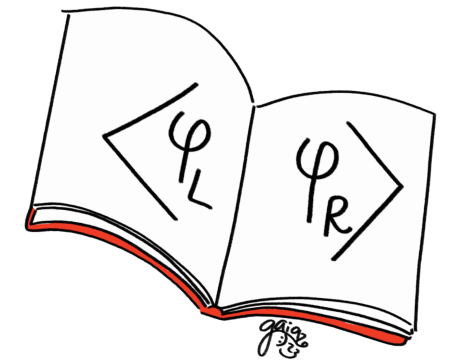
Intersection numbers via $p(z)$-adic expansions



Amplitudes, CERN, August 7, 2023

ALMA MATER STUDIORUM UNIVERSITÀ DI BOLOGNA

Theory and Phenomenology of Fundamental Interactions

UNIVERSITY AND INFN · BOLOGNA

INFN Istituto Nazionale di Fisica Nucleare

erc

Based on **arXiv:2304.14336** in collaboration with **Gaia Fontana**

Tiziano Peraro - University of Bologna and INFN

Loop integrals

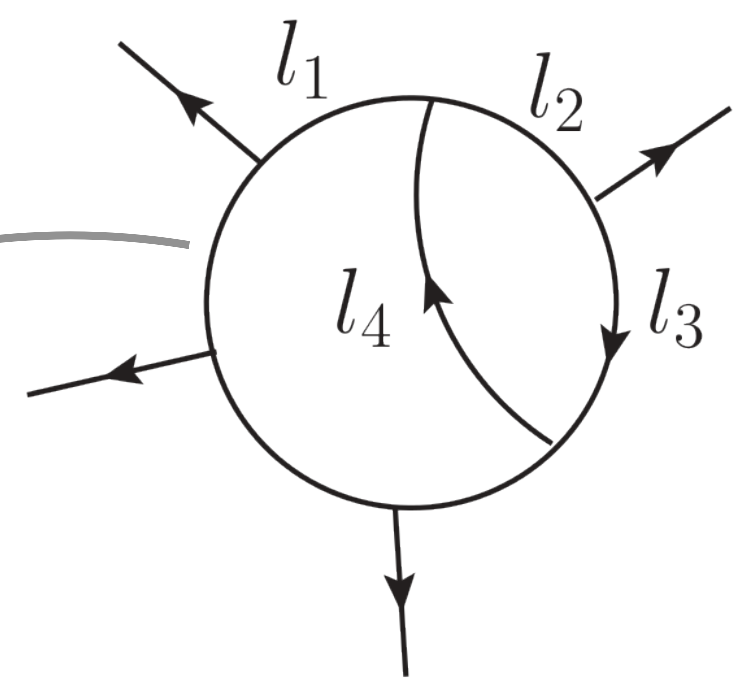
- **Loop integrals** are the **building blocks** of **perturbative QFT**
  - essential for a deeper understanding of **amplitudes**
  - a key component of **phenomenological predictions**

$$I = \int_{-\infty}^{\infty} \left(\prod_{i=1}^{\ell} d^d k_i \right) \frac{1}{D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}}, \qquad \alpha_j \lesseqgtr 0$$

Inverse propagators

$$D_j = l_j^2 - m_j^2$$

+ auxiliaries



Integral decomposition

Chetyrkin, Tkachov (1981), Laporta (2000)

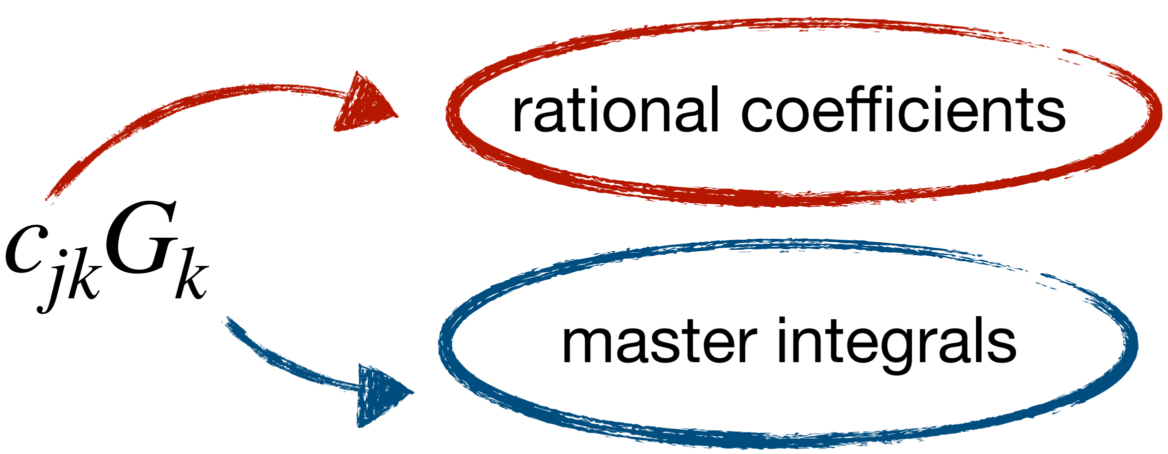
- Feynman integrals obey linear relations, e.g. **IBPs**

$$\int \left(\prod_j d^d k_j\right) \frac{\partial}{\partial k_j^\mu} v^\mu \frac{1}{D_1^{\alpha_1} D_2^{\alpha_2} \cdots} = 0, \qquad v^\mu \in \{p_i^\mu, k_i^\mu\}$$

- Very **large** and sparse linear system

- Solution = **reduction** into a **basis** of linearly independent **master integrals (MIs)** $\{G_j\} \subset \{I_j\}$

$$I_j = \sum c_{jk} G_k$$



IBP reduction

- An **essential** ingredient of higher-order computations…
- …but one of the **main bottlenecks**

Recent improvements:

Finite fields and rational reconstruction
[Kant (2014), von Manteuffel, Schabinger (2014), T.P. (2016)]

- **reconstruct** results from numerical evaluations modulo a prime
- delay reconstruction to amplitude coefficients
- pushed state of art of modern amplitude calculations

Symbolic solutions

- **reduction rules** for **symbolic exponents**
- hybrid methods (e.g. syzygy eq.s + Laporta) (see e.g. **Gluza, Kosower (2010)**)

See also recent developments in:

Fire, Kira+FireFly, FiniteFlow, NeatIBP, Blade, …

“Direct” decomposition methods

Goal
Seek a more “direct” way of projecting out the coefficients of integral decomposition

bypass the **solution** of large **systems**

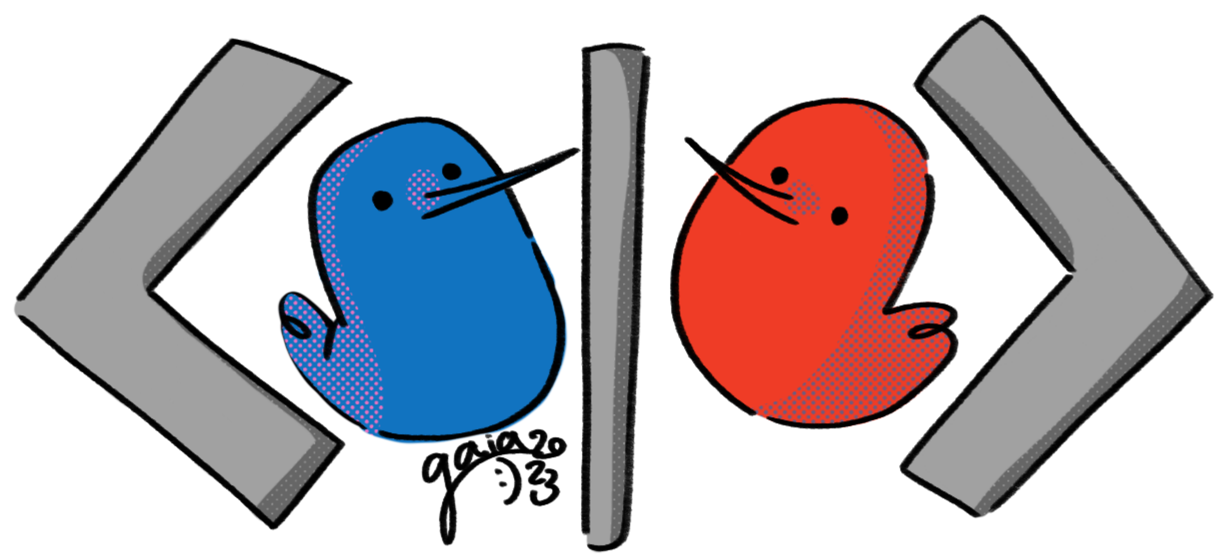
potential for performance gain

but not quite there yet…
¯_(ツ)_/¯

investigate the **vector-space** structure obeyed by **loop integrals** in a family

connections with new areas of **mathematics** (**intersection theory**)

Intersection theory



The main idea

- Reinterpret Feynman integrals as elements of a **vector space**
- **Master integrals (MIs)** form a **basis** w.r.t. IBP relations

$$I = \sum_j c_j G_j$$

- Define **scalar products** (**intersection numbers**)
  - they must be consistent with IBPs!
- **Project** integrals into their components c_j w.r.t. the **basis**

$$c_j = \sum_k (G^{-1})_{jk} (G_k \cdot I) \qquad \text{with} \quad G_{jk} \equiv G_j \cdot G_k$$

The vector space

Mizera (2018), Mastrolia, Mizera (2019), Frellesvig, Gasparotto, Mandal, Mastrolia, Mattiazzi, Mizera (2019)

- We consider **integrals** (or **right integrals**)

$$|\varphi_R\rangle = \int d z_1 \cdots d z_n \, \frac{1}{u(\mathbf{z})} \, \varphi_R(\mathbf{z})$$

- and **dual integrals** (or **left integrals**)

$$\langle \varphi_L | = \int d z_1 \cdots d z_n \, u(\mathbf{z}) \, \varphi_L(\mathbf{z})$$

rational functions

$$u(\mathbf{z}) = \prod_j B_j(\mathbf{z})^{\gamma_j}$$

- *multivalued function*
- *regulates the singularities of* $\varphi_{R,L}$
- B_j *= polynomials,* γ_j *= generic exponents*

- consider a set a integrals with same $u(\mathbf{z})$ and integration domain, but different $\varphi(\mathbf{z})$ (**integral family**)

IBPs

- We assume regulated integrands to vanish at integration boundary

$$\sum_{j=1}^{n} \int d\,z_1 \cdots d\,z_n\, \partial_{z_j}\!\left(\frac{1}{u}\,\xi_j^{(R)}\right) = 0, \qquad \sum_{j=1}^{n} \int d\,z_1 \cdots d\,z_n\, \partial_{z_j}\!\left(u\,\xi_j^{(L)}\right) = 0$$

$$\sum_{j=1}^{n} \left|\left(\partial_{z_j} - (\partial_{z_j} u)/u\right)\xi_j^{(R)}\right\rangle = 0, \qquad \sum_{j=1}^{n} \left\langle\left(\partial_{z_j} + (\partial_{z_j} u)/u\right)\xi_j^{(L)}\right| = 0$$

- we can formally define the vector space via these equations*
- reduction to **bases** $\{\,|e_j^{(R)}\rangle\}_{j=1}^{\nu}$ and $\{\langle e_j^{(L)}|\,\}_{j=1}^{\nu}$ of **MIs** independent modulo IBPs (ν = dimension of vector space)

* Additional identities, such as some symmetry relations, may exist but are formally not taken into account at this stage. They can be easily identified and implemented after the decomposition via IBPs.

Intersection numbers

- **Intersection numbers** are **rational scalar products** btw integrals and their duals

$$\langle \varphi_L | \varphi_R \rangle$$

- They project out integrals into their IBP decomposition

$$|\varphi_R\rangle = \sum_{i=1}^{\nu} c_i^{(R)} |e_i^{(R)}\rangle$$

rational coefficients

$$c_i^{(R)} = \sum_{j=1}^{\nu} (\mathbf{C}^{-1})_{ij} \langle e_j^{(L)} | \varphi_R \rangle$$

metric

$$\mathbf{C}_{ij} \equiv \langle e_i^{(L)} | e_j^{(R)} \rangle$$

master integrals

$$\{ |e_j^{(R)}\rangle \}_{j=1}^{\nu}$$

similar equations for **dual integrals** $\langle \varphi_L |$

Computing intersection numbers

Univariate case

- **One-fold** integrals $\quad |\varphi_R\rangle = \int d\,z\,\frac{1}{u(z)}\,\varphi_R(z), \qquad \langle\varphi_L| = \int d\,z\,u(z)\,\varphi_L(z)$

- Intersection numbers: $\quad \langle\varphi_L|\varphi_R\rangle = \sum_{p\in\mathscr{P}_\omega} \text{Res}_{z=p}(\psi\,\varphi_R)$

  - ψ **local solution** of the **DE** $\quad (\partial_z+\omega)\psi = \varphi_L, \quad \omega \equiv (\partial_z u)/u$
  - $\mathscr{P}_\omega = \{\, z \,|\, z \text{ is a pole of } \omega \,\} \bigcup \{\infty\}$
  - solution for ψ as **Laurent series** around each **pole**

$$\psi = \sum_{i=min}^{max} c_i\,(z-p)^i + O\left((z-p)^{max+1}\right)$$

find c_i via linear algebra

Computing intersection numbers

Multivariate case

- **Recursive** algorithm $\quad |\varphi_R\rangle = \int dz_n \, |\varphi_R\rangle_{n-1}, \qquad \langle\varphi_L| = \int dz_n \, \langle\varphi_R|_{n-1}$

  $(n-1)$-fold integrals

- Similar procedure:
  - **local solutions** of **DEs** around **poles** of rational functions (system of DEs)
  - sums of **residues** at poles
  - depends on $(n-1)$-variate intersection numbers $\langle\varphi_L|\varphi_R\rangle_{n-1}$ and decompositions of $(n-1)$-fold integrals

$$|\varphi_R\rangle_{n-1} = \sum_{j=1}^{\nu_{(n-1)}} \varphi_{R,j} \, |e_j^{(R)}\rangle_{n-1}$$

$(n-1)$-fold basis of MIs

Application to loop integrals

- Use e.g. the **Baikov representation**

$$I = \int \left(\prod_{i=1}^{\ell} d^d k_i \right) \frac{1}{D_1^{\nu_1} \cdots D_n^{\nu_n}} = K \int dz_1 \cdots dz_n \, B(\mathbf{z})^{\gamma} \frac{1}{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}$$

$B =$ Baikov polynomial, $\quad \gamma = (d - \ell - e - 1)/2$

- Identifications: $\quad I \sim |\varphi\rangle = \int d^n z \, \frac{1}{u} \, \varphi$

$$\varphi = \frac{1}{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}, \qquad u = B^{-\gamma} \prod_{j=1}^{n} z_j^{\rho_j}$$

analytic regulators of $z_j \to 0$ singularities

take $\rho_j \to 0$ limit after the decomposition

Pros/Cons

:) Pros

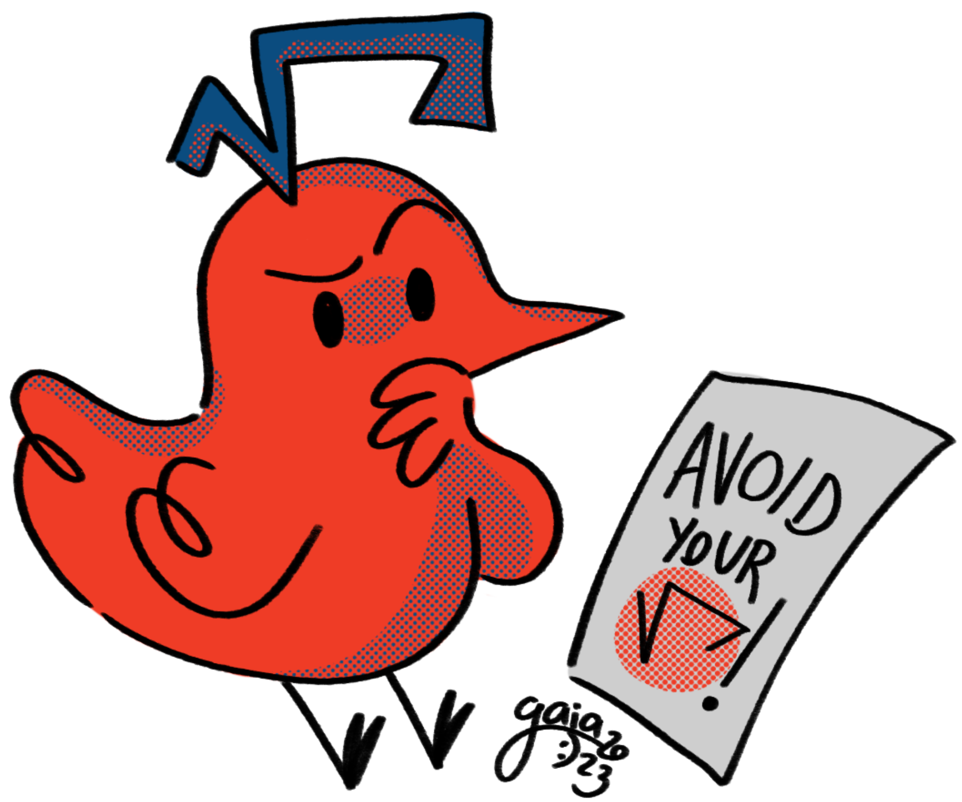
- makes vector-space structure of families of loop integrals manifest
- **"direct"** decomposition
  (not a byproduct of solving a huge system of identities)

:(Cons

- **irrational** contributions in **intermediate** stages of calculation*
  - algebraic bottleneck
  - no easy implementation over **finite fields**
- need of **analytic regulators** ρ_j

* A way out is a reduction to simple poles **[Weinzierl (2020)]** but requires non-trivial sequences of changes of bases and integral transformations

$p(z)$-adic expansions



Polynomial expansions

G. Fontana, T.P. (2023)

$p(z)$-adic expansion

Investigates functions close to roots of polynomials $p(z)$ without irrational operations nor knowing their explicit location

$$f(z) = \sum_{i=min}^{max} c_i(z)\, p^i(z) + \mathcal{O}\left(p(z)^{max+1}\right)$$

rational function

(prime) polynomial over Q

polynomial coefficients $c_i(z)$

$$c_i(z) = \sum_{j=0}^{deg\ p-1} c_{ij}\, z^j$$

It can be obtained via repeated polynomial divisions

$p(z)$-adic expansions and residues

$p(z)$-adic expansion

$$f(z) = \sum_{i=min}^{max} c_i(z)\, p^i(z) + \mathcal{O}\left(p(z)^{max+1}\right) \qquad c_i(z) = \sum_{j=0}^{deg\ p-1} c_{ij}\, z^j$$

Univariate global residue theorem (generalization)

Taking a sum of residues of $f(z)$ at the roots of $p(z)$ from their $p(z)$-adic expansion is trivial and does not require knowing their location

$$\text{Res}_{p(z)}\left(f(z)\right) \equiv \sum_{y\,|\,p(y)=0} \text{Res}_{z=y}\left(f(z)\right) = \frac{c_{-1,deg\ p-1}}{l_c}$$

$l_c \equiv$ leading coefficient of $p(z)$

Back to intersection numbers

Sum over "denominator factors" rather than "poles"

$$\langle \varphi_L | \varphi_R \rangle = \sum_{p(z) \in \mathscr{P}_\omega[z]} \text{Res}_{p(z)} \left(\psi \, \varphi_R \right)$$

- $\mathscr{P}_\omega[z] = \{$factors of the denominator of $\omega\} \bigcup \{\infty\}$
- ψ **local solution** of the **DE** $\quad (\partial_z + \omega)\psi = \varphi_L, \quad \omega \equiv (\partial_z u)/u$
- solution for ψ as $p(z)$**-adic series expasion** around each **factor**

$$\psi = \sum_{i=min}^{max} \sum_{j=0}^{deg\ p(z)-1} c_{ij}\, z^j\, p^i(z) + \mathcal{O}\left(p(z)^{max+1}\right)$$

find c_{ij} via linear algebra

… and similar for multivariate case

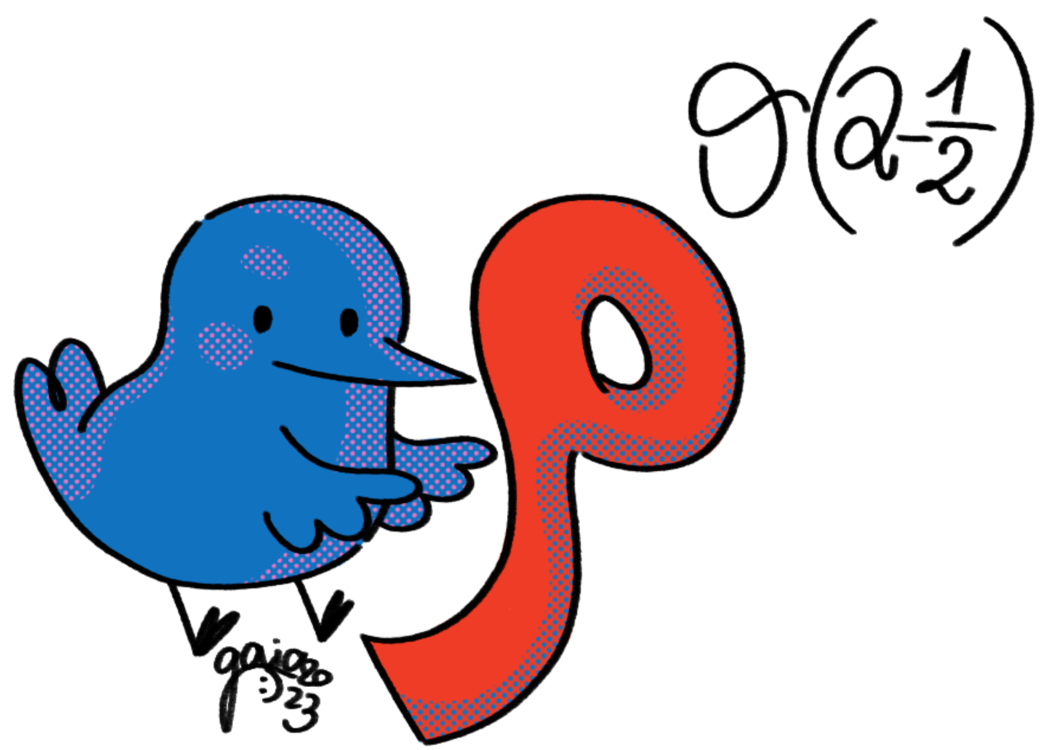
Int. numbers via $p(z)$-adic expansion

- The $p(z)$**-adic expansion** method
  - yields a fully **rational algorithm** for computing int. numbers
  - no integral transformation or change of basis needed

- Proof-of-concept **implementation** over **finite fields**
  - using **FiniteFlow** **[T.P. (2019)]** (in Mathematica)
  - delay full kinematic reconstruction
  - most operations recast as linear algebra problems

➡ **More details on Gaia's poster!**

Dual integrals and analytic regulators



Analytic regulators

Analytic regulators ρ_j, regulate integrands $\varphi_L \sim \dfrac{1}{z_j}$

$$u = B^{-\gamma} \prod_{j=1}^{n} z_j^{\rho_j}$$

- DE for ψ has otherwise no solution
- if $\varphi_L \sim 1/z^{\nu_j}$ $(\nu_j > 0)$ then $\psi \sim 1/\rho_j$

- limit $\rho_j \to 0$ after the decomposition

Drawbacks

- **additional variables** in intermediate stages
- **obscures block-triangular structure** of decompositions
- **more master integrals** in intermediate steps of recursion

Dual integrals

Recall the decomposition:

$$|\varphi_R\rangle = \sum_{i=1}^{\nu} c_i^{(R)} |e_i^{(R)}\rangle, \qquad c_i^{(R)} = \sum_{j=1}^{\nu} (\mathbf{C}^{-1})_{ij} \langle e_j^{(L)} | \varphi_R\rangle$$

Observation

Coefficients $c_i^{(R)}$ are **independent** of the **choice** of the **dual basis** of "left" integrals $\{\langle e_j^{(L)} | \}_{j=1}^{\nu}$

☑ **Idea**

Exploit the **freedom of choice** of the **dual basis** to simplify the calculation

Choice of dual integrals

Two interesting approaches (different formalisms, similar outcomes):

- Alternative formalism for defining dual space of loop integrals
  [Caron-Huot, Pokraka (2021)]
- Simple **choice of dual integrals**

Choose **dual integrals** of the form **[G. Fontana, T.P. (2023)]**

$$\varphi_L(\mathbf{z}) = \rho_1^{\Theta(\alpha_1 - \frac{1}{2})} \cdots \rho_n^{\Theta(\alpha_n - \frac{1}{2})} \frac{1}{z_1^{\alpha_1} \dots z_n^{\alpha_n}}$$

- if there's a **denominator** factor $z_j^{\alpha_j}$ (with $\alpha_j > 0$), **multiply** by ρ_j
- **systematically** work in the **limit** $\rho_j \to 0$
  (i.e. only keep **leading terms** in a $\rho_j \to 0$ expansion in each step)

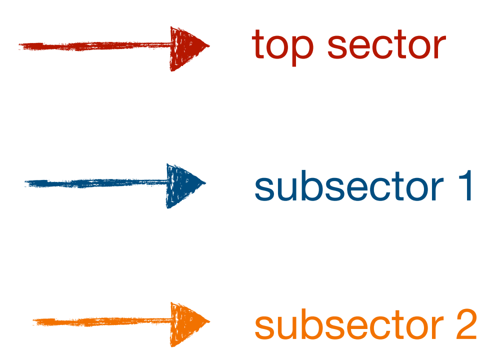
Advantages

- Calculation “effectively” independent of ρ_j
  - working on **leading coefficients** in $\rho_j \to 0$ expansion, often just one
  - over finite fields, never sample or reconstruct ρ_j dependence

- Drastically **simpler** intermediate expressions

- Metric and reduction tables are **block triangular** (blocks ~ sectors)

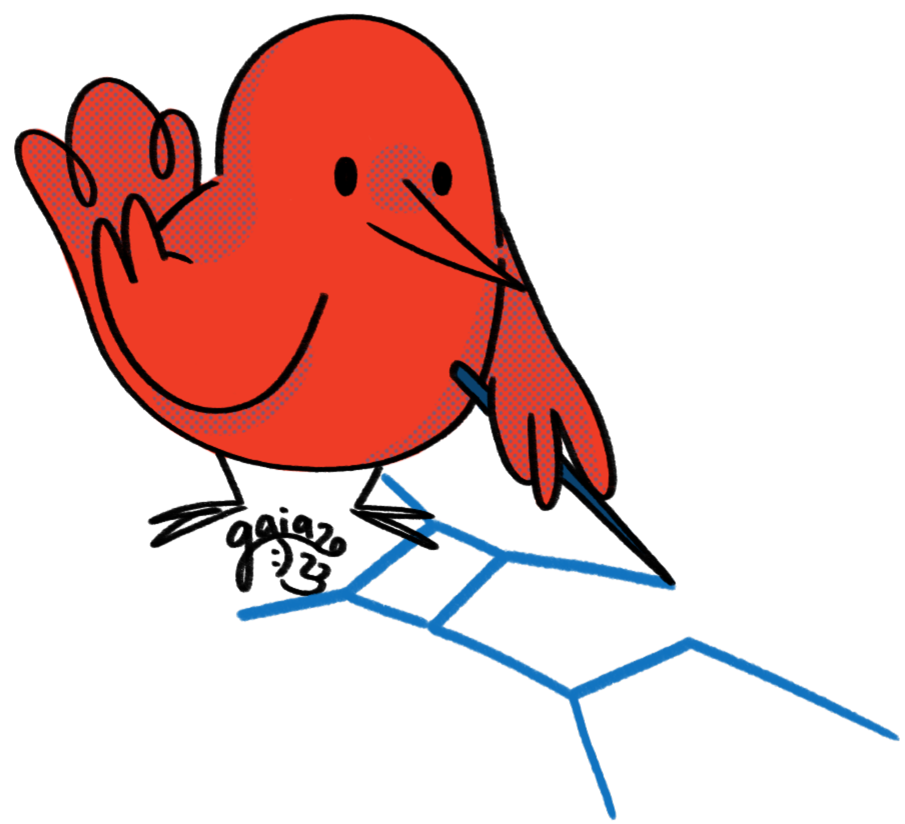
$$\mathbf{C} = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & 0 & 0 & * & * \\ * & * & 0 & 0 & * & * \end{pmatrix}$$



- Many intersection numbers and contributions of poles to them **vanish**
  (ρ_j prefactors must cancel a $1/\rho_j$ singularities, only possible at $z_j \sim 0,\infty$)

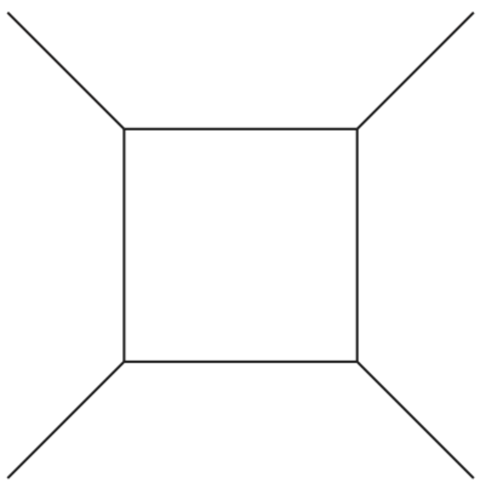
- **Fewer master integrals** in intermediate steps of recursion

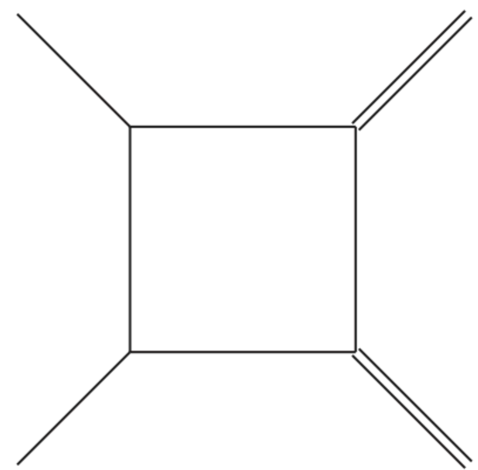
Examples

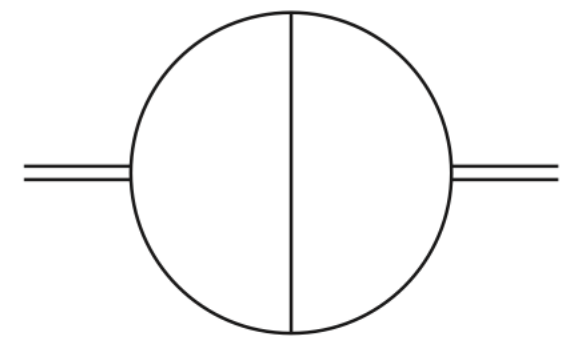


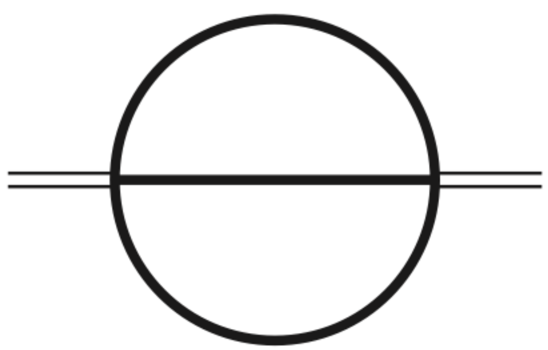
Simple examples and checks

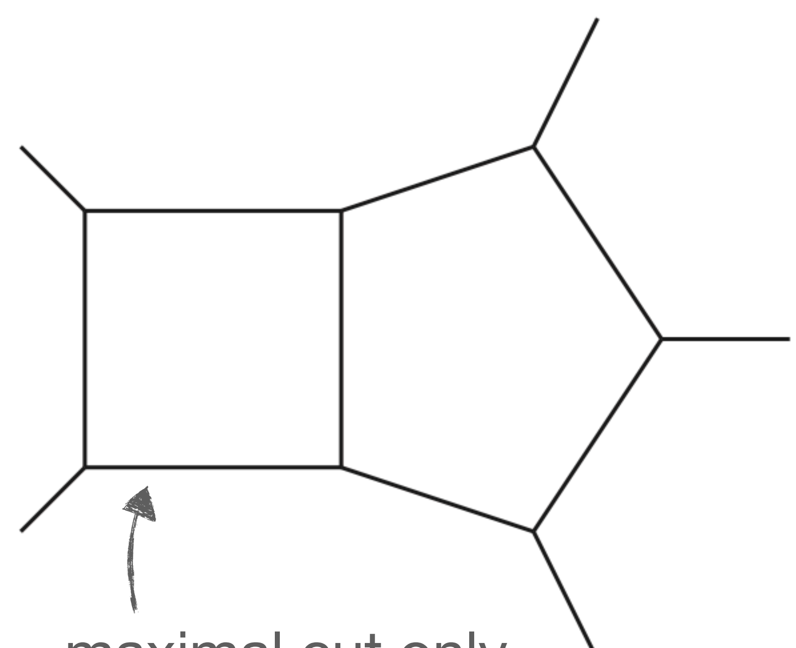
…just checking that things work as expected!











maximal cut only

Conclusions & Outlook

- **Intersection theory** unveils new mathematical structures in loops
- $p(z)$**-adic expansions** simplify study of functions close to singular points
  - avoid algebraic extensions
  - no need to know explicit location of irrational poles
  - avoid bottlenecks and enable finite field technologies

- Future directions:
  - simplifications/optimizations and application to different integral representations (loop-by-loop Baikov, Lee-Pomeransky)
  - Non-recursive multivariate generalization (based on **Chestnov, Frellesvig, Gasparotto, Mandal, Mastrolia (2022)**)