


Physical thresholds and cluster decomposition

Zeno Capatti
ETH Zürich

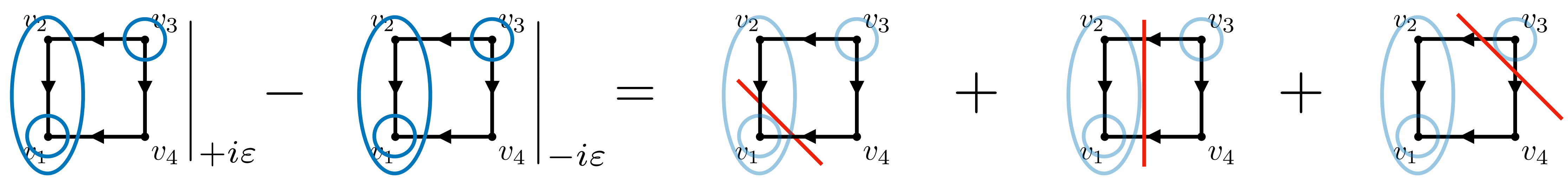
Amplitudes 2023
07/06/2023, CERN, Switzerland

First part: Cross-Free Families

- Sketch the derivation of the representation

$$f_{G_u}^{3\mathrm{d}} = \int \left[\prod_{i=1}^{L} \frac{\mathrm{d}k_i^0}{2\pi}\right] \frac{\mathcal{N}(\{q_e^0\}_{e\in\mathcal{E}})}{\prod_{e\in\mathcal{E}}(q_e^0)^2 - E_e^2}$$

- Highlight role of connectedness by comparison with Time-Ordered Perturbation Theory



(spurious singularities in TOPT)

Acyclic graphs and edge contraction

We start with an arbitrary diagram integrated over loop energies only

$$f^{3\mathrm{d}}_{G_u} = \int \left[\prod_{i=1}^{L} \frac{\mathrm{d}k_i^0}{2\pi}\right] \frac{\mathcal{N}(\{q_e^0\}_{e\in\mathcal{E}})}{\prod_{e\in\mathcal{E}} (q_e^0)^2 - E_e^2} \qquad q_e^0 = p_e^0 + \sum_{i=1}^{L} s_{ei} k_i^0 \qquad E_e = \sqrt{|\vec{q}_e|^2 + m_e^2}$$

G_u undirected graph

$\mathcal{E}$ set of edges of graph

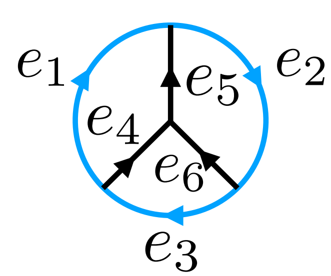
Performing the integrals using residue theorem is a matter of correctly addressing energy conservation (depending on how, get TOPT, LTD, CFF). For CFF (ZC [arXiv:2211.09653]):

$$\frac{1}{(q_e^0)^2 - E_e^2} = \int \mathrm{d}x_e \mathrm{d}\tau_e \frac{e^{i\tau_e(x_e - q_e^0)}}{x_e^2 - E_e^2} = \sum_{\sigma_e \in \{\pm 1\}} \int_0^\infty \frac{\mathrm{d}\tau_e}{2E_e} e^{i\tau_e(E_e - \sigma_e q_e^0)}$$

After insertion of this identity, loop energy integrations are trivial (dependence is only in exponents)

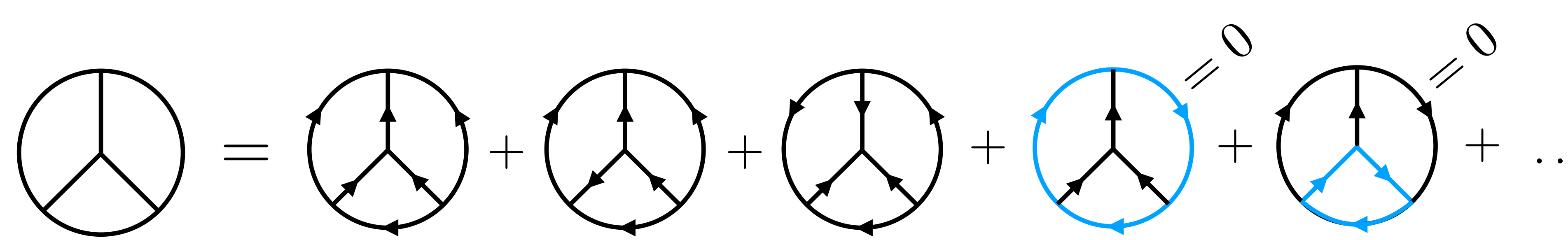
$$f^{3\mathrm{d}}_{G_u} = \sum_{\substack{\text{acyclic}\\ \text{graph } G}} \frac{\mathcal{N}_G}{\prod_{e\in\mathcal{E}} 2E_e} \int_{\mathcal{K}_G} \left[\prod_{e\in\mathcal{E}} \mathrm{d}\tau_e e^{i\tau_e(E_e - p_e^G)}\right] \quad \text{with} \quad \mathcal{K}_G = \left\{ (\tau_e)_{e\in\mathcal{E}} \in \mathbb{R}_+^{|\mathcal{E}|} \,\middle|\, \sum_{e\in\mathcal{E}} s_{ei}^G \tau_e = 0,\ i = 1, \dots, L \right\}$$

($\mathcal{K}_G$ is empty if graph not acyclic)



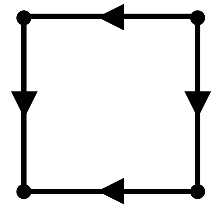
$$\tau_i > 0,\ i = 1, \dots, 6 \quad \tau_2 + \tau_6 + \tau_5 = 0 \quad \tau_1 + \tau_2 + \tau_3 = 0 \quad \tau_3 + \tau_4 - \tau_6 = 0$$

Diagrammatically



Position space: Fourier duality maps acyclic graphs to strongly-connected graphs

(Borinsky, ZC, Laenen, Salas-Bernardez [arXiv:2210.05532])



$$= \int \left[\prod_{j=1}^{4} \frac{\mathrm{d}\tau_j}{2E_j} e^{i\tau_j (E_j^0 - \sigma_j p_j^0)} \Theta(\tau_j) \right] \delta\left(-\tau_1 - \tau_2 + \tau_3 + \tau_4\right)$$

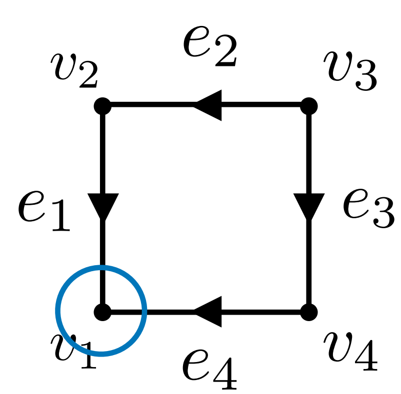
The cone is non-simplicial
Needs triangulation!

$$\mathcal{K}_G = \left\{ (\tau_e)_{e\in\mathcal{E}} \in \mathbb{R}_+^{|\mathcal{E}|} \;\middle|\; \sum_{e\in\mathcal{E}} s_{ei}\tau_e \right\}$$

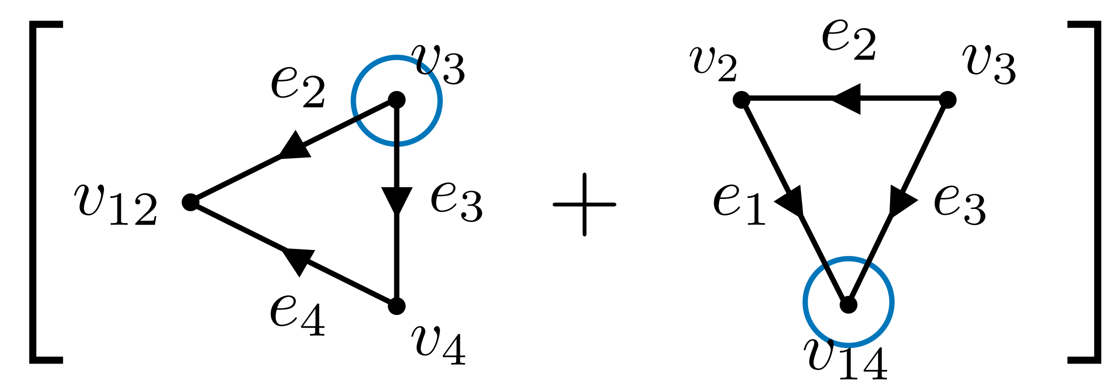
Triangulation introduces spurious singularities or spurious intersections

Edge contraction

How do we perform the remaining integrations (**one for each edge**)? Edge-contraction

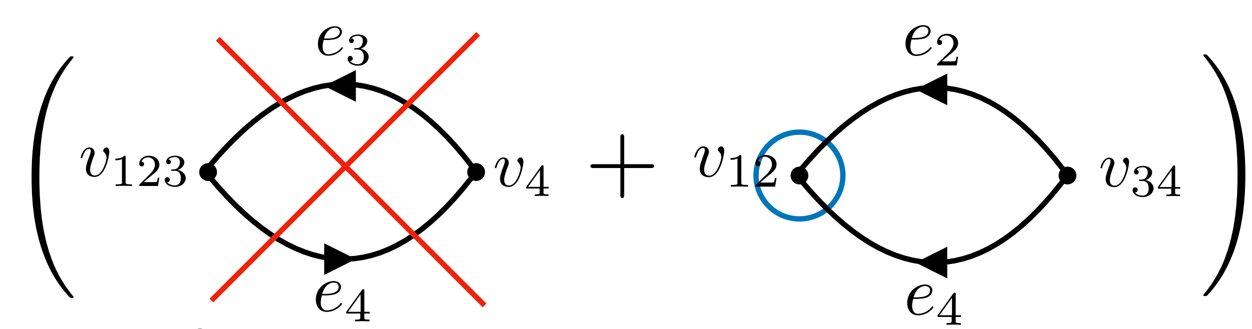


$$= \frac{i}{E_1 + E_4 - p_1^0}$$

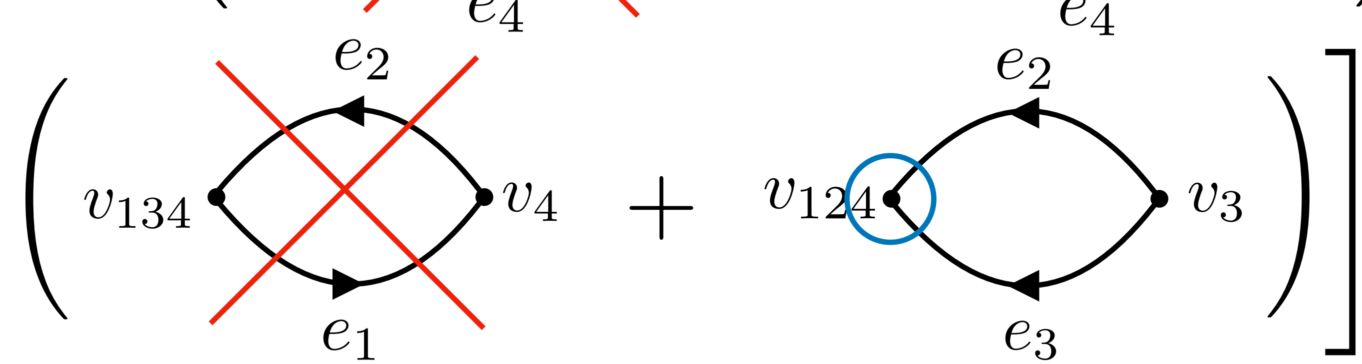


1. Choose sink/source with connected complement
2. Contract one-by-one adjacent edges
3. Multiply by inverse sum of energies of adjacent edges

$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \right.$$



$$\left. + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \right]$$



4. Throw out non-acyclic graphs

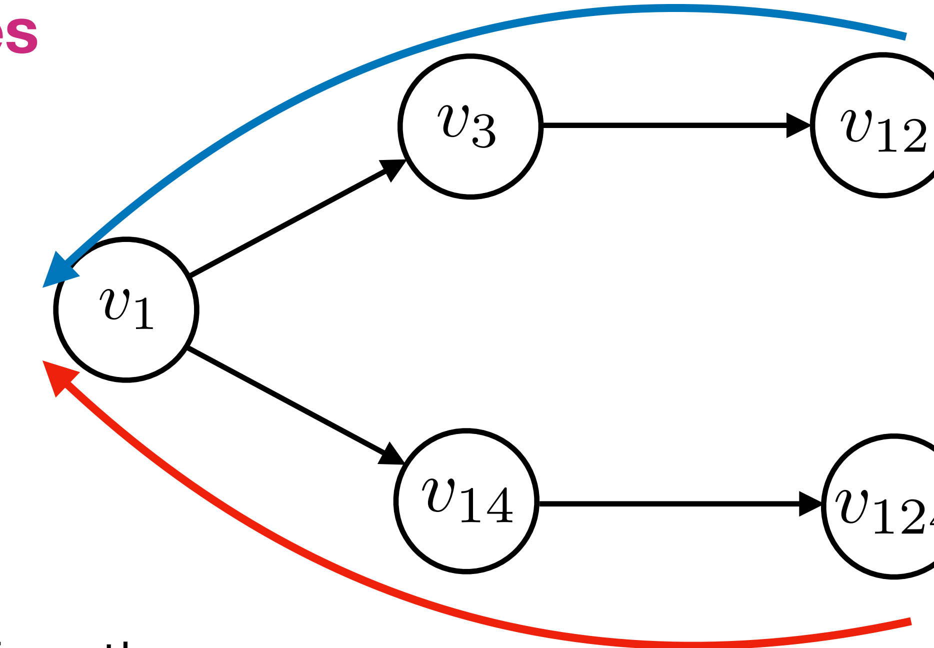
$$= \frac{i}{E_1 + E_4 - p_1^0} \left[\frac{i}{E_2 + E_3 + p_3^0} \frac{i}{E_2 + E_4 - p_1^0 - p_2^0} + \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \frac{i}{E_2 + E_3 + p_3^0} \right]$$

5. Contract parallel edges

All time integrations are performed diagrammatically!

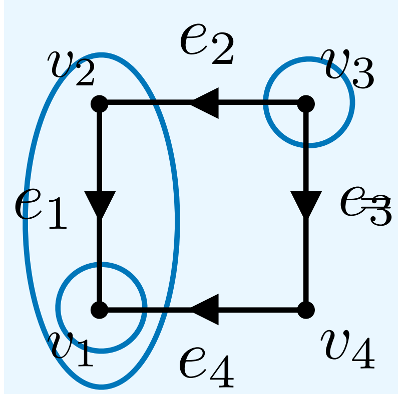
Cross-Free Families

Collecting the chosen vertices and collecting them according to order of choice, we get a decision tree, whose root is the first chosen vertex



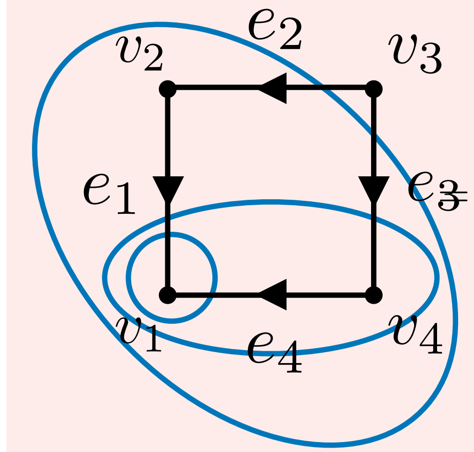
Tracing the route from the leaves to the root gives sets of vertices

$F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$



$$\frac{i}{E_1 + E_4 - p_1^0} \frac{i}{E_2 + E_3 + p_3^0} \frac{i}{E_2 + E_4 - p_1^0 - p_2^0}$$

$F_2 = \{\{v_1\}, \{v_1, v_4\}, \{v_1, v_2, v_4\}\}$



$$\frac{i}{E_1 + E_4 - p_1^0} \frac{i}{E_1 + E_3 - p_1^0 - p_4^0} \frac{i}{E_2 + E_3 + p_3^0}$$

Boundary operator provides nexus e.g. $\partial(\{v_1, v_2\}) = \{e_2, e_4\}$ $\Rightarrow$ $\frac{i}{E_2 + E_4 - p_1^0 - p_2^0}$

We notice some regularities… these families of cuts satisfy

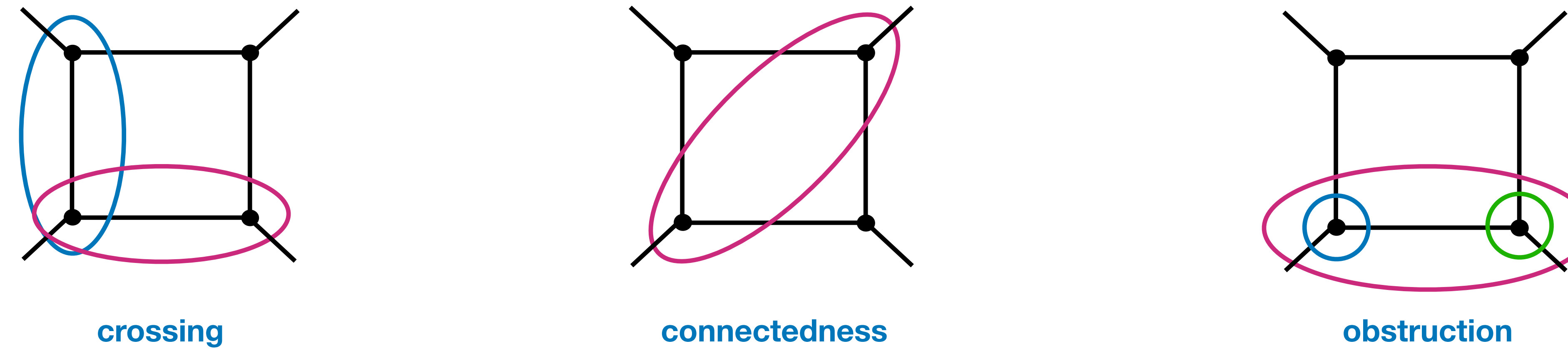
$$S \in F \quad \Rightarrow \quad S,\ V \setminus S \text{ are connected}$$

$$S_1, S_2 \in F \quad \Rightarrow \quad S_1 \subset S_2 \text{ or } S_2 \subset S_1 \text{ or } S_1 \cap S_2 = \emptyset$$

$$S \in F \quad \Rightarrow \quad S \text{ cannot be written as union of other sets in } F$$

Abreu, Britto, Duhr, Gardi arXiv:2010.01068 (2014)
Bloch, Kreimer arXiv:1512.01705 (2015)
Arkani-Hamed, Benincasa, Postnikov arXiv:1709.02813 (2017)
Benincasa, McLeod, Vergu arXiv:2009.03047 (2020)
Capatti, Hirschi, Pelloni, Ruijl, arXiv:2010.01068 (2020)

Forbidden configurations

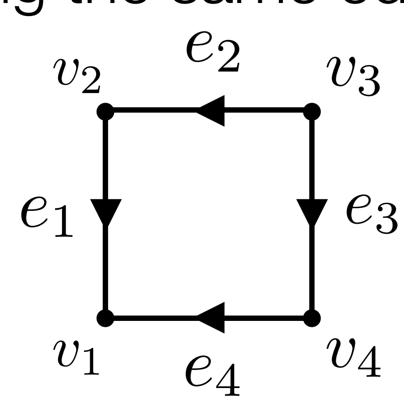


General formula

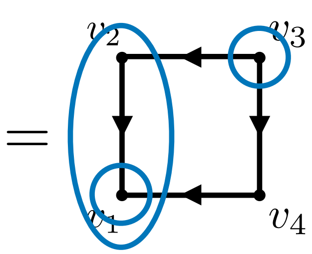
Repeating the same edge-contraction procedure for all acyclic graphs

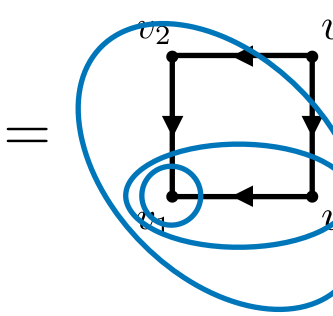
$$f^{3\mathrm{d}}_{G_u} = \sum_{\substack{\text{acyclic}\\ \text{graph } G}} \frac{\mathcal{N}_G}{\prod_e 2E_e} \sum_{F\in\mathcal{F}_G} \frac{1}{\prod_{S\in F}\left[\boldsymbol{E}\cdot\mathbf{1}^{\partial(S)} - \sum_{v\in S} p_v^0\right]}$$

Repeating the same edge-contraction procedure for all acyclic graphs. For our example, we had

$G =$ 

$\mathcal{F}_G = \{F_1, F_2\}$

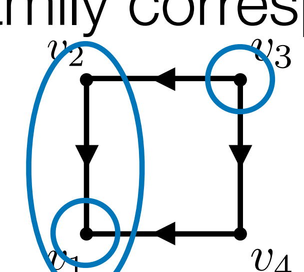
$F_1 =$ 

$F_2 =$ 

Each element of a cross-free family corresponds to a threshold

$F_1 = \{\{v_1\}, \{v_3\}, \{v_1, v_2\}\}$ $\quad \partial(\{v_1, v_2\}) = \{e_2, e_4\}$ $\quad \boldsymbol{E}\cdot\mathbf{1}^{\partial(\{v_1,v_2\})} - \sum_{v\in\{v_1,v_2\}} p_v^0 = E_2 + E_4 - p_1^0 - p_2^0$

And the cross-free family corresponds to a product of thresholds

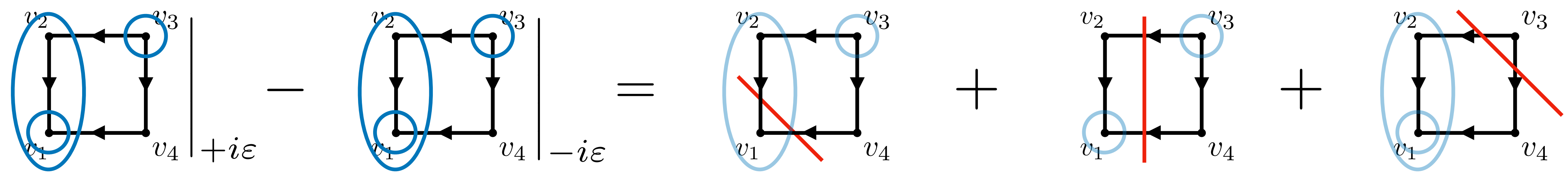


$$= \frac{i}{E_1 + E_4 - p_1^0}\,\frac{i}{E_2 + E_3 + p_3^0}\,\frac{i}{E_2 + E_4 - p_1^0 - p_2^0}$$

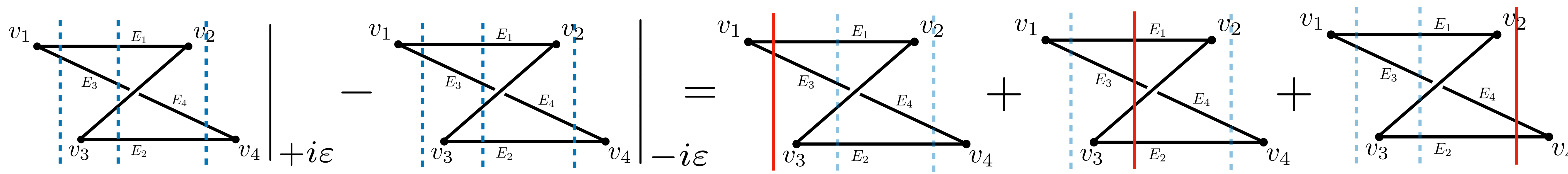
Discontinuities

We can compute discontinuities (Bourjaily, Hannesdottir, McLeod, Schwartz, Vergu [arXiv:2007.13747])

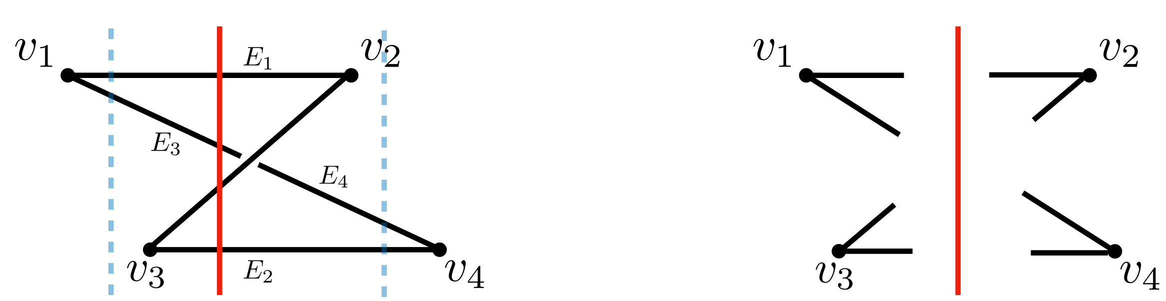
$$\frac{1}{\prod_{S\in F}\left[\boldsymbol{E}\cdot\mathbf{1}^{\partial(S)}-\sum_{v\in S}p_v^0+i\varepsilon\right]}-\frac{1}{\prod_{S\in F}\left[\boldsymbol{E}\cdot\mathbf{1}^{\partial(S)}-\sum_{v\in S}p_v^0-i\varepsilon\right]}=\sum_{S\in F}\frac{\delta\left(\boldsymbol{E}\cdot\mathbf{1}^{\partial(S)}-\sum_{v\in S}p_v^0\right)}{\prod_{S'\in F\setminus\{S\}}\left[\boldsymbol{E}\cdot\mathbf{1}^{\partial(S')}-\sum_{v\in S'}p_v^0\right]}$$



Why use the CFF rep. and not TOPT? Focus on the TOPT term ordering $\{v_1, v_3, v_2, v_4\}$



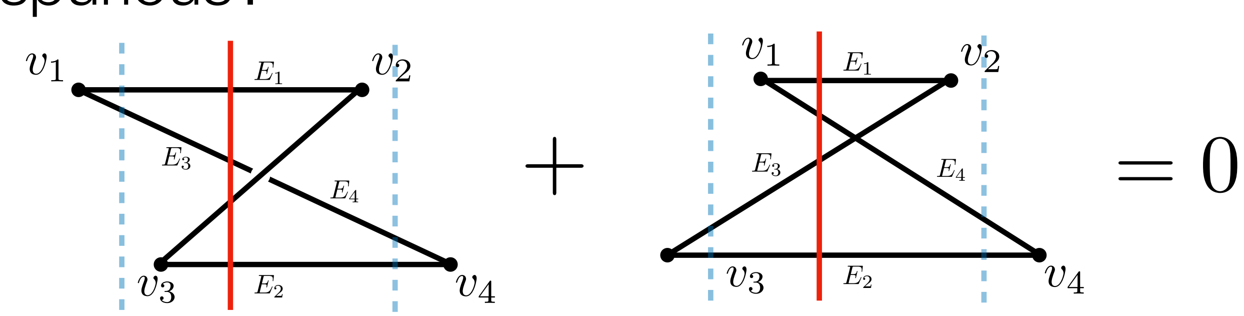
Looking at the second cut



Divides the graph in four connected components, but the CFF representation tells us this is not possible! It is a **spurious threshold**

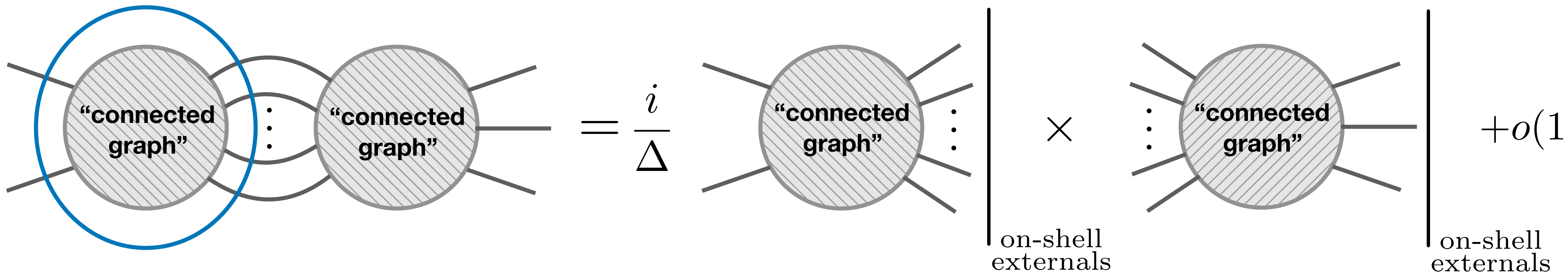
$\delta(E_1+E_2+E_3+E_4-p_1^0-p_3^0)$

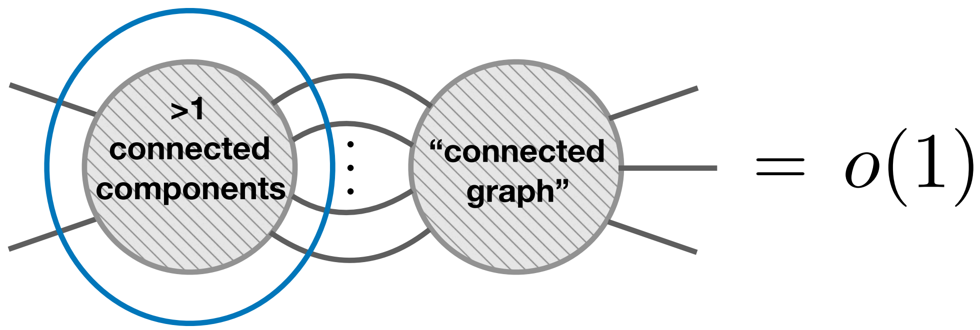
How do we see that it is spurious?



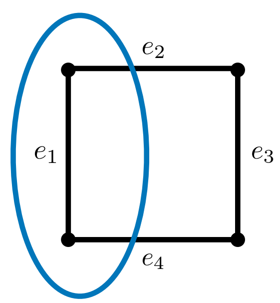
Factorisation formula

We can see that in general from a diagram-level factorisation formula

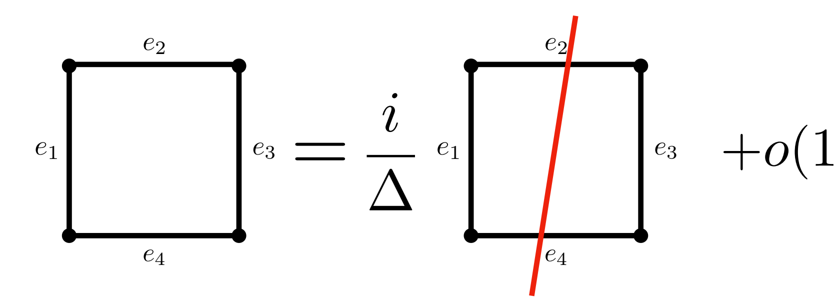


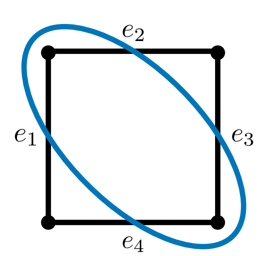


If more than two disconnected components, it is of order one

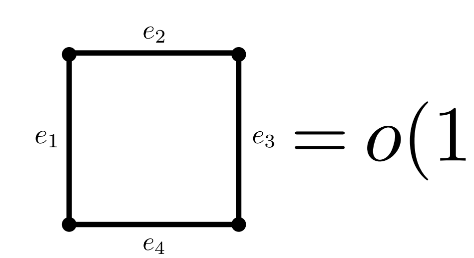


$$\Delta = E_2 + E_4 - p_1^0 - p_4^0 \to 0$$



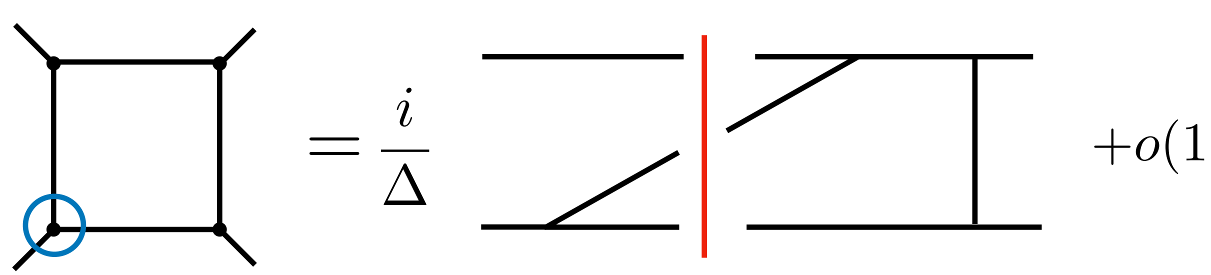


$$\Delta = E_1 + E_2 + E_3 + E_4 - p_1^0 - p_4^0 \to 0$$

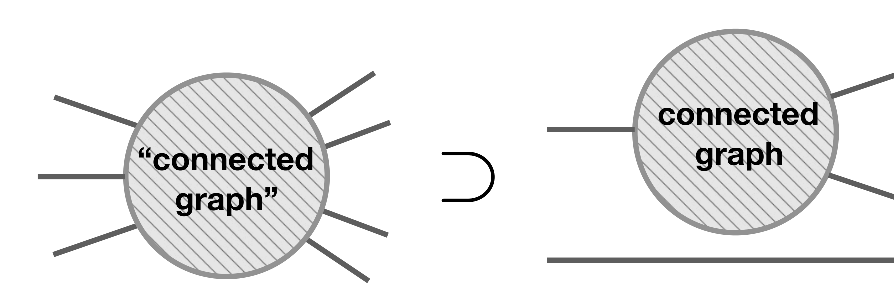


Spectators

What do we really mean by "connected graph"?



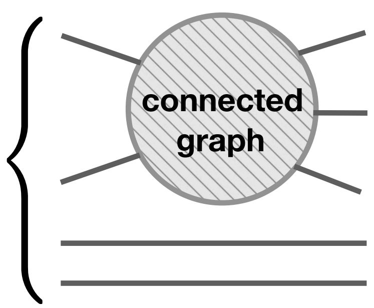
Connected up to spectator particles!



Second part: cluster decomposition and infrared finiteness

- Highlight role of connectedness at the operator level

$$\langle \boldsymbol{\alpha} | \, \mathbf{T}_{\mathrm{c}} \, | \boldsymbol{\beta} \rangle \supset \boldsymbol{\alpha} \left\{ \quad \right\} \boldsymbol{\beta}$$



  Reconstruct unitarity, cluster decomposition principle and infrared finiteness from the diagrammatic analysis

- Use Local Unitarity methods to take advantage of this analysis to numerically evaluate cross-sections

Connected transition matrix

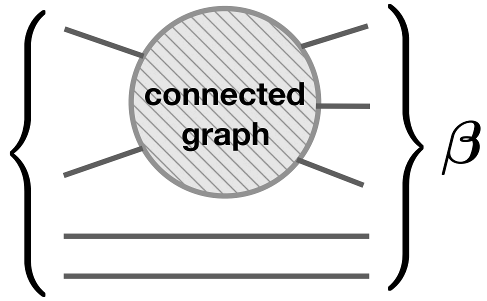
S-matrix

“Unitarity”

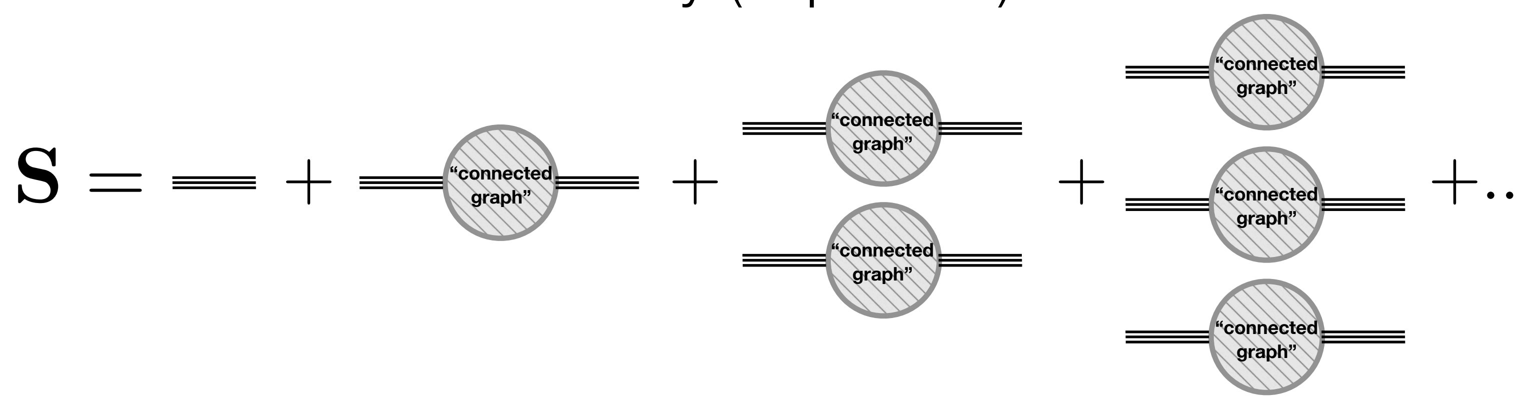
Cluster Decomposition

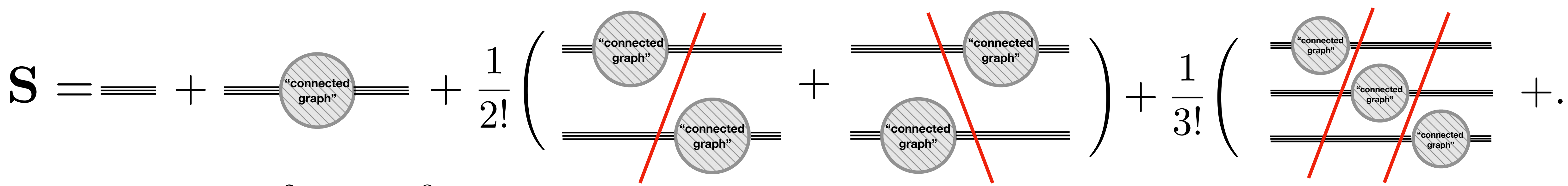
Can we express the role of connectedness at the operator level? Define the connected transition matrix

$$\langle \boldsymbol{\alpha} | \mathbf{T}_{\rm c} | \boldsymbol{\beta} \rangle \supset \boldsymbol{\alpha}$$



The connected transition matrix is not unitary (expected). What is the relationship with S-matrix?





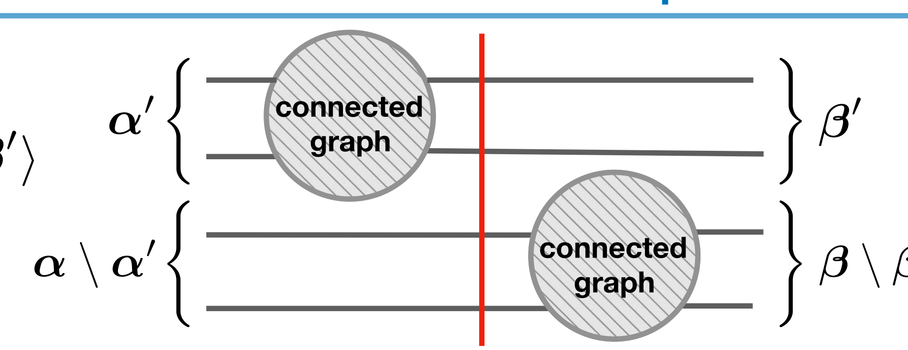
$$\mathbf{S} = 1 + i\mathbf{T}_{\rm c} + \frac{i^2}{2!}\mathbf{T}_{\rm c}^2 + \frac{i^3}{3!}\mathbf{T}_{\rm c}^3 + \ldots = e^{i\mathbf{T}_{\rm c}}$$ (evokes $Z[J] = e^{iW[J]}$)

(See also holomorphic cutting rules: Hannesdottir, Mizera [arXiv:2204.02988])

In order to establish this relation, we need the following formula

cluster decomposition term

$$\underbrace{i\langle \boldsymbol{\alpha} | \mathbf{T}_{\rm c} | \boldsymbol{\beta} \rangle - i\langle \boldsymbol{\alpha} | \mathbf{T}_{\rm c}^\dagger | \boldsymbol{\beta} \rangle = \langle \boldsymbol{\alpha} | \mathbf{T}_{\rm c}\mathbf{T}_{\rm c}^\dagger | \boldsymbol{\beta} \rangle}_{\text{usual unitarity}} - \sum_{\substack{\boldsymbol{\alpha}' \subset \boldsymbol{\alpha} \\ \boldsymbol{\beta}' \subset \boldsymbol{\beta}}} \langle \boldsymbol{\alpha}' | \mathbf{T}_{\rm c} | \boldsymbol{\beta}' \rangle \langle \boldsymbol{\alpha} \setminus \boldsymbol{\alpha}' | \mathbf{T}_{\rm c}^\dagger | \boldsymbol{\beta} \setminus \boldsymbol{\beta}' \rangle$$



Factorisation formula expressed at the operator level!

Cluster decomposition principle

This S-matrix also trivially satisfies the cluster-decomposition principle. Indeed

$$\mathbf{T}_{\rm c}(|\boldsymbol{\alpha}\rangle \otimes |\boldsymbol{\beta}\rangle) = (\mathbf{T}_{\rm c}\,|\boldsymbol{\alpha}\rangle) \otimes |\boldsymbol{\beta}\rangle + |\boldsymbol{\alpha}\rangle \otimes (\mathbf{T}_{\rm c}\,|\boldsymbol{\beta}\rangle) \qquad \Rightarrow \qquad P = P_A P_B$$

for states with large space-like separations.

Transition probabilities

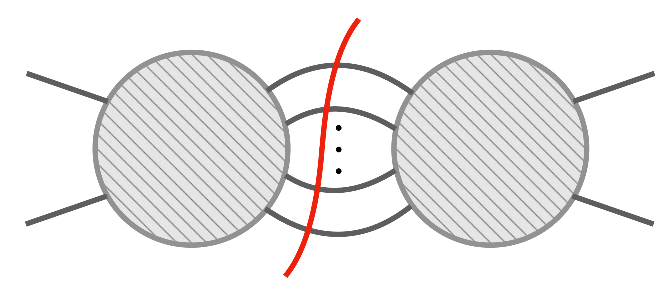
Using it, we can compute transition probabilities

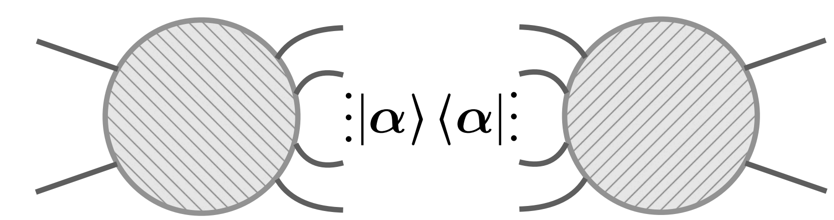
$$P = \mathrm{Tr}[\boldsymbol{\rho}\,\mathbf{S}\,\boldsymbol{P}\,\mathbf{S}^\dagger]$$

$$\underbrace{\boldsymbol{\rho} = \SumInt_{\boldsymbol{\alpha}} \rho_{\boldsymbol{\alpha}}\,|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}| \qquad \boldsymbol{P} = \SumInt_{\boldsymbol{\beta}} \mathcal{P}_{\boldsymbol{\beta}}\,|\boldsymbol{\beta}\rangle\langle\boldsymbol{\beta}|}_{\text{Sum over massless particles requires decoherence}}$$

The decoherence is due to the way we sum contributions with different number of massless particles in the initial and final state

$$P = \sum_m \int \mathrm{d}\Pi_m |\mathcal{A}(pp \to \mathrm{EW} + m\,p)|^2$$





And also the reason why we can write interference diagrams in the first place!

Clusters and infrared finiteness

Finally:

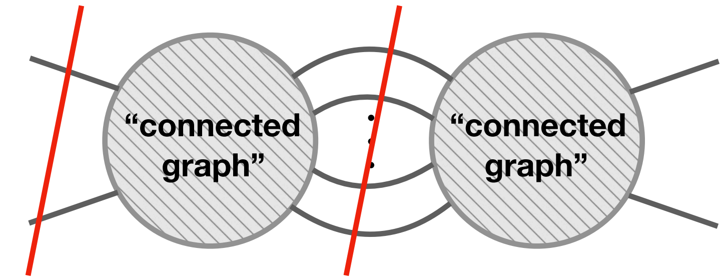
$$P = \mathrm{Tr}[\boldsymbol{\rho}\,\mathbf{S}\,\boldsymbol{P}\,\mathbf{S}] = \sum_{n,m} \frac{i^{n+m}}{n!m!} \underbrace{\mathrm{Tr}[\boldsymbol{\rho}\,\mathbf{T}_{\rm c}^n\,\boldsymbol{P}\,(\mathbf{T}_{\rm c}^\dagger)^m]}_{\text{Infrared finite if density matrix and projector sum over degenerate massless radiation}}$$

Infrared-finiteness follows from the unitarity relation we showed in the previous slide. But we can also look at it at a diagrammatic level.

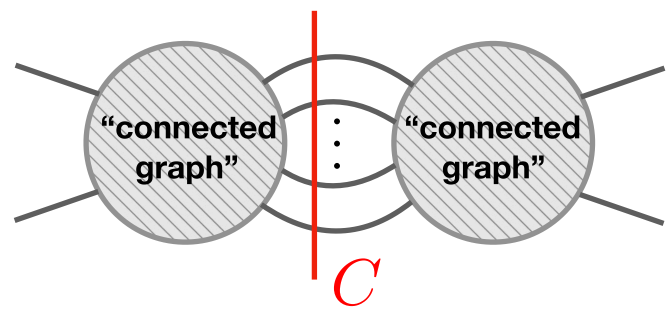
Final-state sum example

How do we show it?

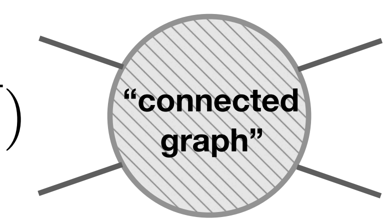
$$\mathrm{Tr}[\boldsymbol{\rho}\,\mathbf{T}_{\mathrm{c}}\,\boldsymbol{P}\,\mathbf{T}_{\mathrm{c}}^{\dagger}] \supset$$



Using the CFF representation we can show that



$$= \lim_{\sqrt{s}\to E_C} (E_C - \sqrt{s})$$

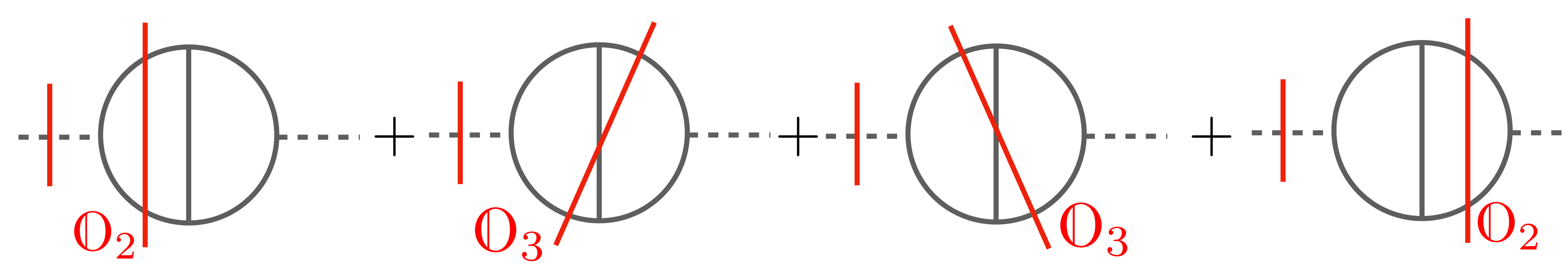


$$\ominus \frac{N(E_C)}{\prod_{C'\neq C}(E_{C'} - E_C)}$$

This relation allows to collect locally interference diagrams

Example: consider massless scalar corrections to the decay of a massive scalar $\boldsymbol{\rho} = |\phi^\star\rangle\langle\phi^\star|$

$$\sigma(\phi^\star \to n\,\mathrm{jets}_\phi) =$$



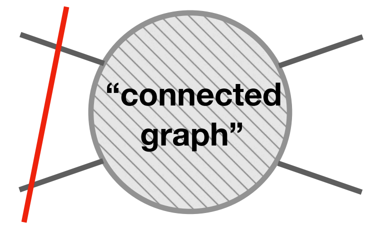
$$= \sum_{i=1}^{4} \int \mathrm{d}\Pi_i f_i$$

Locally finite

$$\ominus \int \mathrm{d}\Pi \sum_{i=1}^{4} g_i = \int \mathrm{d}^3\vec{k}\mathrm{d}^3\vec{l} \sum_{j=1}^{4} \frac{N(E_{C_j})}{\prod_{i\neq j}(E_{C_i} - E_{C_j})}$$

(Capatti, Hirschi, Ruijl, Pelloni [arXiv:2010.01068])

In the preceding example, we fixed a massive initial-state. What if we want to have massless initial states?

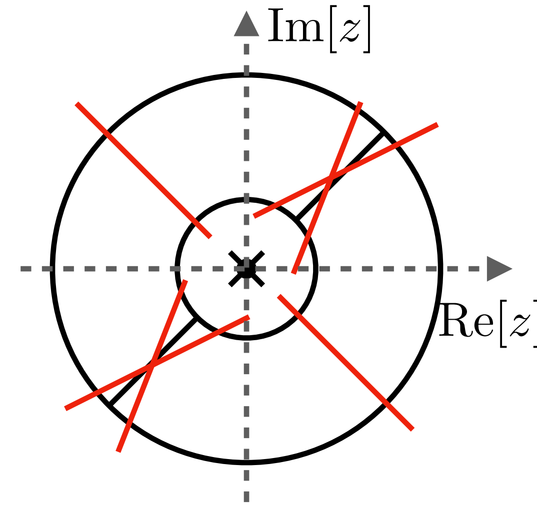


We need to write forward-scattering diagrams as residues of something!

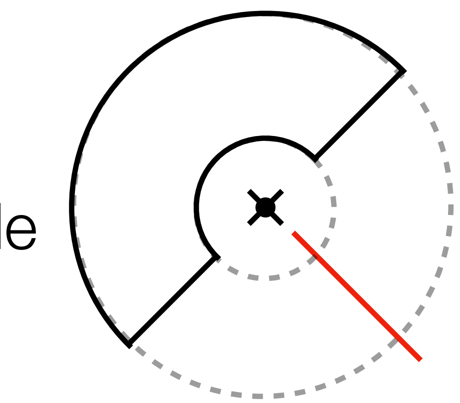
How do we extend to initial-state sums? In other words what are forward-scattering diagrams residues of

Embedding vacuum graph in $\mathbb{C} \setminus \{0\}$

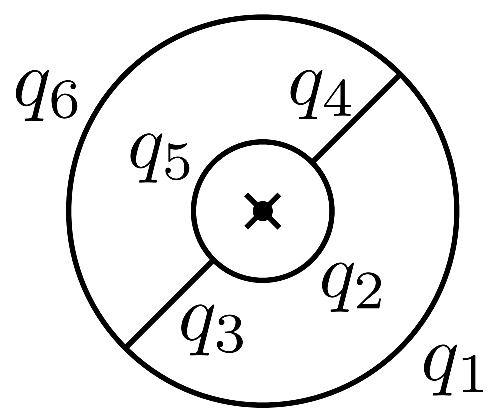
$x^0 \to z$



A Cutkosky cut is a minimal set of edges whose deletion makes the graph contractible

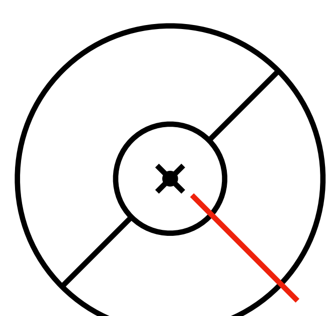


We can construct a three-dimensional representation for embeddings

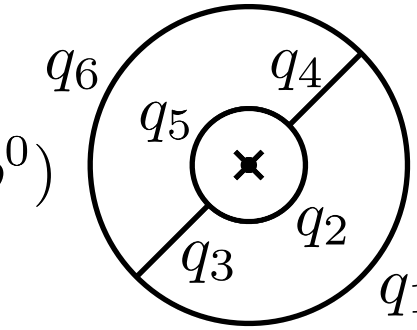


$$= \int \left[\prod_{i=1}^{6} \frac{\mathrm{d}q_i^0}{q_i^2} \right] \delta(q_1 + q_2 - p)\delta(q_1 + q_2 - p)\delta(q_2 + q_3 - q_6 - p)\delta(q_5 + q_6 - p)\delta(q_2 + q_4 + q_6 - p)$$

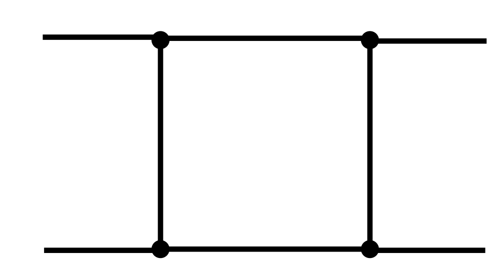
Embeddings have thresholds associated with their Cutkosky cuts



$$= \lim_{p^0 \to E_1 + E_2} (E_1 + E_2 - p^0)$$



$$=$$



This observation allows to extend the Local Unitarity representation to initial states!

Conclusion

- Energy conservation implies a rigid diagrammatic structure for threshold singularities
    - Connectedness
    - Absence of crossing
    - Obstruction-freedom
- These principles are manifest in a novel 3D-representation that holds for any theory (independent of numerator) and at any loop

  Implemented in Mathematica package ⟶ https://github.com/apelloni/cLTD

- The presence of connectedness suggests that, in order to understand IR-finiteness, one should decompose cross-sections according to the degree of connectedness
- 3D representations can be used to express interference diagrams as local residues of forward-scattering diagrams, and forward-scattering diagrams as local residues of vacuum embeddings
- In turn, these local residues can be used to write cross-sections in a way that manifests the KLN cancellation mechanism at the local level (Local Unitarity)
- Local Unitarity can be used to numerically evaluate cross-sections