



Physical thresholds and cluster decomposition

Zeno Capatti
ETH Zürich



Amplitudes 2023
07/06/2023, CERN, Switzerland



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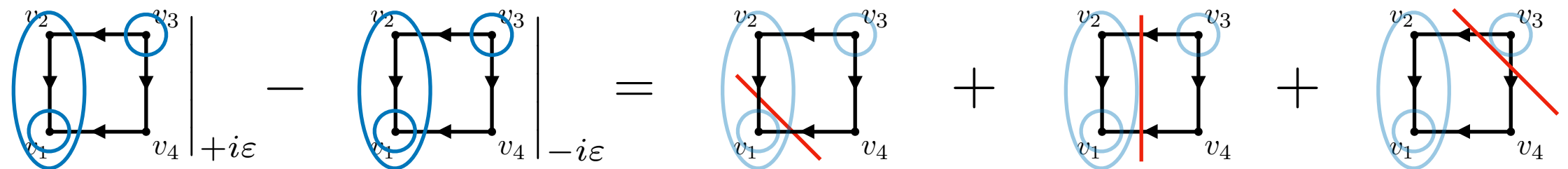


First part: Cross-Free Families

- Sketch the derivation of the representation

$$f_{G_u}^{3d} = \int \left[\prod_{i=1}^L \frac{dk_i^0}{2\pi} \right] \frac{\mathcal{N}(\{q_e^0\}_{e \in \mathcal{E}})}{\prod_{e \in \mathcal{E}} (q_e^0)^2 - E_e^2}$$

- Highlight role of connectedness by comparison with Time-Ordered Perturbation Theory



(spurious singularities in TOPT)

Acyclic graphs and edge contraction

Notation

*Energy
conservation*

Acyclic graphs

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We start with an arbitrary diagram integrated over loop energies only

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$$q_e^0 = p_e^0 + \sum_{i=1}^L s_{ei} k_i^0$$
$$E_e = \sqrt{|\vec{q}_e|^2 + m_e^2}$$

G_u undirected graph
 \mathcal{E} set of edges of graph

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Performing the integrals using residue theorem is a matter of correctly addressing energy conservation (depending on how, get TOPT, LTD, CFF). For CFF ([ZC \[arXiv:2211.09653\]](https://arxiv.org/abs/2211.09653)):

$$\frac{1}{(q_e^0)^2 - E_e^2} = \int dx_e d\tau_e \frac{e^{i\tau_e(x_e - q_e^0)}}{x_e^2 - E_e^2}$$

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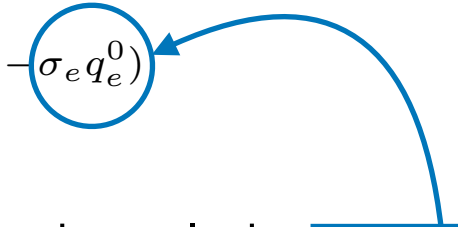
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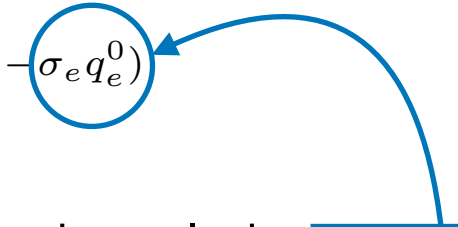
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(\mathcal{K}_G is empty if graph not acyclic)

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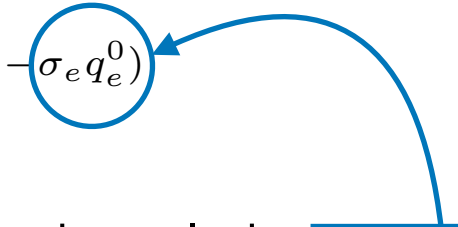
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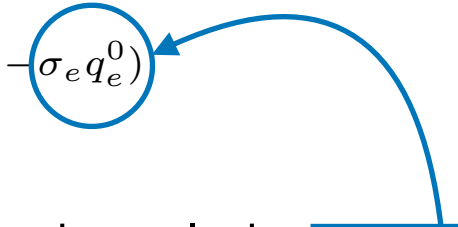
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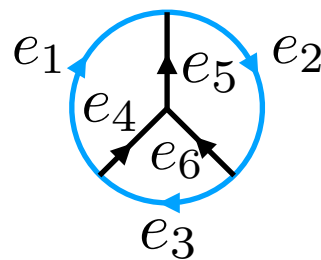
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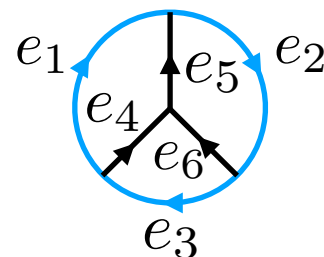
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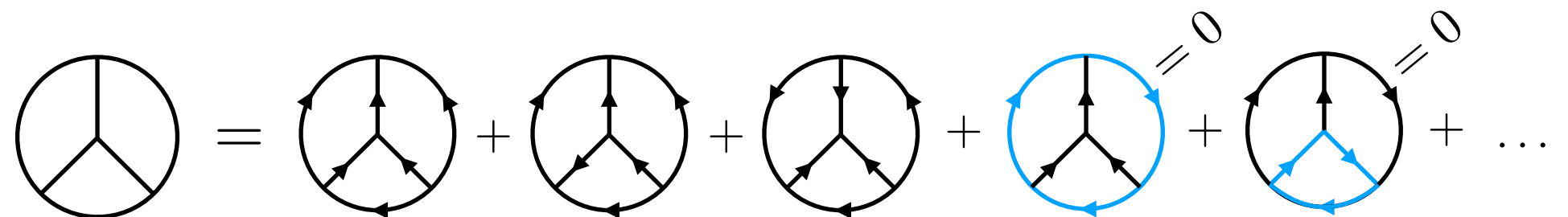
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Diagrammatically



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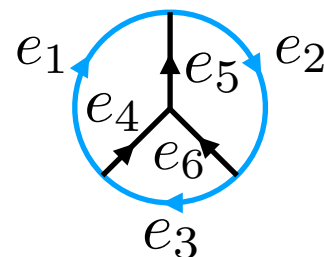
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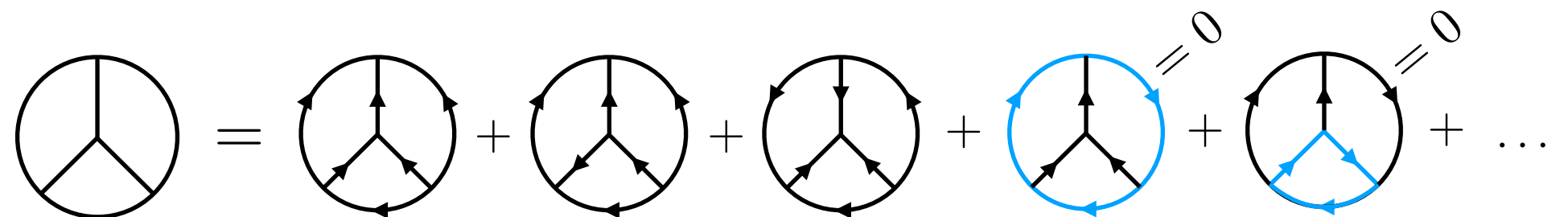
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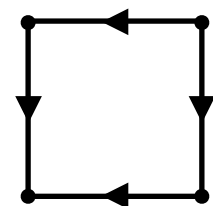


Position space: Fourier duality maps acyclic graphs to strongly-connected graphs

Edge contraction

$$\begin{array}{c} \circ \\ \leftarrow \\ \circ \\ \leftarrow \\ \circ \\ \leftarrow \\ \circ \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

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Triangulation introduces spurious singularities or spurious intersections

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The cone is non-simplicial
Needs triangulation!

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Triangulation introduces spurious singularities or spurious intersections

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$$\begin{array}{c} \square \\ \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{array} = \int \underbrace{\left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right]}_{\mathcal{K}_G} \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

The cone is non-simplicial
Needs triangulation!

$$\mathcal{K}_G = \left\{ (\tau_e)_{e \in \mathcal{E}} \in \mathbb{R}_+^{|\mathcal{E}|} \mid \sum_{e \in \mathcal{E}} s_{ei} \tau_e \right\}$$

Triangulation introduces spurious singularities or spurious intersections

Edge contraction

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{c} \circ \quad \circ \\ \leftarrow \quad \leftarrow \\ \circ \quad \circ \\ \leftarrow \quad \leftarrow \\ \circ \quad \circ \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

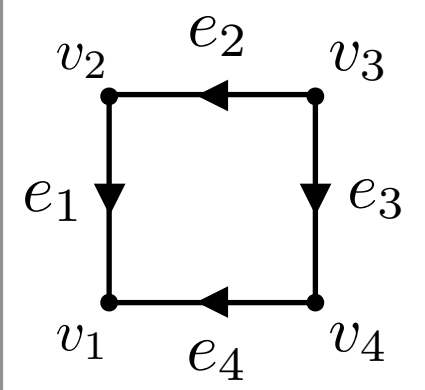
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Edge contraction

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction



$$\begin{array}{c} \text{Square with arrows} \\ \text{= } \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4) \end{array}$$

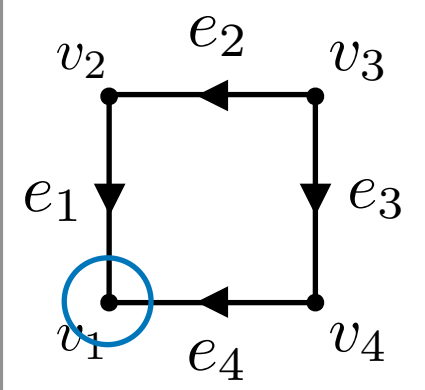
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Triangulation introduces spurious singularities or spurious intersections

Edge contraction

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction



1. Choose sink/source with connected complement

$$\begin{array}{c} \text{Square with arrows} \\ \hline \end{array} = \int \underbrace{\left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right]}_{\mathcal{K}_G} \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

The cone is non-simplicial
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$$\mathcal{K}_G = \left\{ (\tau_e)_{e \in \mathcal{E}} \in \mathbb{R}_+^{|\mathcal{E}|} \mid \sum_{e \in \mathcal{E}} s_{ei} \tau_e \right\}$$

Triangulation introduces spurious singularities or spurious intersections

Edge contraction

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{c} \text{Square with vertices } v_1, v_2, v_3, v_4 \text{ and edges } e_1, e_2, e_3, e_4 \\ \hline \end{array} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{c} \text{Triangle with vertices } v_{12}, v_3, v_4 \text{ and edges } e_2, e_3, e_4 \\ \hline \end{array} + \begin{array}{c} \text{Triangle with vertices } v_2, v_3, v_{14} \text{ and edges } e_1, e_2, e_3 \\ \hline \end{array} \right]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$\begin{array}{c} \text{---} \leftarrow \text{---} \\ | \quad | \\ \text{---} \leftarrow \text{---} \\ | \quad | \\ \text{---} \leftarrow \text{---} \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

The cone is non-simplicial
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$$\mathcal{K}_G = \left\{ (\tau_e)_{e \in \mathcal{E}} \in \mathbb{R}_+^{|\mathcal{E}|} \mid \sum_{e \in \mathcal{E}} s_{ei} \tau_e \right\}$$

Triangulation introduces spurious singularities or spurious intersections

Edge contraction

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{c} v_2 \quad e_2 \quad v_3 \\ | \quad | \\ e_1 \downarrow \quad \downarrow e_3 \\ \text{---} \leftarrow \text{---} \\ | \quad | \\ v_1 \quad e_4 \quad v_4 \end{array} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{c} v_3 \\ \text{---} \leftarrow \text{---} \\ | \quad | \\ v_{12} \quad e_4 \quad v_4 \end{array} + \begin{array}{c} v_2 \quad e_2 \quad v_3 \\ | \quad | \\ e_1 \downarrow \quad \downarrow e_3 \\ v_{14} \end{array} \right]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$\begin{array}{c} \text{Clockwise square loop} \\ \leftarrow \text{ } \rightarrow \\ \leftarrow \text{ } \rightarrow \\ \leftarrow \text{ } \rightarrow \\ \leftarrow \text{ } \rightarrow \end{array} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

The cone is non-simplicial
Needs triangulation!

$$\mathcal{K}_G = \left\{ (\tau_e)_{e \in \mathcal{E}} \in \mathbb{R}_+^{|\mathcal{E}|} \mid \sum_{e \in \mathcal{E}} s_{ei} \tau_e \right\}$$

Triangulation introduces spurious singularities or spurious intersections

Edge contraction

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{c} v_2 \xrightarrow{e_2} v_3 \\ \downarrow e_1 \quad \downarrow e_3 \\ v_1 \xrightarrow{e_4} v_4 \end{array} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{c} v_3 \\ \uparrow e_2 \\ v_{12} \quad \downarrow e_3 \\ \downarrow e_4 \\ v_4 \end{array} + \begin{array}{c} v_2 \xrightarrow{e_2} v_3 \\ \downarrow e_1 \quad \downarrow e_3 \\ v_{14} \end{array} \right]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$\begin{array}{c} \text{Square with arrows} \\ \hline = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4) \end{array}$$

The cone is non-simplicial
Needs triangulation!

$$\mathcal{K}_G = \left\{ (\tau_e)_{e \in \mathcal{E}} \in \mathbb{R}_+^{|\mathcal{E}|} \mid \sum_{e \in \mathcal{E}} s_{ei} \tau_e \right\} \quad \text{Triangulation introduces spurious singularities or spurious intersections}$$

Edge contraction

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
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$$\begin{array}{c} \text{Square with arrows} \\ = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4) \end{array}$$

The cone is non-simplicial
Needs triangulation!

$$\mathcal{K}_G = \left\{ (\tau_e)_{e \in \mathcal{E}} \in \mathbb{R}_+^{|\mathcal{E}|} \mid \sum_{e \in \mathcal{E}} s_{ei} \tau_e \right\}$$

Triangulation introduces spurious singularities or spurious intersections

Edge contraction

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{l} \begin{array}{c} v_2 \xrightarrow{e_2} v_3 \\ \downarrow e_1 \quad \downarrow e_3 \\ v_1 \xrightarrow{e_4} v_4 \end{array} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{c} \begin{array}{c} v_3 \\ \downarrow e_2 \\ v_{12} \end{array} + \begin{array}{c} v_3 \\ \downarrow e_3 \\ v_{14} \end{array} \end{array} \right] \\ \\ = \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \left(\begin{array}{c} \text{Graph 1} + \text{Graph 2} \end{array} \right) + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \left(\begin{array}{c} \text{Graph 3} + \text{Graph 4} \end{array} \right) \right] \end{array}$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges
4. Throw out non-acyclic graphs

$$\begin{array}{c} \text{Square with arrows} \\ = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4) \end{array}$$

The cone is non-simplicial
Needs triangulation!

$$\mathcal{K}_G = \left\{ (\tau_e)_{e \in \mathcal{E}} \in \mathbb{R}_+^{|\mathcal{E}|} \mid \sum_{e \in \mathcal{E}} s_{ei} \tau_e \right\}$$

Triangulation introduces spurious singularities or spurious intersections

Edge contraction

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{l} \begin{array}{c} v_2 \xrightarrow{e_2} v_3 \\ \downarrow e_1 \quad \downarrow e_3 \\ v_1 \xrightarrow{e_4} v_4 \end{array} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{c} \begin{array}{c} v_3 \\ \downarrow e_2 \\ v_{12} \end{array} + \begin{array}{c} v_3 \\ \downarrow e_3 \\ v_{14} \end{array} \end{array} \right] \\ \\ = \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \left(\begin{array}{c} \text{Graph 1} + \text{Graph 2} \end{array} \right) + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \left(\begin{array}{c} \text{Graph 3} + \text{Graph 4} \end{array} \right) \right] \end{array}$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges
4. Throw out non-acyclic graphs

$$\text{Square Loop} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

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Triangulation introduces spurious singularities or spurious intersections

Edge contraction

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\text{Square Loop} = \frac{i}{E_1 + E_4 - p_1^0} \left[\text{Graph 1} + \text{Graph 2} \right]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \left(\text{Graph 3} + \text{Graph 4} \right) + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \left(\text{Graph 5} + \text{Graph 6} \right) \right]$$

4. Throw out non-acyclic graphs

$$\begin{array}{c} \text{Square with arrows} \\ \hline \end{array} = \int \underbrace{\left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right]}_{\mathcal{K}_G} \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

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Triangulation introduces spurious singularities or spurious intersections

Edge contraction

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\begin{array}{c} \text{Square with } v_1 \text{ circled} \\ \hline \end{array} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{c} \text{Triangle } v_{12}v_3v_4 \text{ with } v_3 \text{ circled} \\ + \\ \text{Triangle } v_2v_3v_{14} \text{ with } v_{14} \text{ circled} \end{array} \right]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \left(\begin{array}{c} \text{Graph } v_{123}v_4 \text{ with } e_3, e_4 \text{ crossed out} \\ + \\ \text{Graph } v_{12}v_{34} \text{ with } v_{12} \text{ circled} \end{array} \right) + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \left(\begin{array}{c} \text{Graph } v_{134}v_4 \text{ with } e_2, e_1 \text{ crossed out} \\ + \\ \text{Graph } v_{124}v_3 \text{ with } v_{124} \text{ circled} \end{array} \right) \right]$$

4. Throw out non-acyclic graphs

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \frac{i}{E_2 + E_4 - p_1^0 - p_2^0} + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \frac{i}{E_2 + E_3 + p_3^0} \right]$$

$$\begin{array}{c} \text{Square with arrows} \\ \hline = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4) \end{array}$$

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Edge contraction

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1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$\begin{array}{c} = \frac{i}{E_1 + E_4 - p_1^0} \left[\begin{array}{c} \frac{i}{E_2 + E_3 + p_3^0} \left(\begin{array}{c} \text{Graph } v_{123}v_4 \text{ with } e_3, e_4 \text{ crossed out} \\ + \\ \text{Graph } v_{12}v_{34} \end{array} \right) \\ + \\ \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \left(\begin{array}{c} \text{Graph } v_{134}v_4 \text{ with } e_1, e_2 \text{ crossed out} \\ + \\ \text{Graph } v_{124}v_3 \end{array} \right) \end{array} \right] \end{array}$$

4. Throw out non-acyclic graphs

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \frac{i}{E_2 + E_4 - p_1^0 - p_2^0} + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \frac{i}{E_2 + E_3 + p_3^0} \right]$$

5. Contract parallel edges

$$\text{Square Loop} = \int \left[\prod_{j=1}^4 \frac{d\tau_j}{2E_j} e^{i\tau_j(E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta(-\tau_1 - \tau_2 + \tau_3 + \tau_4)$$

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Triangulation introduces spurious singularities or spurious intersections

Edge contraction

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

$$\text{Square Loop} = \frac{i}{E_1 + E_4 - p_1^0} \left[\text{Graph 1} + \text{Graph 2} \right]$$

1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \left(\text{Graph 3} + \text{Graph 4} \right) + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \left(\text{Graph 5} + \text{Graph 6} \right) \right]$$

4. Throw out non-acyclic graphs

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \frac{i}{E_2 + E_4 - p_1^0 - p_2^0} + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \frac{i}{E_2 + E_3 + p_3^0} \right]$$

5. Contract parallel edges

All time integrations are performed diagrammatically!

Cross-Free Families

Boundary operator

Cross-Free Families

Cross-Free Families

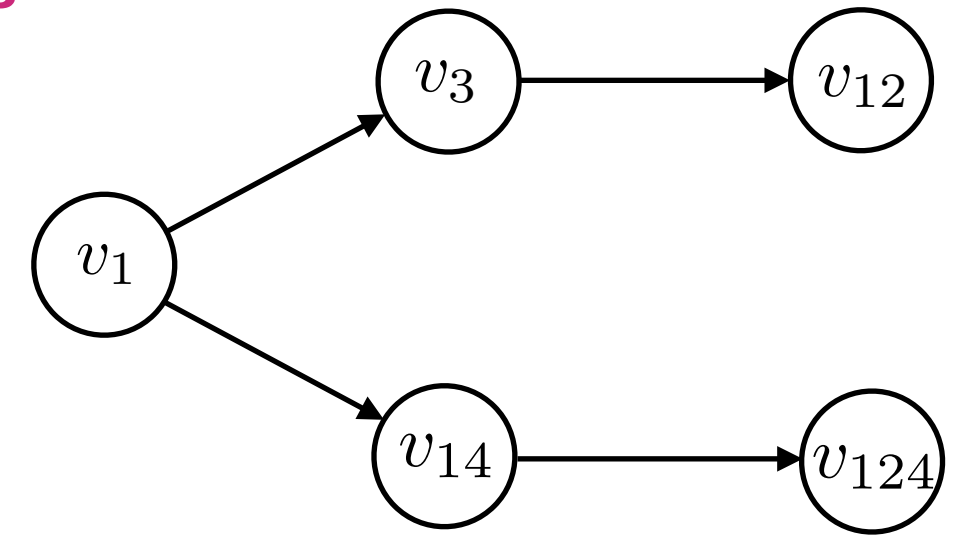
Collecting the chosen vertices and collecting them according to order of choice, we get a decision tree, whose root is the first chosen vertex

Boundary operator

Cross-Free Families

Cross-Free Families

Collecting the chosen vertices and collecting them according to order of choice, we get a decision tree, whose root is the first chosen vertex

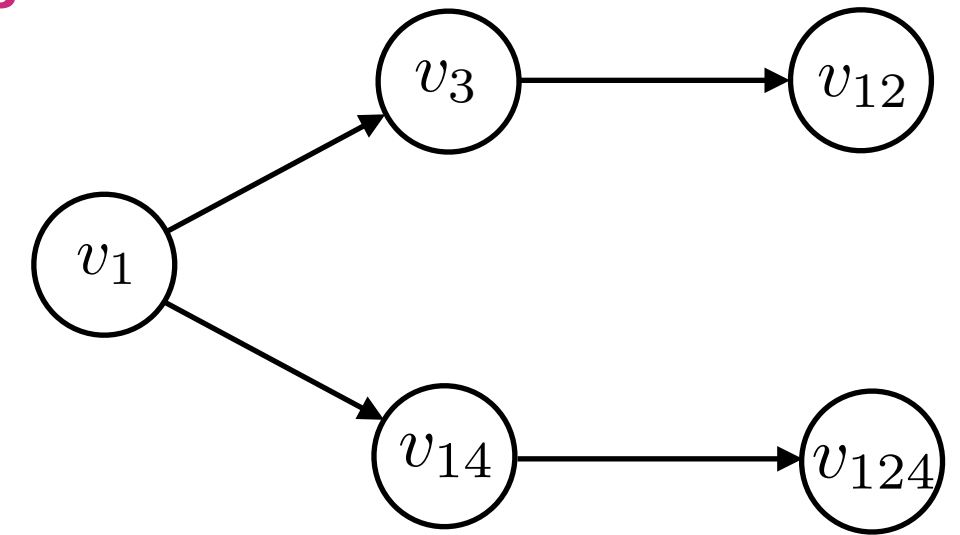


Boundary operator

Cross-Free Families

Cross-Free Families

Collecting the chosen vertices and collecting them according to order of choice, we get a decision tree, whose root is the first chosen vertex



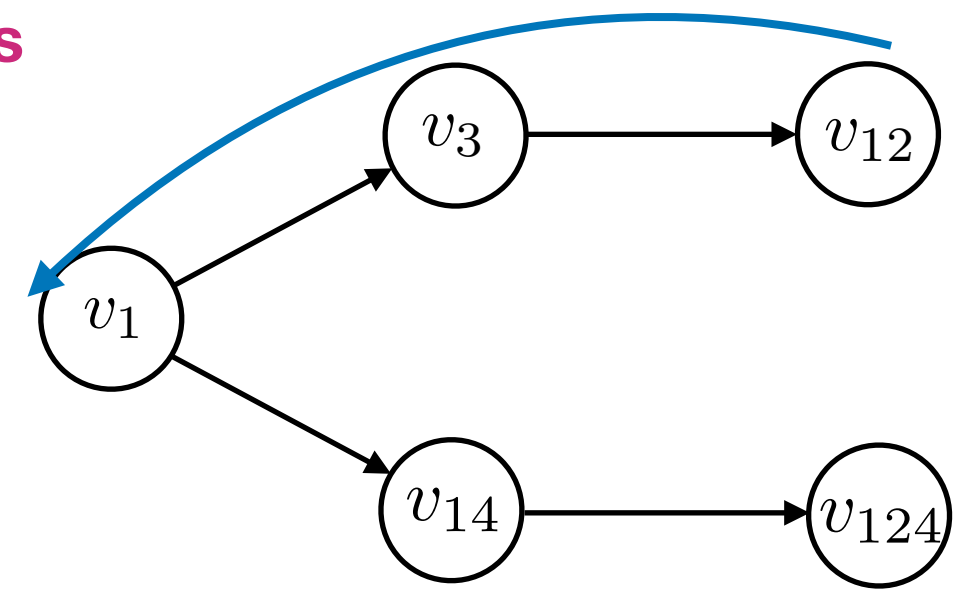
Tracing the route from the leaves to the root gives sets of vertices

Boundary operator

Cross-Free Families

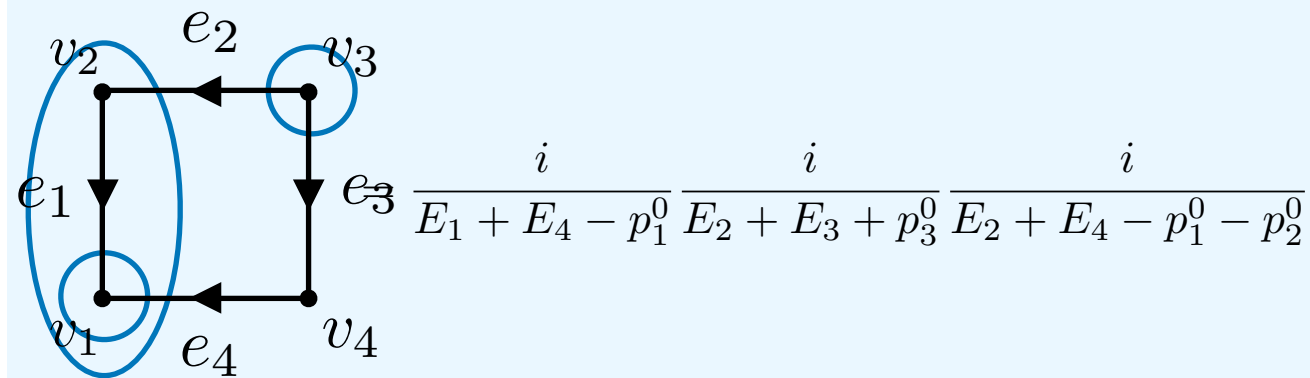
Cross-Free Families

Collecting the chosen vertices and collecting them according to order of choice, we get a decision tree, whose root is the first chosen vertex



Tracing the route from the leaves to the root gives sets of vertices

$$F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$$

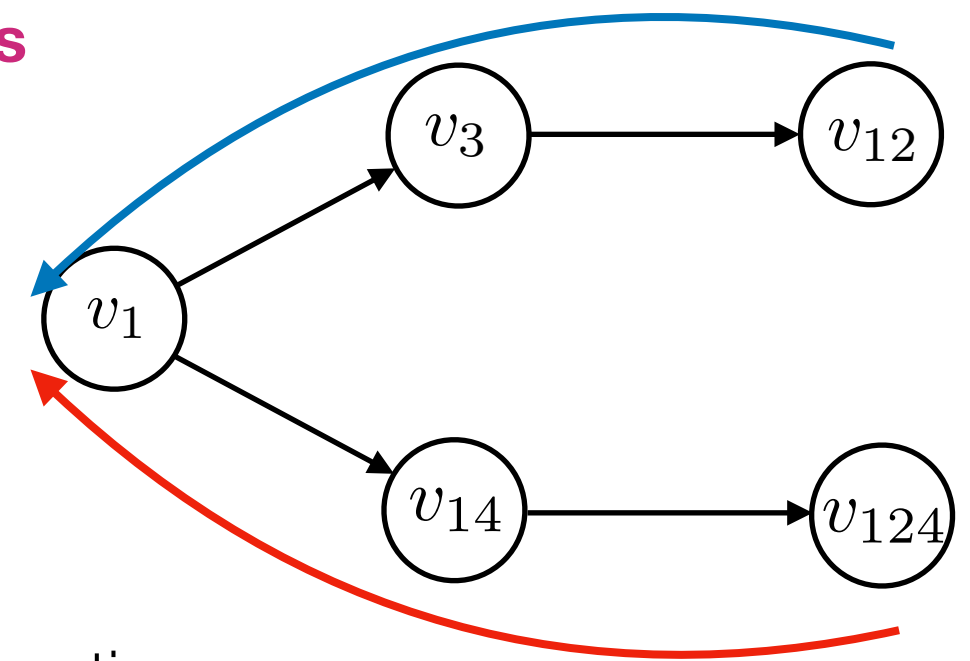


Boundary operator

Cross-Free Families

Cross-Free Families

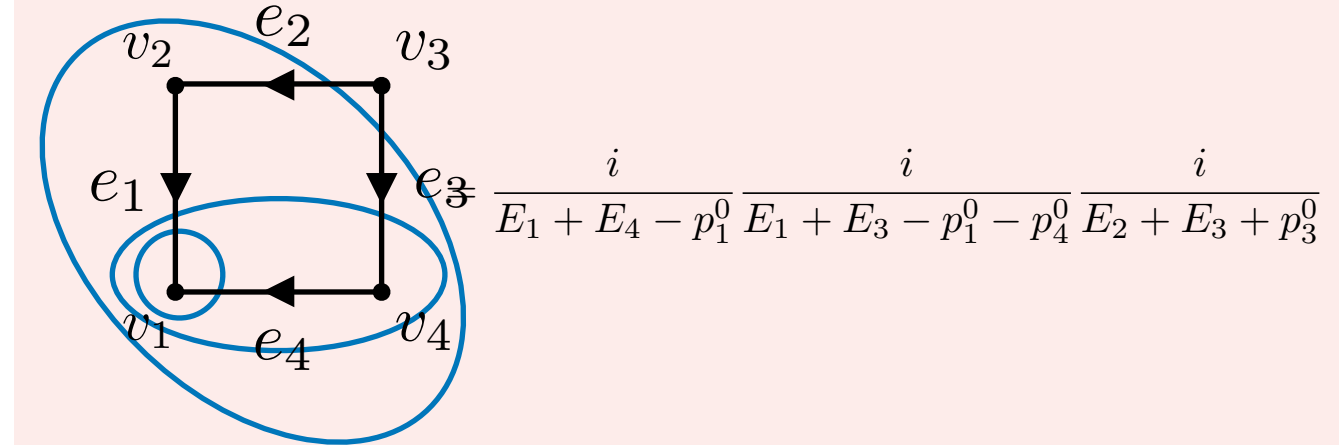
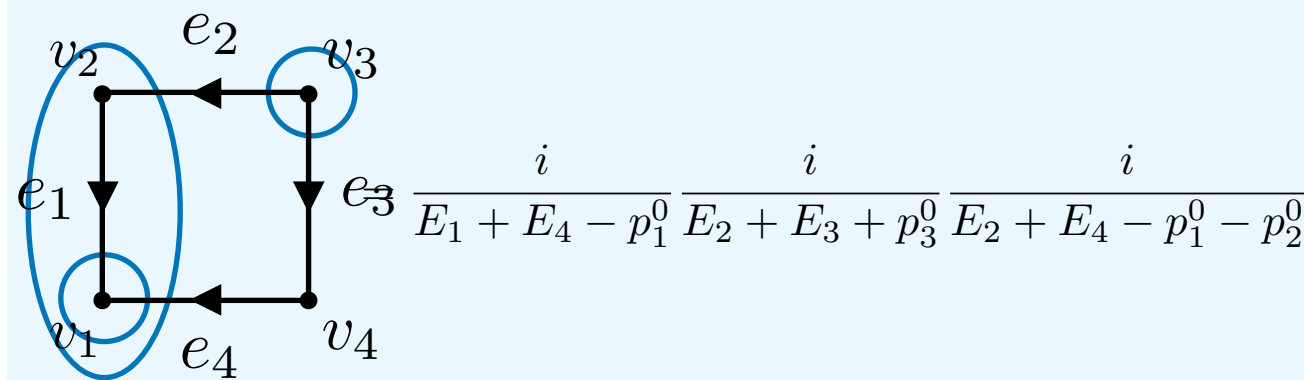
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Tracing the route from the leaves to the root gives sets of vertices

$$F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$$

$$F_2 = \{\{v_1\}, \{v_1, v_4\}, \{v_1, v_2, v_4\}\}$$

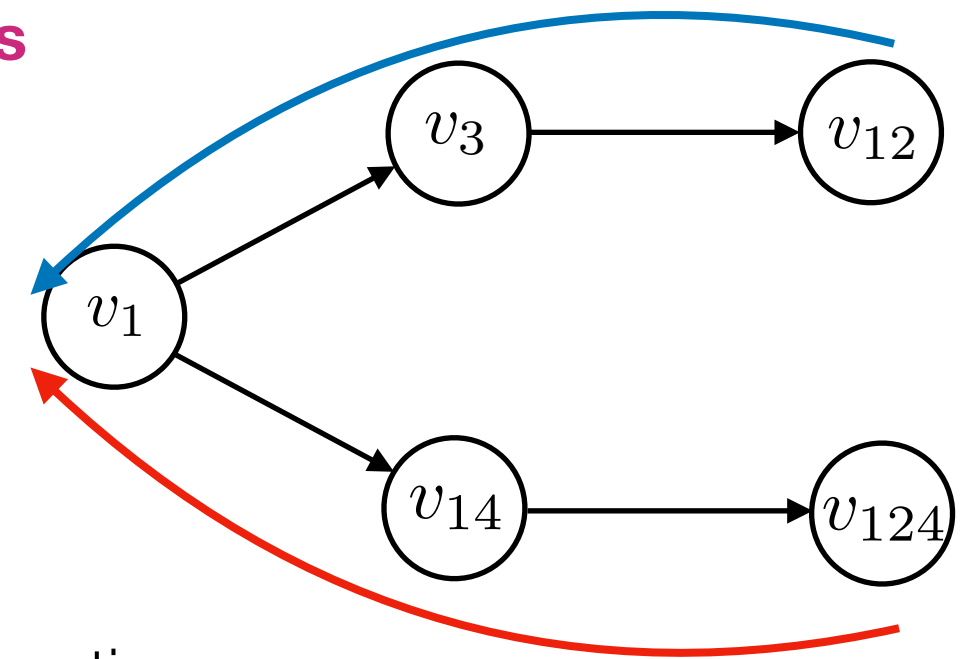


Boundary operator

Cross-Free Families

Cross-Free Families

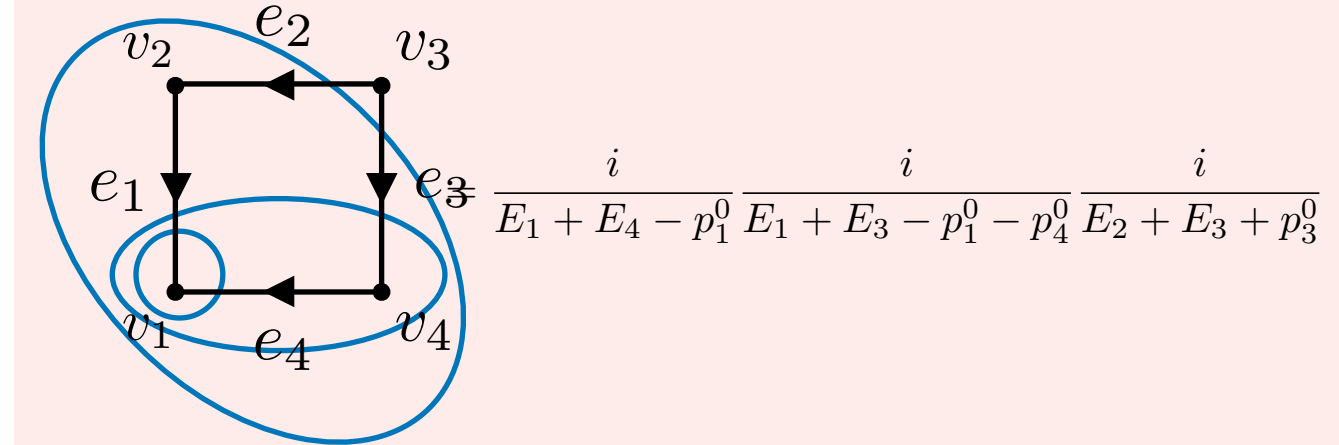
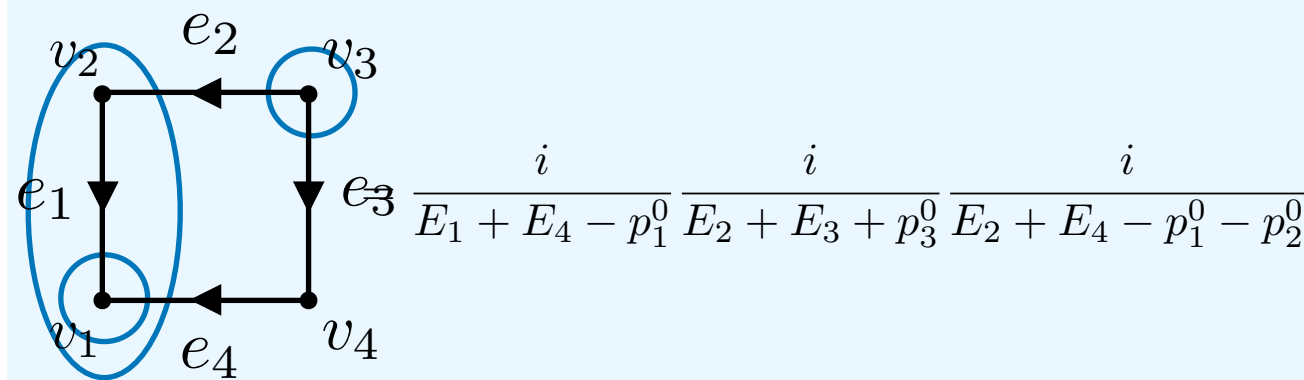
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Tracing the route from the leaves to the root gives sets of vertices

$$F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$$

$$F_2 = \{\{v_1\}, \{v_1, v_4\}, \{v_1, v_2, v_4\}\}$$



Boundary operator

Boundary operator provides nexus

e.g. $\partial(\{v_1, v_2\}) = \{e_2, e_4\}$

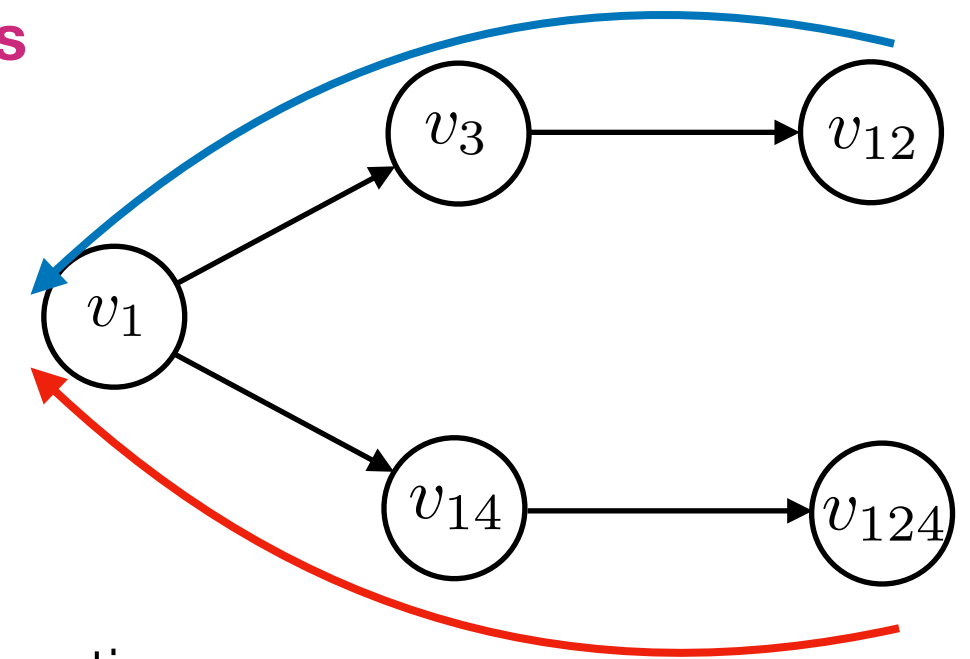
\Rightarrow

$$\frac{i}{E_2 + E_4 - p_1^0 - p_2^0}$$

Cross-Free Families

Cross-Free Families

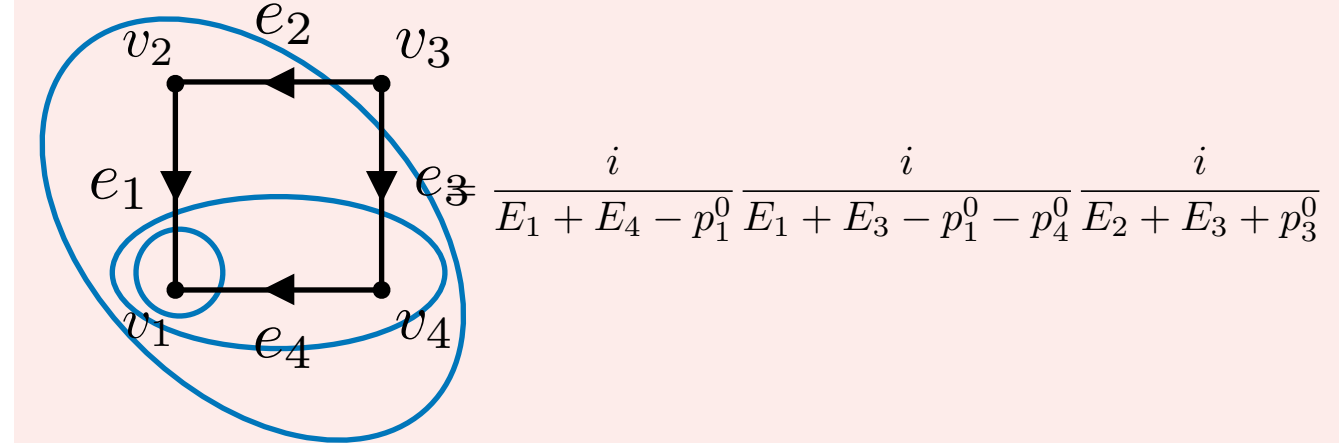
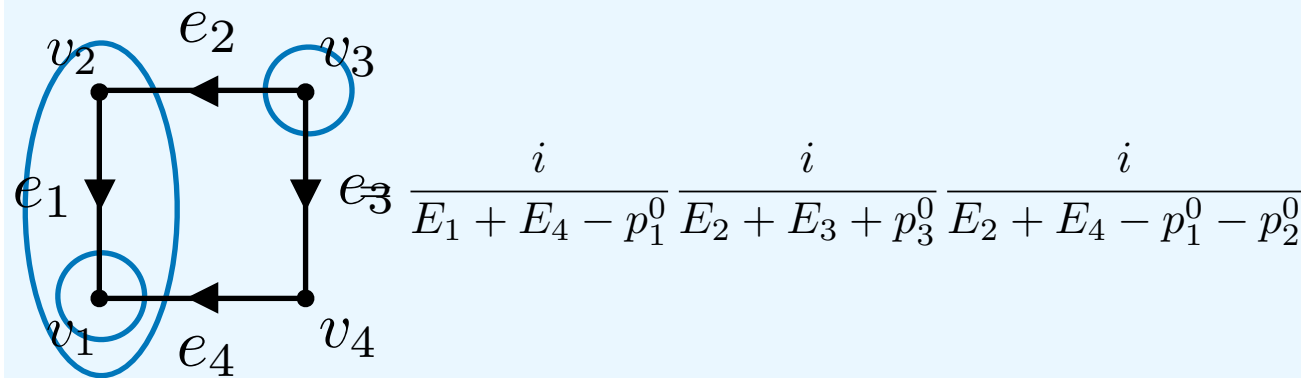
Collecting the chosen vertices and collecting them according to order of choice, we get a decision tree, whose root is the first chosen vertex



Tracing the route from the leaves to the root gives sets of vertices

$$F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$$

$$F_2 = \{\{v_1\}, \{v_1, v_4\}, \{v_1, v_2, v_4\}\}$$



Boundary operator

Boundary operator provides nexus

e.g. $\partial(\{v_1, v_2\}) = \{e_2, e_4\}$

\Rightarrow

$$\frac{i}{E_2 + E_4 - p_1^0 - p_2^0}$$

Cross-Free Families

We notice some regularities... these families of cuts satisfy

$$S \in F \Rightarrow S, V \setminus S \text{ are connected}$$

$$S_1, S_2 \in F \Rightarrow S_1 \subset S_2 \text{ or } S_2 \subset S_1 \text{ or } S_1 \cap S_2 = \emptyset$$

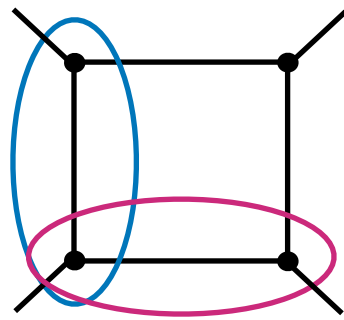
$$S \in F \Rightarrow S \text{ cannot be written as union of other sets in } F$$

Abreu, Britto, Duhr, Gardi arXiv:2010.01068 (2014) Bloch, Kreimer arXiv:1512.01705 (2015)
 Arkani-Hamed, Benincasa, Postnikov arXiv:1709.02813 (2017)
 Benincasa, McLeod, Vergu arXiv:2009.03047 (2020)
 Capatti, Hirschi, Pelloni, Ruijl, arXiv:2010.01068 (2020)

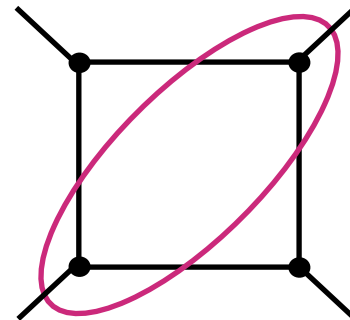
Forbidden configurations

General formula

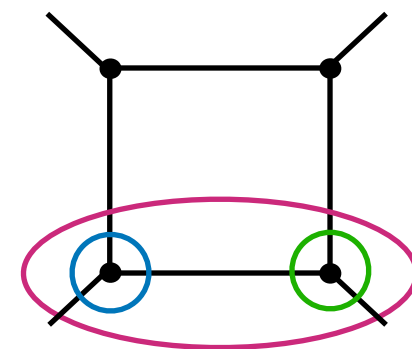
Forbidden configurations



crossing



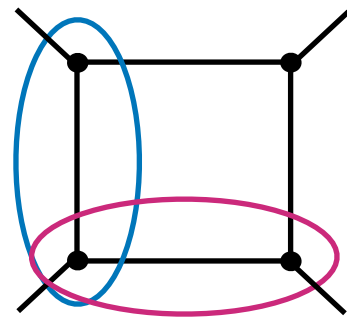
connectedness



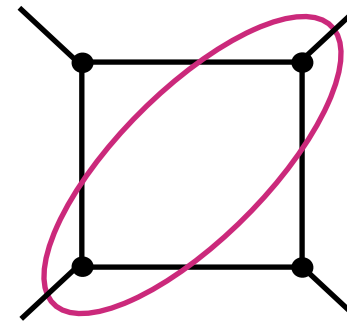
obstruction

General formula

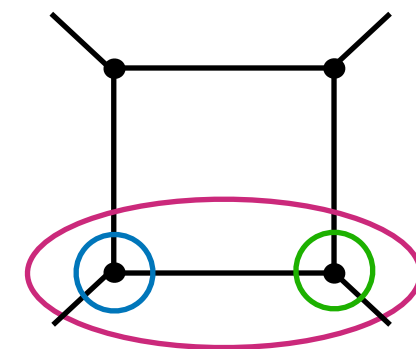
Forbidden configurations



crossing



connectedness



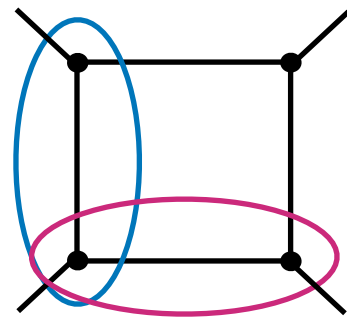
obstruction

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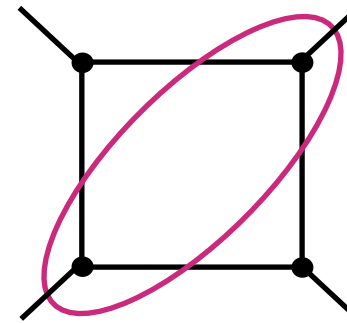
Repeating the same edge-contraction procedure for all acyclic graphs

$$f_{G_u}^{3d} = \sum_{\text{acyclic graph } G} \frac{\mathcal{N}_G}{\prod_e 2E_e} \sum_{F \in \mathcal{F}_G} \frac{1}{\prod_{S \in F} [\mathbf{E} \cdot \mathbf{1}^{\partial(S)} - \sum_{v \in S} p_v^0]}$$

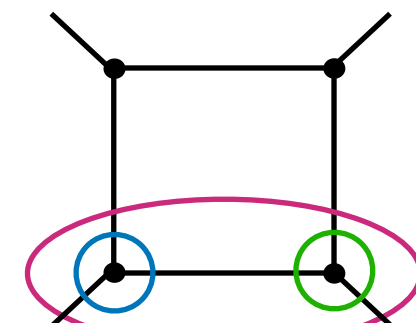
Forbidden configurations



crossing



connectedness



obstruction

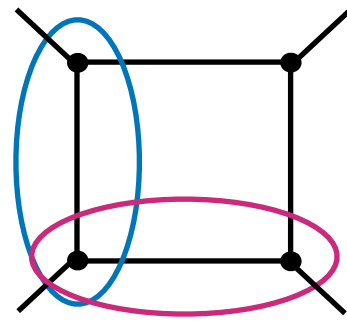
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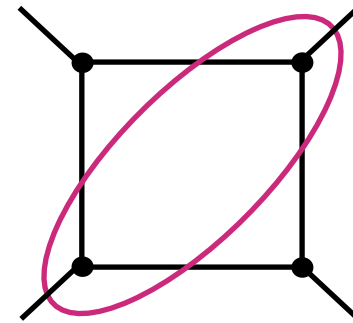
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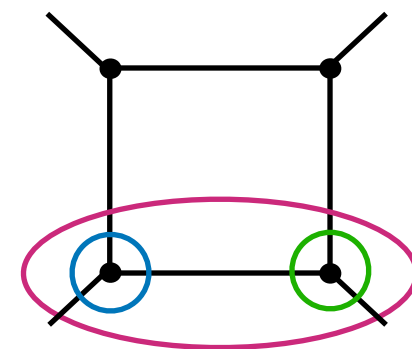
Forbidden configurations



crossing



connectedness



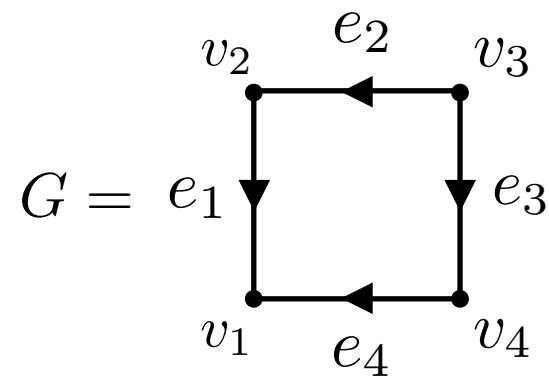
obstruction

General formula

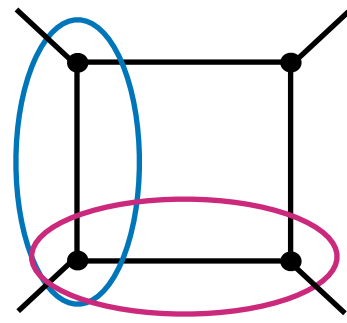
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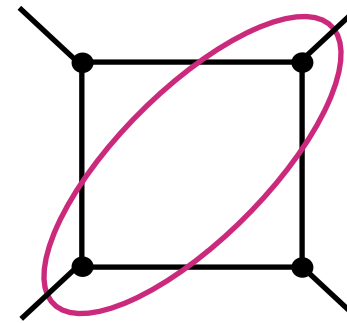
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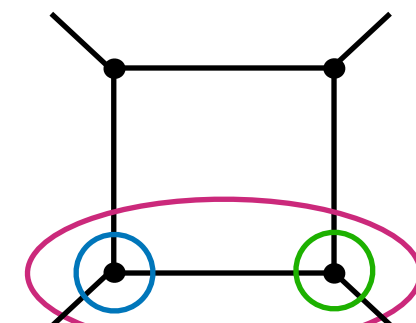
Forbidden configurations



crossing



connectedness



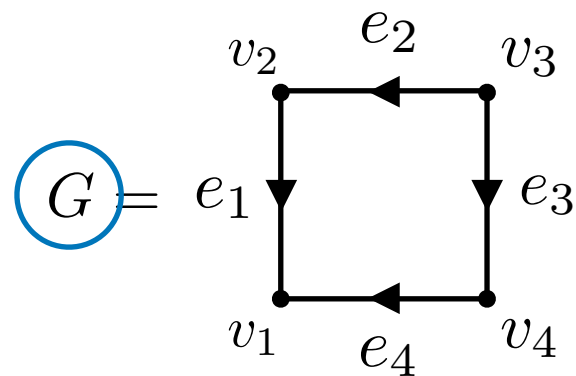
obstruction

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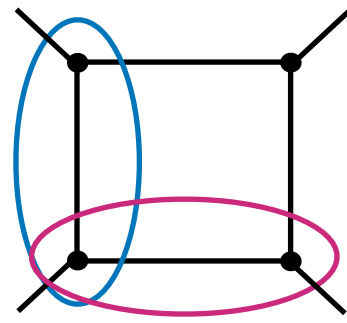
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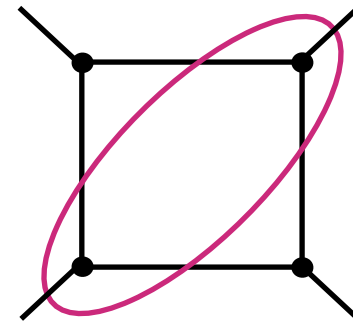
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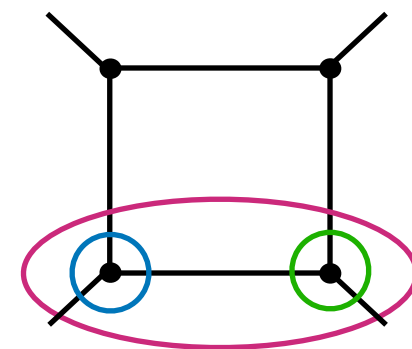
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crossing



connectedness



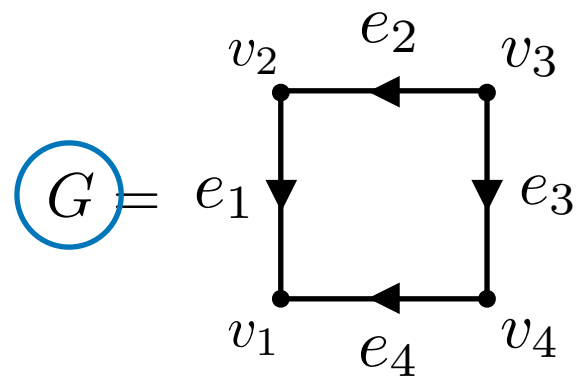
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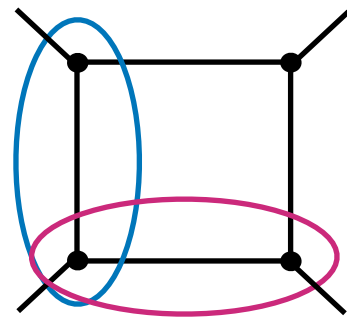
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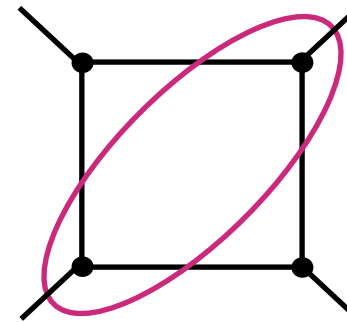


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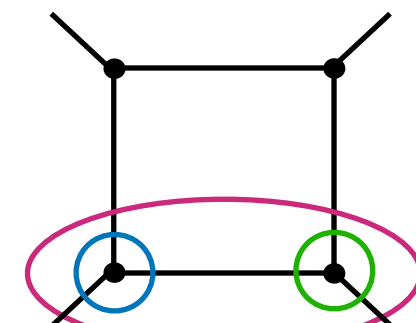
Forbidden configurations



crossing



connectedness



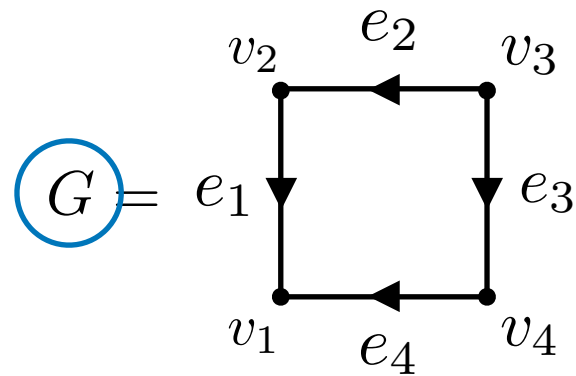
obstruction

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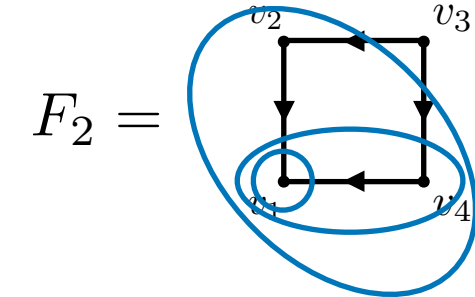
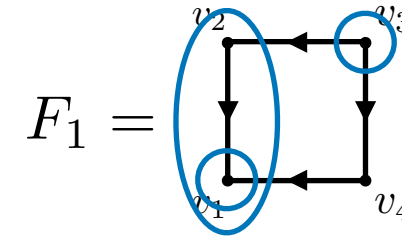
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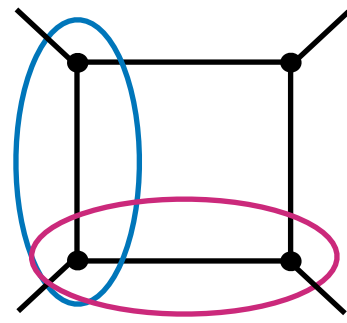
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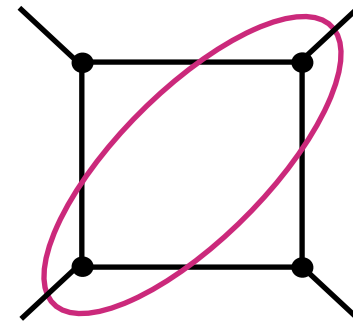
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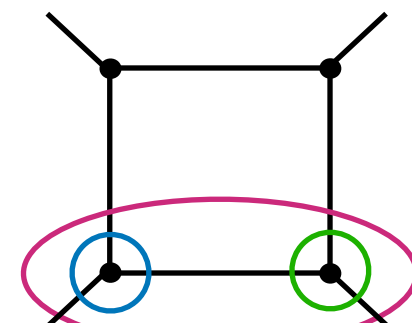
Forbidden configurations



crossing



connectedness



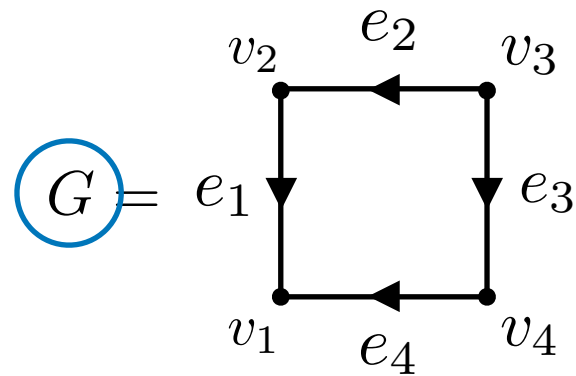
obstruction

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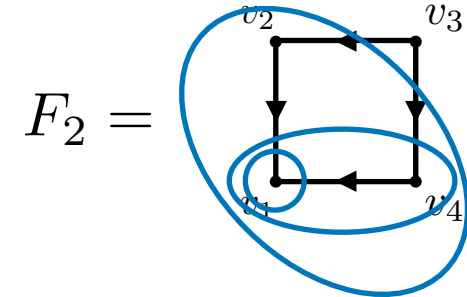
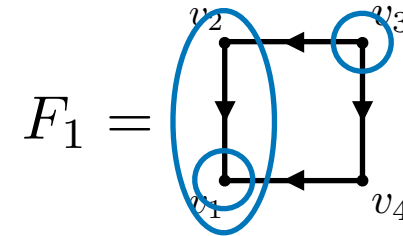
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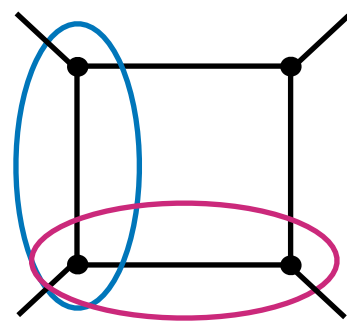
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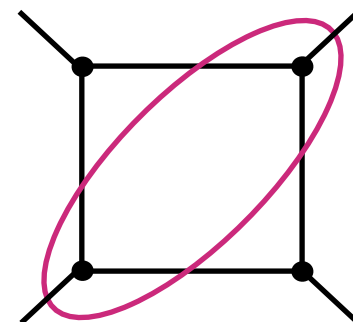
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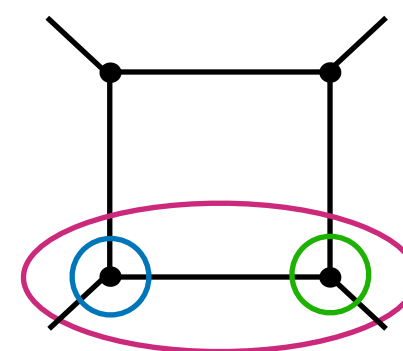
Forbidden configurations



crossing



connectedness



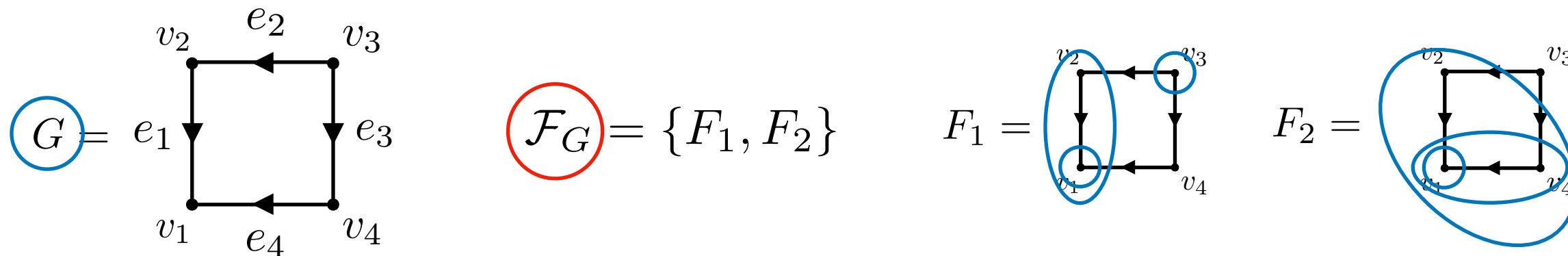
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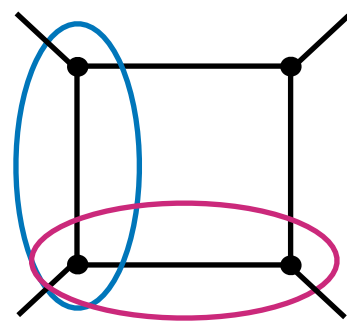
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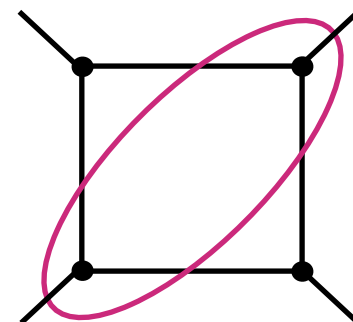
Each element of a cross-free family corresponds to a threshold

$$F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$$

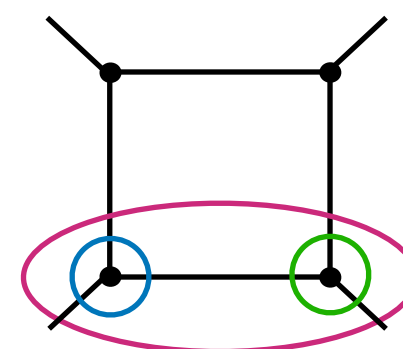
Forbidden configurations



crossing



connectedness



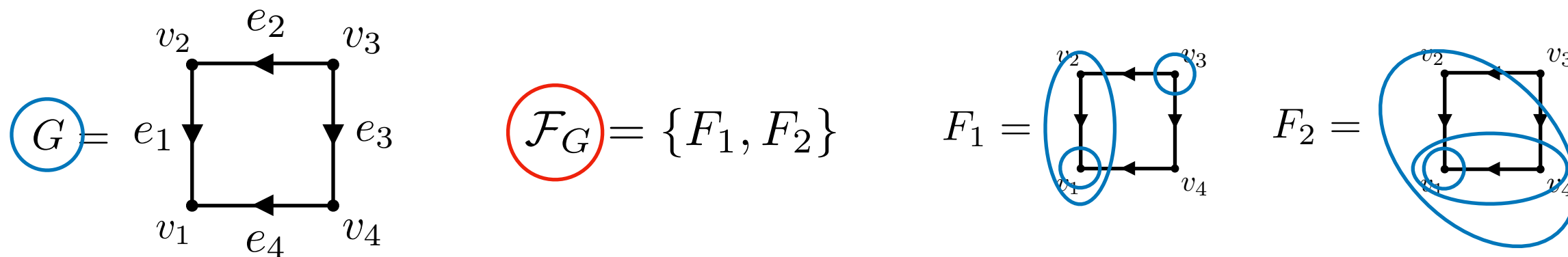
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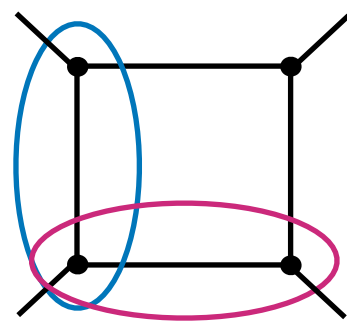
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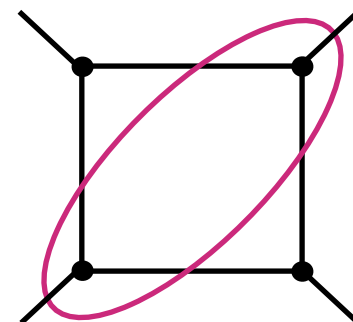
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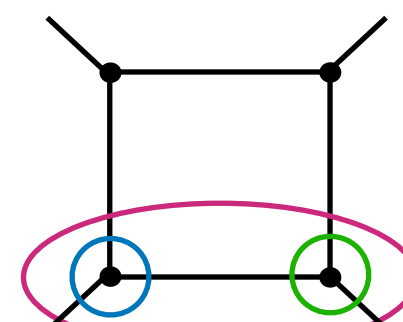
Forbidden configurations



crossing



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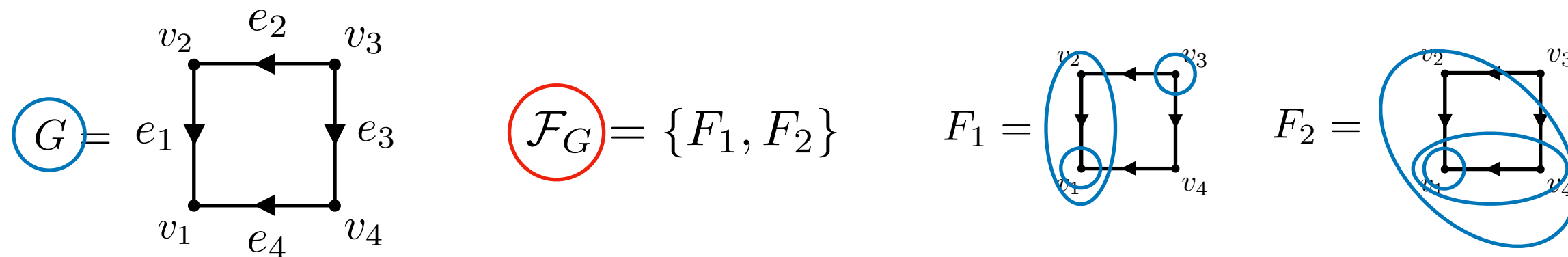
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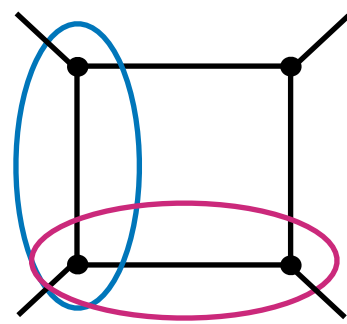
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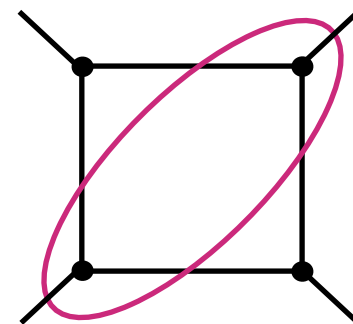
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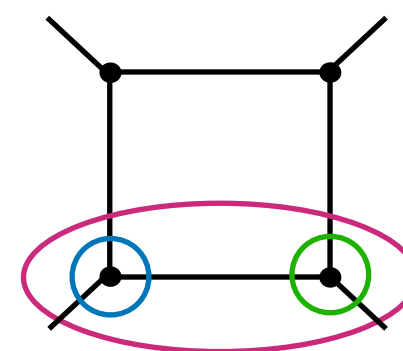
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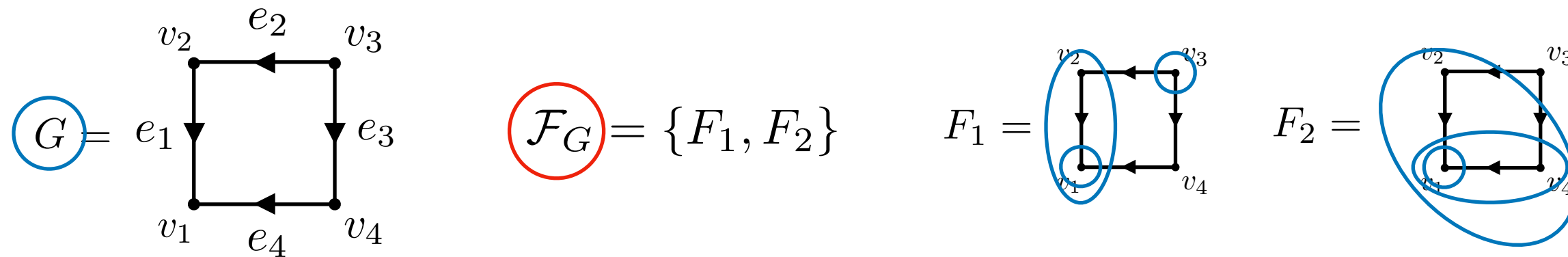
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Each element of a cross-free family corresponds to a threshold

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And the cross-free family corresponds to a product of thresholds

$$= \frac{i}{E_1 + E_4 - p_1^0} \frac{i}{E_2 + E_3 + p_3^0} \frac{i}{E_2 + E_4 - p_1^0 - p_2^0}$$

Discontinuities

Local discontinuities

*Spurious
singularities
in TOPT*

Discontinuities

Local discontinuities

We can compute discontinuities ([Bourjaily, Hannesdottir, McLeod, Schwartz, Vergu \[arXiv:2007.13747\]](#))

$$\frac{1}{\prod_{S \in F} [\mathbf{E} \cdot \mathbf{1}^{\partial(S)} - \sum_{v \in S} p_v^0 + i\varepsilon]} - \frac{1}{\prod_{S \in F} [\mathbf{E} \cdot \mathbf{1}^{\partial(S)} - \sum_{v \in S} p_v^0 - i\varepsilon]} = \sum_{S \in F} \frac{\delta(\mathbf{E} \cdot \mathbf{1}^{\partial(S)} - \sum_{v \in S} p_v^0)}{\prod_{S' \in F \setminus \{S\}} [\mathbf{E} \cdot \mathbf{1}^{\partial(S')} - \sum_{v \in S'} p_v^0]}$$

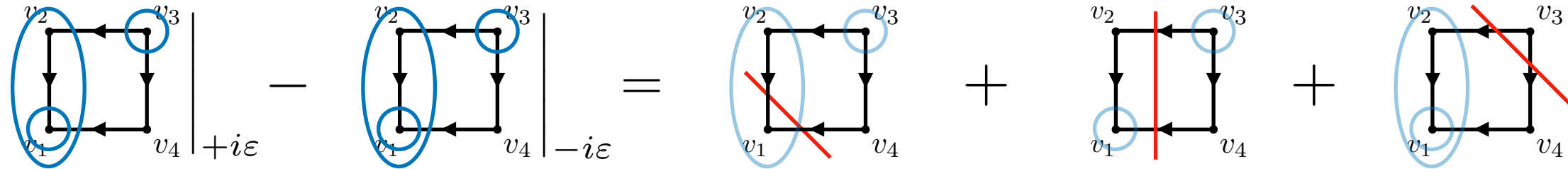
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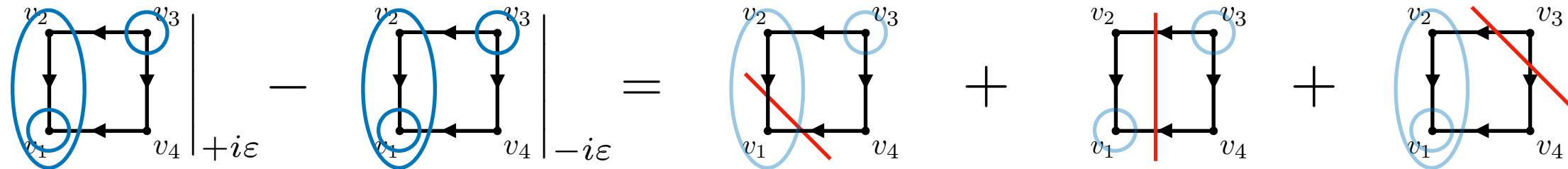
Spurious singularities in TOPT

Discontinuities

Local discontinuities

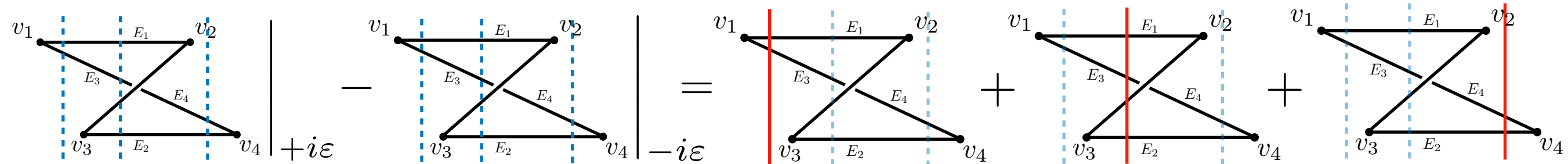
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$$\frac{1}{\prod_{S \in F} [\mathbf{E} \cdot \mathbf{1}^{\partial(S)} - \sum_{v \in S} p_v^0 + i\epsilon]} - \frac{1}{\prod_{S \in F} [\mathbf{E} \cdot \mathbf{1}^{\partial(S)} - \sum_{v \in S} p_v^0 - i\epsilon]} = \sum_{S \in F} \frac{\delta(\mathbf{E} \cdot \mathbf{1}^{\partial(S)} - \sum_{v \in S} p_v^0)}{\prod_{S' \in F \setminus \{S\}} [\mathbf{E} \cdot \mathbf{1}^{\partial(S')} - \sum_{v \in S'} p_v^0]}$$



Spurious singularities in TOPT

Why use the CFF rep. and not TOPT? Focus on the TOPT term ordering $\{v_1, v_3, v_2, v_4\}$

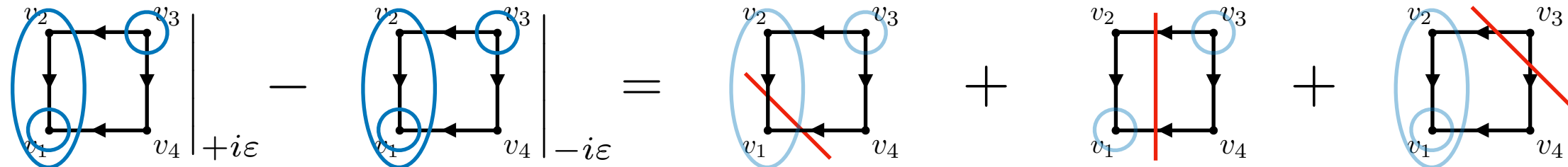


Discontinuities

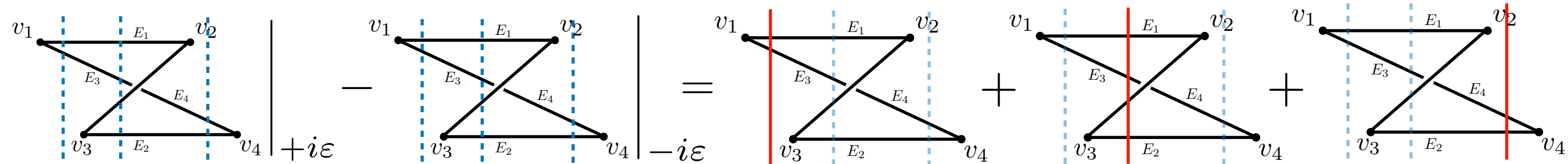
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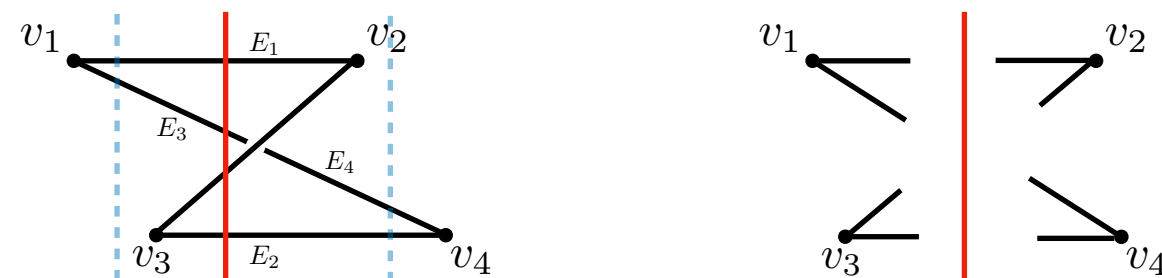
$$\frac{1}{\prod_{S \in F} [\mathbf{E} \cdot \mathbf{1}^{\partial(S)} - \sum_{v \in S} p_v^0 + i\epsilon]} - \frac{1}{\prod_{S \in F} [\mathbf{E} \cdot \mathbf{1}^{\partial(S)} - \sum_{v \in S} p_v^0 - i\epsilon]} = \sum_{S \in F} \frac{\delta(\mathbf{E} \cdot \mathbf{1}^{\partial(S)} - \sum_{v \in S} p_v^0)}{\prod_{S' \in F \setminus \{S\}} [\mathbf{E} \cdot \mathbf{1}^{\partial(S')} - \sum_{v \in S'} p_v^0]}$$



Why use the CFF rep. and not TOPT? Focus on the TOPT term ordering $\{v_1, v_3, v_2, v_4\}$



Looking at the second cut



Divides the graph in four connected components, but the CFF representation tells us this is not possible! It is a **spurious threshold**

$$\delta(E_1 + E_2 + E_3 + E_4 - p_1^0 - p_3^0)$$

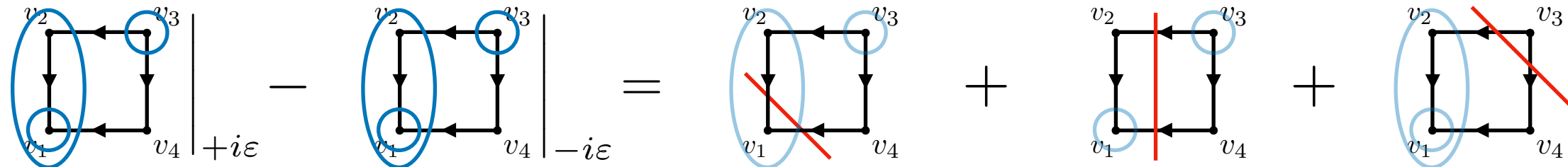
Spurious singularities in TOPT

Discontinuities

Local discontinuities

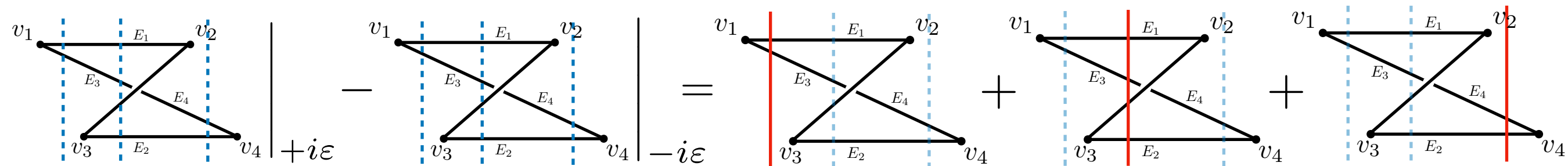
We can compute discontinuities (Bourjaily, Hannesdottir, McLeod, Schwartz, Vergu [arXiv:2007.13747])

$$\frac{1}{\prod_{S \in F} [\mathbf{E} \cdot \mathbf{1}^{\partial(S)} - \sum_{v \in S} p_v^0 + i\epsilon]} - \frac{1}{\prod_{S \in F} [\mathbf{E} \cdot \mathbf{1}^{\partial(S)} - \sum_{v \in S} p_v^0 - i\epsilon]} = \sum_{S \in F} \frac{\delta(\mathbf{E} \cdot \mathbf{1}^{\partial(S)} - \sum_{v \in S} p_v^0)}{\prod_{S' \in F \setminus \{S\}} [\mathbf{E} \cdot \mathbf{1}^{\partial(S')} - \sum_{v \in S'} p_v^0]}$$

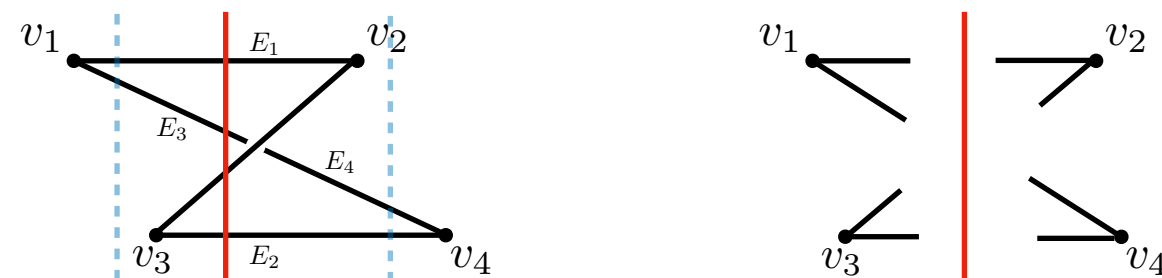


Spurious singularities in TOPT

Why use the CFF rep. and not TOPT? Focus on the TOPT term ordering $\{v_1, v_3, v_2, v_4\}$



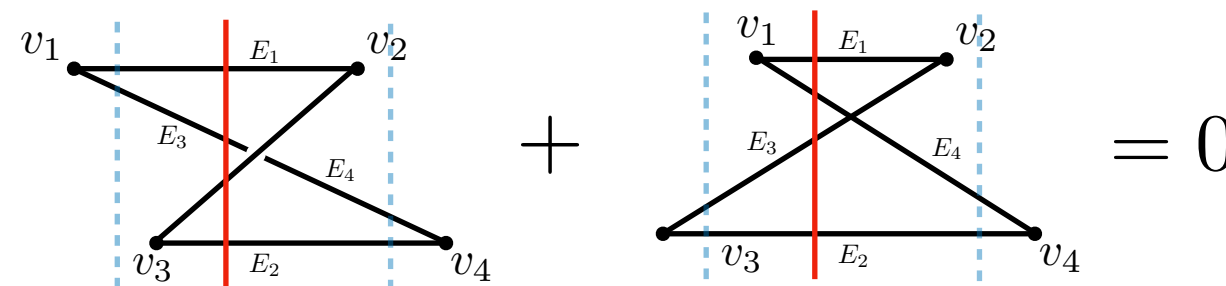
Looking at the second cut



Divides the graph in four connected components, but the CFF representation tells us this is not possible! It is a **spurious threshold**

$$\delta(E_1 + E_2 + E_3 + E_4 - p_1^0 - p_3^0)$$

How do we see that it is spurious?



*Factorisation
formula*

Spectators

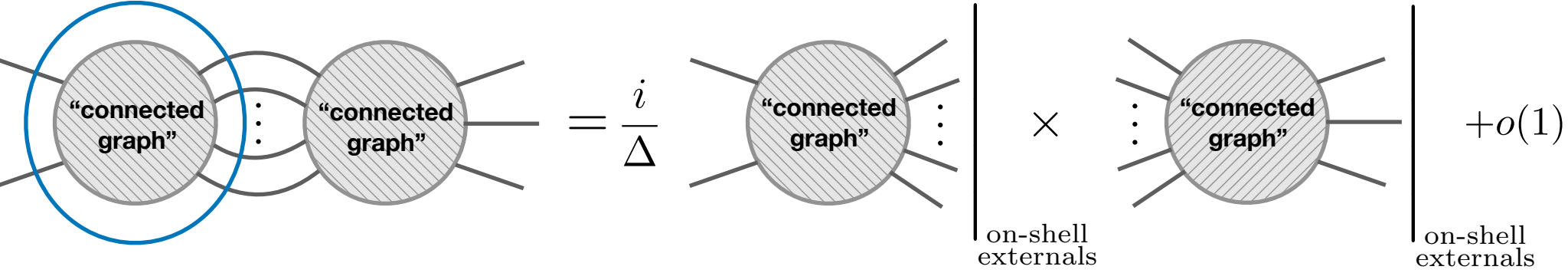
***Factorisation
formula***

We can see that in general from a diagram-level factorisation formula

Spectators

Factorisation formula

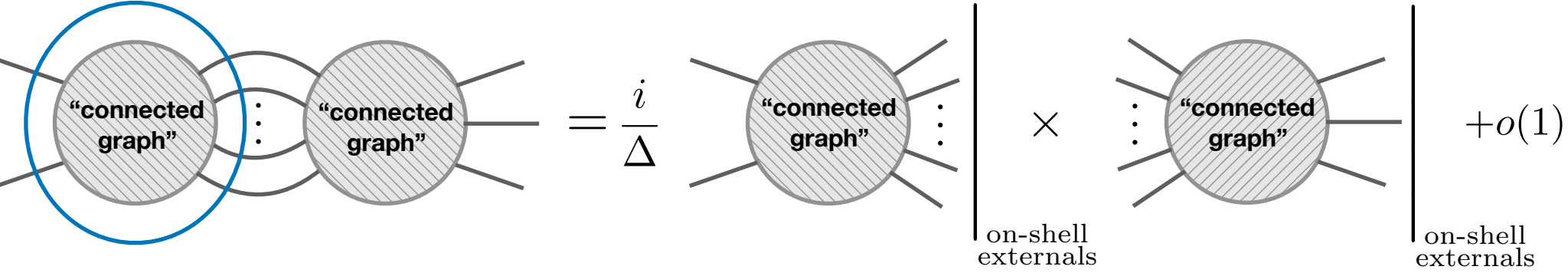
We can see that in general from a diagram-level factorisation formula



Spectators

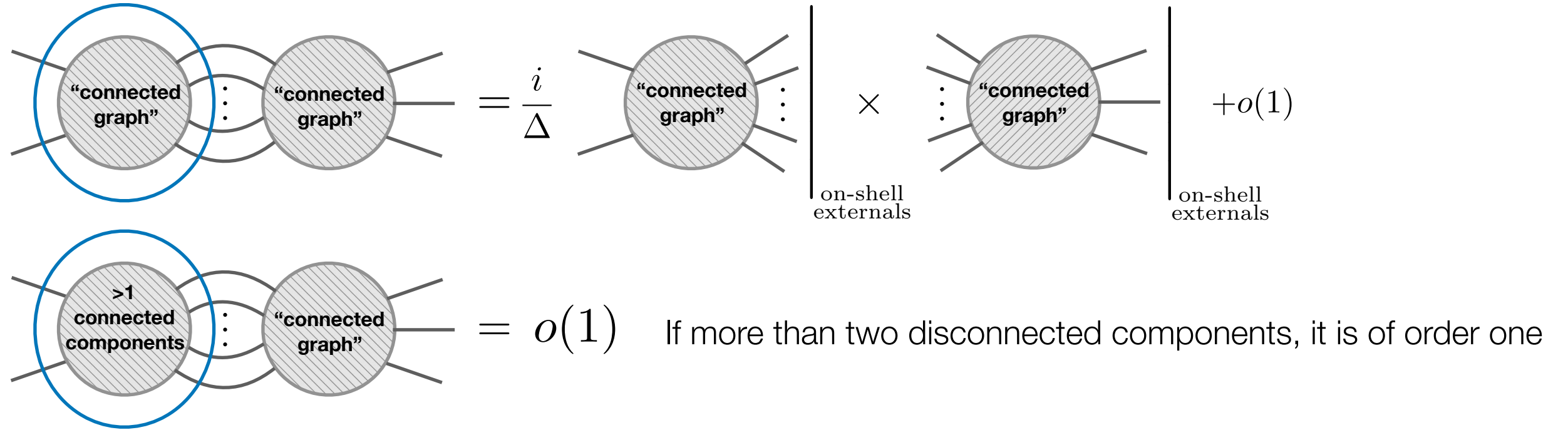
Factorisation formula

We can see that in general from a diagram-level factorisation formula



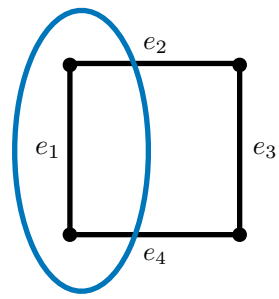
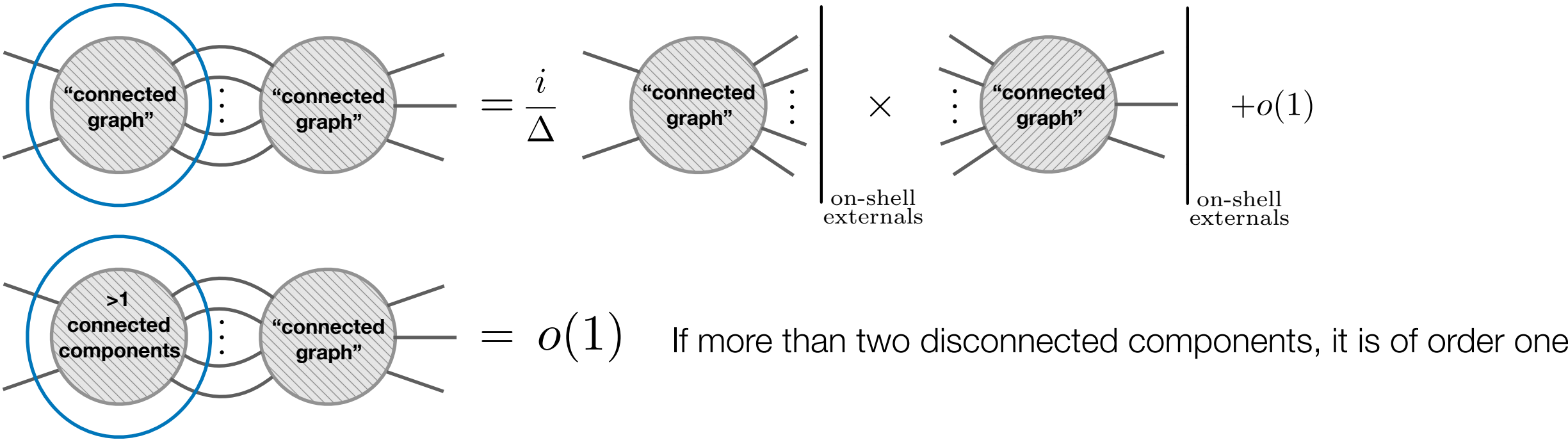
Spectators

We can see that in general from a diagram-level factorisation formula

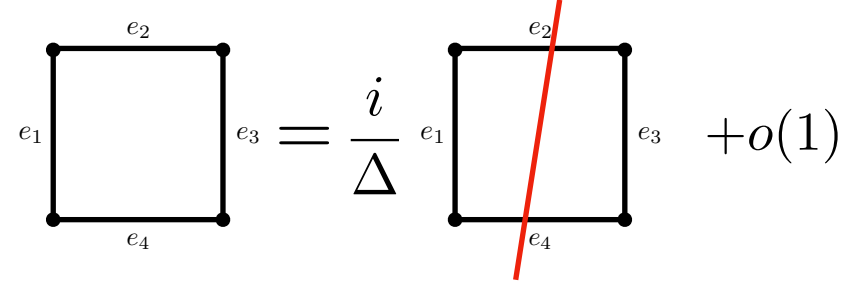


Factorisation formula

We can see that in general from a diagram-level factorisation formula



$$\Delta = E_2 + E_4 - p_1^0 - p_4^0 \rightarrow 0$$



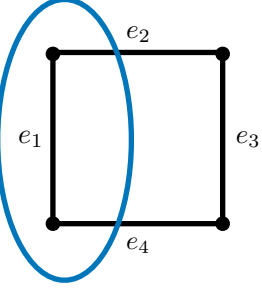
Spectators

Factorisation formula

We can see that in general from a diagram-level factorisation formula

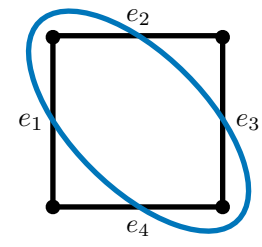
$$\text{“connected graph”} \text{---} \text{“connected graph”} = \frac{i}{\Delta} \left[\text{“connected graph”} \right]_{\text{on-shell externals}} \times \left[\text{“connected graph”} \right]_{\text{on-shell externals}} + o(1)$$

$$\text{“>1 connected components”} \text{---} \text{“connected graph”} = o(1) \quad \text{If more than two disconnected components, it is of order one}$$



$$\Delta = E_2 + E_4 - p_1^0 - p_4^0 \rightarrow 0$$

$$\text{Square} = \frac{i}{\Delta} \text{Square} + o(1)$$



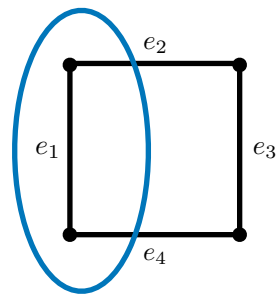
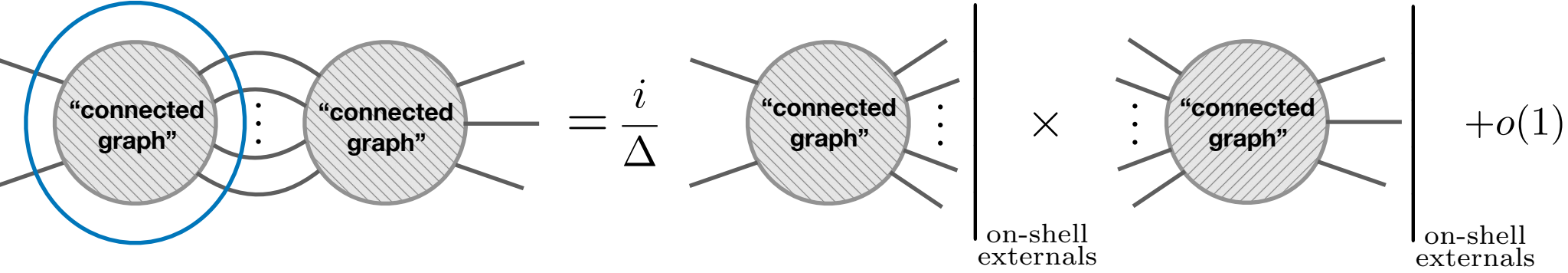
$$\Delta = E_1 + E_2 + E_3 + E_4 - p_1^0 - p_4^0 \rightarrow 0$$

$$\text{Square} = o(1)$$

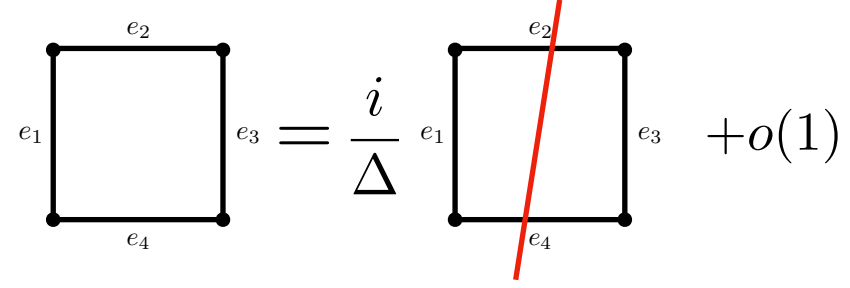
Spectators

Factorisation formula

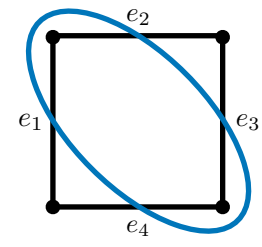
We can see that in general from a diagram-level factorisation formula



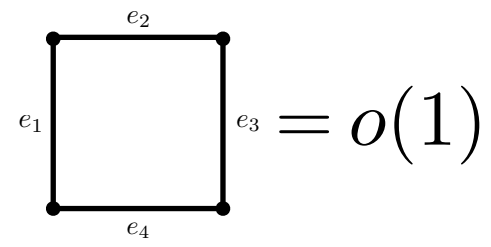
$$\Delta = E_2 + E_4 - p_1^0 - p_4^0 \rightarrow 0$$



$$= \frac{i}{\Delta} \text{ (diagonal square) } + o(1)$$



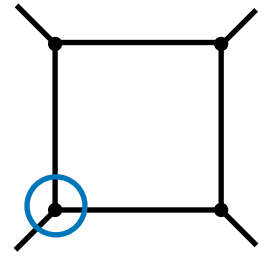
$$\Delta = E_1 + E_2 + E_3 + E_4 - p_1^0 - p_4^0 \rightarrow 0$$



$$= o(1)$$

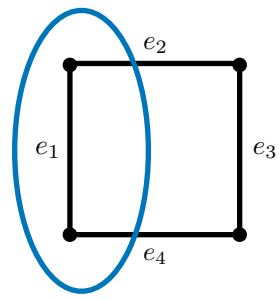
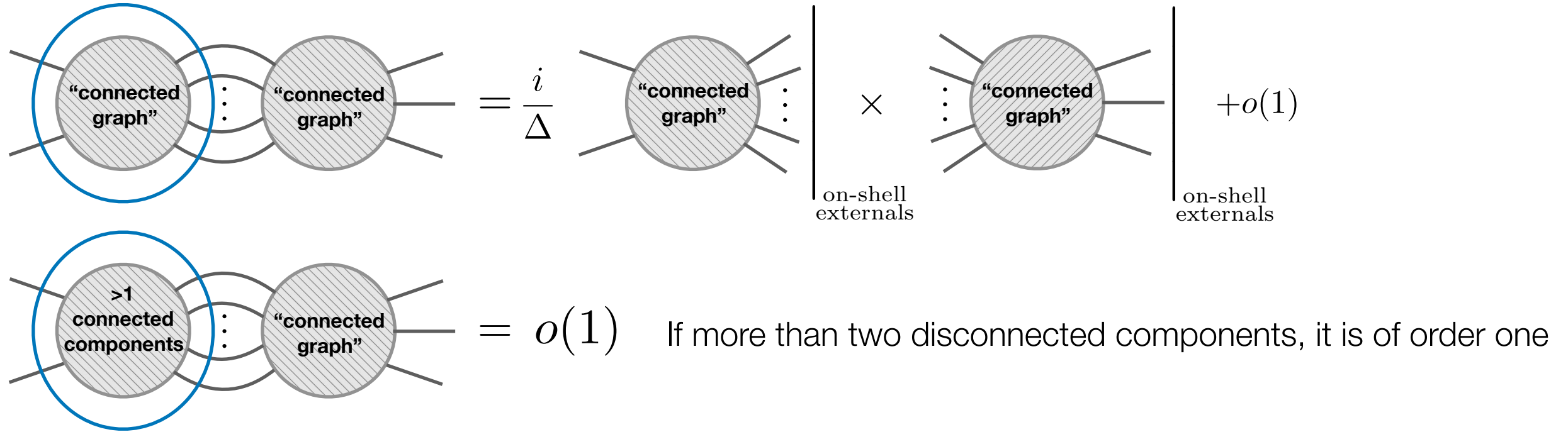
Spectators

What do we really mean by "connected graph"?

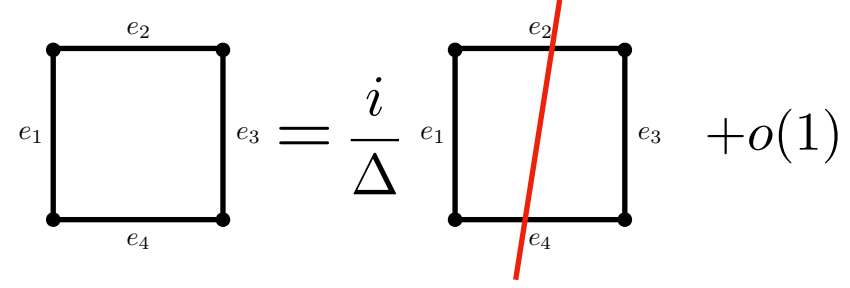


Factorisation formula

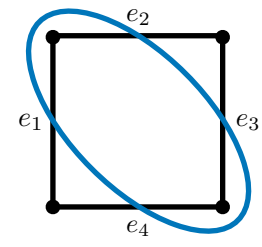
We can see that in general from a diagram-level factorisation formula



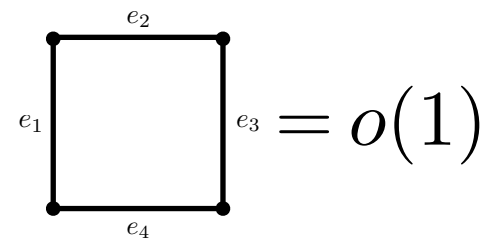
$$\Delta = E_2 + E_4 - p_1^0 - p_4^0 \rightarrow 0$$



$$= \frac{i}{\Delta} \text{Square with } e_2, e_4 \text{ crossed out} + o(1)$$



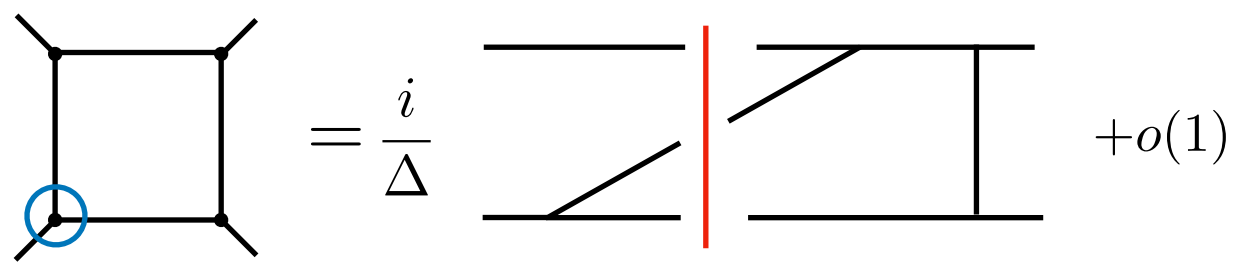
$$\Delta = E_1 + E_2 + E_3 + E_4 - p_1^0 - p_4^0 \rightarrow 0$$



$$= o(1)$$

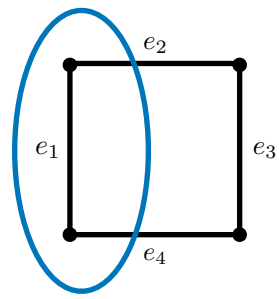
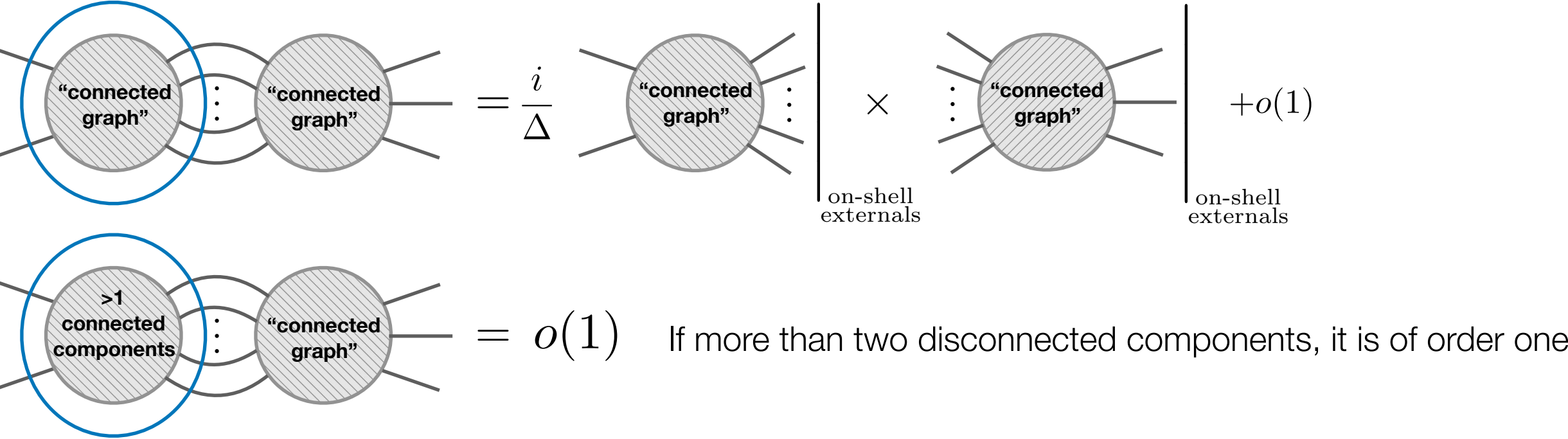
Spectators

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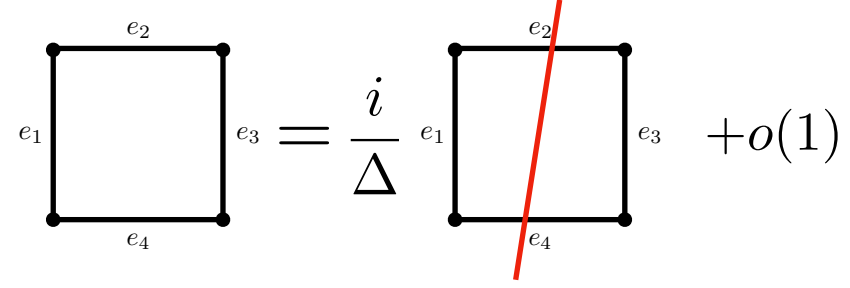


Factorisation formula

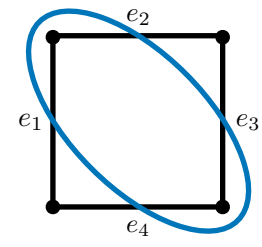
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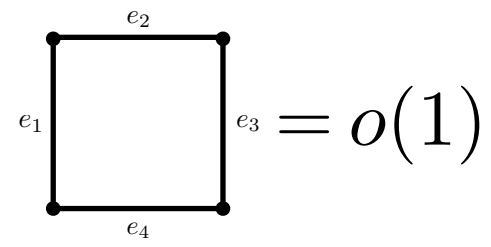
$$\Delta = E_2 + E_4 - p_1^0 - p_4^0 \rightarrow 0$$



$$= \frac{i}{\Delta} + o(1)$$



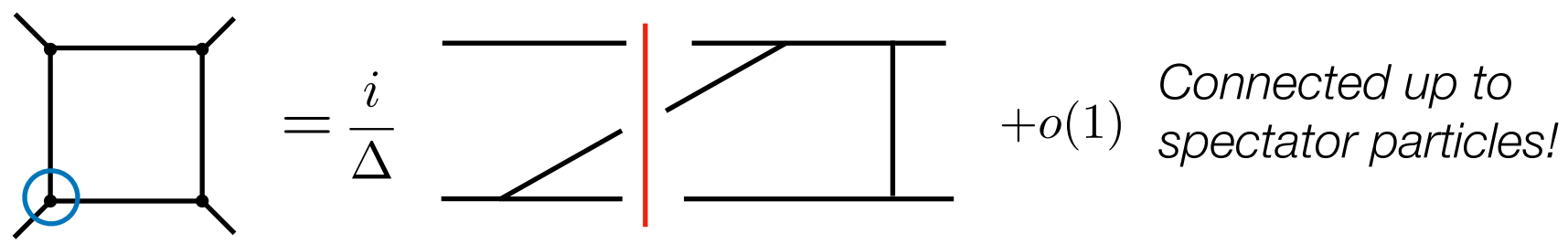
$$\Delta = E_1 + E_2 + E_3 + E_4 - p_1^0 - p_4^0 \rightarrow 0$$



$$= o(1)$$

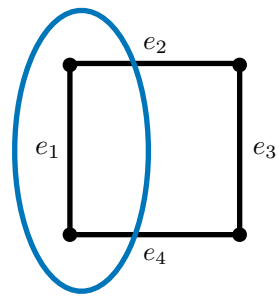
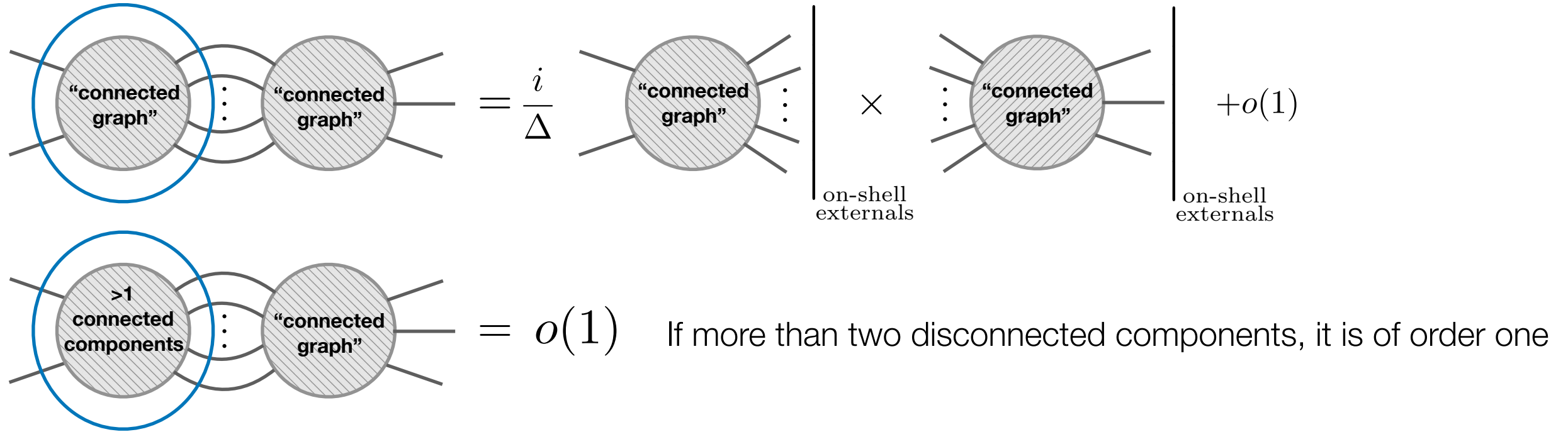
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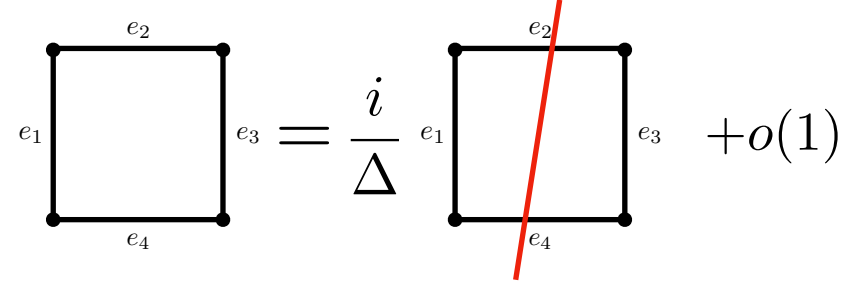


Factorisation formula

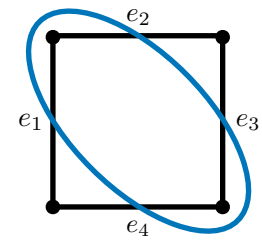
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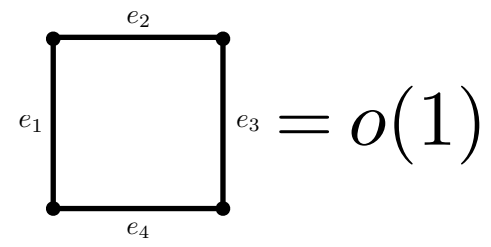
$$\Delta = E_2 + E_4 - p_1^0 - p_4^0 \rightarrow 0$$



$$= \frac{i}{\Delta} \text{Square with red diagonal} + o(1)$$



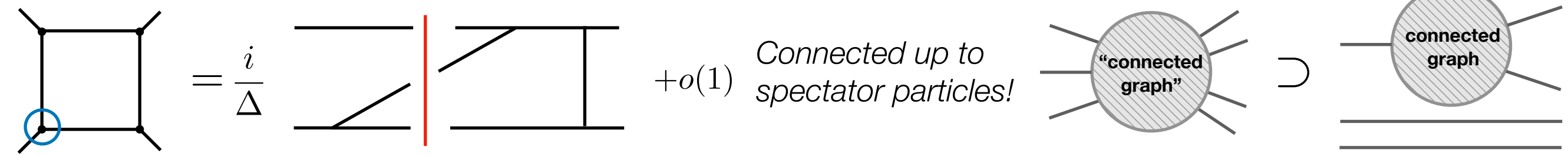
$$\Delta = E_1 + E_2 + E_3 + E_4 - p_1^0 - p_4^0 \rightarrow 0$$



$$= o(1)$$

Spectators

What do we really mean by "connected graph"?



Second part: cluster decomposition and infrared finiteness

- Highlight role of connectedness at the operator level

$$\langle \alpha | \mathbf{T}_c | \beta \rangle \supset \alpha \left\{ \begin{array}{c} \text{connected} \\ \text{graph} \\ \hline \hline \end{array} \right\} \beta$$

Reconstruct unitarity, cluster decomposition principle and infrared finiteness from the diagrammatic analysis

- Use Local Unitarity methods to take advantage of this analysis to numerically evaluate cross-sections

Cluster Decomposition

*Connected
transition matrix*

S-matrix

“Unitarity”

Cluster Decomposition

*Connected
transition matrix*

Can we express the role of connectedness at the operator level? Define the connected transition matrix

S-matrix

“Unitarity”

Cluster Decomposition

Can we express the role of connectedness at the operator level? Define the connected transition matrix

$$\langle \alpha | \mathbf{T}_c | \beta \rangle \supset \alpha \left\{ \begin{array}{c} \text{connected} \\ \text{graph} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} \beta$$

Connected transition matrix

S-matrix

“Unitarity”

Cluster Decomposition

Can we express the role of connectedness at the operator level? Define the connected transition matrix

$$\langle \alpha | \mathbf{T}_c | \beta \rangle \supset \alpha \left\{ \begin{array}{c} \text{connected} \\ \text{graph} \\ \hline \hline \end{array} \right\} \beta$$

The connected transition matrix is not unitary (expected). What is the relationship with S-matrix?

Connected transition matrix

S-matrix

“Unitarity”

Connected transition matrix

Cluster Decomposition

Can we express the role of connectedness at the operator level? Define the connected transition matrix

$$\langle \alpha | \mathbf{T}_c | \beta \rangle \supset \alpha \left\{ \begin{array}{c} \text{connected graph} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} \beta$$

S-matrix

The connected transition matrix is not unitary (expected). What is the relationship with S-matrix?

$$\mathbf{S} = \text{---} + \text{---} \text{ "connected graph" } \text{---} + \begin{array}{c} \text{---} \text{ "connected graph" } \text{---} \\ \text{---} \text{ "connected graph" } \text{---} \end{array} + \begin{array}{c} \text{---} \text{ "connected graph" } \text{---} \\ \text{---} \text{ "connected graph" } \text{---} \\ \text{---} \text{ "connected graph" } \text{---} \end{array} + \dots$$

"Unitarity"

Cluster Decomposition

Can we express the role of connectedness at the operator level? Define the connected transition matrix

$$\langle \alpha | \mathbf{T}_c | \beta \rangle \supset \alpha \left\{ \begin{array}{c} \text{connected graph} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} \beta$$

The connected transition matrix is not unitary (expected). What is the relationship with S-matrix?

$$\mathbf{S} = \text{---} + \text{---} \text{---} \text{---} + \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \dots$$

$$\mathbf{S} = \text{---} + \text{---} \text{---} \text{---} + \frac{1}{2!} \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right) + \frac{1}{3!} \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \dots \right)$$

Cluster Decomposition

Can we express the role of connectedness at the operator level? Define the connected transition matrix

$$\langle \alpha | \mathbf{T}_c | \beta \rangle \supset \alpha \left\{ \begin{array}{c} \text{connected graph} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} \beta$$

The connected transition matrix is not unitary (expected). What is the relationship with S-matrix?

$$\mathbf{S} = \text{---} + \text{---} \text{---} \text{---} + \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \dots$$

$$\mathbf{S} = \text{---} + \text{---} \text{---} \text{---} + \frac{1}{2!} \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right) + \frac{1}{3!} \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \dots \right)$$

$$\mathbf{S} = 1 + i\mathbf{T}_c + \frac{i^2}{2!} \mathbf{T}_c^2 + \frac{i^3}{3!} \mathbf{T}_c^3 + \dots = e^{i\mathbf{T}_c}$$

Cluster Decomposition

Can we express the role of connectedness at the operator level? Define the connected transition matrix

$$\langle \alpha | \mathbf{T}_c | \beta \rangle \supset \alpha \left\{ \begin{array}{c} \text{connected graph} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} \beta$$

The connected transition matrix is not unitary (expected). What is the relationship with S-matrix?

$$\mathbf{S} = \text{---} + \text{---} \text{---} \text{---} + \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \dots$$

$$\mathbf{S} = \text{---} + \text{---} \text{---} \text{---} + \frac{1}{2!} \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right) + \frac{1}{3!} \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \dots \right)$$

$$\mathbf{S} = 1 + i\mathbf{T}_c + \frac{i^2}{2!} \mathbf{T}_c^2 + \frac{i^3}{3!} \mathbf{T}_c^3 + \dots = e^{i\mathbf{T}_c} \quad (\text{evokes } Z[J] = e^{iW[J]})$$

Cluster Decomposition

Can we express the role of connectedness at the operator level? Define the connected transition matrix

$$\langle \alpha | \mathbf{T}_c | \beta \rangle \supset \alpha \left\{ \begin{array}{c} \text{connected graph} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} \beta$$

The connected transition matrix is not unitary (expected). What is the relationship with S-matrix?

$$\mathbf{S} = \text{---} + \text{---} \text{---} \text{---} \text{---} + \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \dots$$

$$\mathbf{S} = \text{---} + \text{---} \text{---} \text{---} \text{---} + \frac{1}{2!} \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right) + \frac{1}{3!} \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \dots \right)$$

$$\mathbf{S} = 1 + i\mathbf{T}_c + \frac{i^2}{2!} \mathbf{T}_c^2 + \frac{i^3}{3!} \mathbf{T}_c^3 + \dots = e^{i\mathbf{T}_c} \quad (\text{evokes } Z[J] = e^{iW[J]})$$

(See also holomorphic cutting rules: Hannesdottir, Mizera [arXiv:2204.02988])

Cluster Decomposition

Can we express the role of connectedness at the operator level? Define the connected transition matrix

$$\langle \alpha | \mathbf{T}_c | \beta \rangle \supset \alpha \left\{ \begin{array}{c} \text{connected graph} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} \beta$$

The connected transition matrix is not unitary (expected). What is the relationship with S-matrix?

$$\mathbf{S} = \text{---} + \text{---} \text{---} \text{---} \text{---} + \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \dots$$

$$\mathbf{S} = \text{---} + \text{---} \text{---} \text{---} \text{---} + \frac{1}{2!} \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right) + \frac{1}{3!} \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} + \dots \right)$$

$$\mathbf{S} = 1 + i\mathbf{T}_c + \frac{i^2}{2!} \mathbf{T}_c^2 + \frac{i^3}{3!} \mathbf{T}_c^3 + \dots = e^{i\mathbf{T}_c} \quad (\text{evokes } Z[J] = e^{iW[J]})$$

(See also holomorphic cutting rules: Hannesdottir, Mizera [arXiv:2204.02988])

In order to establish this relation, we need the following formula

Cluster Decomposition

Can we express the role of connectedness at the operator level? Define the connected transition matrix

$$\langle \alpha | \mathbf{T}_c | \beta \rangle \supset \alpha \left\{ \begin{array}{c} \text{connected graph} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} \beta$$

The connected transition matrix is not unitary (expected). What is the relationship with S-matrix?

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α'

$\left\{ \begin{array}{c} \text{connected graph} \\ \text{---} \\ \text{---} \end{array} \right\}$

β'

$\alpha \setminus \alpha'$

$\left\{ \begin{array}{c} \text{---} \\ \text{---} \left(\text{connected graph} \right) \text{---} \\ \text{---} \end{array} \right\}$

$\beta \setminus \beta'$

cluster decomposition term

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cluster decomposition term

Factorisation formula expressed at the operator level!

Cluster decomposition principle

Transition probabilities

Clusters and infrared finiteness

***Cluster
decomposition
principle***

This S-matrix also trivially satisfies the cluster-decomposition principle. Indeed

$$\mathbf{T}_c(|\alpha\rangle \otimes |\beta\rangle) = (\mathbf{T}_c |\alpha\rangle) \otimes |\beta\rangle + |\alpha\rangle \otimes (\mathbf{T}_c |\beta\rangle) \quad \Rightarrow \quad P = P_A P_B$$

for states with large space-like separations.

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Using it, we can compute transition probabilities

$$P = \text{Tr}[\rho \mathbf{S} P \mathbf{S}^\dagger]$$

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Sum over massless particles requires decoherence

Clusters and infrared finiteness

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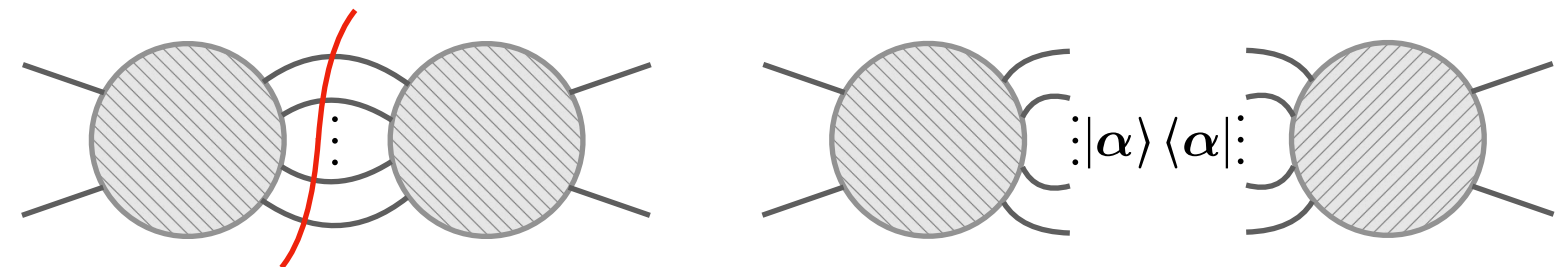
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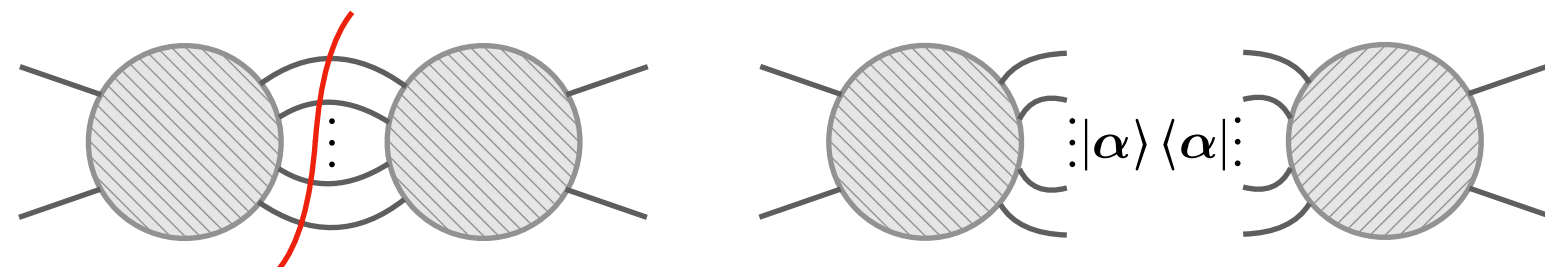
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And also the reason why we can write interference diagrams in the first place!

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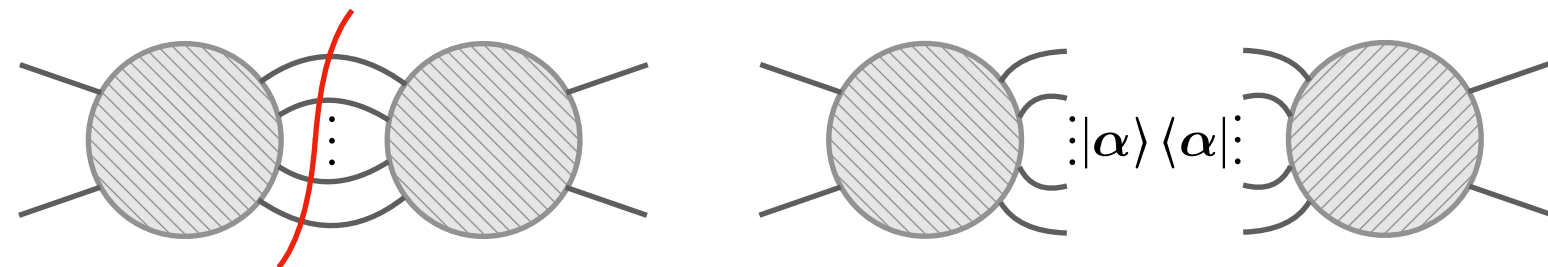
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Clusters and infrared finiteness

Finally:

$$P = \text{Tr}[\rho \mathbf{S} P \mathbf{S}] = \sum_{n,m} \frac{i^{n+m}}{n!m!} \text{Tr}[\rho \mathbf{T}_c^n P (\mathbf{T}_c^\dagger)^m]$$

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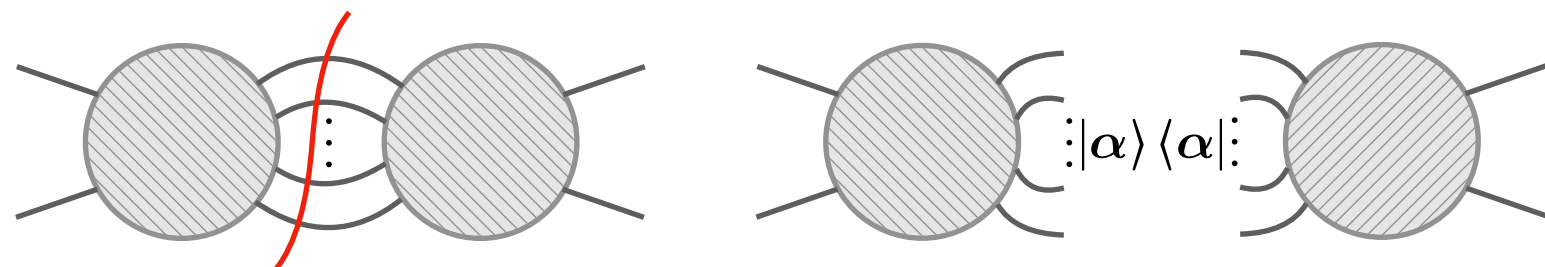
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$$P = \text{Tr}[\rho \mathbf{S} P \mathbf{S}] = \sum_{n,m} \frac{i^{n+m}}{n!m!} \underbrace{\text{Tr}[\rho \mathbf{T}_c^n P (\mathbf{T}_c^\dagger)^m]}_{\text{Infrared finite if density matrix and projector sum over degenerate massless radiation}}$$

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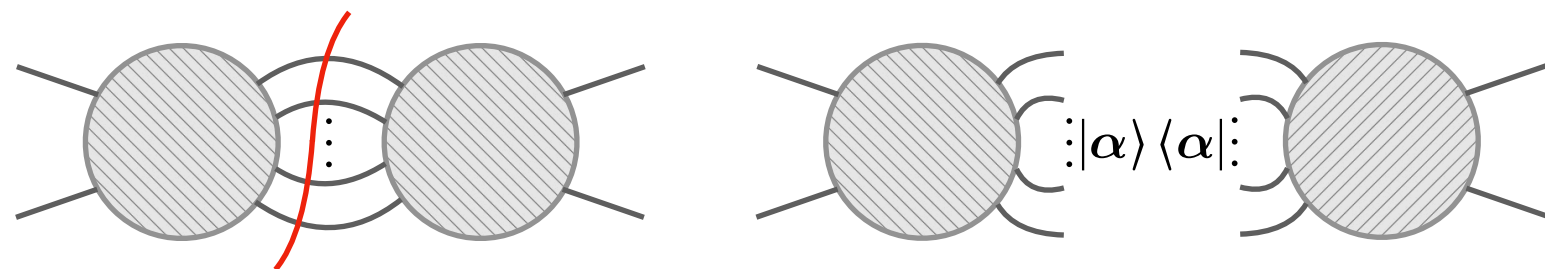
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Infrared finite if density matrix and projector sum over degenerate massless radiation

Infrared-finiteness follows from the unitarity relation we showed in the previous slide. But we can also look at it at a diagrammatic level.

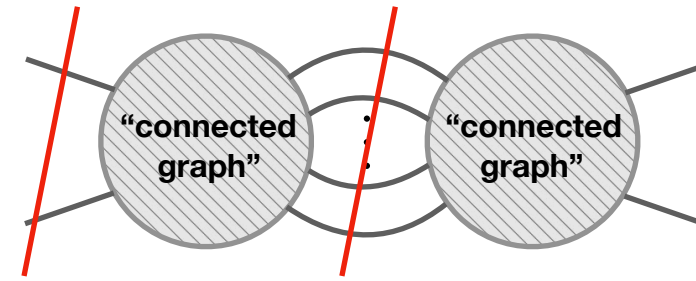
*Final-state sum
example*

How do we show it?

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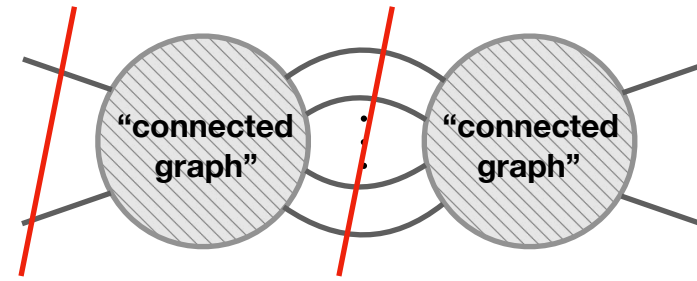
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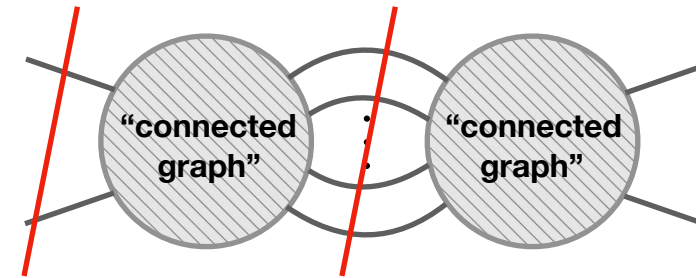


*Final-state sum
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Using the CFF representation we can show that

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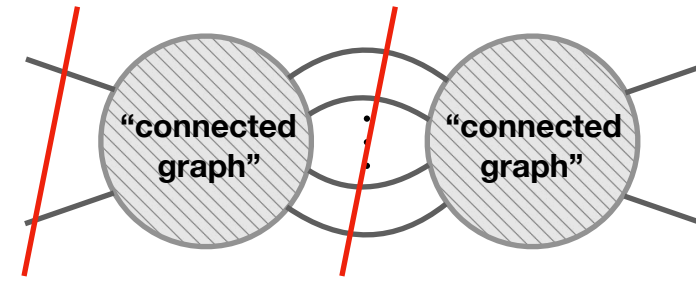
Using the CFF representation we can show that

$$= \lim_{\sqrt{s} \rightarrow E_C} (E_C - \sqrt{s})$$

A diagram showing two circular nodes, each labeled "connected graph" and filled with diagonal hatching. They are connected by two horizontal arcs. A vertical red line passes through the center of the two nodes, with a red letter 'C' below it. To the right of this diagram is an equals sign followed by a limit expression: $\lim_{\sqrt{s} \rightarrow E_C} (E_C - \sqrt{s})$. To the right of the limit expression is a single circular node labeled "connected graph" with diagonal hatching and four external lines.

How do we show it?

$$\text{Tr}[\rho \mathbf{T}_c P \mathbf{T}_c^\dagger] \supset$$



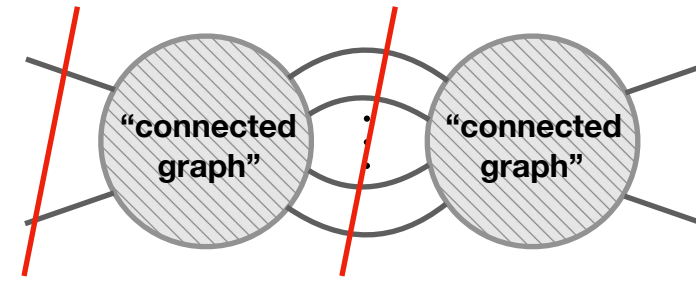
Final-state sum example

Using the CFF representation we can show that

$$\begin{aligned}
 & \text{Diagram with two 'connected graph' nodes connected by a vertical line, with a red vertical line labeled 'C' cutting through the connection.} \\
 &= \lim_{\sqrt{s} \rightarrow E_C} (E_C - \sqrt{s}) \text{Diagram with one 'connected graph' node.} \\
 &= \frac{N(E_C)}{\prod_{C' \neq C} (E_{C'} - E_C)}
 \end{aligned}$$

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$$\begin{array}{c} \text{"connected graph"} \\ \vdots \\ \text{"connected graph"} \end{array} \begin{array}{c} \text{"connected graph"} \\ \vdots \\ \text{"connected graph"} \end{array} = \lim_{\sqrt{s} \rightarrow E_C} (E_C - \sqrt{s}) \begin{array}{c} \text{"connected graph"} \end{array} = \frac{N(E_C)}{\prod_{C' \neq C} (E_{C'} - E_C)}$$

The diagram shows two "connected graph" nodes connected by a vertical line, with a red 'C' below the line. This is followed by an equals sign and a limit expression, then a single "connected graph" node, and another equals sign and a fraction.

This relation allows to collect locally interference diagrams

How do we show it?

$$\text{Tr}[\rho \mathbf{T}_c P \mathbf{T}_c^\dagger] \supset \text{Diagram with two "connected graph" nodes connected by a vertical line, with red slashes on the external lines.$$

Final-state sum example

Using the CFF representation we can show that

$$\text{Diagram with two "connected graph" nodes connected by a vertical line labeled } C \text{ with a red slash} = \lim_{\sqrt{s} \rightarrow E_C} (E_C - \sqrt{s}) \text{Diagram with one "connected graph" node} = \frac{N(E_C)}{\prod_{C' \neq C} (E_{C'} - E_C)}$$

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Example: consider massless scalar corrections to the decay of a massive scalar $\rho = |\phi^*\rangle \langle \phi^*|$

How do we show it?

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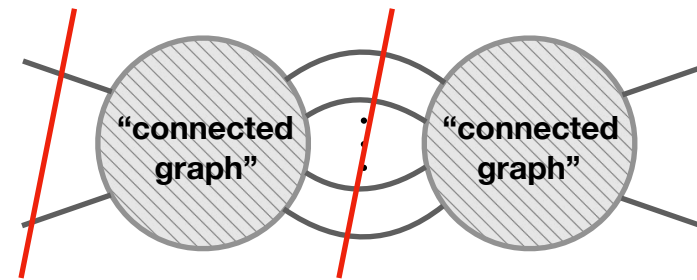
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$$\sigma(\phi^* \rightarrow n \text{ jets}_\phi) =$$

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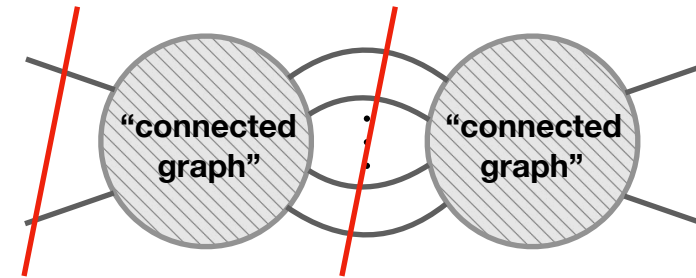
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$$\begin{array}{c} \text{“connected graph”} \\ \vdots \\ \text{“connected graph”} \end{array} \quad = \lim_{\sqrt{s} \rightarrow E_C} (E_C - \sqrt{s}) \quad \text{“connected graph”} \quad = \frac{N(E_C)}{\prod_{C' \neq C} (E_{C'} - E_C)}$$

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Example: consider massless scalar corrections to the decay of a massive scalar $\rho = |\phi^*\rangle \langle \phi^*|$

$$\sigma(\phi^* \rightarrow n \text{ jets}_\phi) = \text{---} \left[\text{circle with } \mathbb{O}_2 \text{ cut} \right] \text{---} + \text{---} \left[\text{circle with } \mathbb{O}_3 \text{ cut} \right] \text{---} + \text{---} \left[\text{circle with } \mathbb{O}_3 \text{ cut} \right] \text{---} + \text{---} \left[\text{circle with } \mathbb{O}_2 \text{ cut} \right] \text{---} = \sum_{i=1}^4 \int d\Pi_i f_i$$

How do we show it?

$$\text{Tr}[\rho \mathbf{T}_c P \mathbf{T}_c^\dagger] \supset \text{Diagram with two "connected graph" circles connected by a vertical line, with red slashes on the top and bottom lines.$$

Final-state sum example

Using the CFF representation we can show that

$$\text{Diagram with two "connected graph" circles connected by a vertical line labeled } C = \lim_{\sqrt{s} \rightarrow E_C} (E_C - \sqrt{s}) \text{ Diagram with one "connected graph" circle} = \frac{N(E_C)}{\prod_{C' \neq C} (E_{C'} - E_C)}$$

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Example: consider massless scalar corrections to the decay of a massive scalar $\rho = |\phi^*\rangle \langle \phi^*|$

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How do we show it?

$$\text{Tr}[\rho \mathbf{T}_c P \mathbf{T}_c^\dagger] \supset \text{Diagram with two "connected graph" nodes connected by a vertical line, with red slashes on the external lines.$$

Final-state sum
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Using the CFF representation we can show that

$$\text{Diagram with two "connected graph" nodes connected by a vertical line labeled } C \text{ with a red slash} = \lim_{\sqrt{s} \rightarrow E_C} (E_C - \sqrt{s}) \text{Diagram with one "connected graph" node} \stackrel{\ominus}{=} \frac{N(E_C)}{\prod_{C' \neq C} (E_{C'} - E_C)}$$

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Diagrams 1-4 show a circle with a vertical line and a red slash on the external lines, labeled \mathbb{O}_2 and \mathbb{O}_3 .

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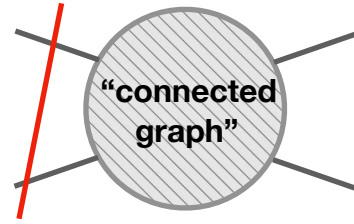
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Locally finite

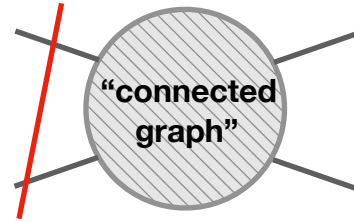
In the preceding example, we fixed a massive initial-state. What if we want to have massless initial states?

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We need to write forward-scattering diagrams as residues of something!

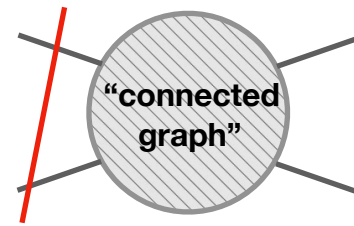
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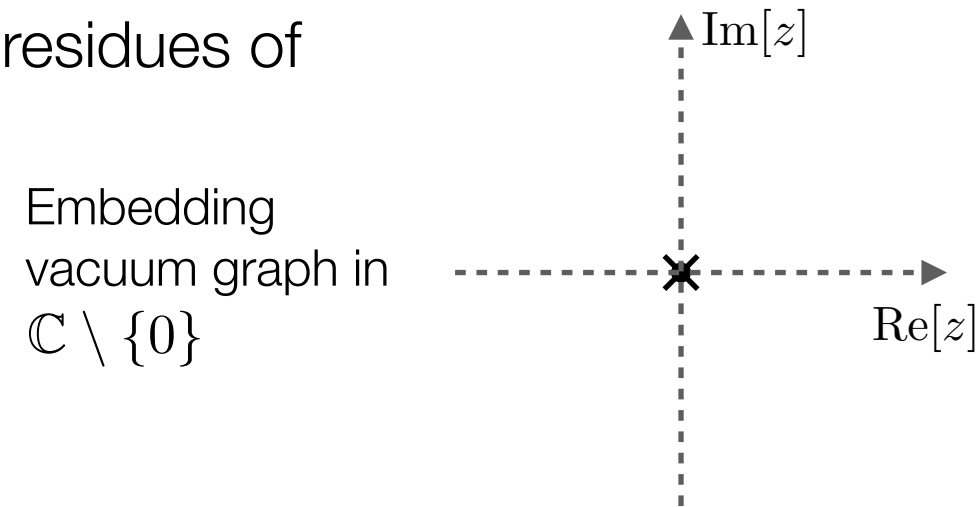
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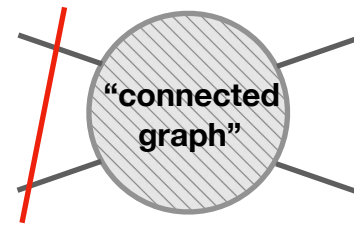


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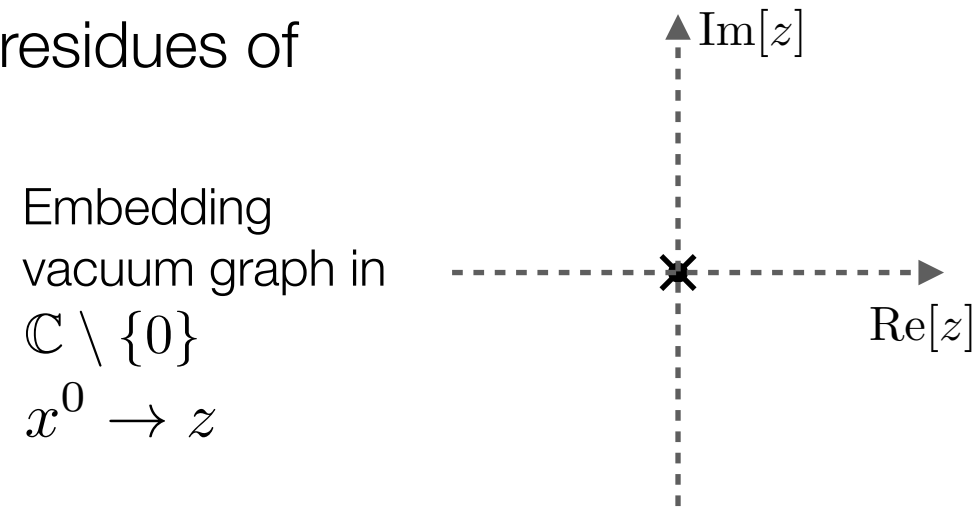


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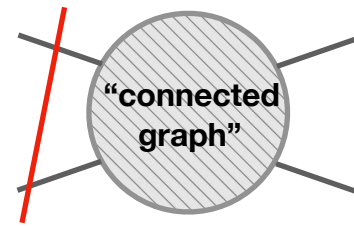


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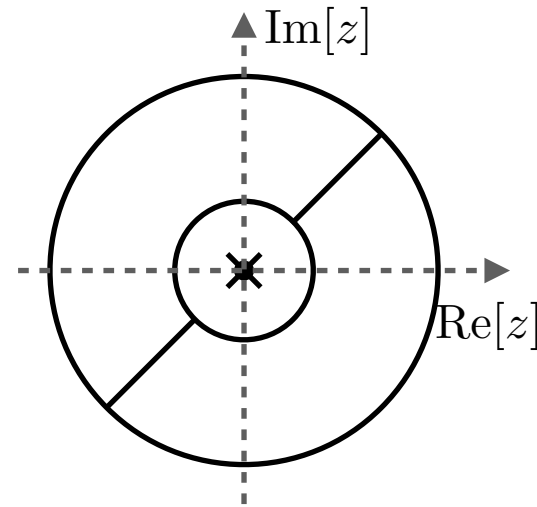
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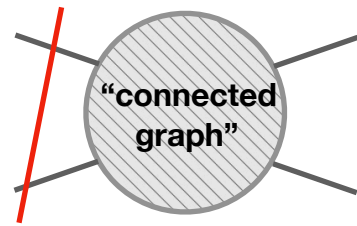
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Embedding
vacuum graph in
 $\mathbb{C} \setminus \{0\}$
 $x^0 \rightarrow z$



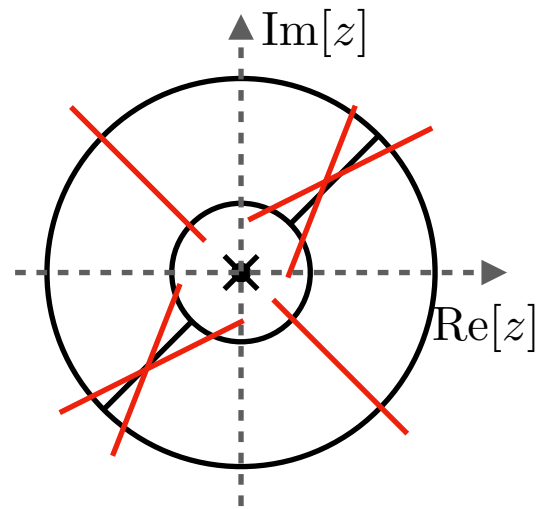
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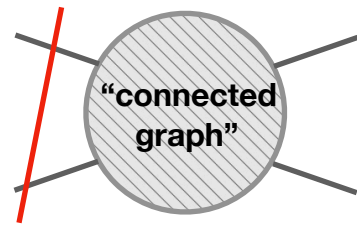
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A Cutkosky cut is a minimal set of edges whose deletion makes the graph contractible

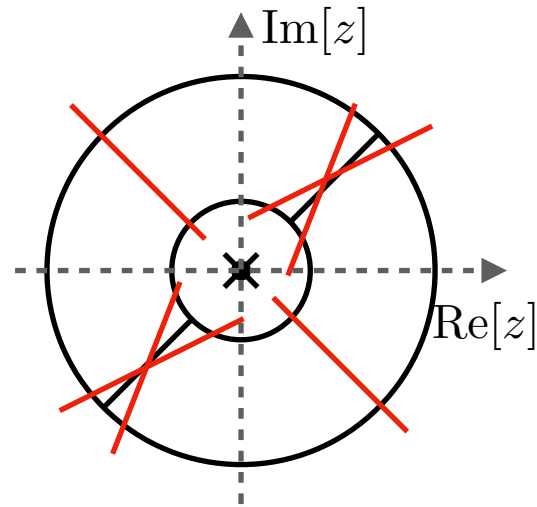
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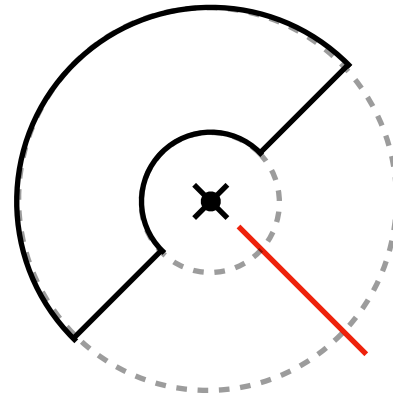
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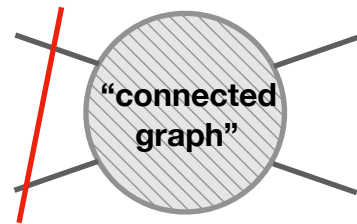
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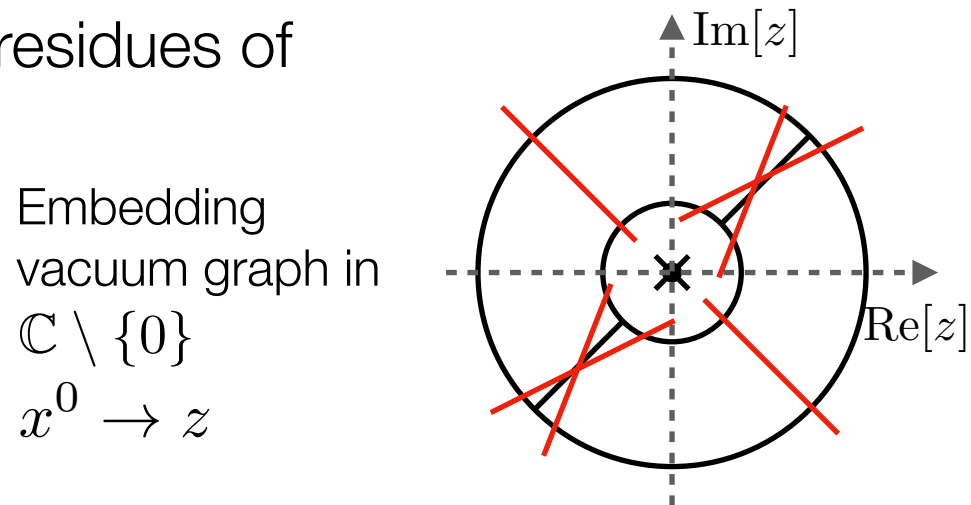


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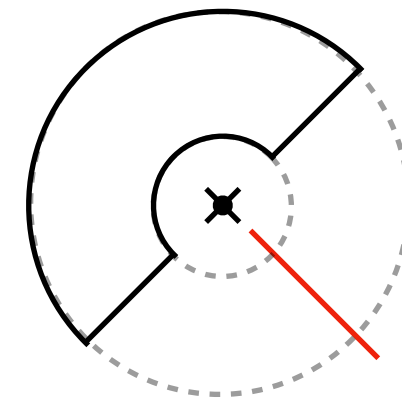
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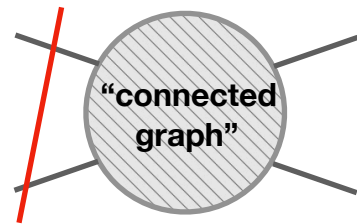
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We can construct a three-dimensional representation for embeddings

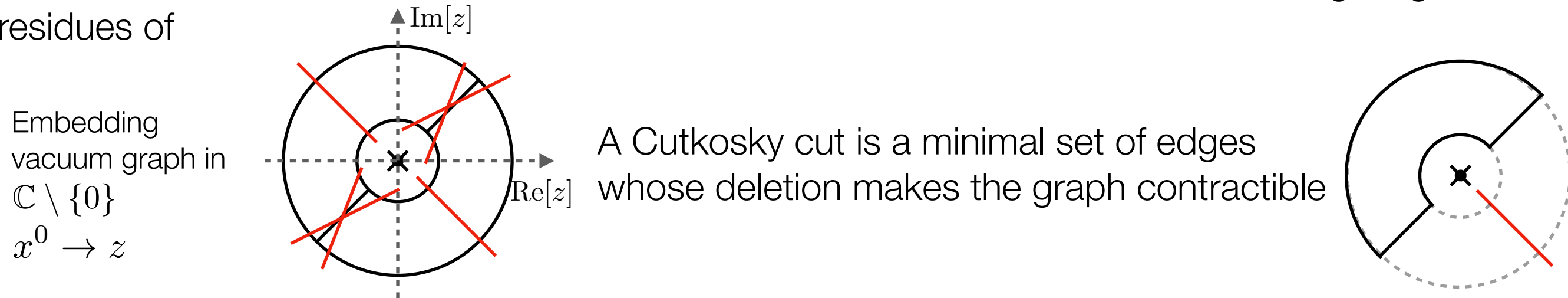
$$= \int \left[\prod_{i=1}^6 \frac{dq_i^0}{q_i^2} \right] \delta(q_1 + q_2 - p) \delta(q_1 + q_2 - p) \delta(q_2 + q_3 - q_6 - p) \delta(q_5 + q_6 - p) \delta(q_2 + q_4 + q_6 - p)$$

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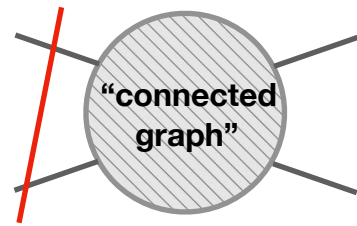
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Embeddings have thresholds associated with their Cutkosky cuts

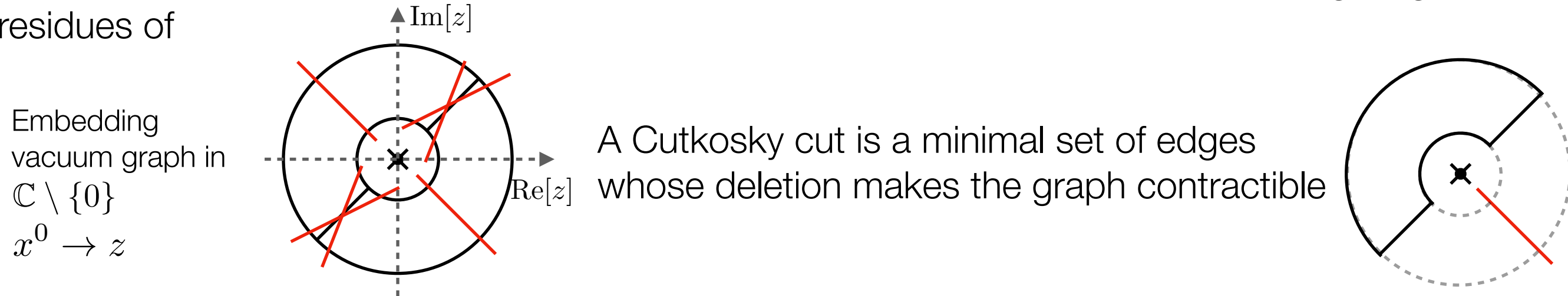
$$= \lim_{p^0 \rightarrow E_1 + E_2} (E_1 + E_2 - p^0)$$

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Embeddings have thresholds associated with their Cutkosky cuts

This observation allows to extend the Local Unitarity representation to initial states!

Conclusion

- Energy conservation implies a rigid diagrammatic structure for threshold singularities
 - Connectedness
 - Absence of crossing
 - Obstruction-freedom

- These principles are manifest in a novel 3D-representation that holds for any theory (independent of numerator) and at any loop

Implemented in
Mathematica package



<https://github.com/apelloni/cLTD>

- The presence of connectedness suggests that, in order to understand IR-finiteness, one should decompose cross-sections according to the degree of connectedness
- 3D representations can be used to express interference diagrams as local residues of forward-scattering diagrams, and forward-scattering diagrams as local residues of vacuum embeddings
- In turn, these local residues can be used to write cross-sections in a way that manifests the KLN cancellation mechanism at the local level (Local Unitarity)
- Local Unitarity can be used to numerically evaluate cross-sections