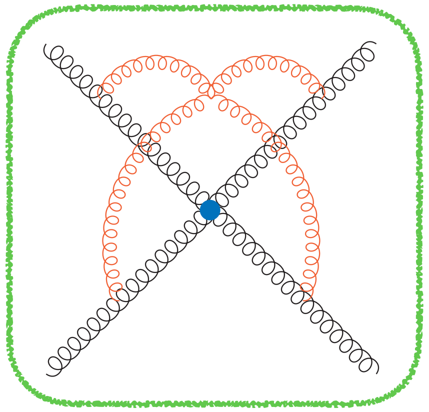
$$\Gamma(\mu) = \underline{\Gamma_{dipole}(\mu)} + \underline{\Delta_4}$$



$$\mathbf{\Gamma}_{\rm dipole}(\{p\},\mu) = \sum_{1\leq i<j\leq 4} \mathbf{T}_i^a\, \mathbf{T}_j^a\, \gamma^{\rm cusp}(\alpha_{\rm s})\, \log\left(\frac{\mu^2}{-s_{ij}-i\delta}\right) + \sum_{i=1}^{4} \gamma^i(\alpha_{\rm s})$$

		2 loop	3 loop
1. conformal invariant cross ratios	$\frac{s_{ij}s_{hk}}{s_{ih}s_{jk}}$	✗	✓
2. constant		✗	✓

Non-planar



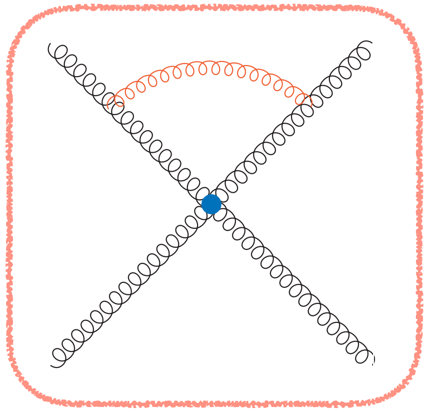
Pure gauge

Dixon: 0901.3414

$$\mathbf{\Delta}_4^{(3)} = 128\, f_{abe}\, f_{cde} \left[\mathbf{T}_1^a\, \mathbf{T}_2^c\, \mathbf{T}_3^b\, \mathbf{T}_4^d\, D_1(x) - \mathbf{T}_4^a\, \mathbf{T}_1^b\, \mathbf{T}_2^c\, \mathbf{T}_3^d\, D_2(x)\right]$$
$$-\, 16\, C\, f_{abe}\, f_{cde} \sum_{i=1}^{4} \sum_{\substack{1\leq j<k\leq 4\\ j,k\neq i}} \left\{\mathbf{T}_i^a, \mathbf{T}_i^d\right\} \mathbf{T}_j^b\, \mathbf{T}_k^c\,,$$

Almelid, Duhr, Gardi
1507.00047

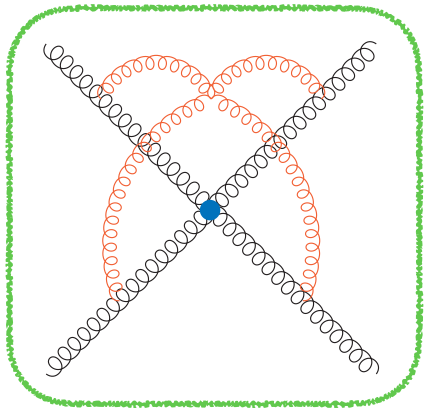
$$\Gamma(\mu) = \underline{\Gamma_{dipole}}(\mu) + \underline{\Delta_4}$$



$$\mathbf{\Gamma}_{\text{dipole}}(\{p\},\mu) = \sum_{1\le i<j\le 4} \mathbf{T}_i^a\, \mathbf{T}_j^a\, \gamma^{\text{cusp}}(\alpha_{\text{s}})\, \log\left(\frac{\mu^2}{-s_{ij}-i\delta}\right) + \sum_{i=1}^{4} \gamma^i(\alpha_{\text{s}})$$

		2 loop	3 loop
1. conformal invariant cross ratios	$\frac{s_{ij}s_{hk}}{s_{ih}s_{jk}}$	✗	✓
2. constant		✗	✓

Non-planar



Pure gauge

Dixon: 0901.3414

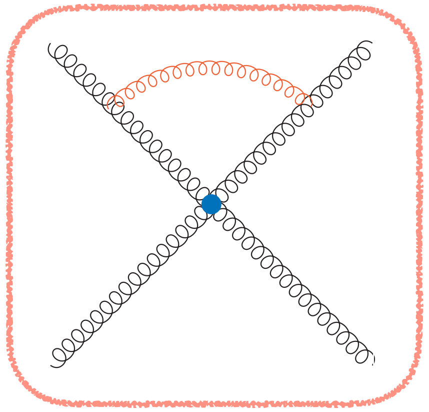
$$\mathbf{\Delta}_4^{(3)} = 128\, f_{abe}\, f_{cde} \left[\mathbf{T}_1^a\, \mathbf{T}_2^c\, \mathbf{T}_3^b\, \mathbf{T}_4^d\, D_1(x) - \mathbf{T}_4^a\, \mathbf{T}_1^b\, \mathbf{T}_2^c\, \mathbf{T}_3^d\, D_2(x)\right]$$

$$- 16\, C\, f_{abe}\, f_{cde} \sum_{i=1}^{4} \sum_{\substack{1\le j<k\le 4\\ j,k\neq i}} \left\{\mathbf{T}_i^a, \mathbf{T}_i^d\right\} \mathbf{T}_j^b\, \mathbf{T}_k^c\, ,$$

Almelid, Duhr, Gardi 1507.00047

$\mathcal{N}=4$ Henn, Mistlberger 1608.00850

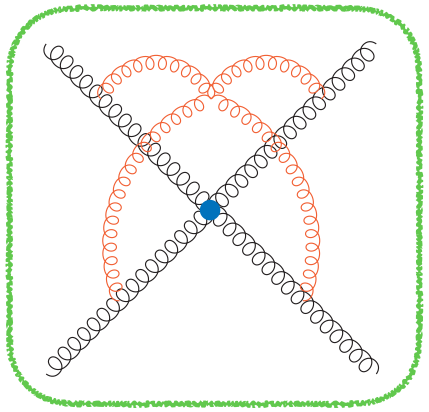
$$\Gamma(\mu) = \underline{\Gamma_{dipole}}(\mu) + \underline{\Delta_4}$$



$$\mathbf{\Gamma}_{\rm dipole}(\{p\},\mu) = \sum_{1\le i<j\le 4} \mathbf{T}_i^a\,\mathbf{T}_j^a\,\gamma^{\rm cusp}(\alpha_s)\;\log\left(\frac{\mu^2}{-s_{ij}-i\delta}\right) + \sum_{i=1}^{4}\gamma^i(\alpha_s)$$

		2 loop	3 loop
1. conformal invariant cross ratios	$\frac{s_{ij}s_{hk}}{s_{ih}s_{jk}}$	✗	✓
2. constant		✗	✓

Non-planar



$$\mathbf{\Delta}_4^{(3)} = 128\,f_{abe}\,f_{cde}\left[\mathbf{T}_1^a\,\mathbf{T}_2^c\,\mathbf{T}_3^b\,\mathbf{T}_4^d\,D_1(x) - \mathbf{T}_4^a\,\mathbf{T}_1^b\,\mathbf{T}_2^c\,\mathbf{T}_3^d\,D_2(x)\right]$$

$$-\,16\,C\,f_{abe}\,f_{cde}\sum_{i=1}^{4}\sum_{\substack{1\le j<k\le 4\\ j,k\neq i}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\,\mathbf{T}_k^c\,,$$

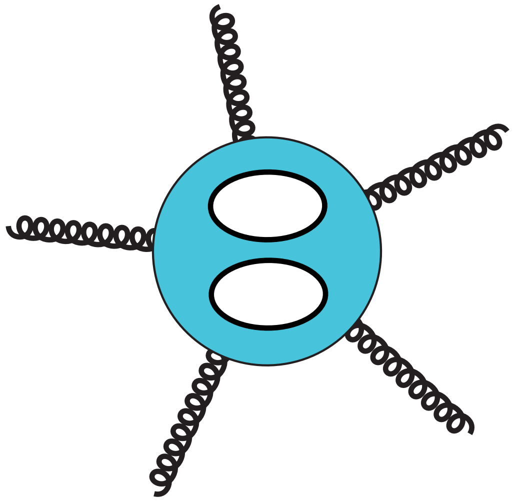
Almelid, Duhr, Gardi
1507.00047

Pure gauge
Dixon: 0901.3414

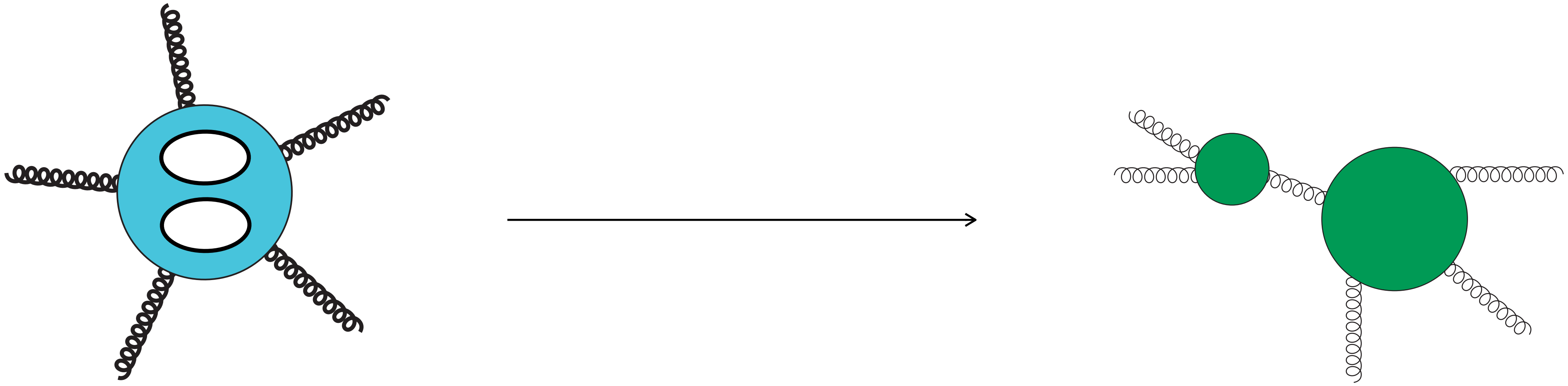
$\mathcal{N}=4$ Henn, Mistlberger 1608.00850

QCD Chakraborty, Caola, GG, Tancredi,
von Manteuffel: 2112.11097

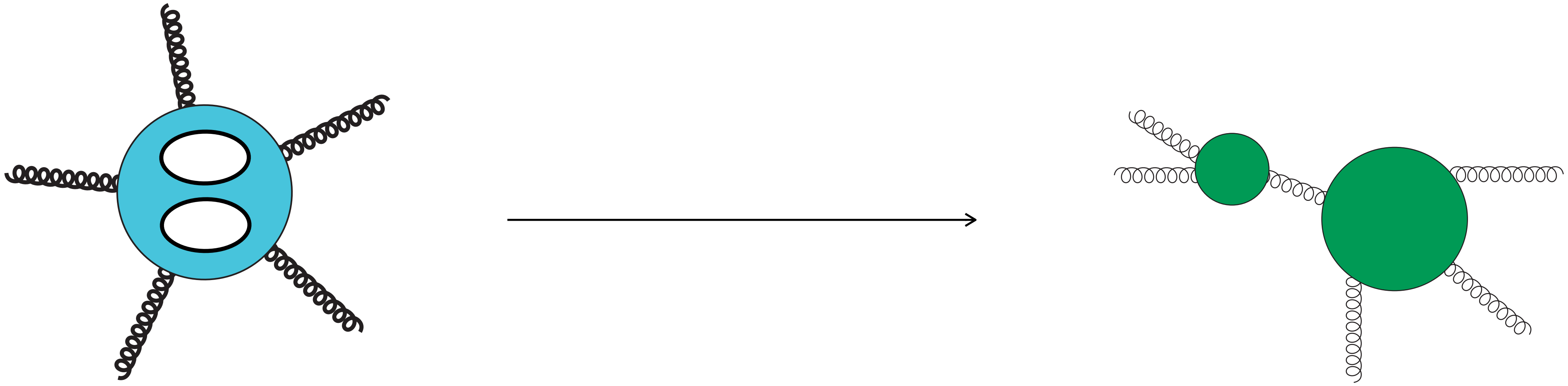
Soft-Collinear Limit



Soft-Collinear Limit

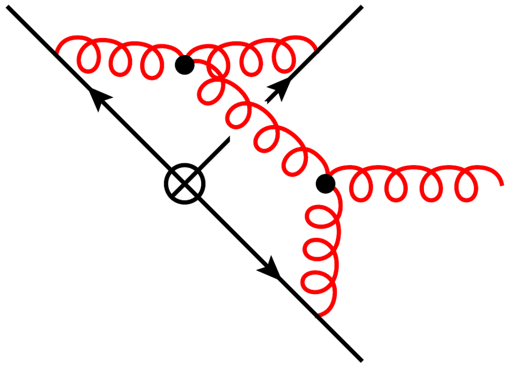


Soft-Collinear Limit

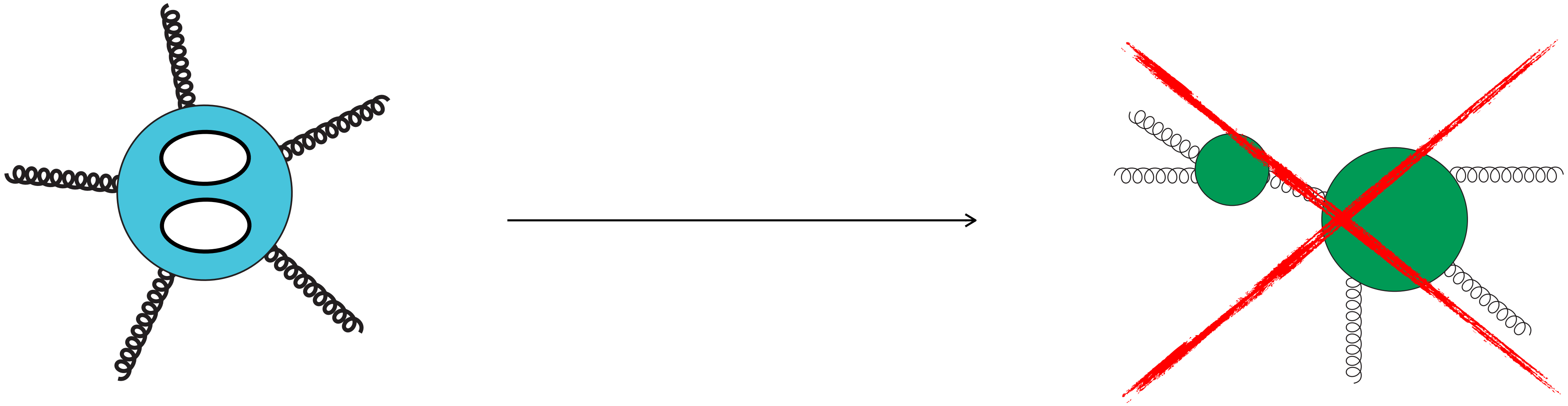


Catani, de Florian, Rodrigo : 1112.4405
Dixon, Herrmann, Yan, Zhu: 1912.09370

$$\boldsymbol{S}^{+,(2)}_{a,ikj} = \left(V^{q}_{ij}\right)^{2} f^{a a_k b} f^{b a_i a_j} \boldsymbol{T}^{a_i}_{i} \boldsymbol{T}^{a_j}_{j} \boldsymbol{T}^{a_k}_{k} \left[\frac{\langle ik \rangle}{\langle iq \rangle \langle qk \rangle} F(z^{ij}_{k}, \epsilon) - \frac{\langle jk \rangle}{\langle jq \rangle \langle qk \rangle} F(z^{ji}_{k}, \epsilon) \right]$$

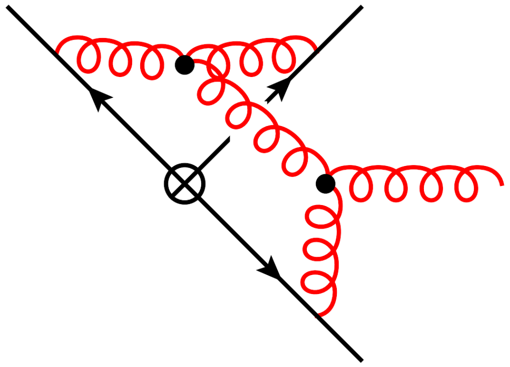


Soft-Collinear Limit

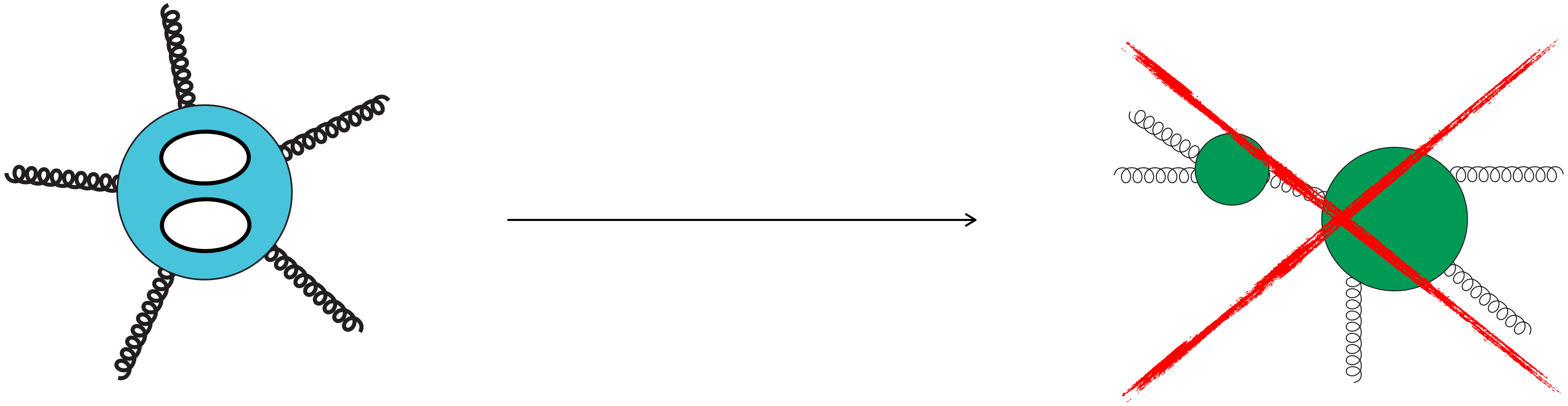


Catani, de Florian, Rodrigo : 1112.4405
Dixon, Herrmann, Yan, Zhu: 1912.09370

$$\boldsymbol{S}^{+,(2)}_{a,ikj} = \left(V^{q}_{ij}\right)^2 f^{aa_kb} f^{ba_ia_j} \boldsymbol{T}^{a_i}_{i} \boldsymbol{T}^{a_j}_{j} \boldsymbol{T}^{a_k}_{k} \left[\frac{\langle ik\rangle}{\langle iq\rangle\langle qk\rangle} F(z^{ij}_{k},\epsilon) - \frac{\langle jk\rangle}{\langle jq\rangle\langle qk\rangle} F(z^{ji}_{k},\epsilon) \right]$$

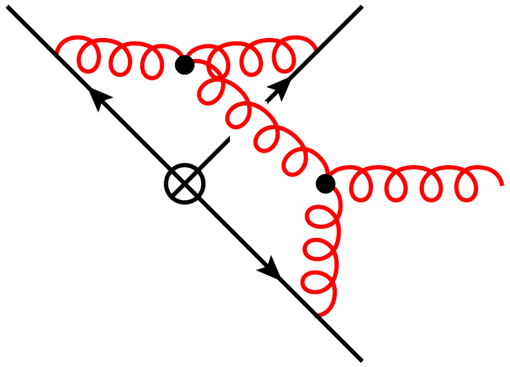


Soft-Collinear Limit



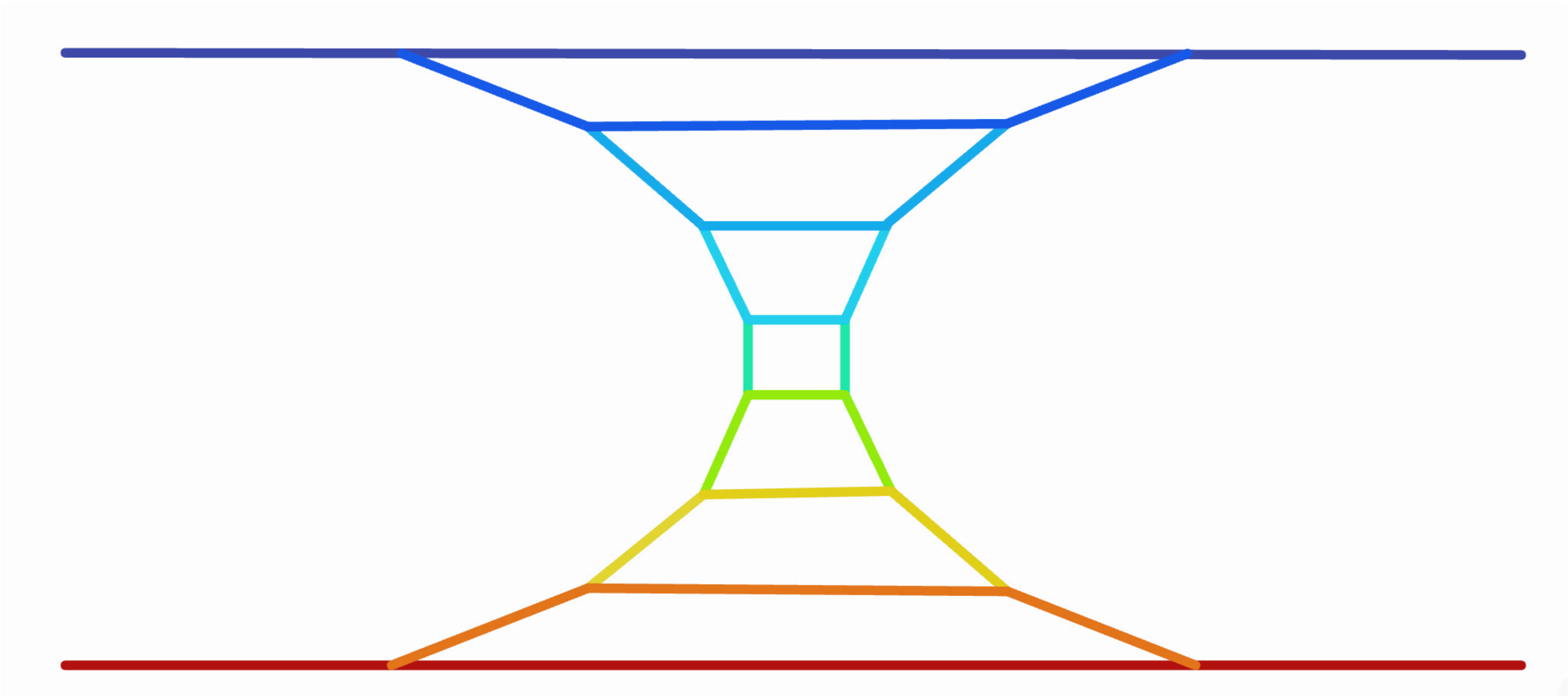
Catani, de Florian, Rodrigo : 1112.4405
Dixon, Herrmann, Yan, Zhu: 1912.09370

$$\boldsymbol{S}^{+,(2)}_{a,ikj} = \left(V^{q}_{ij}\right)^2 f^{aa_kb} f^{ba_ia_j} \boldsymbol{T}^{a_i}_{i} \boldsymbol{T}^{a_j}_{j} \boldsymbol{T}^{a_k}_{k} \left[\frac{\langle ik\rangle}{\langle iq\rangle\langle qk\rangle} F(z^{ij}_{k},\epsilon) - \frac{\langle jk\rangle}{\langle jq\rangle\langle qk\rangle} F(z^{ji}_{k},\epsilon) \right]$$

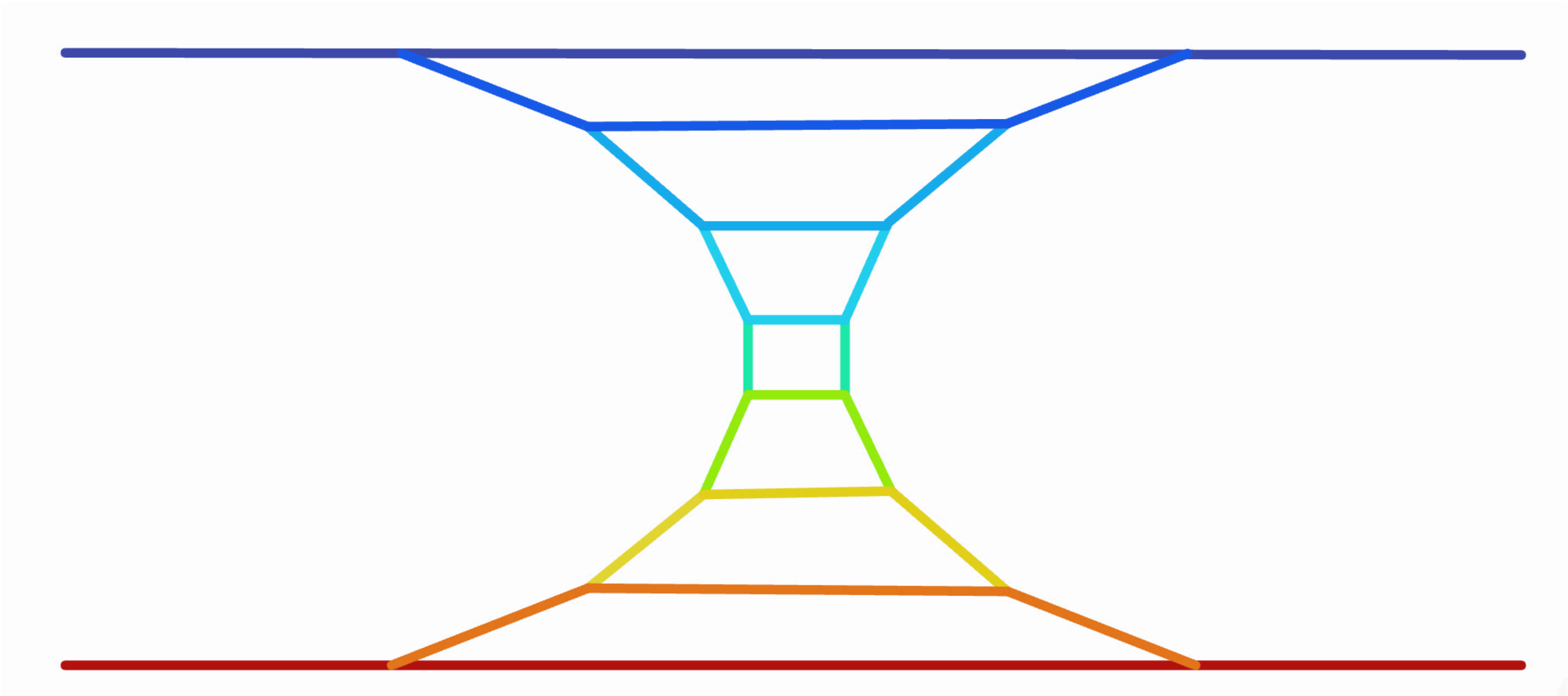


Purely non-planar!

Regge Limit

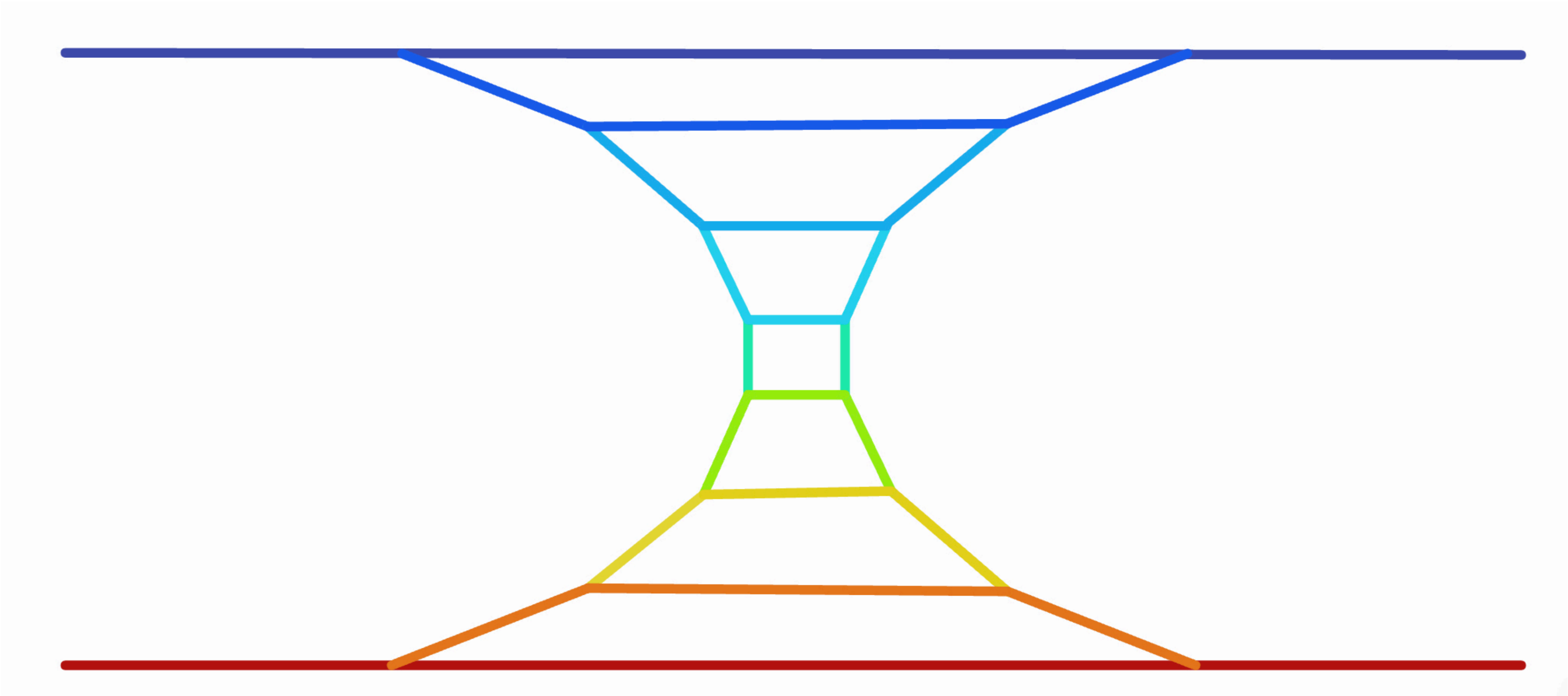


Regge Limit



$$s \gg |t|$$

Regge Limit

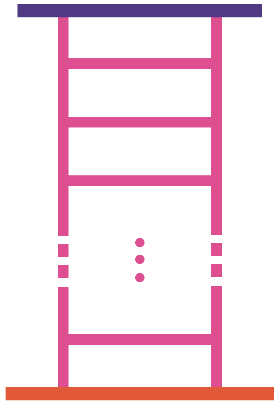


$$s \gg |t|$$

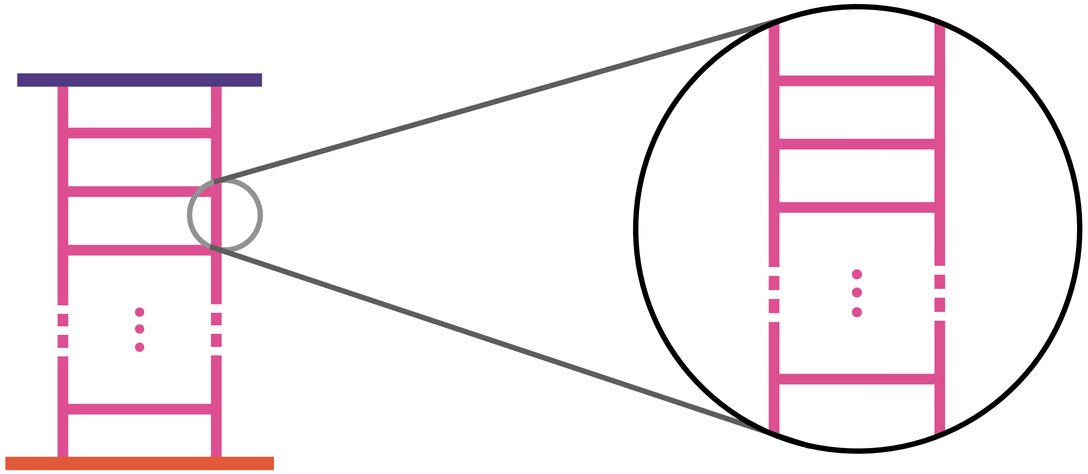
[see Einan's Talk]

$$\log\left(\frac{s}{-t}\right) \to \infty$$

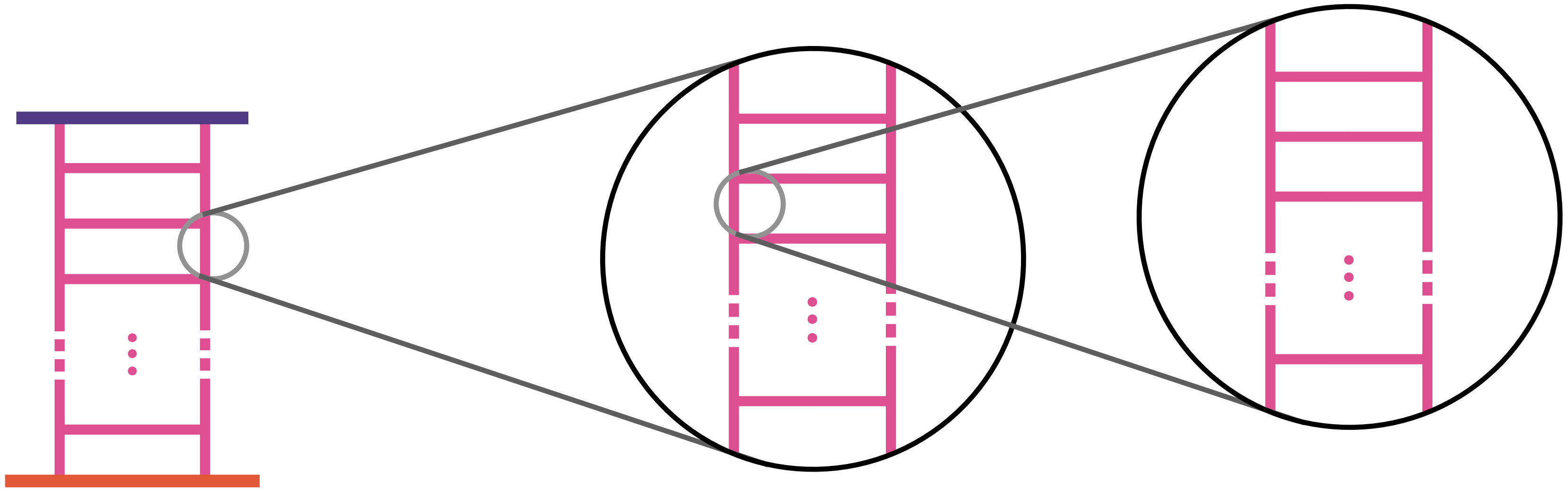
$$\log\left(\frac{s}{-t}\right) \to \infty$$



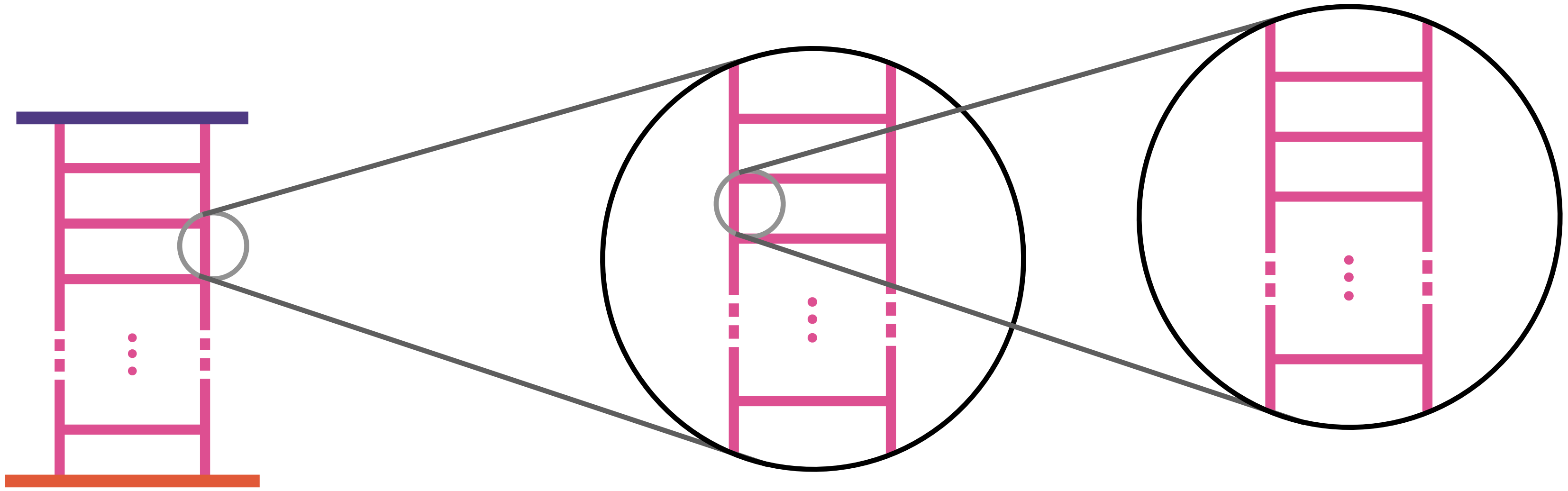
$$\log\left(\frac{s}{-t}\right) \to \infty$$

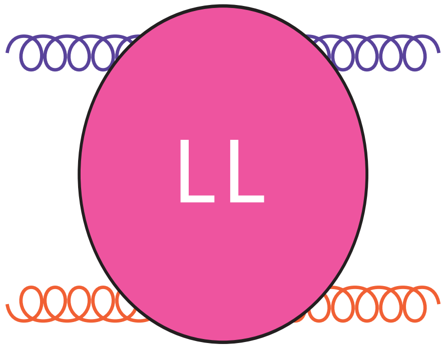


$$\log\left(\frac{s}{-t}\right) \to \infty$$

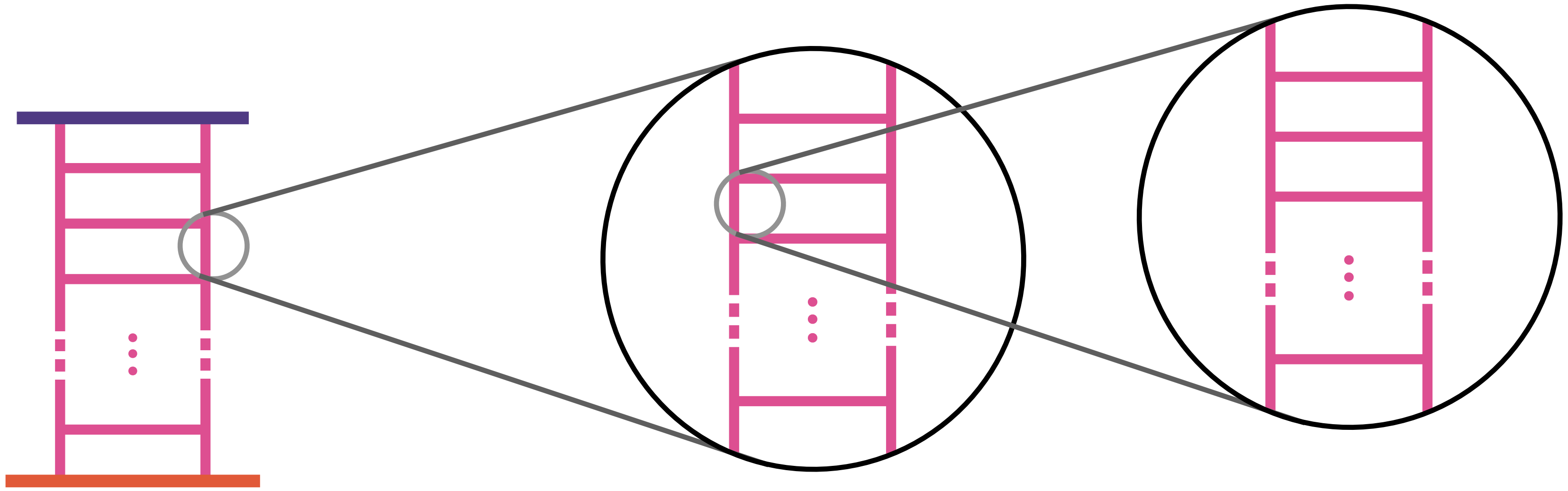


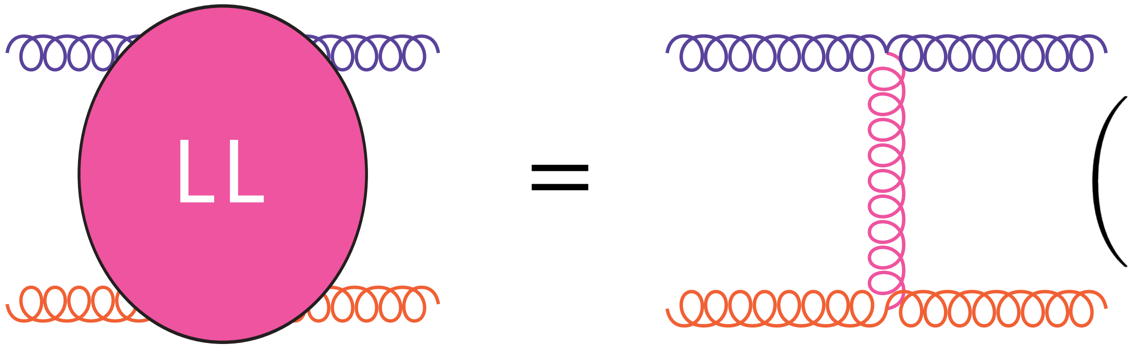
$$\log\left(\frac{s}{-t}\right) \to \infty$$





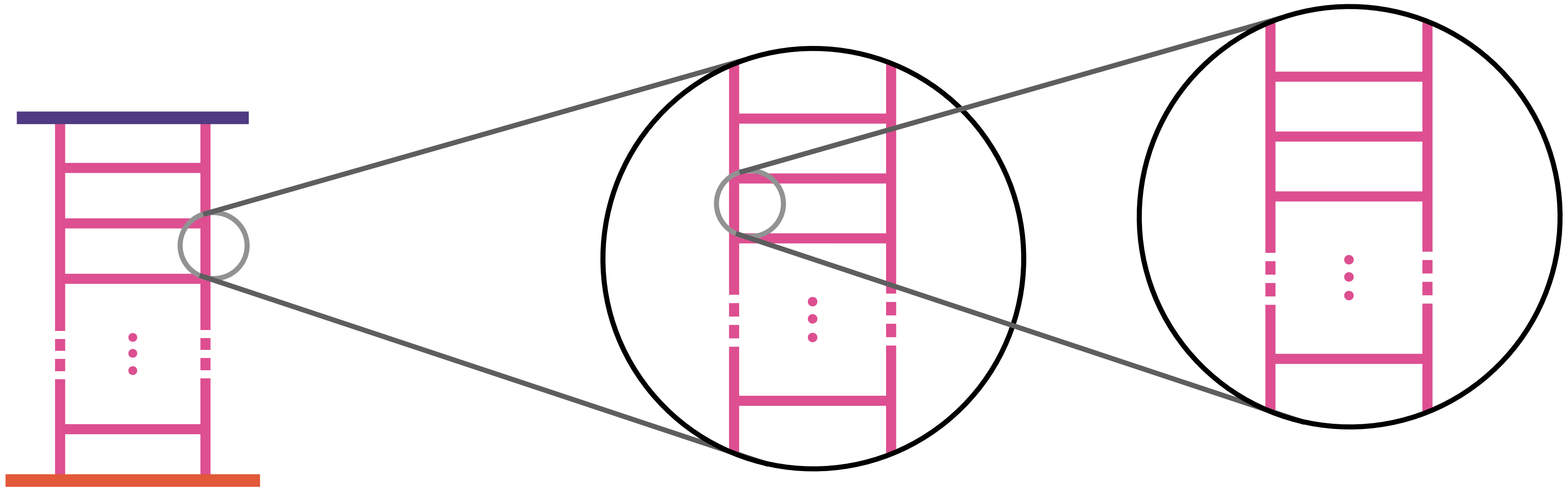
$$\log\left(\frac{s}{-t}\right) \to \infty$$

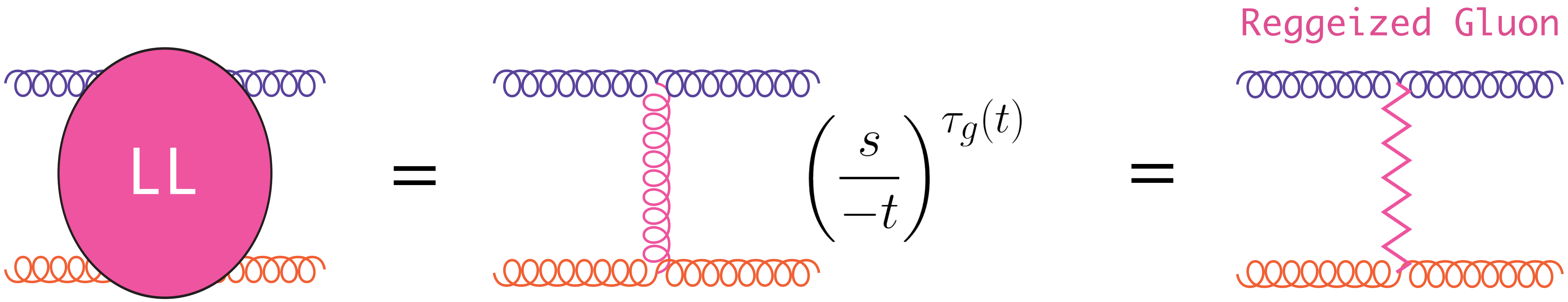


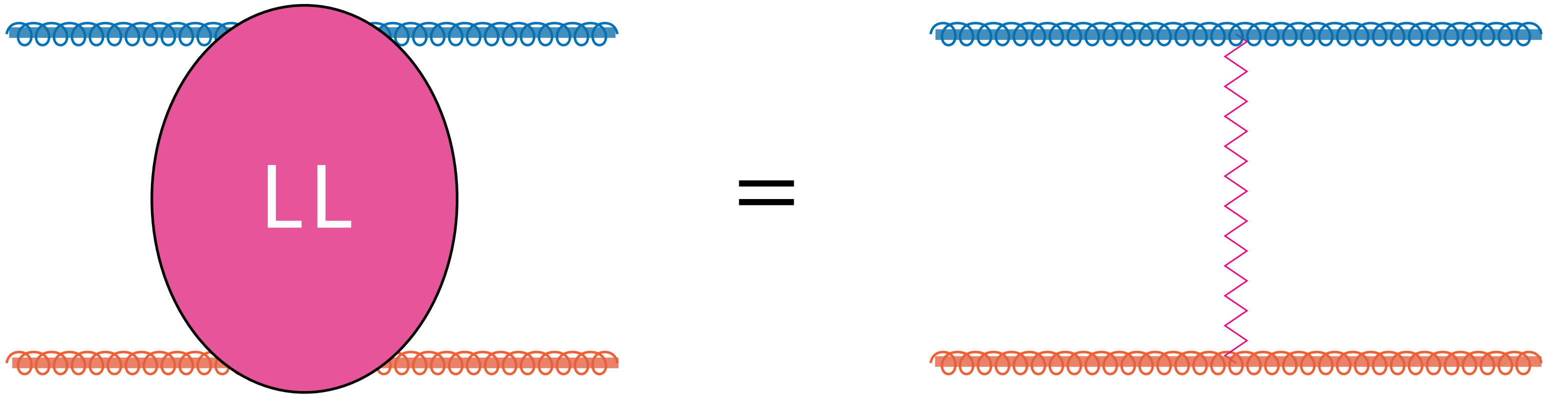


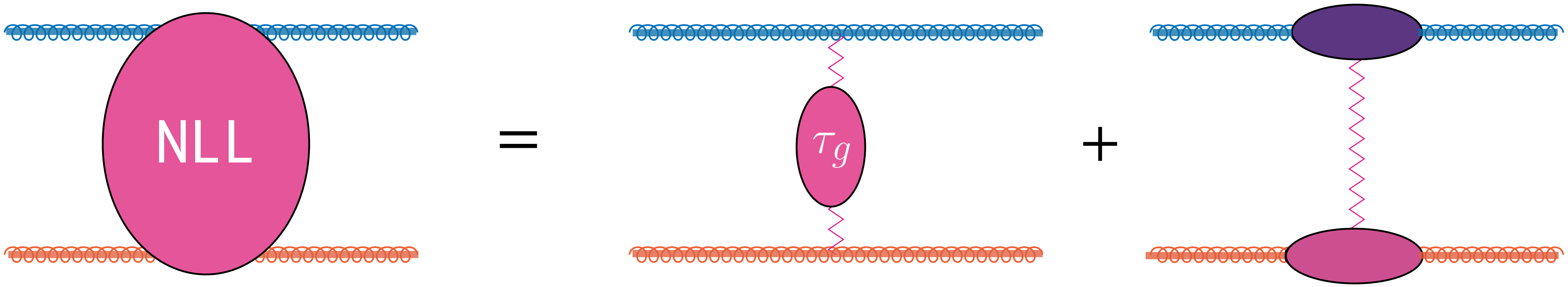
$$\left(\frac{s}{-t}\right)^{\tau_g(t)}$$

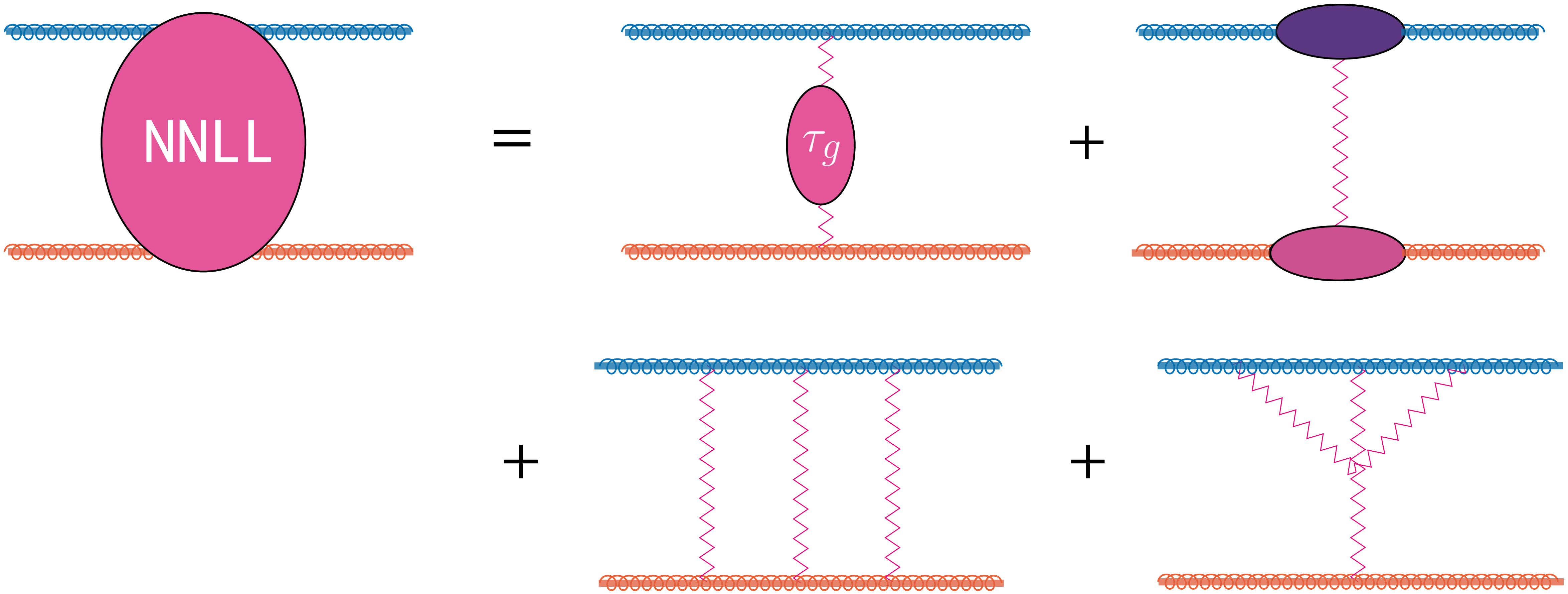
$$\log\left(\frac{s}{-t}\right) \to \infty$$

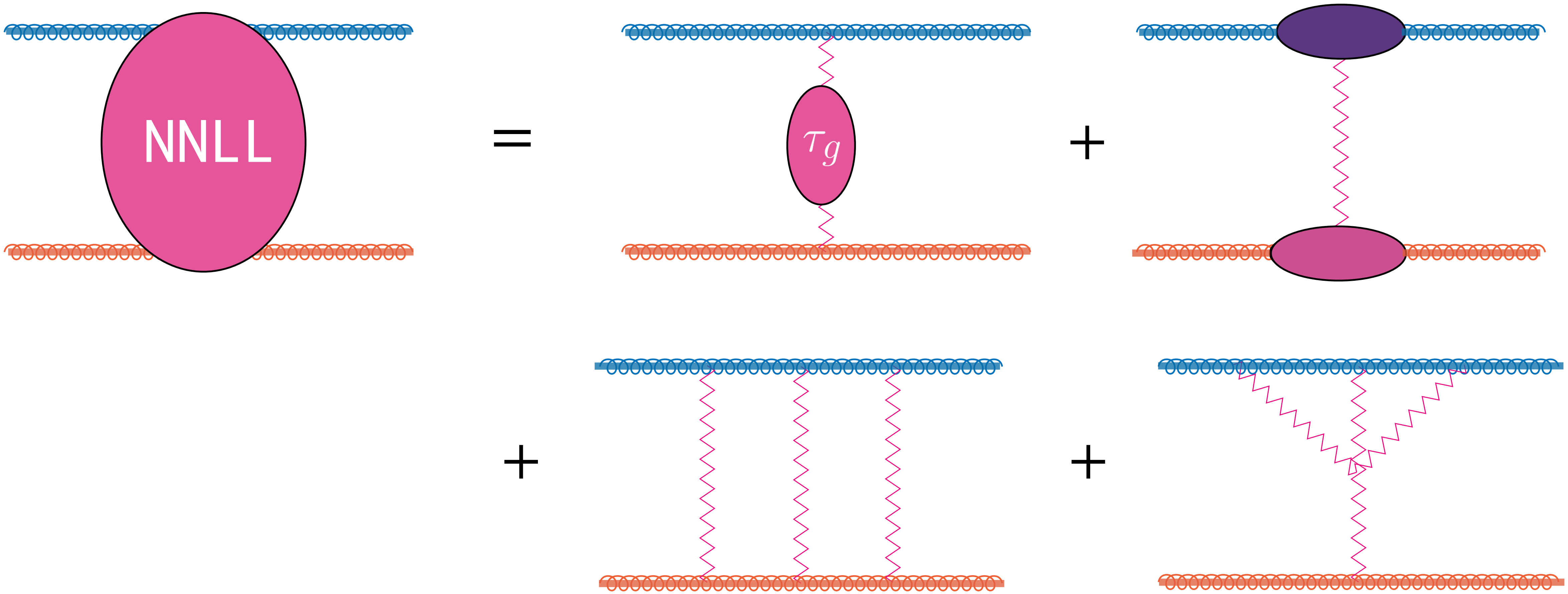




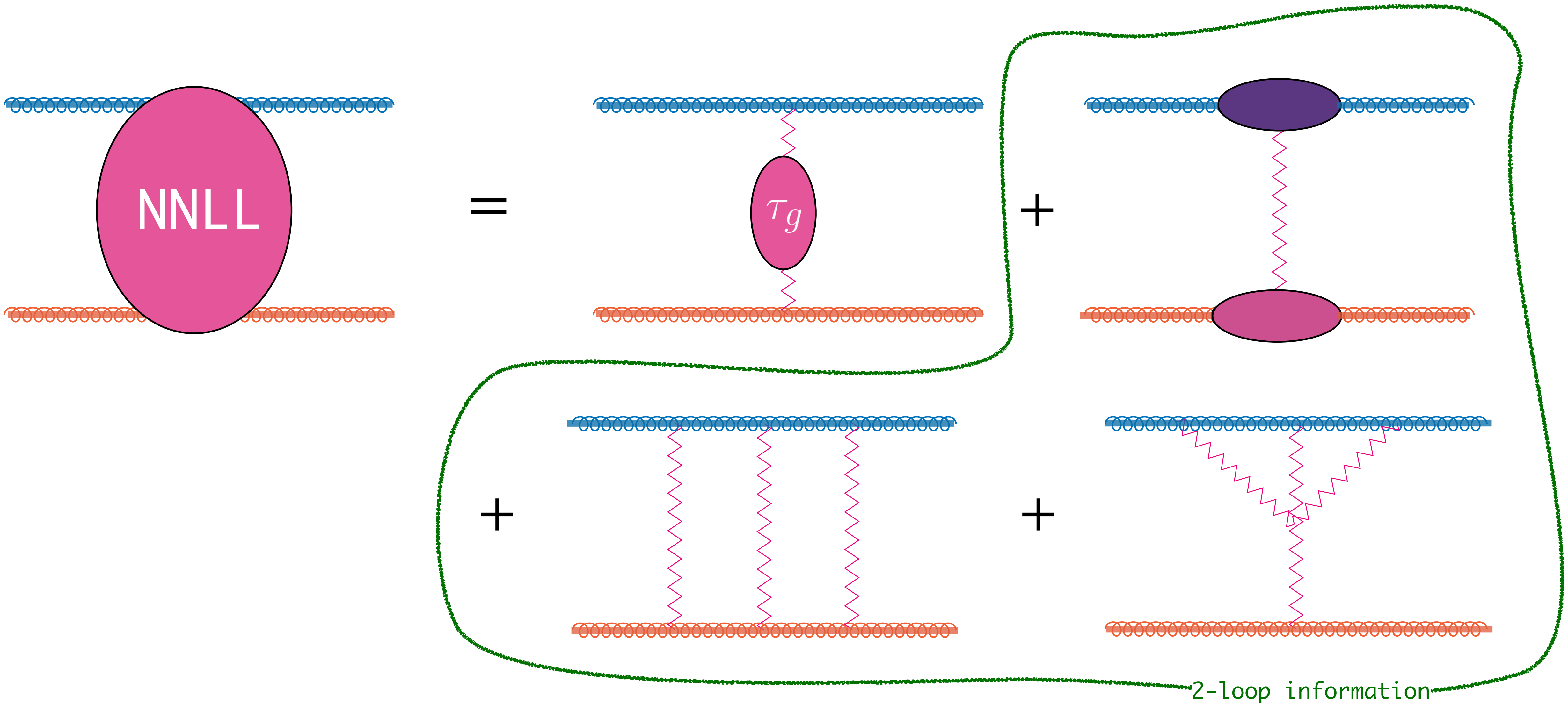

LL
=


NLL
=
τ_g
+


NNLL
=
τ_g
+
+
+



Caron-Huot: 1309.6521
Fadin, Lipatov: 1712.09805
Caron-Huot, Gardi, Vernazza :1701.05241
Del Duca, Marzucca, Verbeek: 2111.14265
Falcioni, Gardi, Maher, Milloy, Vernazza: 2112.11098



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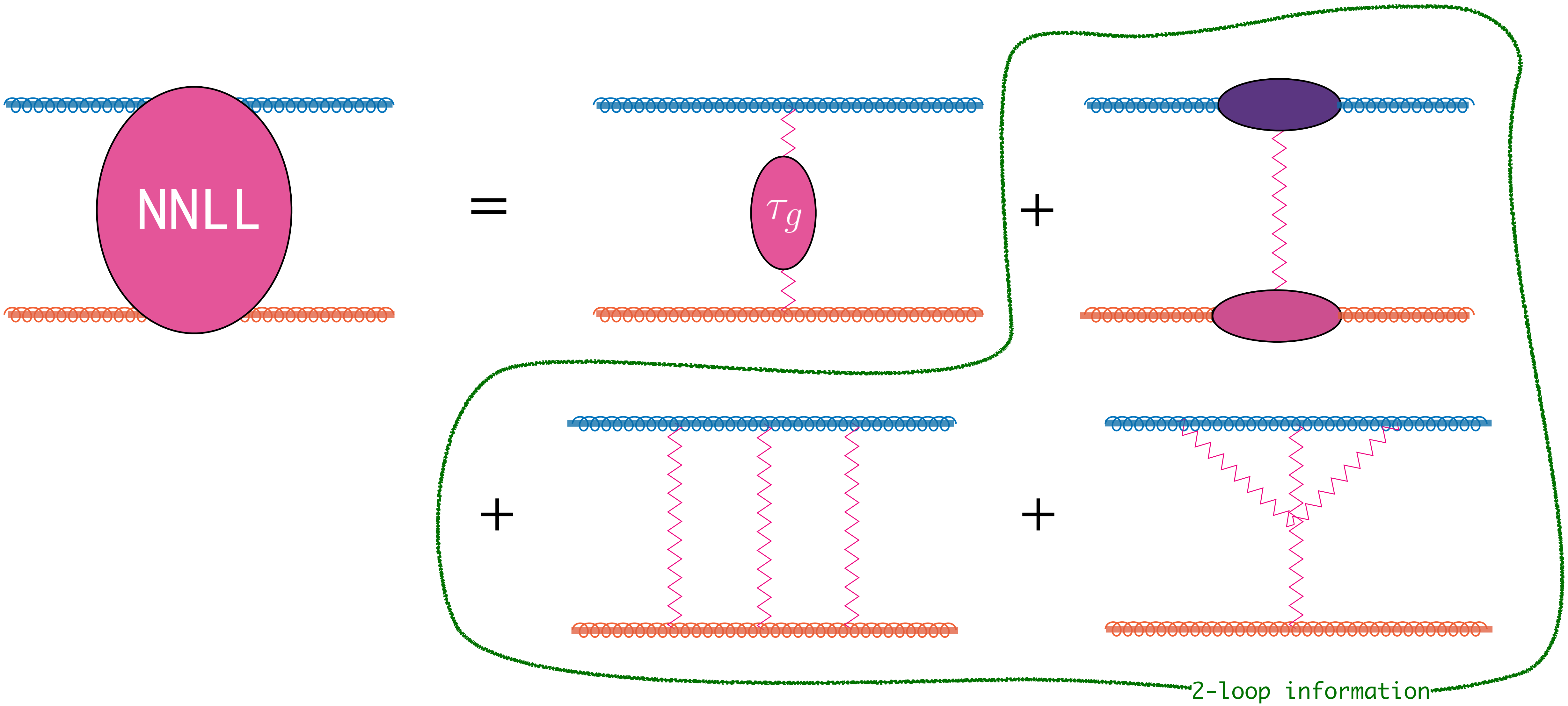
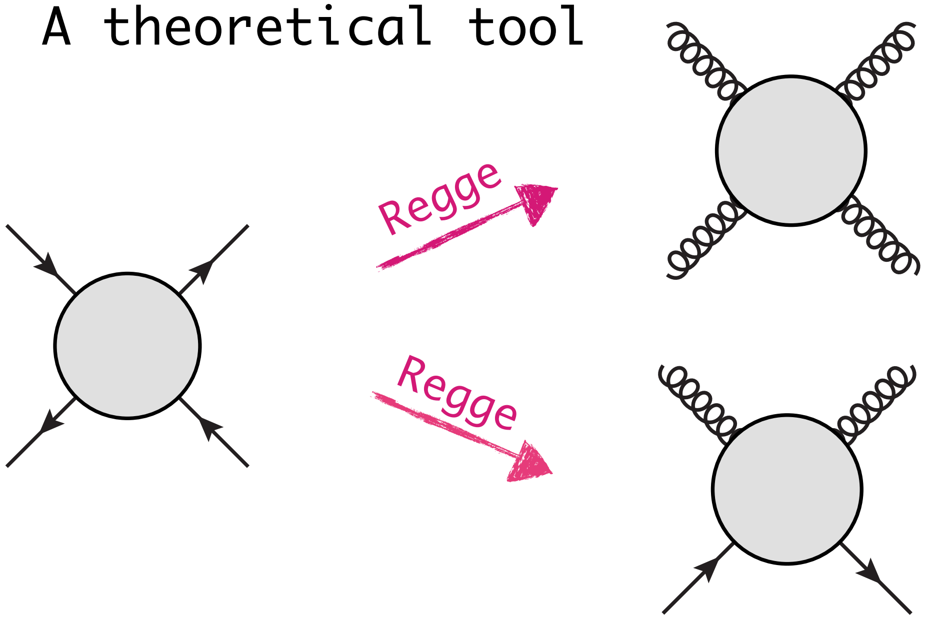


A theoretical tool



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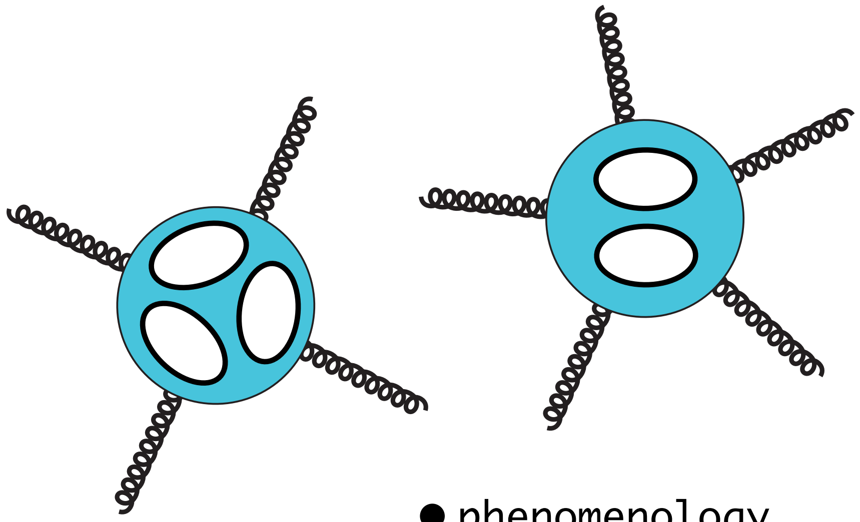
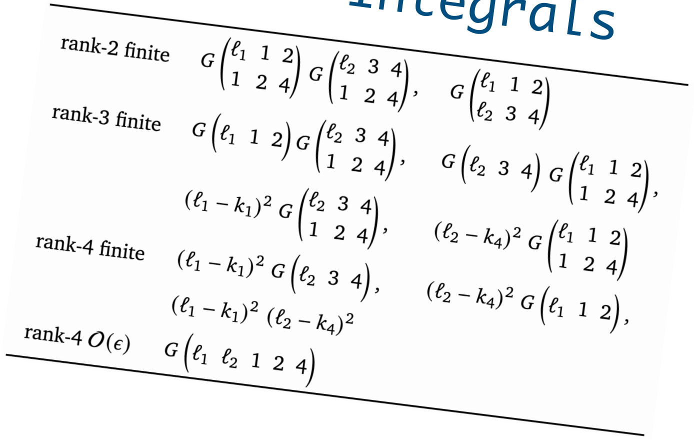


- phenomenology
- soft&collinear limits

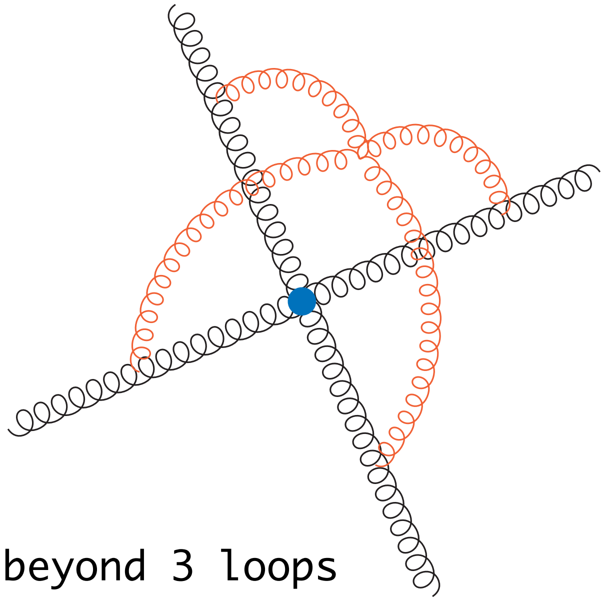
Finite Integrals



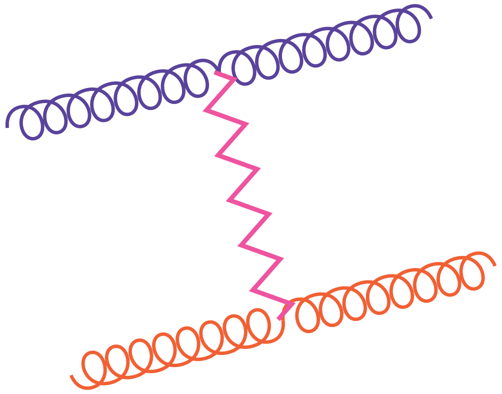
rank-2 finite	$G\begin{pmatrix}\ell_1 & 1 & 2\\ 1 & 2 & 4\end{pmatrix} G\begin{pmatrix}\ell_2 & 3 & 4\\ 1 & 2 & 4\end{pmatrix}$,	$G\begin{pmatrix}\ell_1 & 1 & 2\\ \ell_2 & 3 & 4\end{pmatrix}$
rank-3 finite	$G\begin{pmatrix}\ell_1 & 1 & 2\end{pmatrix} G\begin{pmatrix}\ell_2 & 3 & 4\\ 1 & 2 & 4\end{pmatrix}$,	$G\begin{pmatrix}\ell_2 & 3 & 4\end{pmatrix} G\begin{pmatrix}\ell_1 & 1 & 2\\ 1 & 2 & 4\end{pmatrix}$,
	$(\ell_1 - k_1)^2\, G\begin{pmatrix}\ell_2 & 3 & 4\\ 1 & 2 & 4\end{pmatrix}$,	$(\ell_2 - k_4)^2\, G\begin{pmatrix}\ell_1 & 1 & 2\\ 1 & 2 & 4\end{pmatrix}$
rank-4 finite	$(\ell_1 - k_1)^2\, G\begin{pmatrix}\ell_2 & 3 & 4\end{pmatrix}$,	$(\ell_2 - k_4)^2\, G\begin{pmatrix}\ell_1 & 1 & 2\end{pmatrix}$,
	$(\ell_1 - k_1)^2\, (\ell_2 - k_4)^2$	
rank-4 $\mathcal{O}(\epsilon)$	$G\begin{pmatrix}\ell_1 & \ell_2 & 1 & 2 & 4\end{pmatrix}$	

- more integrals
- interaction with IBPs
- classify the full tower in ϵ

Thank you!!



- beyond 3 loops
- role in cross sections

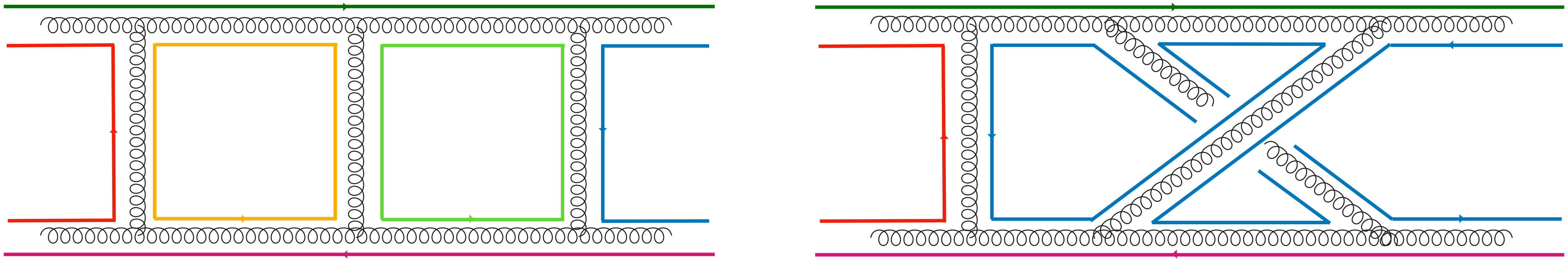


- next-to-leading power

Backup Slides

The planar limit

$SU(N_c)$



't Hooft coupling

$$\lambda = g^2\, N_c$$

$$\lambda^{\text{loops}}\, N_c^{\chi}$$

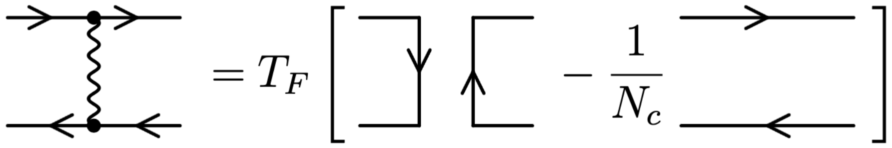
Sphere $\chi = 2$

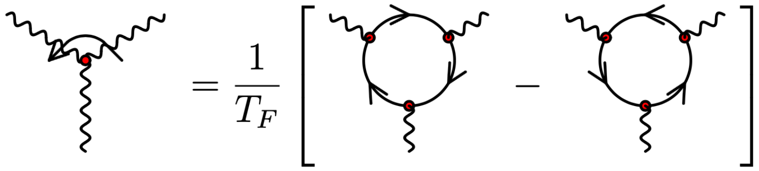
Other Manifolds $\chi = 2 - \text{holes}$

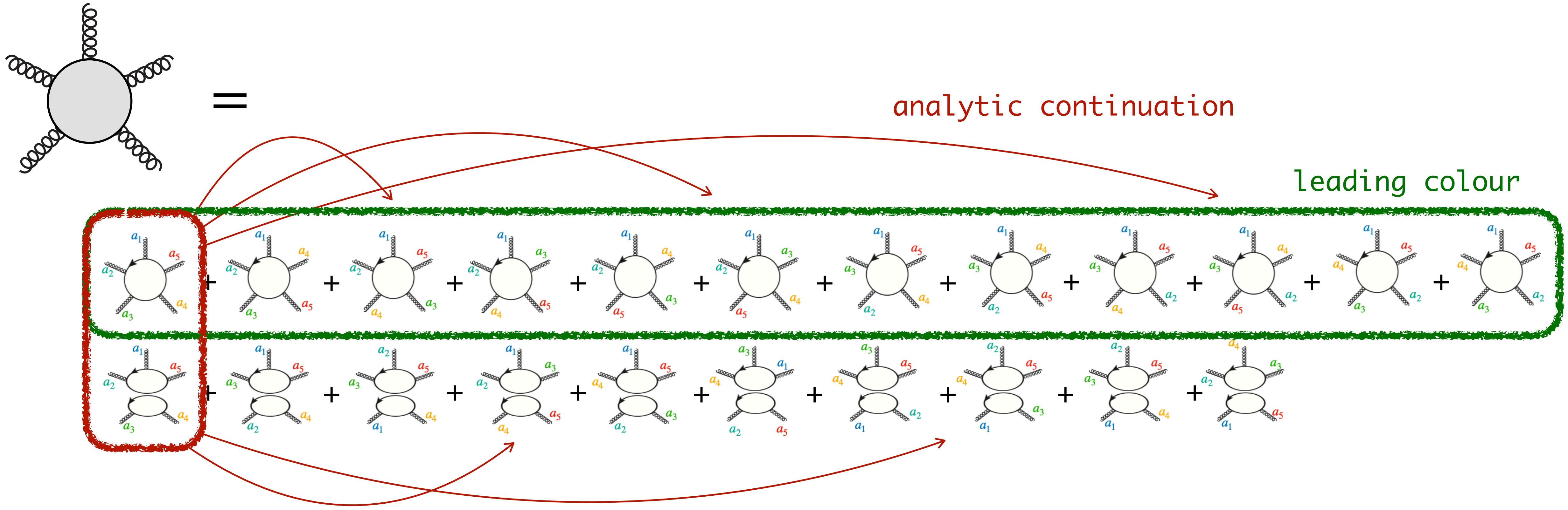
large N_c limit ↔ planar diagrams

fewer diagrams, natural ordering, simpler kinematics feynman diagrams → **colour decomposition** → helicity projection → integration by parts → master integrals computation

$$A = \Sigma\, A_c\, \mathcal{C}_c$$







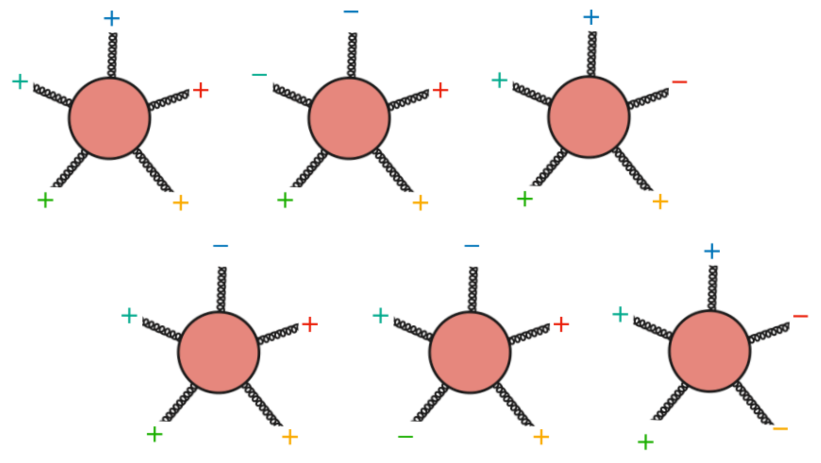
$$A_c = \Sigma\, N_c{}^a\, n_f{}^b \boxed{A_c^{ab}}$$

individually gauge invariant

feynman diagrams → colour decomposition → **helicity projection** → integration by parts → master integrals computation

$A = \Sigma \, A_c \, \mathcal{C}_c$

$$A_c = \sum F_i \, \mathcal{T}_i$$



$$\epsilon_q^{\mu}(p) \rightarrow \frac{[q|\mu|p\rangle}{\sqrt{2}[pq]}$$

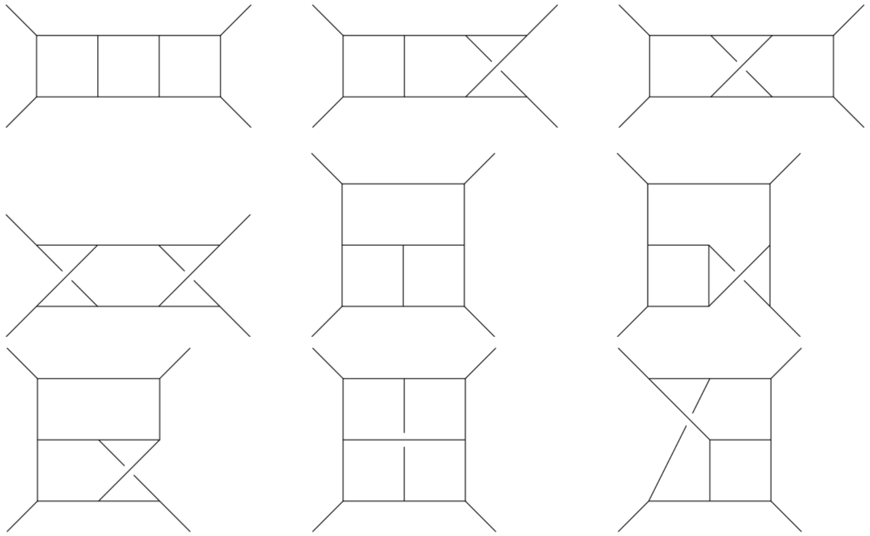
$$u(p) \rightarrow |p\rangle \text{ or } |p]$$

't Hooft-Veltman scheme + Tancredi, Peraro: 1906.03298, 2012.00820

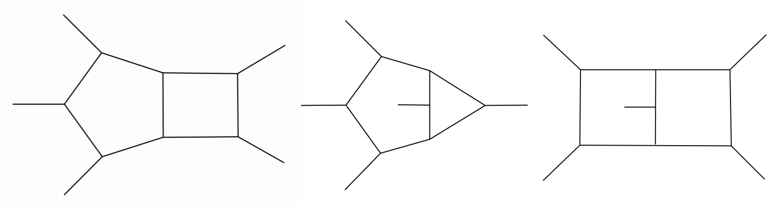
$$A_c = \Sigma_{i=1}^{m} \, \overline{F}_i \, \overline{\mathcal{T}}_i + \Sigma_{i=m+1}^{n} \, \overline{F}_i \, \boxed{\overline{\mathcal{T}}_i}$$

$$\downarrow$$

$$A_c = \Sigma_{i=1}^{m} \, H_c^i \, \overline{\mathcal{T}}_i \qquad\qquad \mathcal{O}(\epsilon)$$

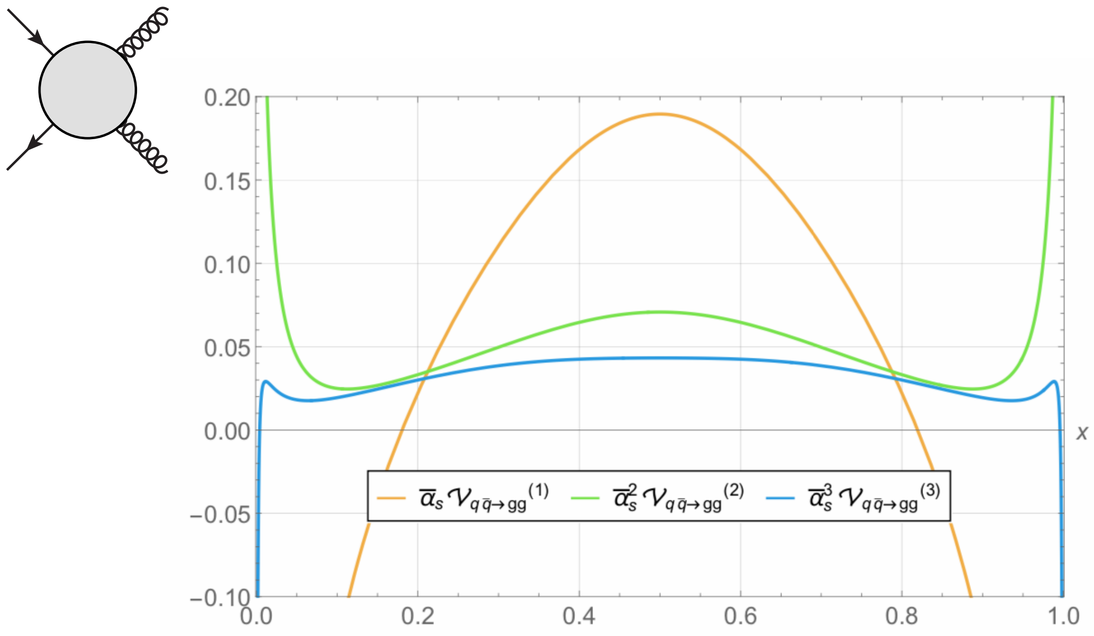


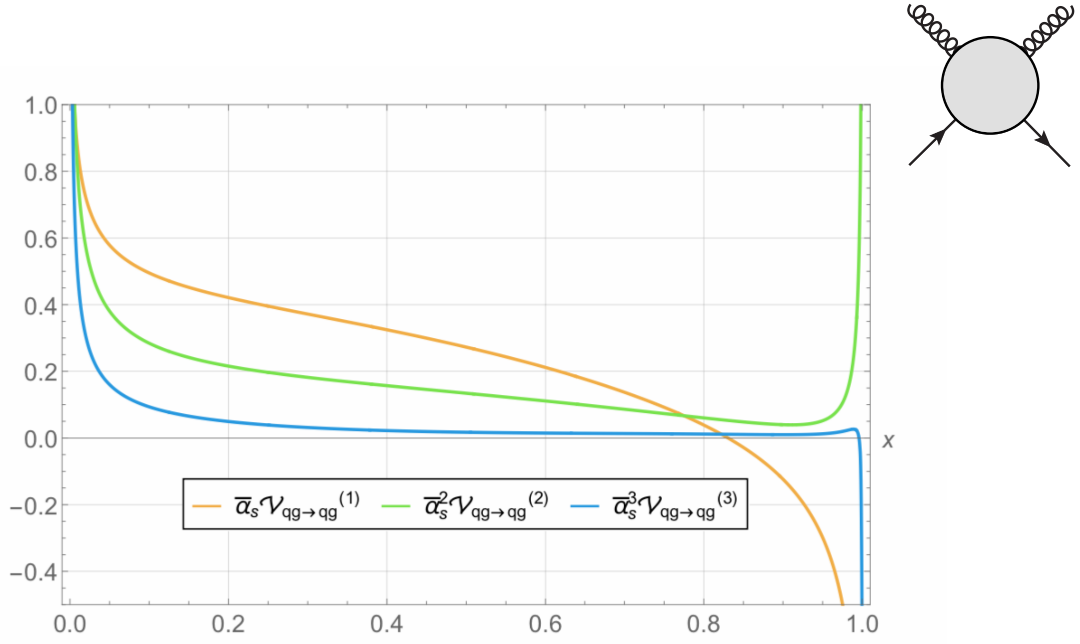
Henn, Mistlberger, V.A.Smirnov, Wasser:
2002.09492

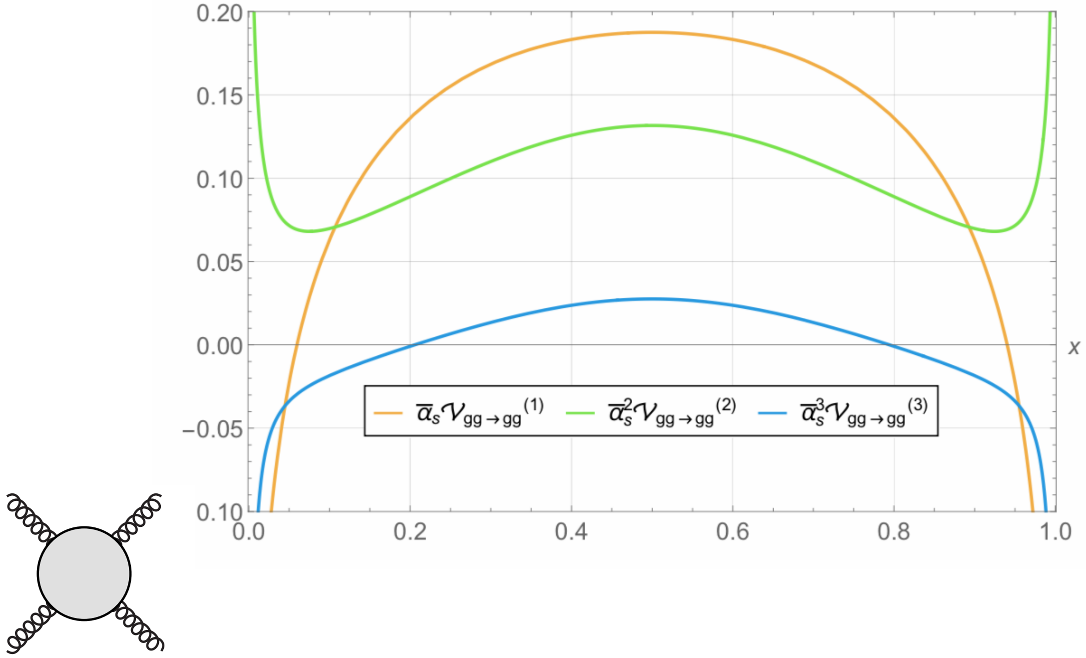


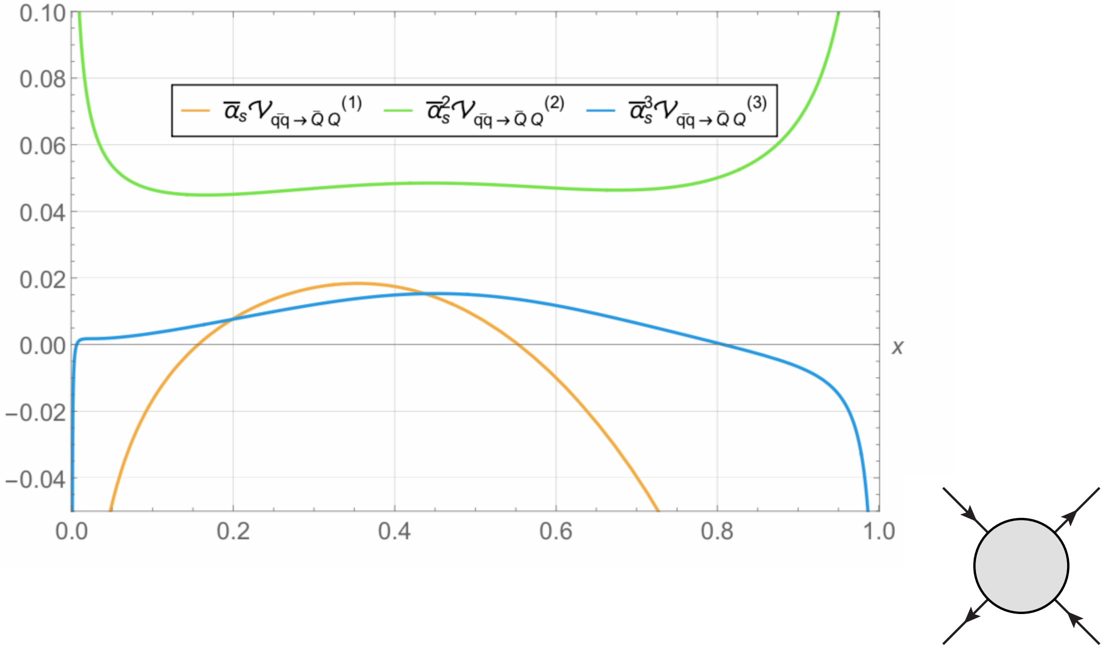
Chicherin, Sotnikov:
2009.07803

$$\mathcal{M}_m = \Sigma \, \epsilon^n \, c_{n,i} \, \mathbb{T}_i$$

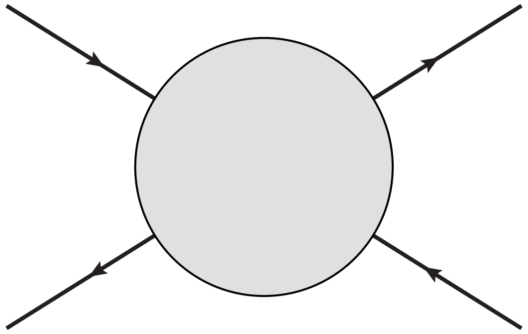
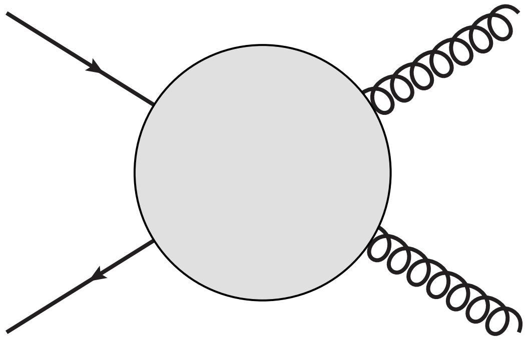
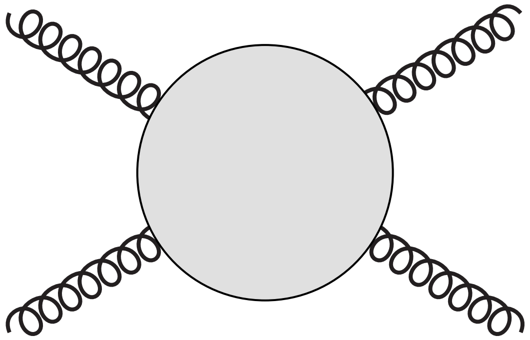






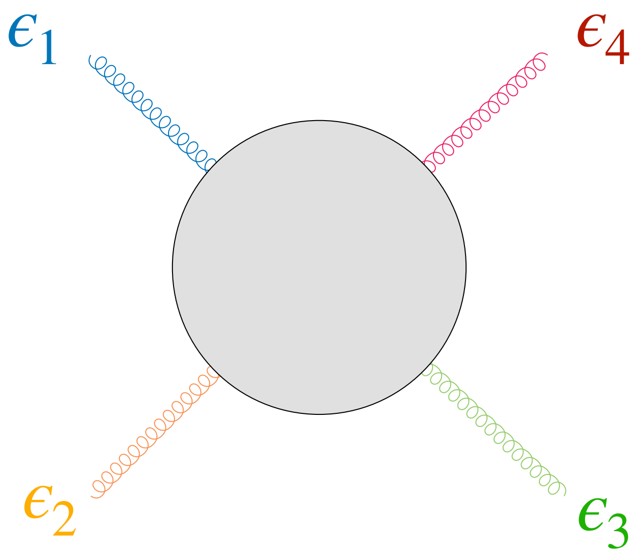


`5 point coming soon..`

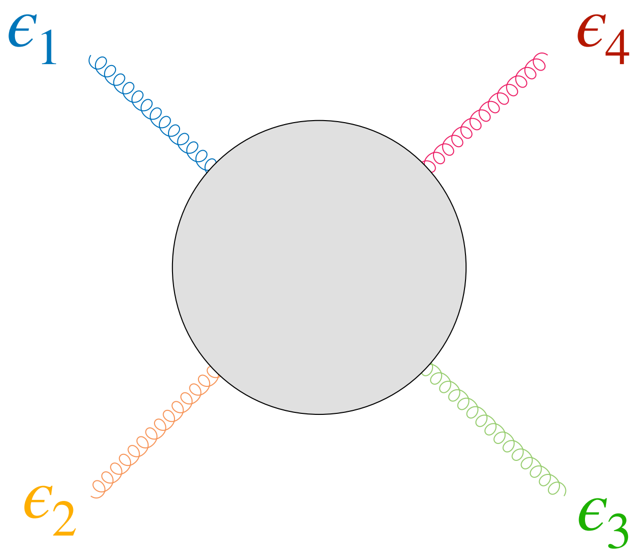
# of QCD Feynman diagrams			
tree level	1	3	4
1-loop	9	30	81
2-loop	158	595	1771
3-loop	3584	14971	48723

!?!

Spin

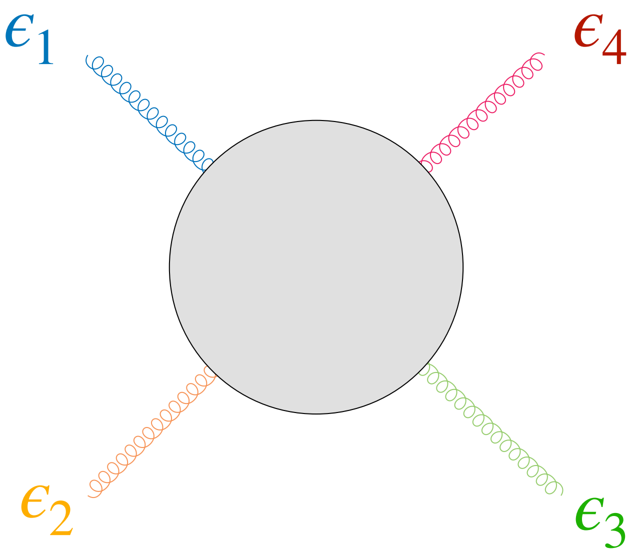


Spin



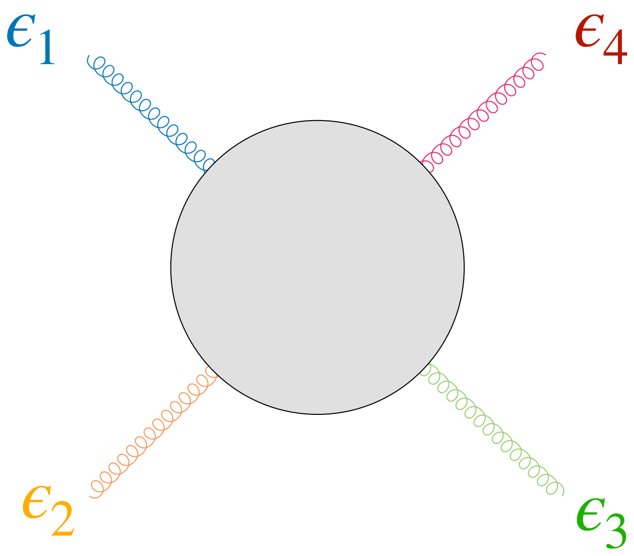
$$A_c = A_c^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

Spin



$$\begin{aligned} A_c &= A_c^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4} \\ &= \sum_j F_c^j \; T_j^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4} \end{aligned}$$

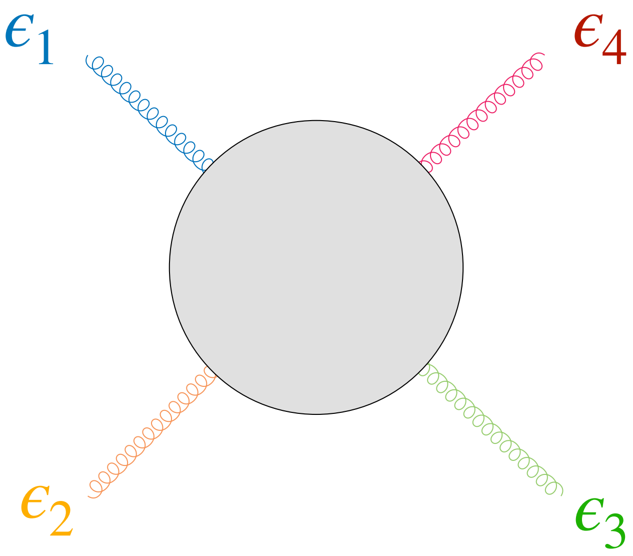
Spin



$$A_c = A_c^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$
$$= \sum_j F_c^j \; T_j^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$$p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_3^{\mu_4}$$

Spin



$$A_c = A_c^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$$= \sum_j F_c^j \; T_j^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$$p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_3^{\mu_4}$$

$$p_2^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$$

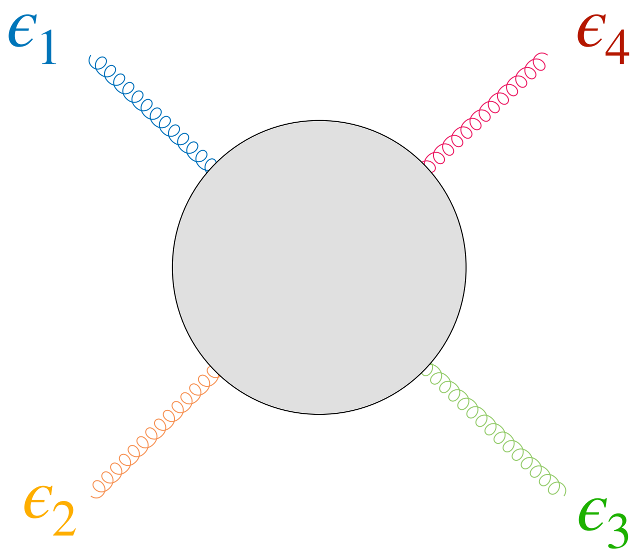
$$p_3^{\mu_1}p_1^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$$

$$p_2^{\mu_1}p_1^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$$

$$p_3^{\mu_1}p_2^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$$

$\vdots$ **81**

Spin



$$A_c = A_c^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$$= \sum_j F_c^j \; T_j^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_3^{\mu_4}$

$p_2^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$

$p_3^{\mu_1}p_1^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$

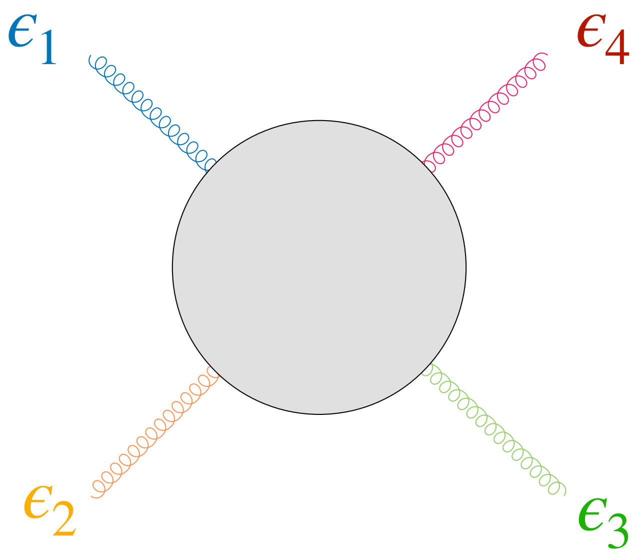
$p_2^{\mu_1}p_1^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$

$p_3^{\mu_1}p_2^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$

$\vdots$ **81**

$p_3^{\mu_1}p_3^{\mu_2}g^{\mu_3\mu_4}$

Spin



$$A_c = A_c^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$$= \sum_j F_c^j \; T_j^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_3^{\mu_4}$

$p_2^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$

$p_3^{\mu_1}p_1^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$

$p_2^{\mu_1}p_1^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$

$p_3^{\mu_1}p_2^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$

$\vdots$ **81**

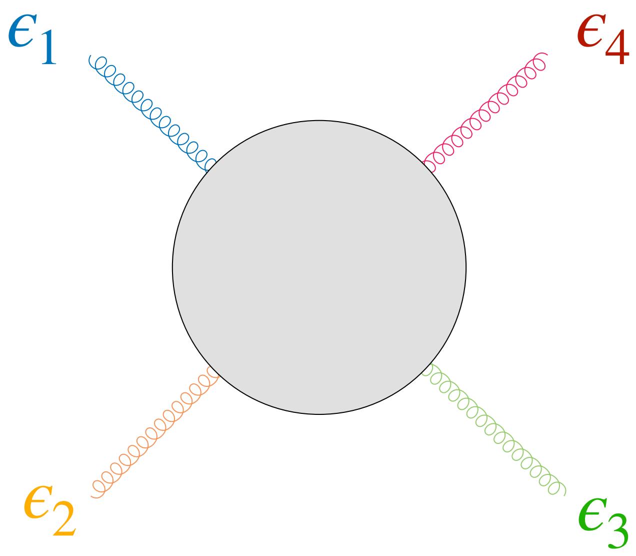
$p_3^{\mu_1}p_3^{\mu_2}g^{\mu_3\mu_4}$

$g^{\mu_1\mu_2}p_1^{\mu_3}p_1^{\mu_4}$

$g^{\mu_1\mu_3}p_1^{\mu_2}p_2^{\mu_4}$

$\vdots$ **54**

Spin



$$A_c = A_c^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$$= \sum_j F_c^j \; T_j^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$$p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_3^{\mu_4}$$

$$p_2^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$$

$$p_3^{\mu_1}p_1^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$$

$$p_2^{\mu_1}p_1^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$$

$$p_3^{\mu_1}p_2^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$$

⋮ **81**

$$p_3^{\mu_1}p_3^{\mu_2}g^{\mu_3\mu_4}$$

$$g^{\mu_1\mu_2}p_1^{\mu_3}p_1^{\mu_4}$$

$$g^{\mu_1\mu_3}p_1^{\mu_2}p_2^{\mu_4}$$

⋮ **54**

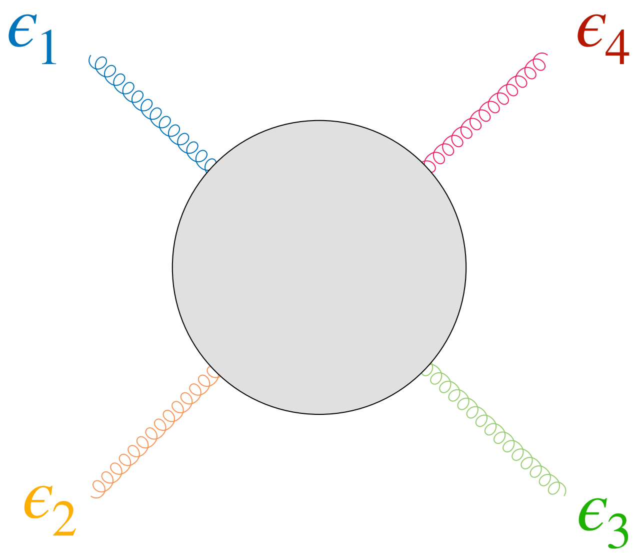
$$g^{\mu_1\mu_2}g^{\mu_3\mu_4}$$

$$g^{\mu_1\mu_3}g^{\mu_2\mu_4}$$

$$g^{\mu_1\mu_4}g^{\mu_2\mu_3}$$

3

Spin



$$A_c = A_c^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$$= \sum_j F_c^j \; T_j^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$$p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_3^{\mu_4}$$
$$p_2^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$$
$$p_3^{\mu_1}p_1^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$$
$$p_2^{\mu_1}p_1^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$$
$$p_3^{\mu_1}p_2^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$$

⋮ **81**

$$p_3^{\mu_1}p_3^{\mu_2}g^{\mu_3\mu_4}$$
$$g^{\mu_1\mu_2}p_1^{\mu_3}p_1^{\mu_4}$$
$$g^{\mu_1\mu_3}p_1^{\mu_2}p_2^{\mu_4}$$

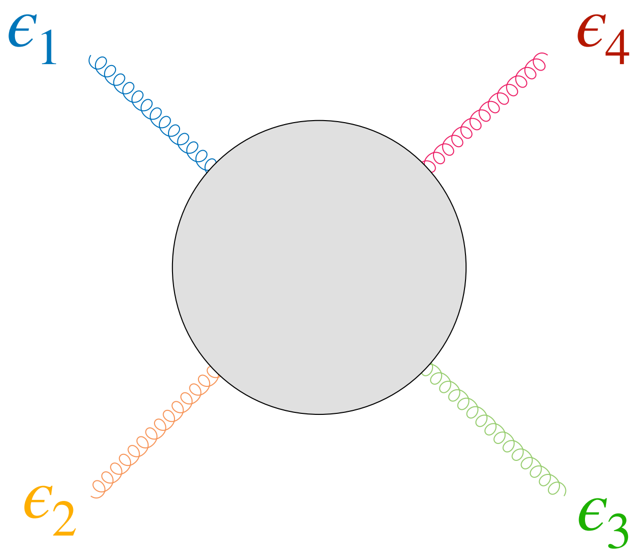
⋮ **54**

$$g^{\mu_1\mu_2}g^{\mu_3\mu_4}$$
$$g^{\mu_1\mu_3}g^{\mu_2\mu_4}$$
$$g^{\mu_1\mu_4}g^{\mu_2\mu_3}$$

3

138

Spin



$$A_c = A_c^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$$= \sum_j F_c^j\ T_j^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$$p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_3^{\mu_4}$$
$$p_2^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$$
$$p_3^{\mu_1}p_1^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$$
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$$p_3^{\mu_1}p_2^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$$
$$\vdots$$

81

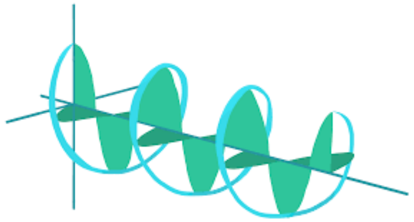
$$p_3^{\mu_1}p_3^{\mu_2}g^{\mu_3\mu_4}$$
$$g^{\mu_1\mu_2}p_1^{\mu_3}p_1^{\mu_4}$$
$$g^{\mu_1\mu_3}p_1^{\mu_2}p_2^{\mu_4}$$
$$\vdots$$

54

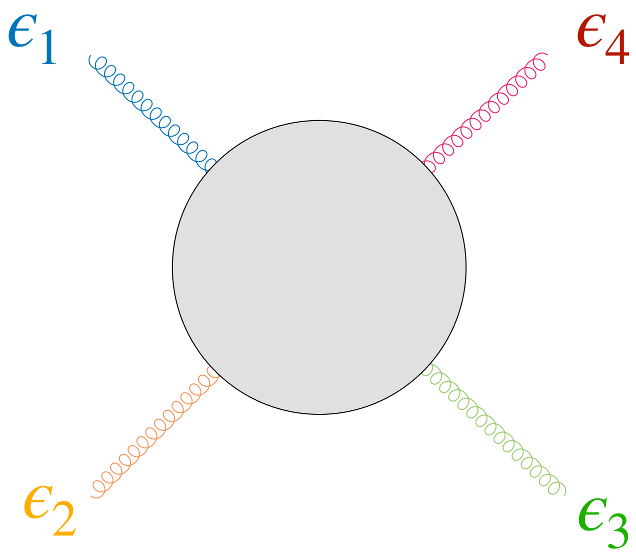
$$g^{\mu_1\mu_2}g^{\mu_3\mu_4}$$
$$g^{\mu_1\mu_3}g^{\mu_2\mu_4}$$
$$g^{\mu_1\mu_4}g^{\mu_2\mu_3}$$

3

Transversality $\epsilon_i \cdot p_i = 0$



Spin



$$A_c = A_c^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$$= \sum_j F_c^j \; T_j^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$\cancel{p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} p_3^{\mu_4}}$

$p_2^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} p_2^{\mu_4}$

$p_3^{\mu_1} p_1^{\mu_2} p_1^{\mu_3} p_2^{\mu_4}$

$p_2^{\mu_1} p_1^{\mu_2} p_3^{\mu_3} p_2^{\mu_4}$

$p_3^{\mu_1} p_2^{\mu_2} p_1^{\mu_3} p_2^{\mu_4}$

$\vdots$ **81**

$p_3^{\mu_1} p_3^{\mu_2} g^{\mu_3\mu_4}$

$g^{\mu_1\mu_2} p_1^{\mu_3} p_1^{\mu_4}$

$g^{\mu_1\mu_3} p_1^{\mu_2} p_2^{\mu_4}$

$\vdots$ **54**

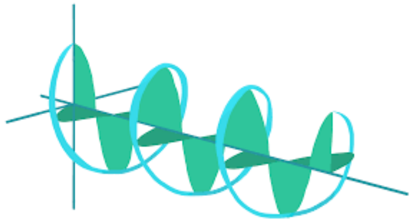
$g^{\mu_1\mu_2} g^{\mu_3\mu_4}$

$g^{\mu_1\mu_3} g^{\mu_2\mu_4}$

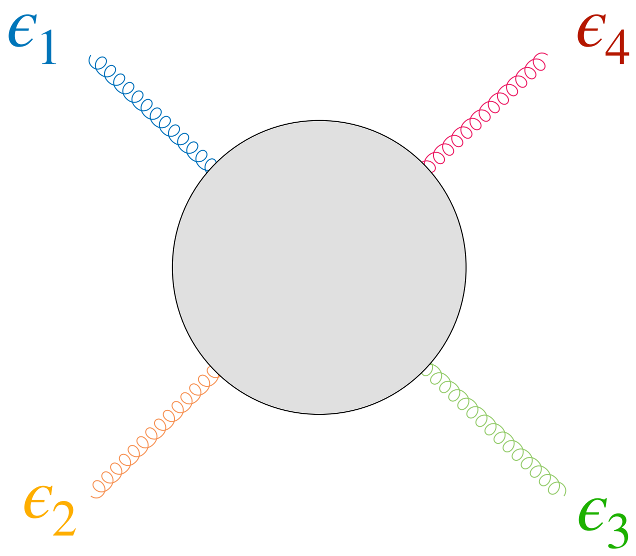
$g^{\mu_1\mu_4} g^{\mu_2\mu_3}$

3

Transversality $\epsilon_i \cdot p_i = 0$



Spin



$$A_c = A_c^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$$= \sum_j F_c^j \; T_j^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$\cancel{p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_3^{\mu_4}}$

$\cancel{p_2^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}}$

$p_3^{\mu_1}p_1^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$

$p_2^{\mu_1}p_1^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$

$\cancel{p_3^{\mu_1}p_2^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}}$

⋮ **81**

$p_3^{\mu_1}p_3^{\mu_2}g^{\mu_3\mu_4}$

$g^{\mu_1\mu_2}p_1^{\mu_3}p_1^{\mu_4}$

$g^{\mu_1\mu_3}p_1^{\mu_2}p_2^{\mu_4}$

⋮ **54**

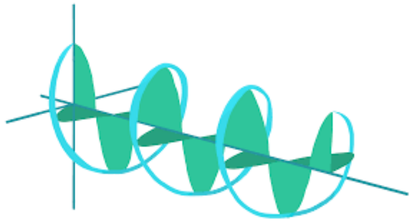
$g^{\mu_1\mu_2}g^{\mu_3\mu_4}$

$g^{\mu_1\mu_3}g^{\mu_2\mu_4}$

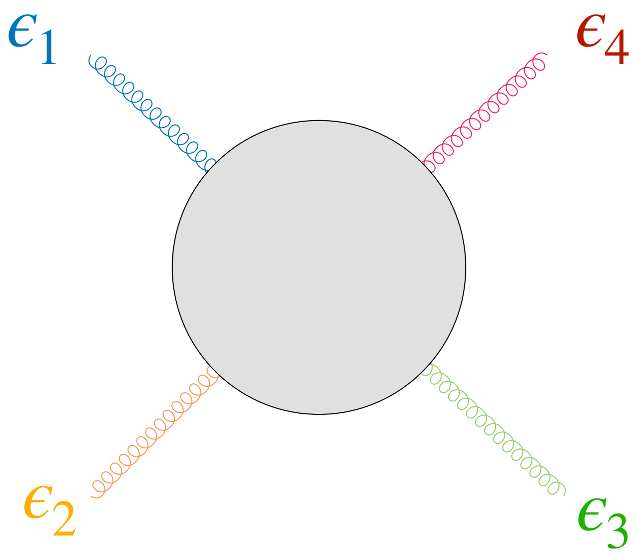
$g^{\mu_1\mu_4}g^{\mu_2\mu_3}$

3

Transversality $\epsilon_i \cdot p_i = 0$



Spin



$$A_c = A_c^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$
$$= \sum_j F_c^j \; T_j^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$\cancel{p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_3^{\mu_4}}$

$\cancel{p_2^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}}$

$p_3^{\mu_1}p_1^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$

$p_2^{\mu_1}p_1^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$

$\cancel{p_3^{\mu_1}p_2^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}}$

⋮ **81**

$p_3^{\mu_1}p_3^{\mu_2}g^{\mu_3\mu_4}$

$g^{\mu_1\mu_2}p_1^{\mu_3}p_1^{\mu_4}$

$g^{\mu_1\mu_3}p_1^{\mu_2}p_2^{\mu_4}$

⋮ **54**

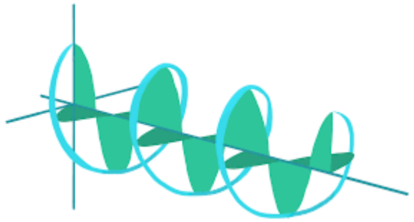
$g^{\mu_1\mu_2}g^{\mu_3\mu_4}$

$g^{\mu_1\mu_3}g^{\mu_2\mu_4}$

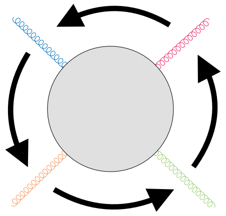
$g^{\mu_1\mu_4}g^{\mu_2\mu_3}$

3

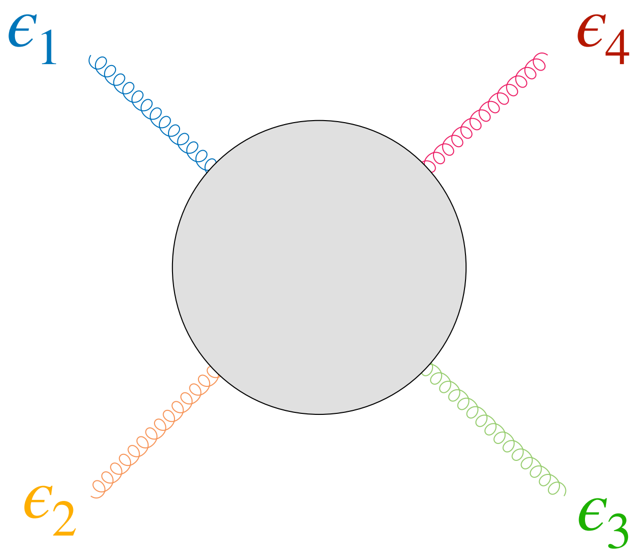
Transversality $\epsilon_i \cdot p_i = 0$



Reference choice $\epsilon_i \cdot p_{i+1} = 0$



Spin



$$A_c = A_c^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

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$$\cancel{p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_3^{\mu_4}}$$

$$\cancel{p_2^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}}$$

$$p_3^{\mu_1}p_1^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$$

$$p_2^{\mu_1}p_1^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$$

$$\cancel{p_3^{\mu_1}p_2^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}}$$

$\vdots$ **81**

$$\cancel{p_3^{\mu_1}p_3^{\mu_2}g^{\mu_3\mu_4}}$$

$$g^{\mu_1\mu_2}p_1^{\mu_3}p_1^{\mu_4}$$

$$g^{\mu_1\mu_3}p_1^{\mu_2}p_2^{\mu_4}$$

$\vdots$ **54**

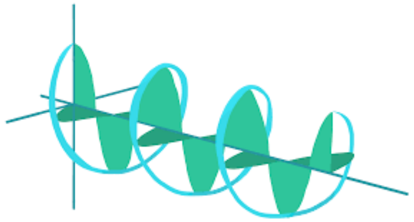
$$g^{\mu_1\mu_2}g^{\mu_3\mu_4}$$

$$g^{\mu_1\mu_3}g^{\mu_2\mu_4}$$

$$g^{\mu_1\mu_4}g^{\mu_2\mu_3}$$

3

Transversality $\epsilon_i \cdot p_i = 0$



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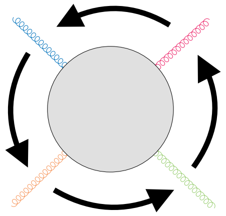
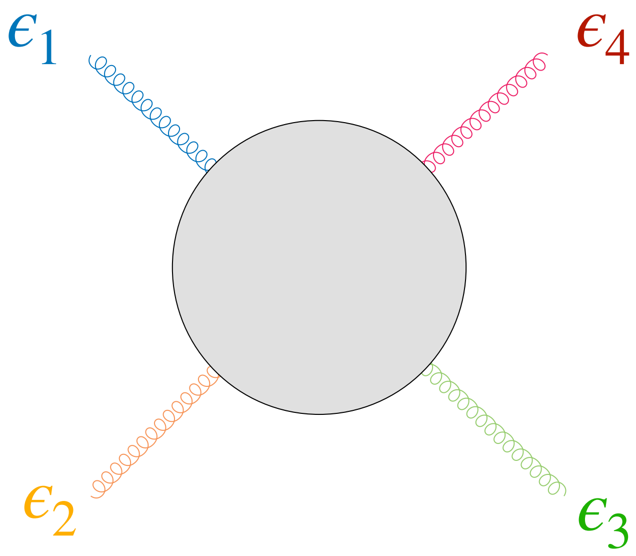


Spin



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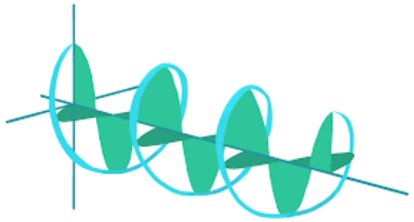
$\cancel{p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_3^{\mu_4}}$	$\cancel{p_3^{\mu_1}p_3^{\mu_2}g^{\mu_3\mu_4}}$	$g^{\mu_1\mu_2}g^{\mu_3\mu_4}$
$\cancel{p_2^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}}$	$\cancel{g^{\mu_1\mu_2}p_1^{\mu_3}p_1^{\mu_4}}$	$g^{\mu_1\mu_3}g^{\mu_2\mu_4}$
$p_3^{\mu_1}p_1^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$	$g^{\mu_1\mu_3}p_1^{\mu_2}p_2^{\mu_4}$	$g^{\mu_1\mu_4}g^{\mu_2\mu_3}$
$\cancel{p_2^{\mu_1}p_1^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}}$	$\vdots$ **54**	**3**
$\cancel{p_3^{\mu_1}p_2^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}}$		
$\vdots$ **81**		

Transversality $\epsilon_i \cdot p_i = 0$



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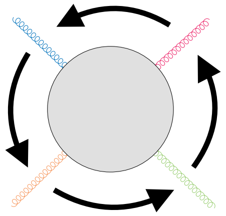
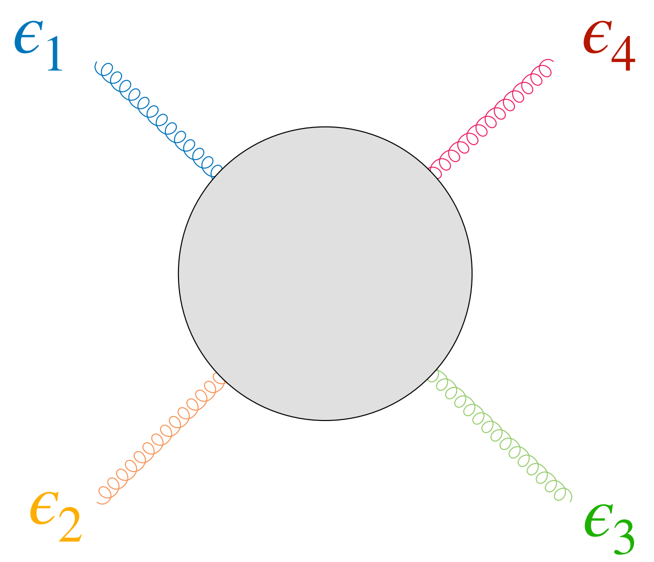


Spin



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$\cancel{p_2^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} p_2^{\mu_4}}$

$p_3^{\mu_1} p_1^{\mu_2} p_1^{\mu_3} p_2^{\mu_4}$

$\cancel{p_2^{\mu_1} p_1^{\mu_2} p_3^{\mu_3} p_2^{\mu_4}}$

$\cancel{p_3^{\mu_1} p_2^{\mu_2} p_1^{\mu_3} p_2^{\mu_4}}$

⋮ **81**

$\cancel{p_3^{\mu_1} p_3^{\mu_2} g^{\mu_3\mu_4}}$

$\cancel{g^{\mu_1\mu_2} p_1^{\mu_3} p_1^{\mu_4}}$

$g^{\mu_1\mu_3} p_1^{\mu_2} p_2^{\mu_4}$

⋮ **54**

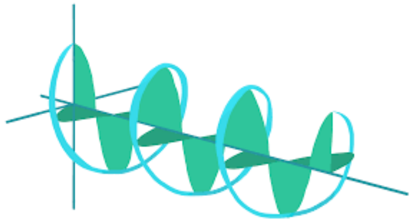
$g^{\mu_1\mu_2} g^{\mu_3\mu_4}$

$g^{\mu_1\mu_3} g^{\mu_2\mu_4}$

$g^{\mu_1\mu_4} g^{\mu_2\mu_3}$

3

Transversality $\epsilon_i \cdot p_i = 0$



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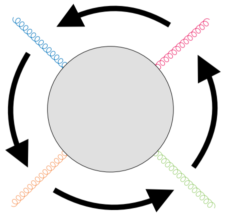
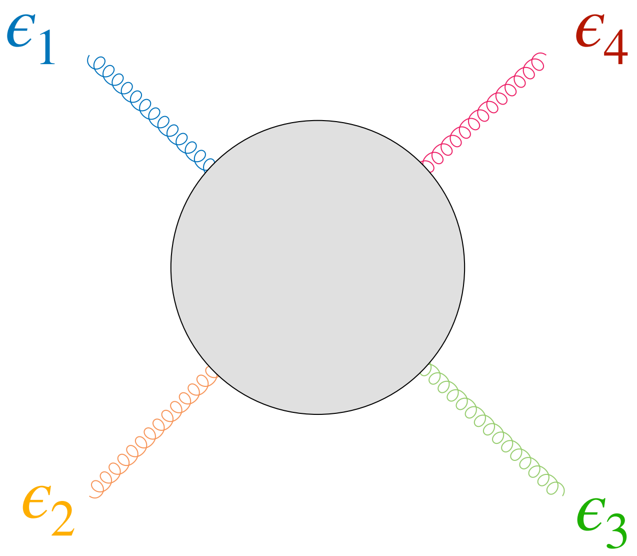


138 → 10

Spin



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$$p_3^{\mu_1}p_1^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$$

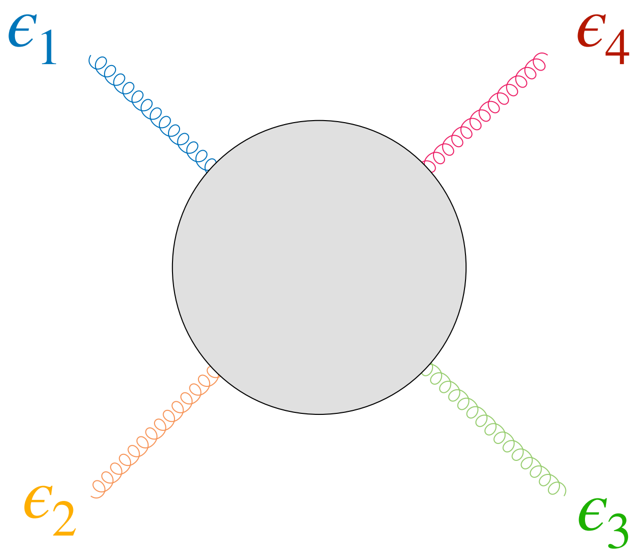
$$p_3^{\mu_1}p_1^{\mu_2}g^{\mu_3\mu_4} \qquad p_3^{\mu_1}p_2^{\mu_4}g^{\mu_2\mu_3} \qquad p_1^{\mu_2}p_1^{\mu_3}g^{\mu_1\mu_4}$$

$$p_3^{\mu_1}p_1^{\mu_3}g^{\mu_2\mu_4} \qquad p_1^{\mu_2}p_2^{\mu_4}g^{\mu_1\mu_3} \qquad p_1^{\mu_3}p_3^{\mu_4}g^{\mu_1\mu_2}$$

$$g^{\mu_1\mu_2}g^{\mu_3\mu_4} \qquad g^{\mu_1\mu_3}g^{\mu_2\mu_4} \qquad g^{\mu_1\mu_4}g^{\mu_2\mu_3}$$

138 → 10

Spin



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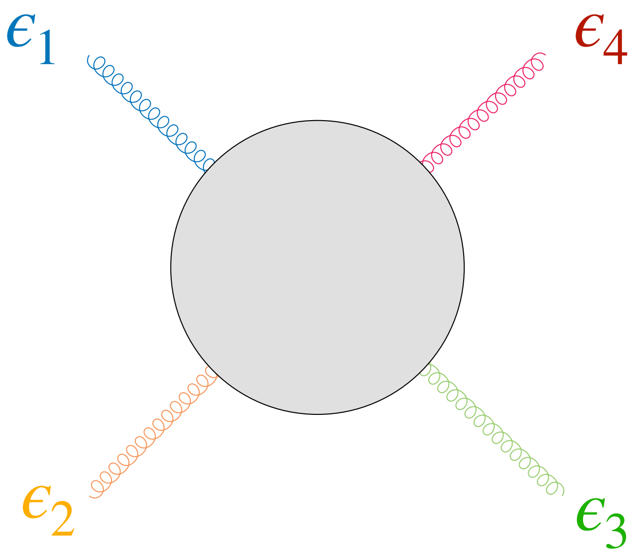
$$p_3^{\mu_1}p_1^{\mu_3}g^{\mu_2\mu_4} \qquad p_1^{\mu_2}p_2^{\mu_4}g^{\mu_1\mu_3} \qquad p_1^{\mu_3}p_3^{\mu_4}g^{\mu_1\mu_2}$$

$$g^{\mu_1\mu_2}g^{\mu_3\mu_4} \qquad g^{\mu_1\mu_3}g^{\mu_2\mu_4} \qquad g^{\mu_1\mu_4}g^{\mu_2\mu_3}$$

138 → 10

Tancredi, Peraro: 1906.03298, 2012.00820

Spin



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$$= \sum_j F_c^j \; T_j^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$$p_3^{\mu_1}p_1^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$$

$$p_3^{\mu_1}p_1^{\mu_2}g^{\mu_3\mu_4} \qquad p_3^{\mu_1}p_2^{\mu_4}g^{\mu_2\mu_3} \qquad p_1^{\mu_2}p_1^{\mu_3}g^{\mu_1\mu_4}$$

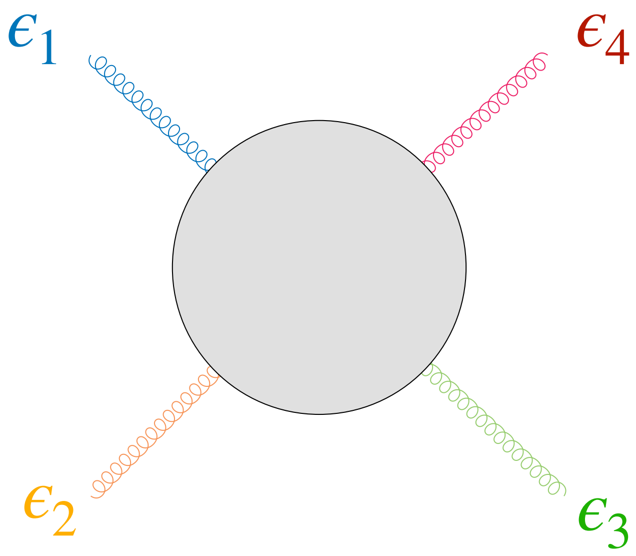
$$p_3^{\mu_1}p_1^{\mu_3}g^{\mu_2\mu_4} \qquad p_1^{\mu_2}p_2^{\mu_4}g^{\mu_1\mu_3} \qquad p_1^{\mu_3}p_3^{\mu_4}g^{\mu_1\mu_2}$$

$$g^{\mu_1\mu_2}g^{\mu_3\mu_4} + g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3}$$

138 → 10

Tancredi, Peraro: 1906.03298, 2012.00820

Spin



$$A_c = A_c^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$
$$= \sum_j F_c^j \; T_j^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$$p_3^{\mu_1}p_1^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$$

$$p_3^{\mu_1}p_1^{\mu_2}g^{\mu_3\mu_4} \qquad p_3^{\mu_1}p_2^{\mu_4}g^{\mu_2\mu_3} \qquad p_1^{\mu_2}p_1^{\mu_3}g^{\mu_1\mu_4}$$

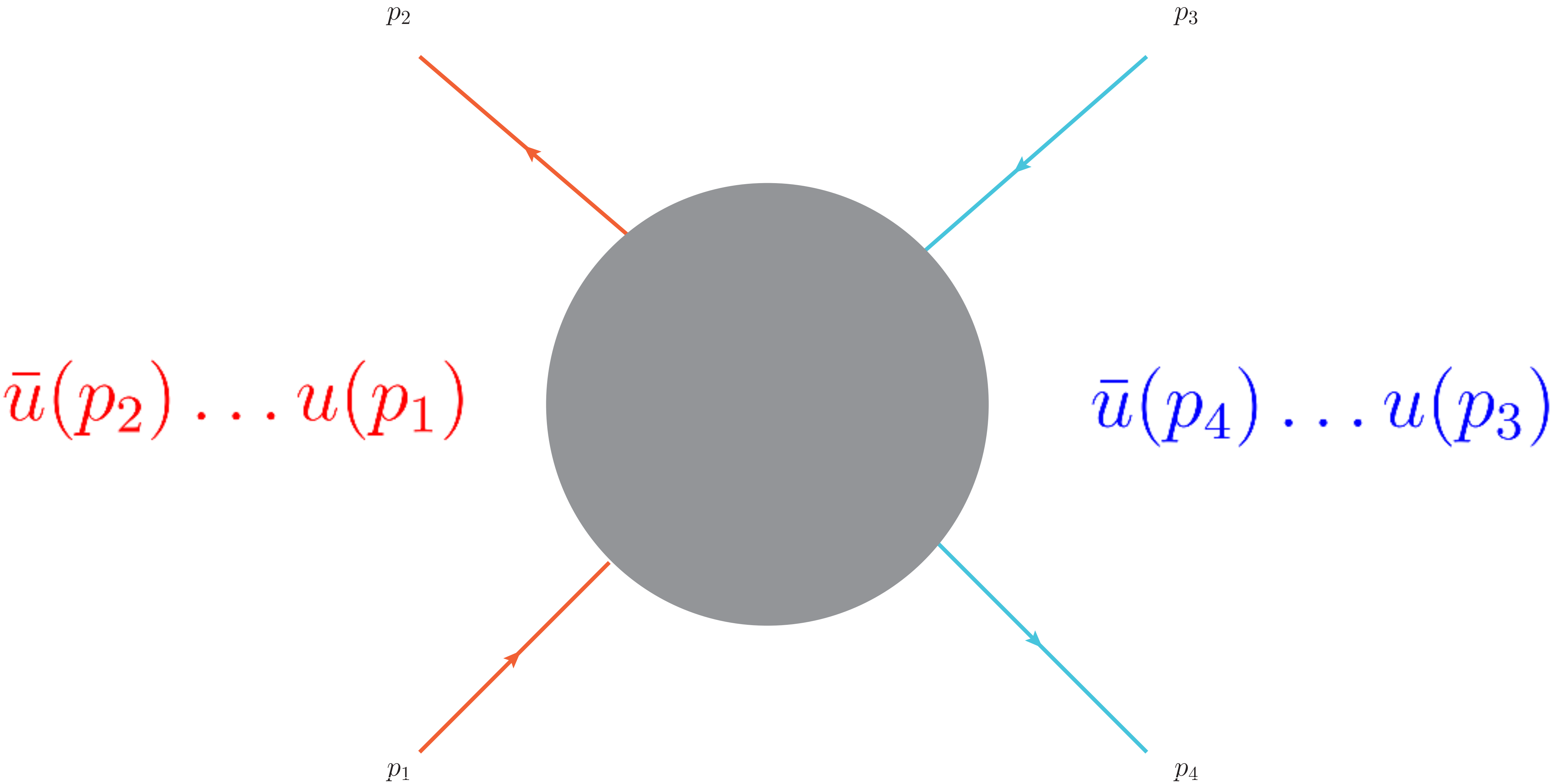
$$p_3^{\mu_1}p_1^{\mu_3}g^{\mu_2\mu_4} \qquad p_1^{\mu_2}p_2^{\mu_4}g^{\mu_1\mu_3} \qquad p_1^{\mu_3}p_3^{\mu_4}g^{\mu_1\mu_2}$$

$$g^{\mu_1\mu_2}g^{\mu_3\mu_4} + g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3}$$

138 → 10 → 8

Tancredi, Peraro: 1906.03298, 2012.00820

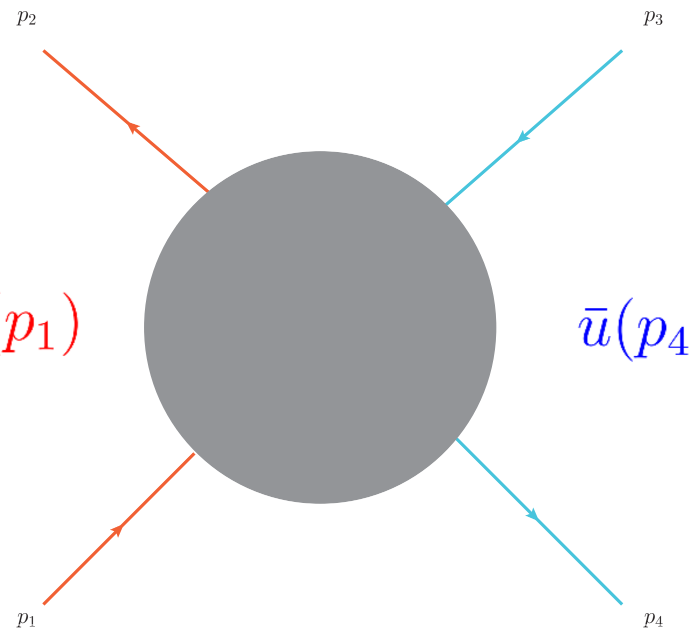
qQ → qQ



$$\bar{u}(p_2)\ldots u(p_1)$$

$$\bar{u}(p_4)\ldots u(p_3)$$

qQ → qQ

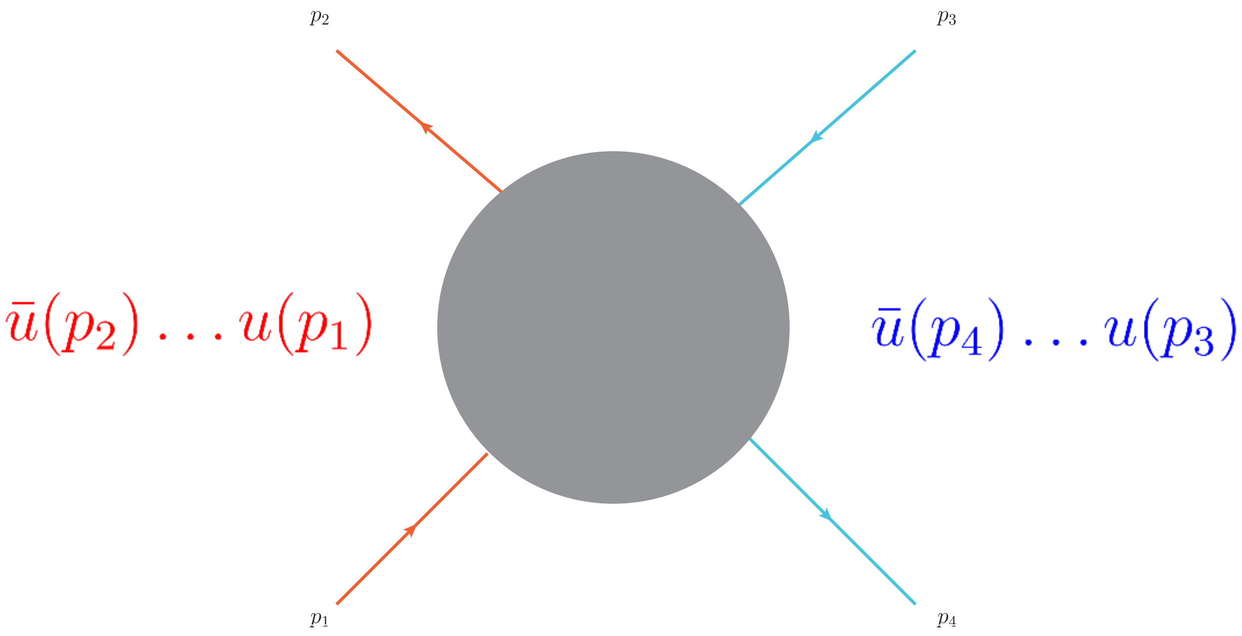


$\bar{u}(p_2)\ldots u(p_1)$

$\bar{u}(p_4)\ldots u(p_3)$

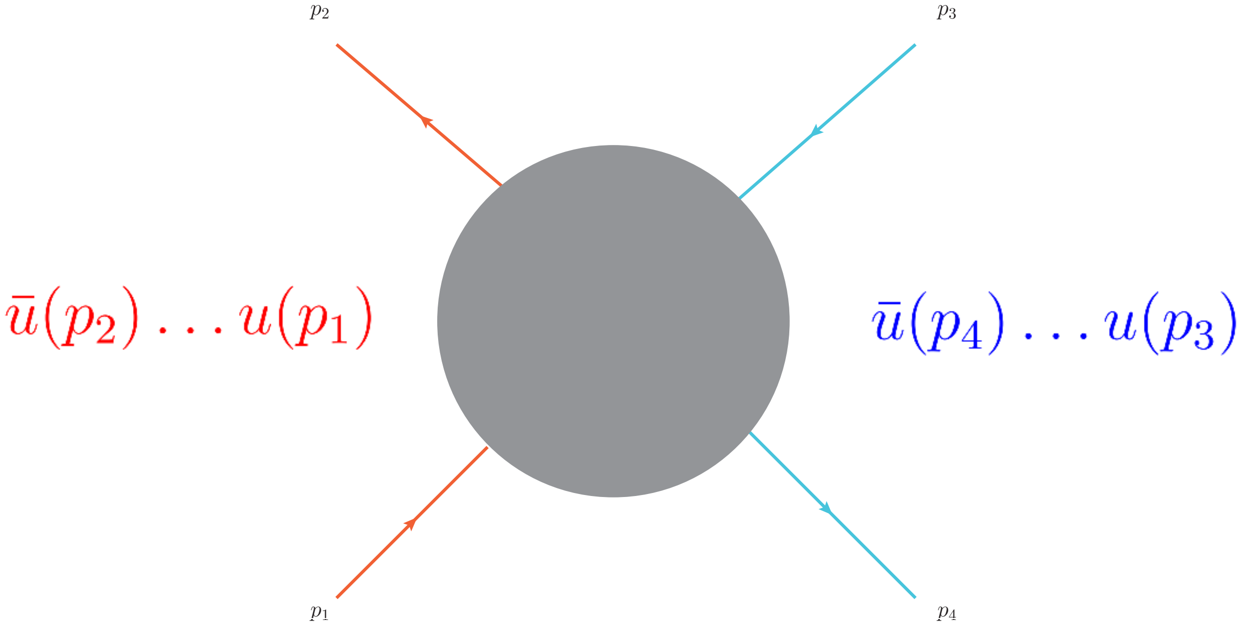
qQ → qQ

$$\boxed{A^X = \sum_{i=1}^{N} F_i T_i^X}$$



qQ → qQ

$$\boxed{A^X = \sum_{i=1}^{N} F_i T_i^X}$$



$$T_1 = \bar{u}(p_2)\gamma_{\mu_1}u(p_1) \;\times\; \bar{u}(p_4)\gamma^{\mu_1}u(p_3)$$

$$T_2 = \bar{u}(p_2)\not{p}_3 u(p_1) \;\times\; \bar{u}(p_4)\not{p}_1 u(p_3)$$

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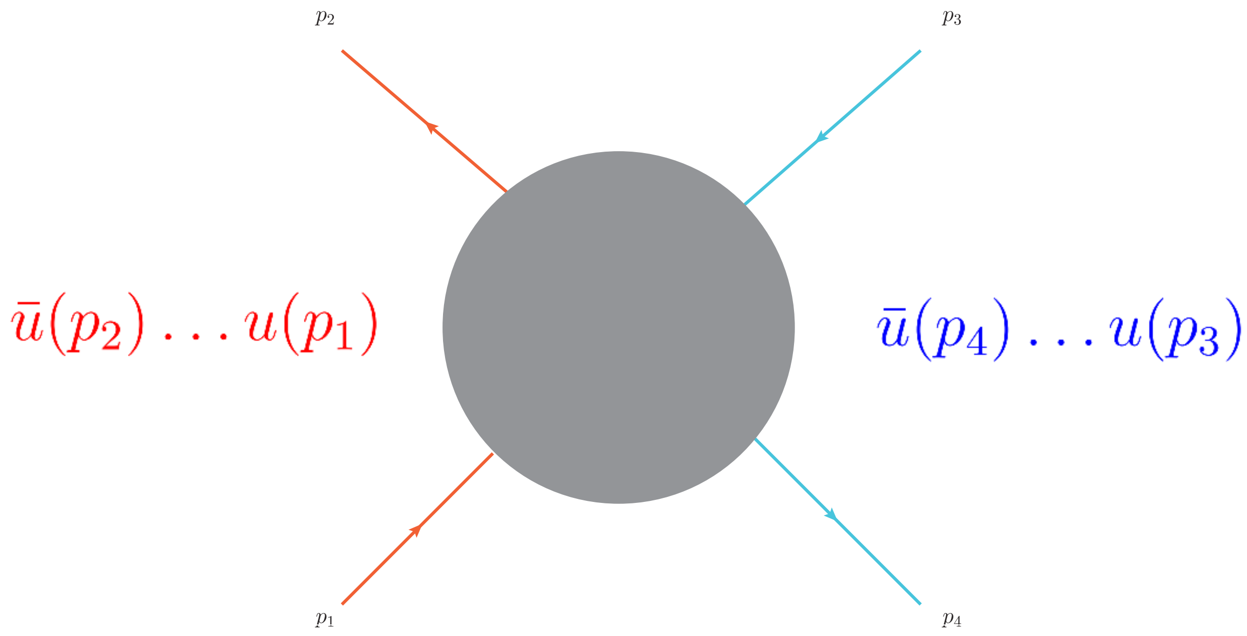

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$$T_4 = \bar{u}(p_2)\gamma_{\mu_1}\not{p}_3\gamma_{\mu_3}u(p_1) \;\times\; \bar{u}(p_4)\gamma^{\mu_1}\not{p}_1\gamma^{\mu_3}u(p_3)$$

qQ → qQ

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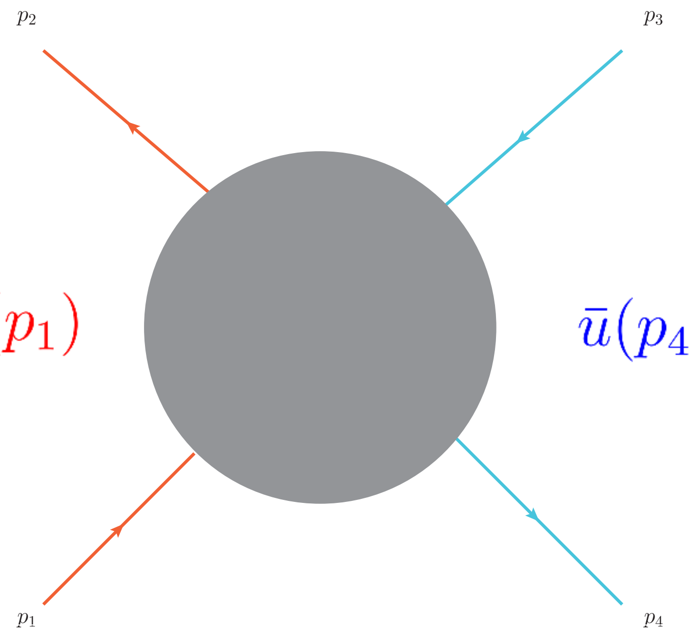

$\bar{u}(p_2)\ldots u(p_1)$

$\bar{u}(p_4)\ldots u(p_3)$

$$T_1 = \bar{u}(p_2)\gamma_{\mu_1}u(p_1) \quad \times \quad \bar{u}(p_4)\gamma^{\mu_1}u(p_3)$$

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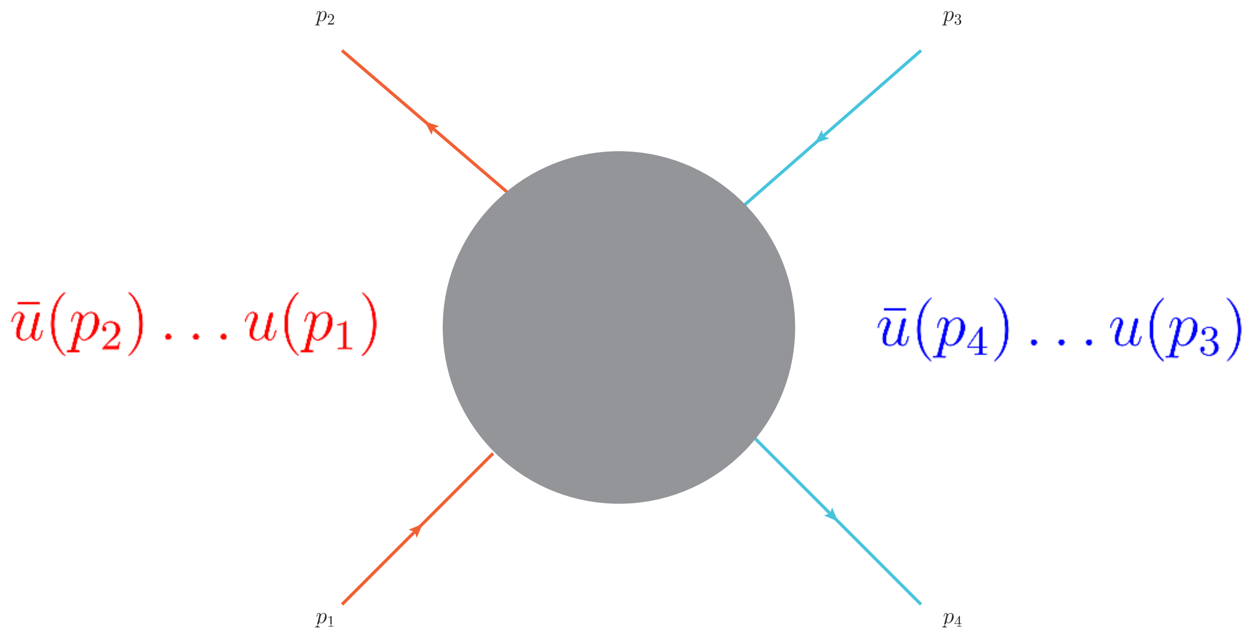
$$T_5 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}u(p_1) \quad \times \quad \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_5}u(p_3)$$

$$T_6 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_3\gamma_{\mu_4}\gamma_{\mu_5}u(p_1) \quad \times \quad \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\not{p}_1\gamma^{\mu_4}\gamma^{\mu_5}u(p_3)$$

qQ → qQ

$$\boxed{A^X = \sum_{i=1}^{N} F_i T_i^X}$$



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$$T_8 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\not{p}_3\gamma_{\mu_5}\gamma_{\mu_6}\gamma_{\mu_7}u(p_1) \;\times\; \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\not{p}_1\gamma^{\mu_5}\gamma^{\mu_6}\gamma^{\mu_7}u(p_3)$$

qQ → qQ

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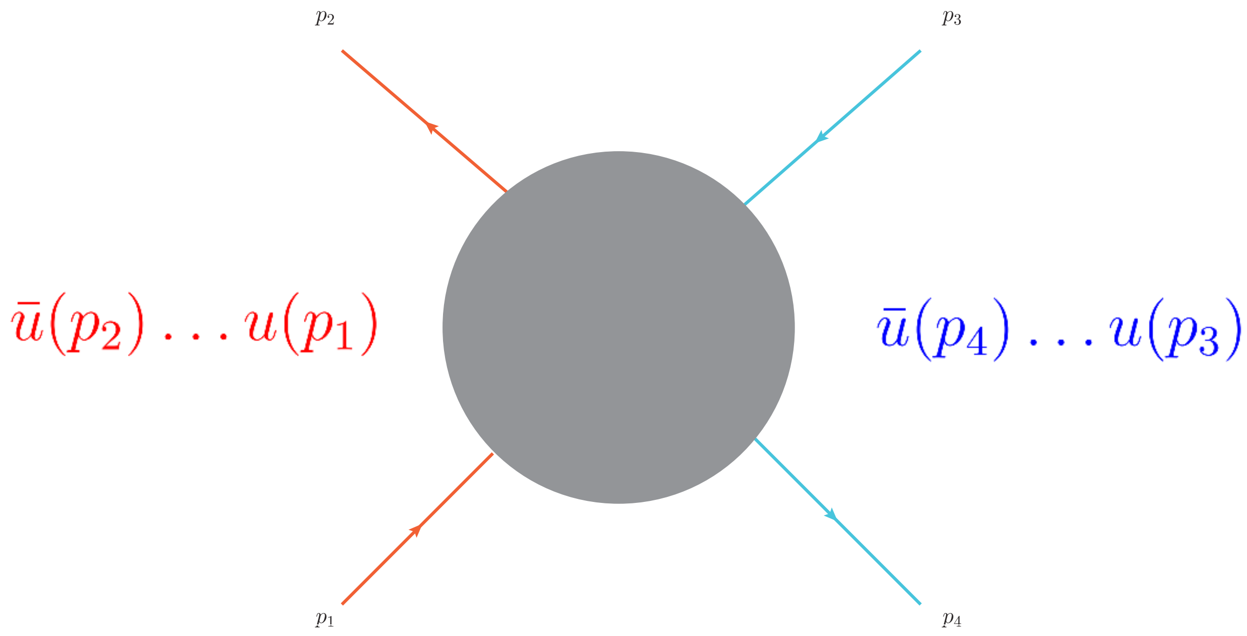

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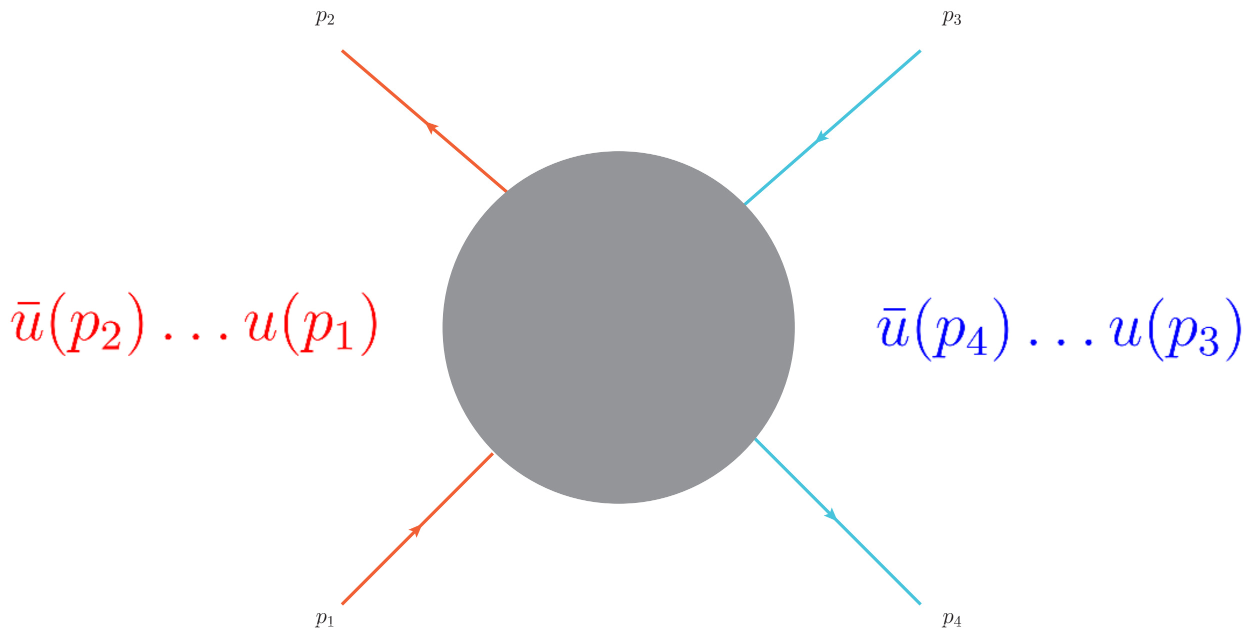

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in d=4

$$T_3 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}u(p_3)$$

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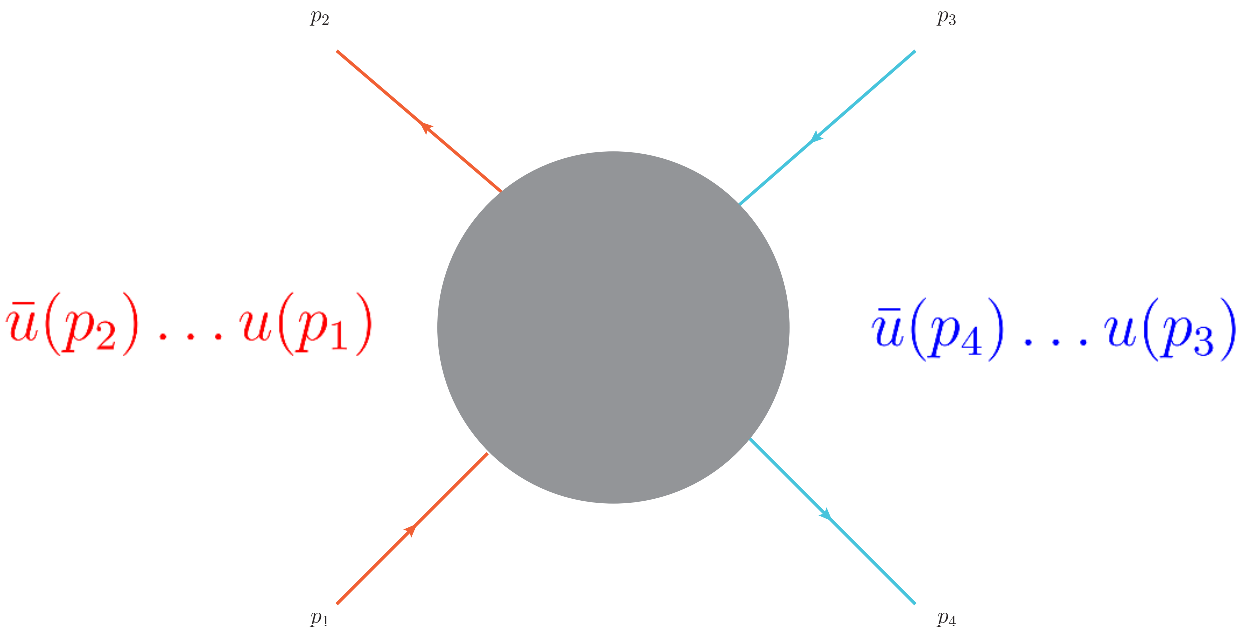
$\vdots$

L. Tancredi, T. Peraro: arXiv:1906.03298, arXiv:2012.00820

qQ → qQ

$$\boxed{A^X = \sum_{i=1}^{N} F_i T_i^X}$$



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$$T_3 = T_3^4 + (d-4)T_3^{-2\epsilon}$$

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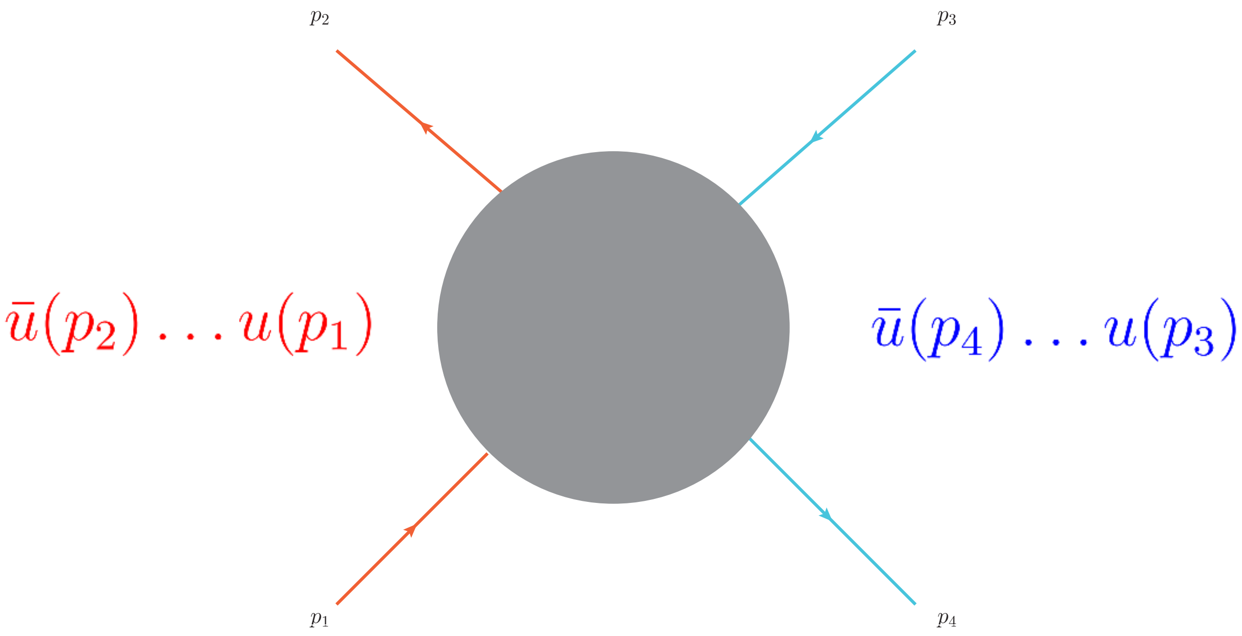

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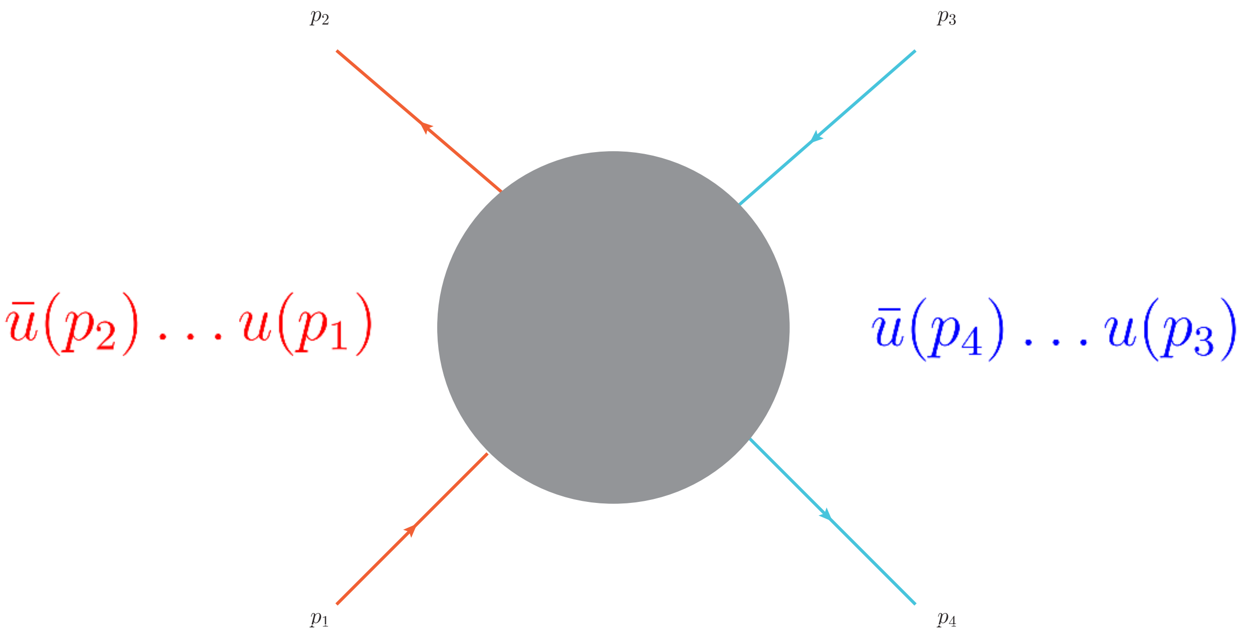

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L. Tancredi, T. Peraro: arXiv:1906.03298, arXiv:2012.00820

qQ → qQ

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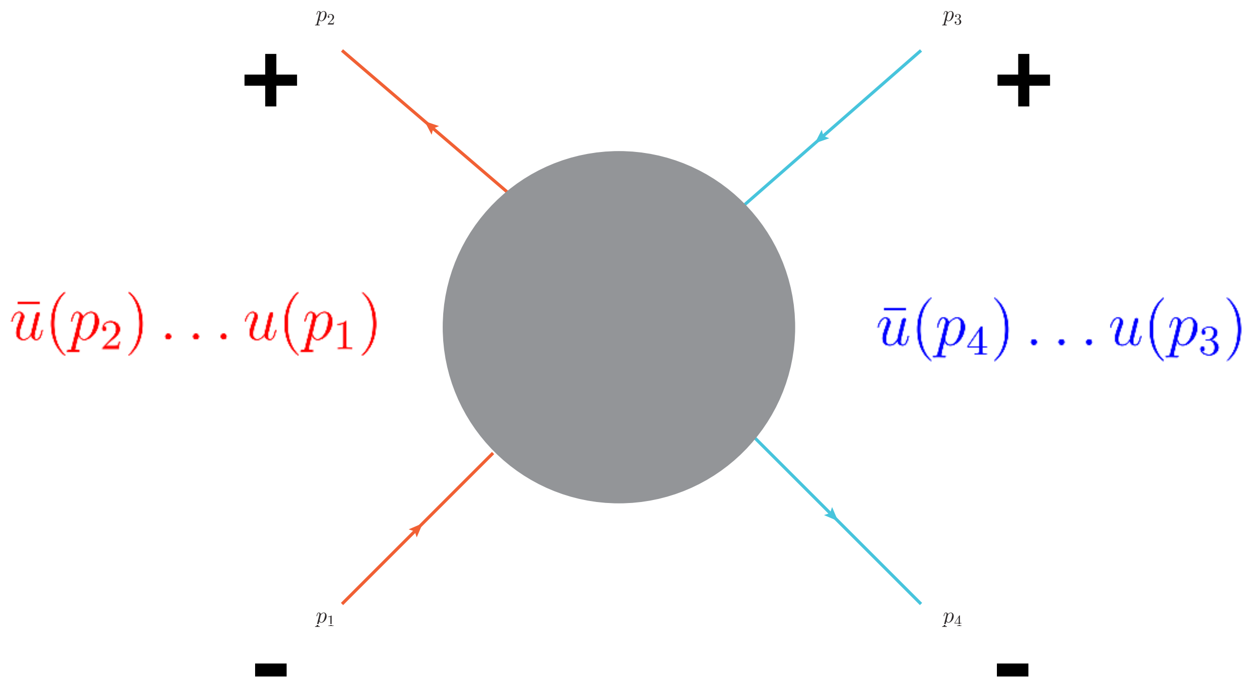

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L. Tancredi, T. Peraro: arXiv:1906.03298, arXiv:2012.00820

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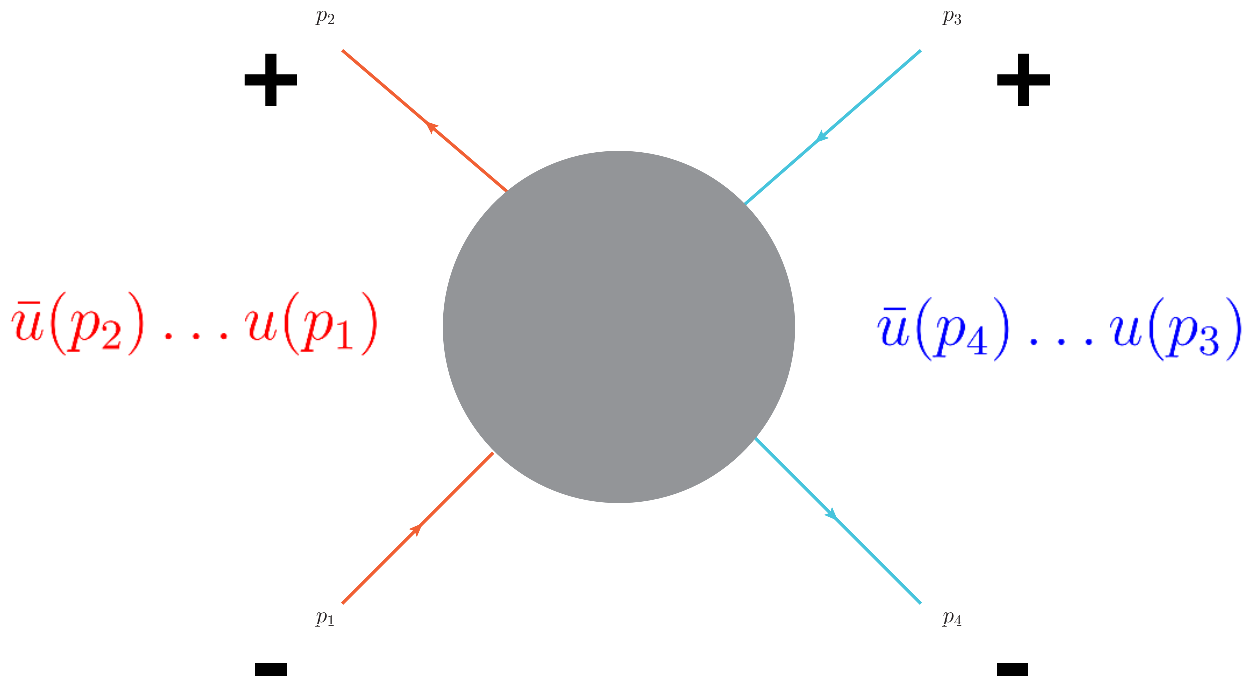

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\overline{T}_3 &= (d-4)T_3^{-2\epsilon}\\
\overline{T}_4 &= (d-4)T_4^{-2\epsilon}\\
\overline{T}_5 &= (d-4)T_5^{-2\epsilon}\\
\overline{T}_6 &= (d-4)T_6^{-2\epsilon}\\
\overline{T}_7 &= (d-4)T_7^{-2\epsilon}\\
\overline{T}_8 &= (d-4)T_8^{-2\epsilon}
\end{aligned}}$$

Orthogonal
&
zero in d=4 !!

L. Tancredi, T. Peraro: arXiv:1906.03298, arXiv:2012.00820

qQ → qQ

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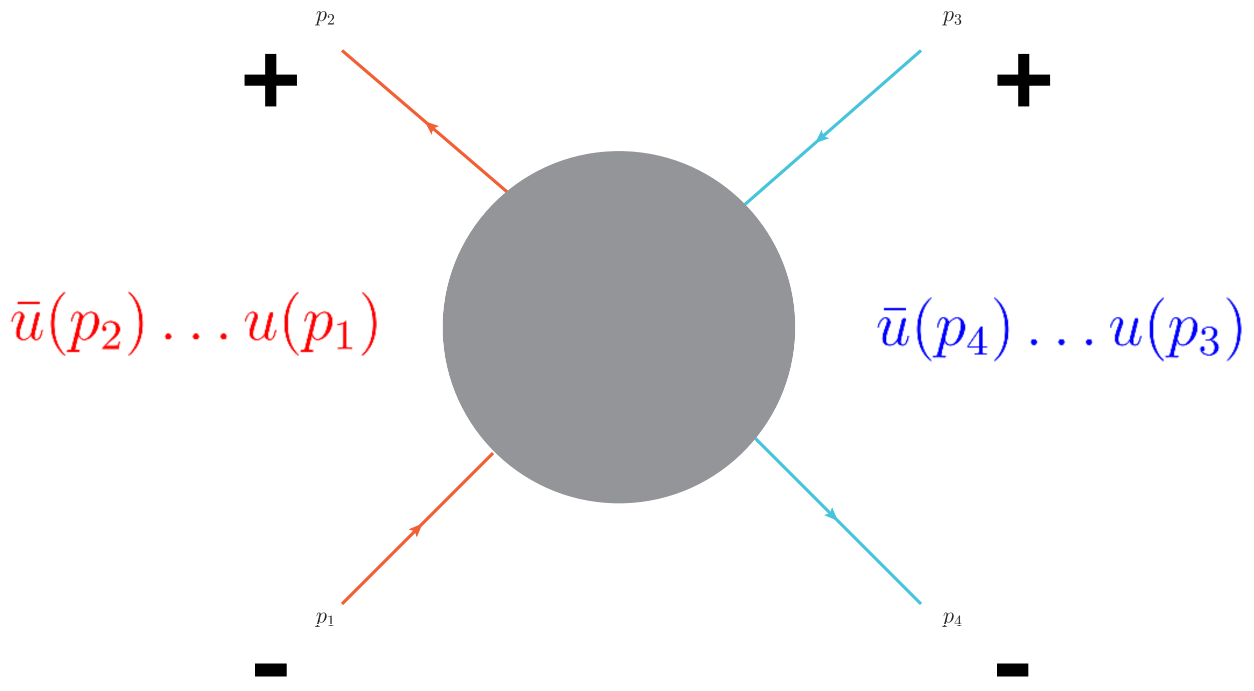

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Orthogonal
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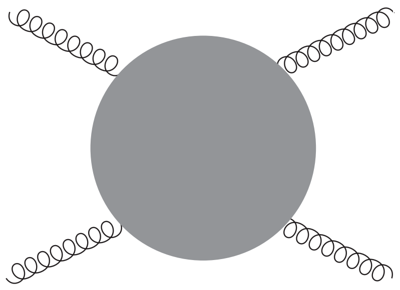
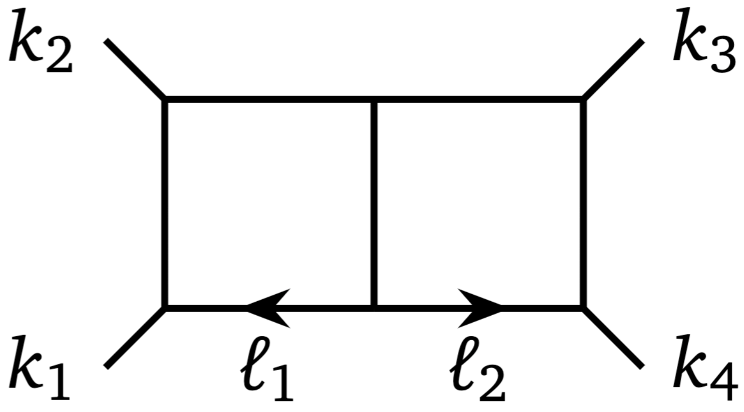
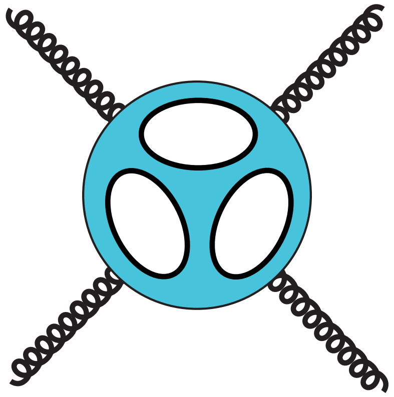


From 138 to 8 tensors!

L. Tancredi, T. Peraro: arXiv:1906.03298, arXiv:2012.00820



rank-2 finite	$G\begin{pmatrix}\ell_1 & 1 & 2\\ 1 & 2 & 4\end{pmatrix}G\begin{pmatrix}\ell_2 & 3 & 4\\ 1 & 2 & 4\end{pmatrix},$	$G\begin{pmatrix}\ell_1 & 1 & 2\\ \ell_2 & 3 & 4\end{pmatrix}$
rank-3 finite	$G\begin{pmatrix}\ell_1 & 1 & 2\end{pmatrix}G\begin{pmatrix}\ell_2 & 3 & 4\\ 1 & 2 & 4\end{pmatrix},$	$G\begin{pmatrix}\ell_2 & 3 & 4\end{pmatrix}G\begin{pmatrix}\ell_1 & 1 & 2\\ 1 & 2 & 4\end{pmatrix},$
	$(\ell_1 - k_1)^2\ G\begin{pmatrix}\ell_2 & 3 & 4\\ 1 & 2 & 4\end{pmatrix},$	$(\ell_2 - k_4)^2\ G\begin{pmatrix}\ell_1 & 1 & 2\\ 1 & 2 & 4\end{pmatrix}$
rank-4 finite	$(\ell_1 - k_1)^2\ G\begin{pmatrix}\ell_2 & 3 & 4\end{pmatrix},$	$(\ell_2 - k_4)^2\ G\begin{pmatrix}\ell_1 & 1 & 2\end{pmatrix},$
	$(\ell_1 - k_1)^2\ (\ell_2 - k_4)^2$	
rank-4 $\mathcal{O}(\epsilon)$	$G\begin{pmatrix}\ell_1 & \ell_2 & 1 & 2 & 4\end{pmatrix}$	

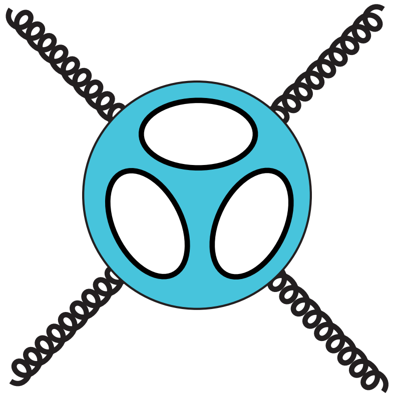


$$= \underbrace{\frac{1}{\epsilon^6}H_{-6} + \frac{1}{\epsilon^5}H_{-5} + \frac{1}{\epsilon^4}H_{-4} + \frac{1}{\epsilon^3}H_{-3} + \frac{1}{\epsilon^2}H_{-2} + \frac{1}{\epsilon}H_{-1}}_{\text{predicted by lower loops!}} + H_0$$

$$\mathscr{H}_{hard} = \mathscr{Z}_{IR}^{-1}(\mu) \cdot \mathscr{H}_{ren}(\mu) \qquad \mathscr{Z}_{IR}(\mu) = \mathbb{P}\exp\left[\int_{\mu}^{\infty} \frac{\mathrm{d}\mu'}{\mu'}\mathbf{\Gamma}(\mu')\right]$$

$$\boxed{\begin{aligned} \mathscr{Z}_{IR}^{-1} &= 1 - \alpha_s \mathbf{I}_1 - \alpha_s^2 \mathbf{I}_2 - \alpha_s^3 \mathbf{I}_3 + \dots \\ \mathscr{H}_{ren} &= \mathscr{H}_0 + \alpha_s \mathscr{H}_1 + \alpha_s^2 \mathscr{H}_2 + \alpha_s^3 \mathscr{H}_3 + \dots \end{aligned}}$$

$$\begin{aligned} \mathscr{H}_{hard} = {} & \mathscr{H}_0 + && \text{tree} \\ & \alpha_s\left(\mathscr{H}_1 - \mathbf{I}_1 \cdot \mathscr{H}_0\right) + && \text{1-loop} \\ & \alpha_s^2\left(\mathscr{H}_2 - \mathbf{I}_1 \cdot \mathscr{H}_1 - \mathbf{I}_2 \cdot \mathscr{H}_0\right) + && \text{2-loop} \\ & \alpha_s^3\left(\mathscr{H}_3 - \mathbf{I}_1 \cdot \mathscr{H}_2 - \mathbf{I}_2 \cdot \mathscr{H}_1 - \mathbf{I}_3 \cdot \mathscr{H}_0\right) && \text{3-loop} \end{aligned}$$



$$= \underbrace{\frac{1}{\epsilon^6}H_{-6} + \frac{1}{\epsilon^5}H_{-5} + \frac{1}{\epsilon^4}H_{-4} + \frac{1}{\epsilon^3}H_{-3} + \frac{1}{\epsilon^2}H_{-2} + \frac{1}{\epsilon}H_{-1}}_{\text{predicted by lower loops!}} + H_0$$

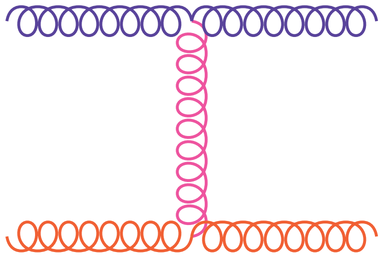
$$\mathscr{H}_{hard} = \mathscr{Z}_{IR}^{-1}(\mu) \cdot \mathscr{H}_{ren}(\mu) \qquad \mathscr{Z}_{IR}(\mu) = \mathbb{P}\exp\left[\int_{\mu}^{\infty} \frac{\mathrm{d}\mu'}{\mu'}\mathbf{\Gamma}(\mu')\right]$$

$$\boxed{\begin{aligned} \mathscr{Z}_{IR}^{-1} &= 1 - \alpha_s \mathbf{I}_1 - \alpha_s^2 \mathbf{I}_2 - \alpha_s^3 \mathbf{I}_3 + \ldots \\ \mathscr{H}_{ren} &= \mathscr{H}_0 + \alpha_s \mathscr{H}_1 + \alpha_s^2 \mathscr{H}_2 + \alpha_s^3 \mathscr{H}_3 + \ldots \end{aligned}}$$

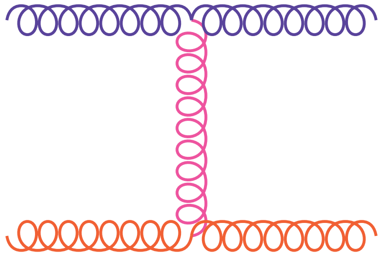
$$\begin{aligned} \mathscr{H}_{hard} = \mathscr{H}_0 &+ && \texttt{tree} \\ &\alpha_s\left(\mathscr{H}_1 - \mathbf{I}_1 \cdot \mathscr{H}_0\right) + && \texttt{1-loop} \\ &\alpha_s^2\left(\mathscr{H}_2 - \mathbf{I}_1 \cdot \mathscr{H}_1 - \mathbf{I}_2 \cdot \mathscr{H}_0\right) + && \texttt{2-loop} \\ &\alpha_s^3\left(\mathscr{H}_3 - \mathbf{I}_1 \cdot \mathscr{H}_2 - \mathbf{I}_2 \cdot \mathscr{H}_1 - \mathbf{I}_3 \cdot \mathscr{H}_0\right) && \texttt{3-loop} \end{aligned}$$

Tree level

Tree level



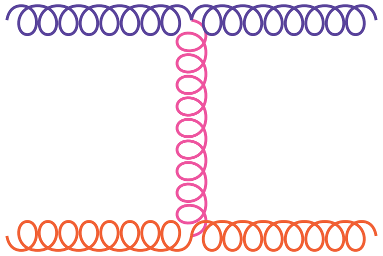
Tree level



→

$$A^{(0)} = 8\pi i \alpha_s \frac{s}{t}\, \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\; f^{abe} f^{cde}$$

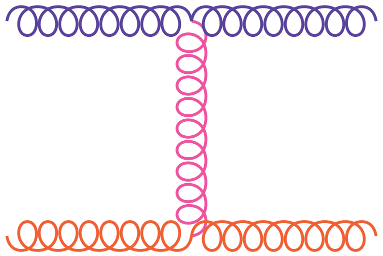
Tree level



$$\Rightarrow \quad A^{(0)} = 8\pi i \alpha_s \, \frac{s}{t} \, \delta_{\lambda_1 \lambda_3} \delta_{\lambda_2 \lambda_4} \, f^{abe} f^{cde}$$

One loop

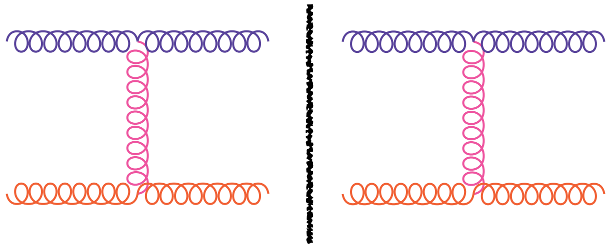
Tree level



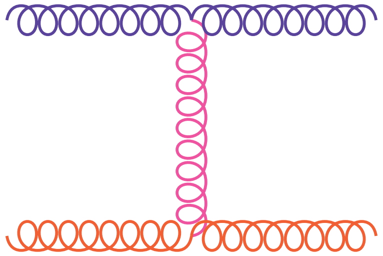
$$A^{(0)} = 8\pi i \alpha_s \, \frac{s}{t} \, \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \, f^{abe} f^{cde}$$

One loop

$$\mathrm{Im}\, A^{(1)} = \int \mathrm{d}\Pi_2$$



Tree level



$$\boxed{A^{(0)} = 8\pi i\alpha_s\,\frac{s}{t}\,\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\;f^{abe}f^{cde}}$$

One loop

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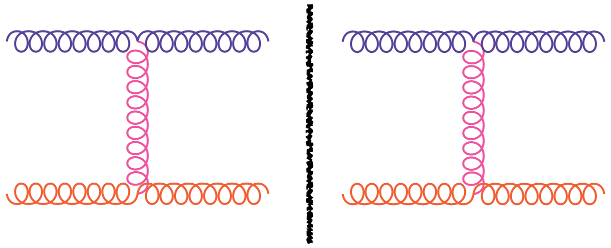
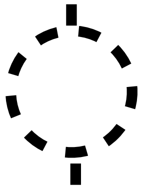
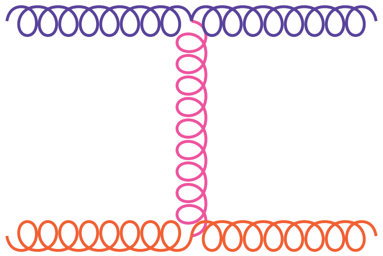


$$= f^{age}f^{beh}f^{cgf}f^{dhf}\;4\alpha_s^2\;s$$



Tree level



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One loop

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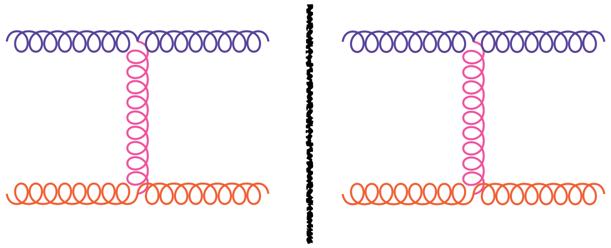
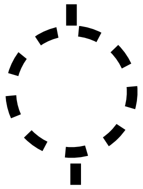
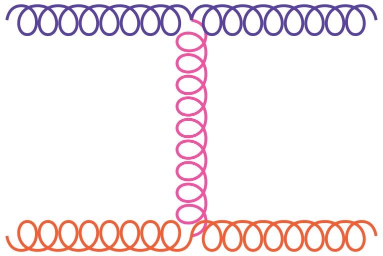


$$= f^{age}f^{beh}f^{cgf}f^{dhf}\; 4\alpha_s^2\, s$$



$$= \int \mathrm{d}^2k_\perp \frac{q_\perp^2}{k_\perp^2(k_\perp - q_\perp)^2}$$

Tree level



→

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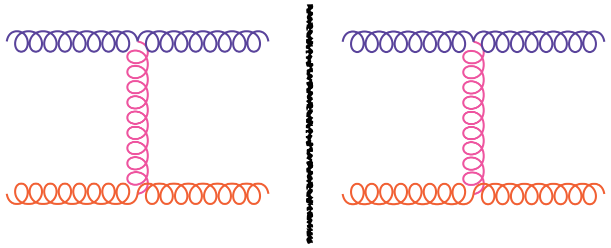
One loop

$$\mathrm{Im}\, A^{(1)} = \int \mathrm{d}\Pi_2$$



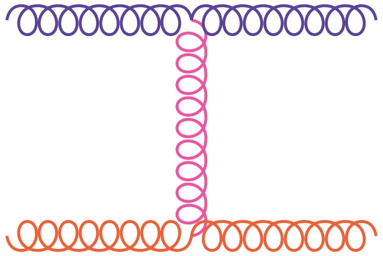
$$= f^{age} f^{beh} f^{cgf} f^{dhf}\; 4\alpha_s^2\, s \; \text{◌}$$

$$\text{◌} = \int \mathrm{d}^2 k_\perp \frac{q_\perp^2}{k_\perp^2 (k_\perp - q_\perp)^2}$$

→

$$A^{(1)} = \frac{N_c \alpha_s}{4\pi^2}\; \text{◌}\; \log\left(\frac{s}{q_\perp^2}\right)\, A^{(0)}$$

Tree level



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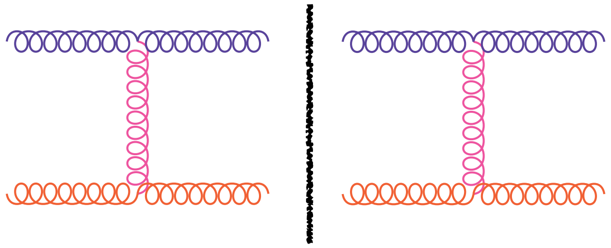
$$A^{(0)} = 8\pi i \alpha_s \, \frac{s}{t} \, \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \, f^{abe} f^{cde}$$

One loop

$$\mathrm{Im}\, A^{(1)} = \int \mathrm{d}\Pi_2$$



$$= f^{age} f^{beh} f^{cgf} f^{dhf} \, 4\alpha_s^2 \, s \, \bigcirc$$

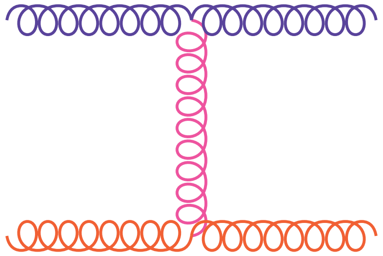
$$\bigcirc = \int \mathrm{d}^2 k_\perp \, \frac{q_\perp^2}{k_\perp^2 (k_\perp - q_\perp)^2}$$

→

$$A^{(1)} = \frac{N_c \alpha_s}{4\pi^2} \bigcirc \log\left(\frac{s}{q_\perp^2}\right) A^{(0)}$$

Two loop

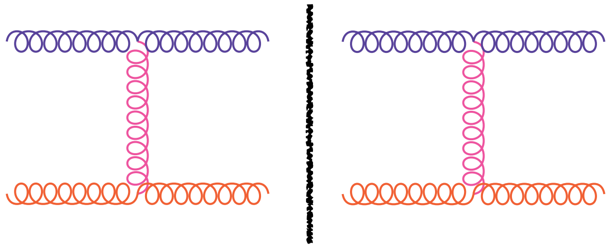
Tree level



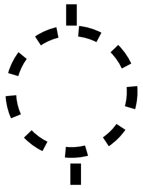
$$\Rightarrow \quad \boxed{A^{(0)} = 8\pi i \alpha_s \, \frac{s}{t} \, \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \, f^{abe} f^{cde}}$$

One loop

$$\mathrm{Im}\, A^{(1)} = \int \mathrm{d}\Pi_2$$

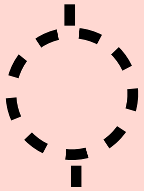


$$= f^{age} f^{beh} f^{cgf} f^{dhf} \; 4\alpha_s^2 \, s$$



$$= \int \mathrm{d}^2 k_\perp \, \frac{q_\perp^2}{k_\perp^2 (k_\perp - q_\perp)^2}$$

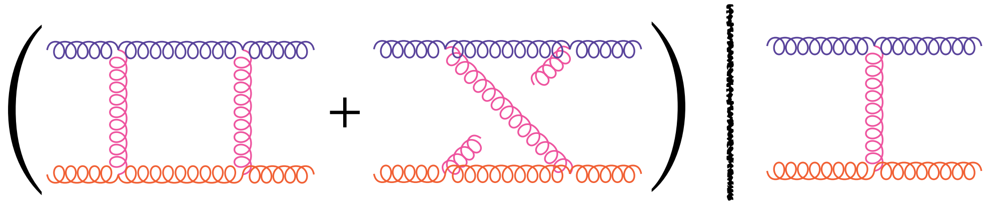
$$\Rightarrow \quad A^{(1)} \; = \; \frac{N_c \alpha_s}{4\pi^2}$$



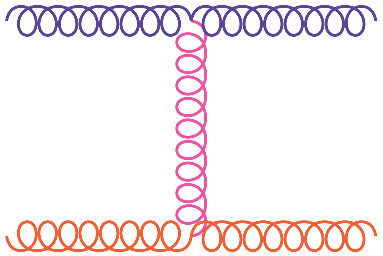
$$\log\left(\frac{s}{q_\perp^2}\right) \; A^{(0)}$$

Two loop

$$\mathrm{Im}\, A^{(2)} = \int d\Pi_2$$



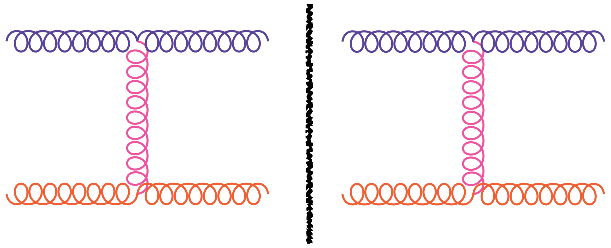
Tree level



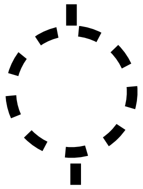
$$\Rightarrow \quad \boxed{A^{(0)} = 8\pi i\alpha_s \frac{s}{t}\,\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\; f^{abe} f^{cde}}$$

One loop

$$\mathrm{Im}\,A^{(1)} = \int \mathrm{d}\Pi_2$$



$$= f^{age} f^{beh} f^{cgf} f^{dhf}\; 4\alpha_s^2\, s$$



$$= \int \mathrm{d}^2 k_\perp \frac{q_\perp^2}{k_\perp^2 (k_\perp - q_\perp)^2}$$

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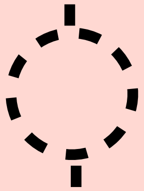
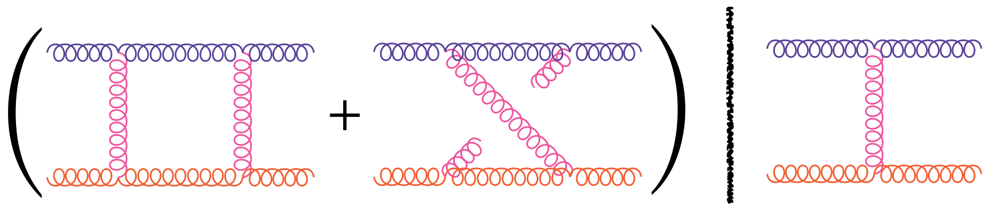


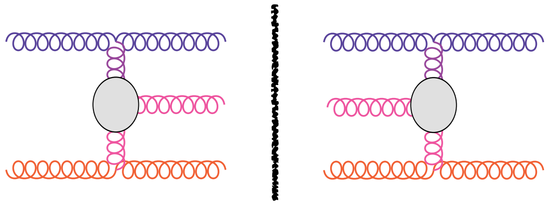
$$\log\left(\frac{s}{q_\perp^2}\right) A^{(0)}$$

Two loop

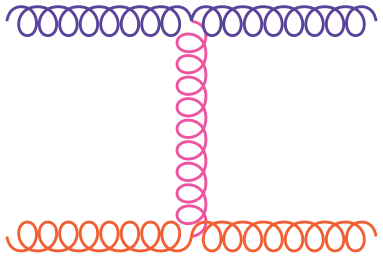
$$\mathrm{Im}\,A^{(2)} = \int d\Pi_2$$



$$+ \int d\Pi_3$$



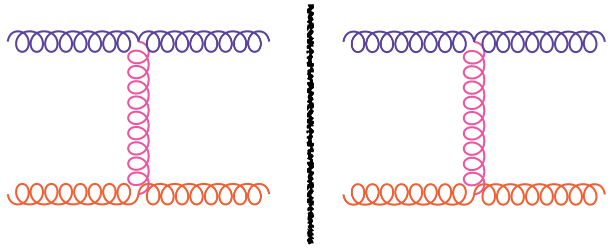
Tree level



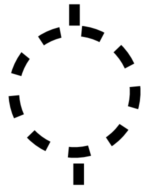
$$A^{(0)} = 8\pi i\alpha_s \frac{s}{t}\,\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\; f^{abe}f^{cde}$$

One loop

$$\mathrm{Im}\,A^{(1)} = \int \mathrm{d}\Pi_2$$

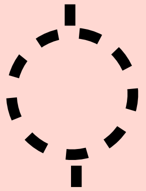


$$= f^{age}f^{beh}f^{cgf}f^{dhf}\;4\alpha_s^2\,s$$



$$= \int \mathrm{d}^2k_\perp \frac{q_\perp^2}{k_\perp^2(k_\perp - q_\perp)^2}$$

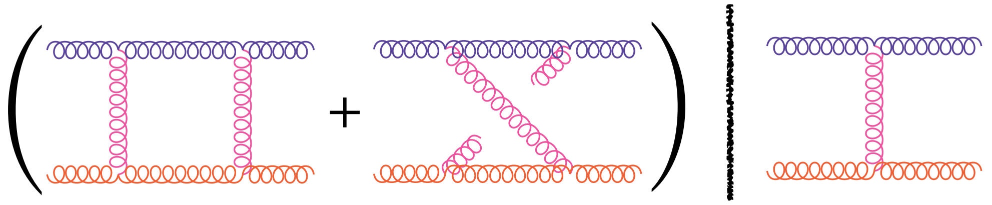
$$A^{(1)} = \frac{N_c\alpha_s}{4\pi^2}$$



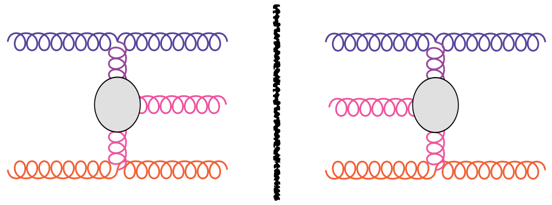
$$\log\left(\frac{s}{q_\perp^2}\right)\; A^{(0)}$$

Two loop

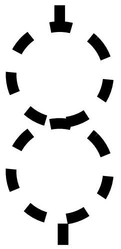
$$\mathrm{Im}\,A^{(2)} = \int d\Pi_2$$



$$+\int d\Pi_3$$

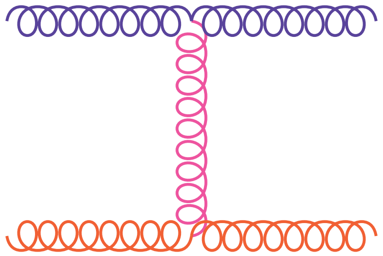


$$= -\frac{\pi}{2}\left(\frac{N_c\alpha_s}{4\pi^2}\right)^2$$



$$\log\left(\frac{s}{q_\perp^2}\right)\; A^{(0)}$$

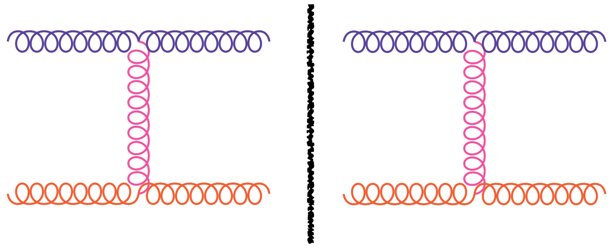
Tree level



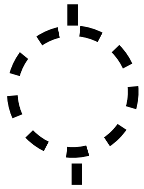
$$A^{(0)} = 8\pi i\alpha_s \, \frac{s}{t} \, \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \, f^{abe} f^{cde}$$

One loop

$\mathrm{Im}\ A^{(1)} = \int \mathrm{d}\Pi_2$

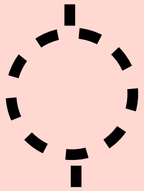


$= f^{age} f^{beh} f^{cgf} f^{dhf} \; 4\alpha_s^2 \, s$



$= \int \mathrm{d}^2 k_\perp \, \frac{q_\perp^2}{k_\perp^2 (k_\perp - q_\perp)^2}$

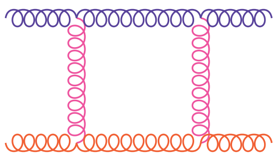
$A^{(1)} \; = \; \frac{N_c \alpha_s}{4\pi^2}$



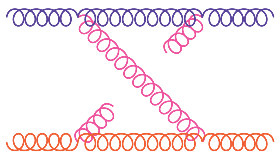
$\log\left(\frac{s}{q_\perp^2}\right) A^{(0)}$

Two loop

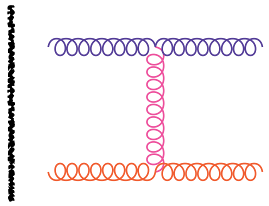
$\mathrm{Im}\ A^{(2)} = \int d\Pi_2 \Bigg($



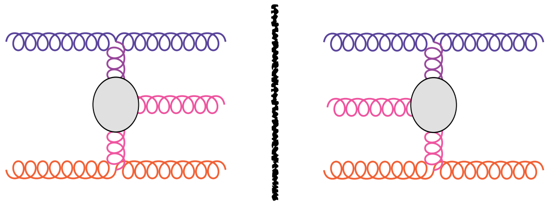
$+$



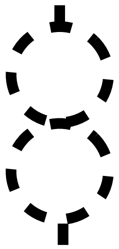
$\Bigg)$



$+ \int d\Pi_3$

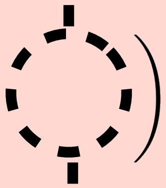


$= -\frac{\pi}{2} \left(\frac{N_c \alpha_s}{4\pi^2}\right)^2$



$\log\left(\frac{s}{q_\perp^2}\right) \; A^{(0)}$

$A^{(2)} = \frac{1}{2} \Bigg(\frac{N_c \alpha_s}{4\pi^2}$



$\Bigg)^2 \log^2\left(\frac{s}{q_\perp^2}\right) A^{(0)}$