Loops of loops expansion in the Amplituhedron

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JHEP 03 (2022) 108 (Nima Arkani-Hamed, Johannes Henn)

to appear soon (Taro Brown, Umut Oktem, Shruti Paranjape)

CERN, August 8, 2023

Object of interest:

Four-point amplitude in planar N=4 SYM theory

MOTIVATION

Planar N=4 SYM: playground for new ideas in scattering amplitudes

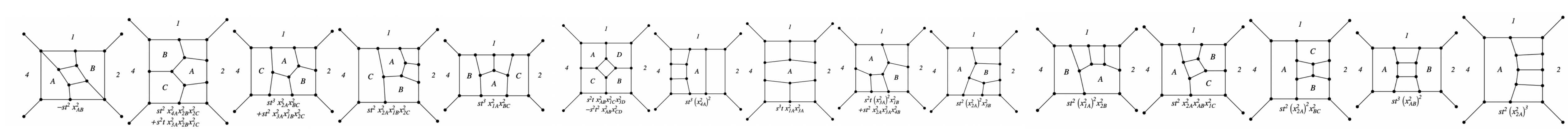
Four-point amplitudes: restricted kinematics, powerful symmetries

Bern-Dixon-Smirnov (BDS) ansatz

$$\mathcal{M}_n \equiv 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon) = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right] \qquad \text{for } n{=}4,5$$

kinematical part fully fixed, **leading IR divergence** predicted by integrability

The complexity of loop integrand grows fast with the loop order



MOTIVATION

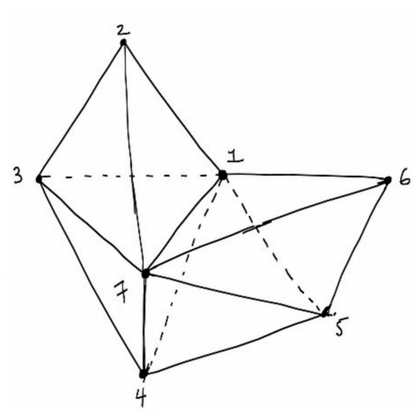
Why is the integrand so complicated while the final amplitude is so simple?

The integrand must be complicated because it contains a lot of "data", infinite number of cuts that must be satisfied

Are these data lost after integration (how do they transform into numbers)?

Can we extract some IR finite object from the integrand?

Amplituhedron: new geometric definition for the all-loop integrand



Can we use it to calculate the integrand to all loops?

If yes, can we integrate, resum and explore strong coupling?

THIS TALK

Using the **Amplituhedron** we define a new geometric expansion "**loops of loops**" for the integrand

At the leading "**tree approximation**" calculate integrand to all loops, integration, **resummation**, strong coupling

Published paper with Nima and Johannes from 2022

Calculate the integrand at the sub-leading "**one-loop**" to all loops, systematize the expansion

Paper to appear with Taro, Umut and Shruti

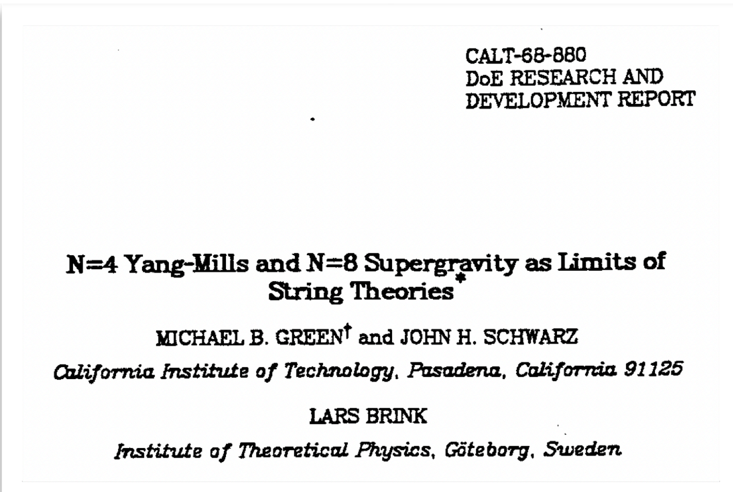
Integrate and resum sub-leading order, towards higher (or all) orders

Future work

INTRODUCTION

EARLY RESULTS

One-loop amplitude calculated in 1982, (full) two-loop in 1997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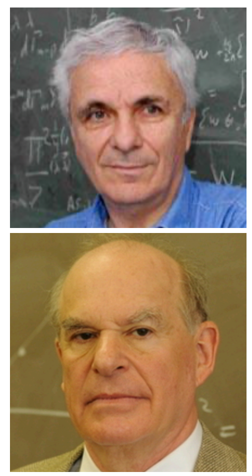
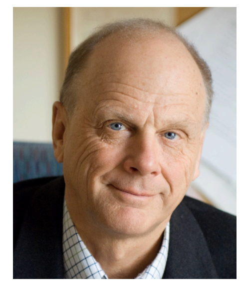
DoE RESEARCH AND
DEVELOPMENT REPORT

N=4 Yang-Mills and N=8 Supergravity as Limits of String Theories*

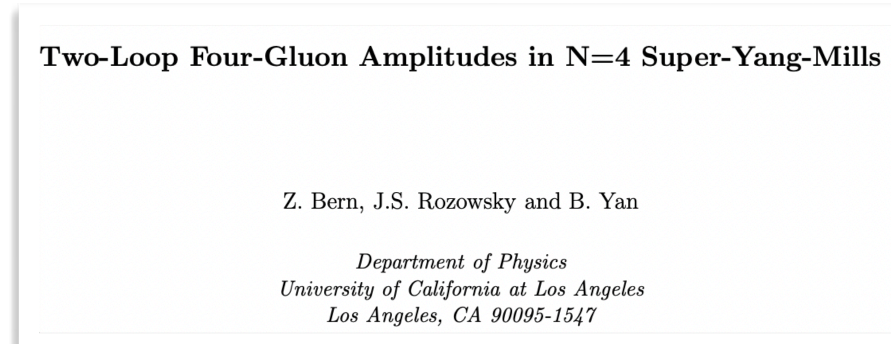
MICHAEL B. GREEN† and JOHN H. SCHWARZ
California Institute of Technology, Pasadena, California 91125

LARS BRINK
Institute of Theoretical Physics, Göteborg, Sweden



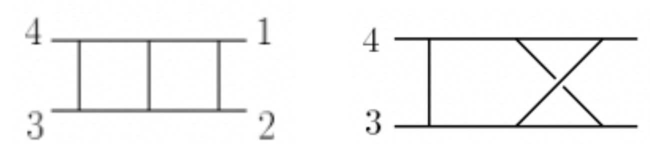


Lars Brink
1943-2022

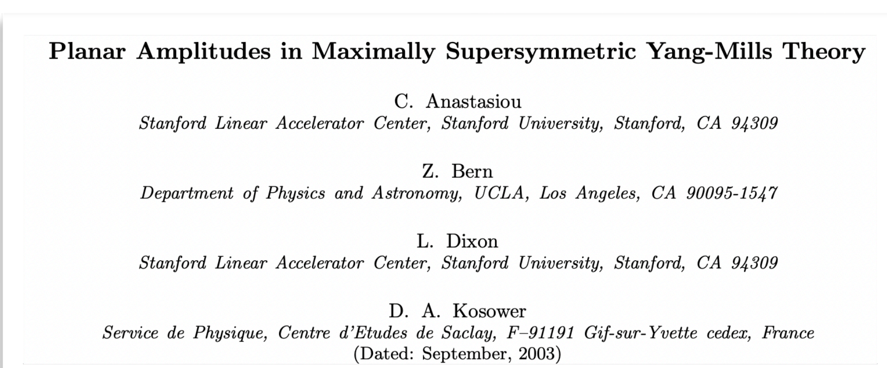

Two-Loop Four-Gluon Amplitudes in N=4 Super-Yang-Mills

Z. Bern, J.S. Rozowsky and B. Yan

Department of Physics
University of California at Los Angeles
Los Angeles, CA 90095-1547



In 2003 Anastasiou, Bern, Dixon and Kosower:
planar sector (large N limit) of the amplitude


Planar Amplitudes in Maximally Supersymmetric Yang-Mills Theory

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L. Dixon
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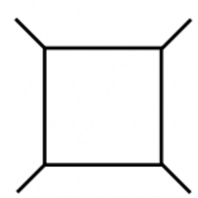
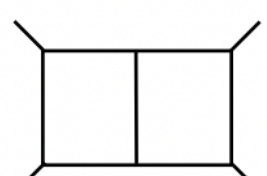
D. A. Kosower
Service de Physique, Centre d'Etudes de Saclay, F-91191 Gif-sur-Yvette cedex, France

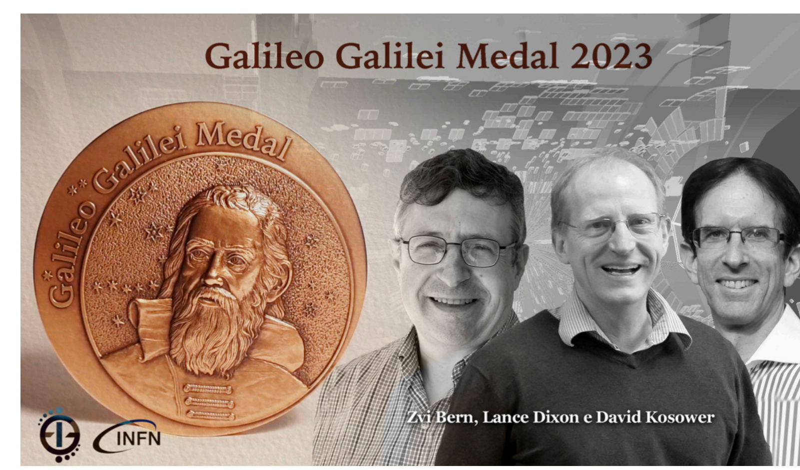
(Dated: September, 2003)



(a)



(b)

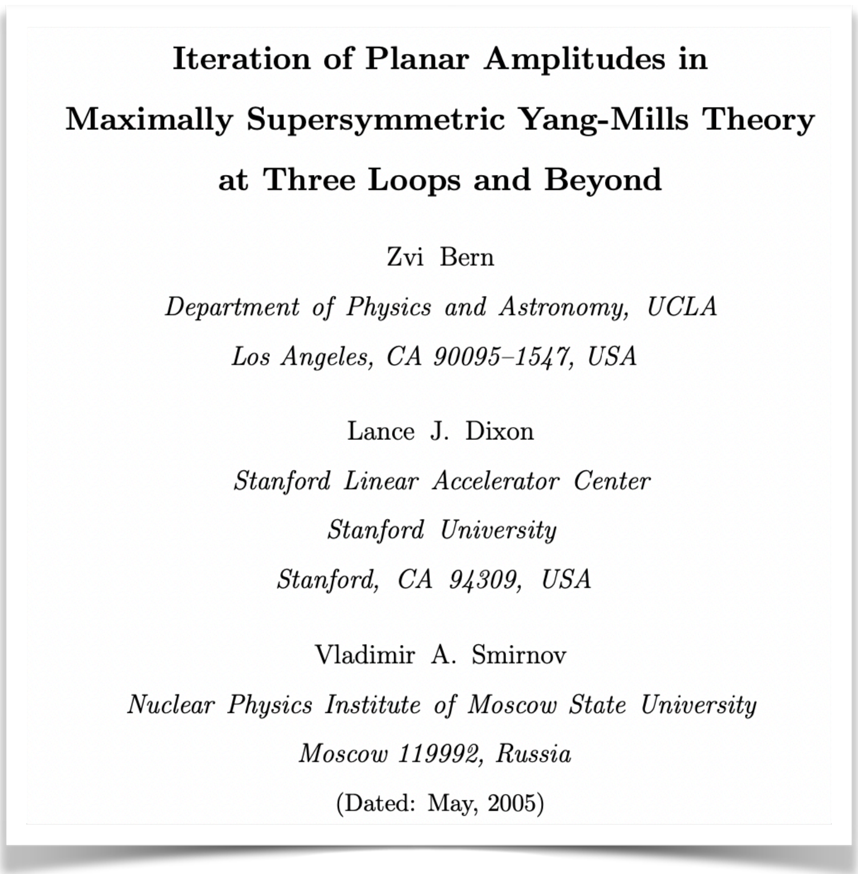


Observed **relation** between two-loop and one-loop in dimensional regularization

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f(\epsilon)\, M_n^{(1)}(2\epsilon) - \frac{5}{4}\zeta_4$$

BDS ANSATZ

In 2005, Bern, Dixon and Smirnov calculated 3-loop amplitude



Iteration of Planar Amplitudes in Maximally Supersymmetric Yang-Mills Theory at Three Loops and Beyond

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Vladimir A. Smirnov
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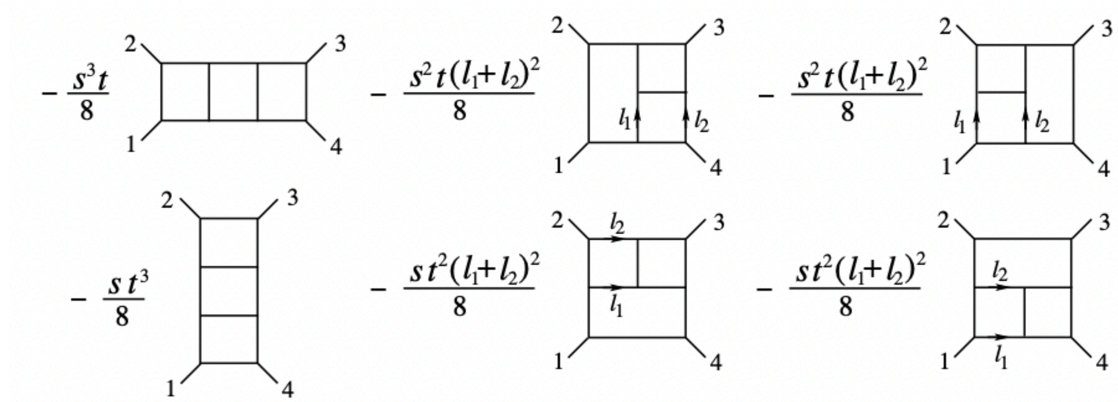
Moscow 119992, Russia

(Dated: May, 2005)

$$M_4^{(3)}(\epsilon) = -\frac{1}{8}st\left(s^2\, I_4^{(3)a}(s,t) + 2s\, I_4^{(3)b}(t,s) + t^2\, I_4^{(3)a}(t,s) + 2t\, I_4^{(3)b}(s,t)\right)$$



integrand already given in 1997 paper

The integrand obtained using **unitarity methods**, after integration they found the same iterative structure

$$M_4^{(3)}(\epsilon) \;=\; -\frac{1}{3}\left[M_4^{(1)}(\epsilon)\right]^3 + M_4^{(1)}(\epsilon)\, M_4^{(2)}(\epsilon) + f^{(3)}(\epsilon)\, M_4^{(1)}(3\,\epsilon) + C^{(3)} + \mathcal{O}(\epsilon)$$

Conjecture:

$$\mathcal{M}_n \equiv 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon) = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right]$$

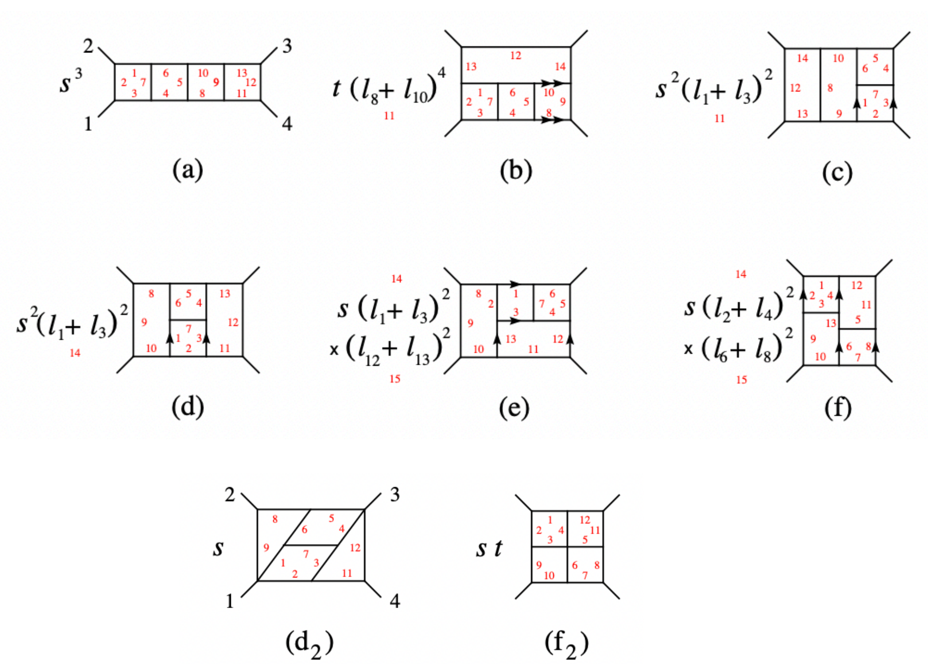
cusp-anomalous dimension calculated in 2006 by Beisert, Eden and Staudacher from integrability

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$$

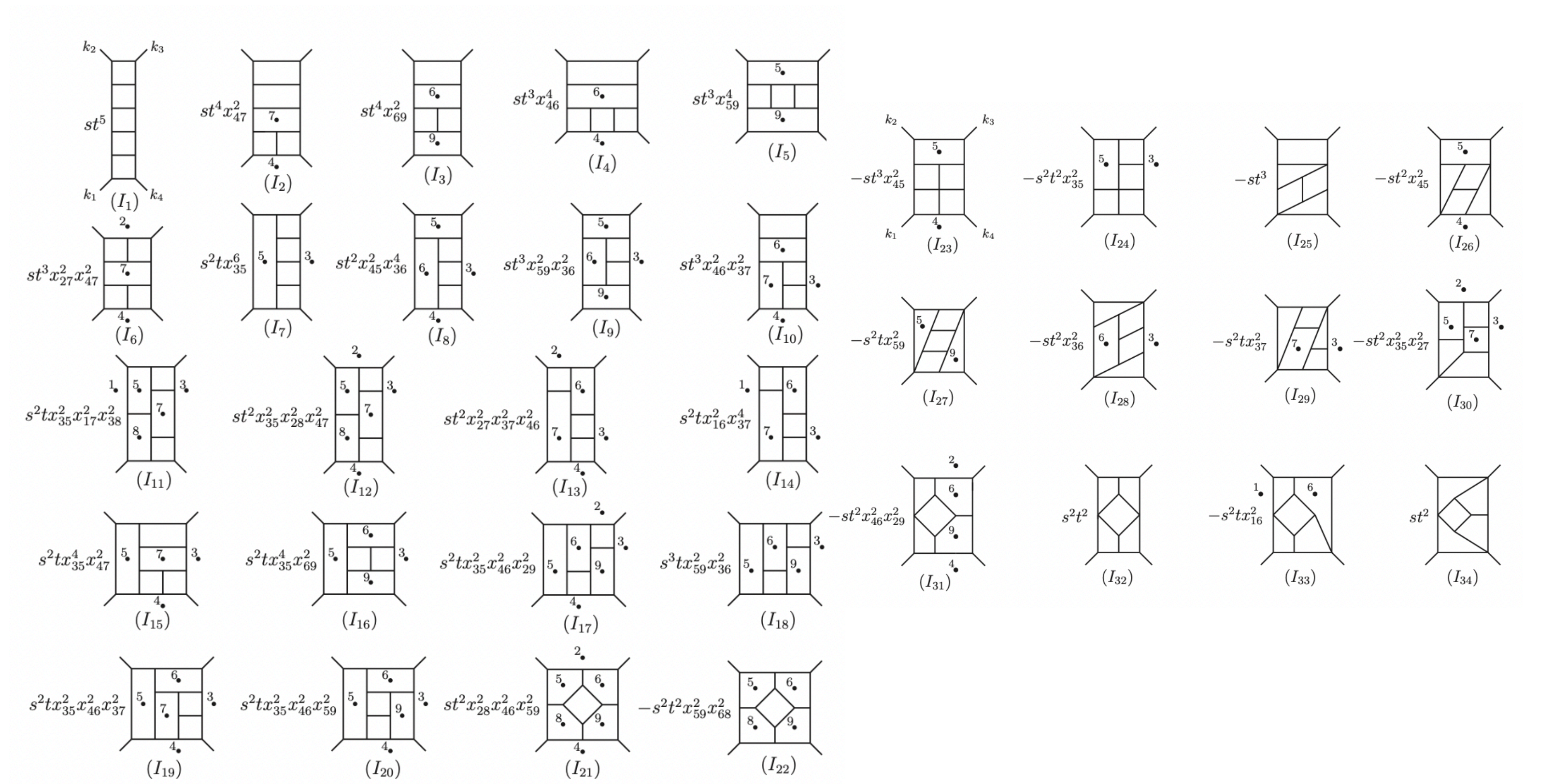
FOUR AND FIVE-LOOPS

In next two years, 4-loop and 5-loop integrands were constructed

(Bern, Czakon, Dixon, Kosower, Smirnov, 2006)



(Bern, Carrasco, Johansson, Kosower, 2007)



- analytic results not known, leading IR divergence verified at 4-loops numerically
- numerators can be chosen to be invariant under dual conformal symmetry

(Drummond, Henn, Korchemsky, Sokatchev, 2008)

LOOP INTEGRAND

In 2010, we took the planar integrand seriously and formulated recursion relations for N=4 SYM in momentum twistor space (Hodges, 2009)

- the integrand is a **unique rational function**

$$\mathcal{I}^{\ell-\text{loop}}_{n,k}(AB_1, AB_2, \ldots, AB_\ell, Z_1, Z_2, \ldots, Z_n)$$

The All-Loop Integrand For Scattering Amplitudes in Planar $\mathcal{N}=4$ SYM

N. Arkani-Hamed[a], J. Bourjaily[a,b], F. Cachazo[a,c], S. Caron-Huot[a], J. Trnka[a,b]

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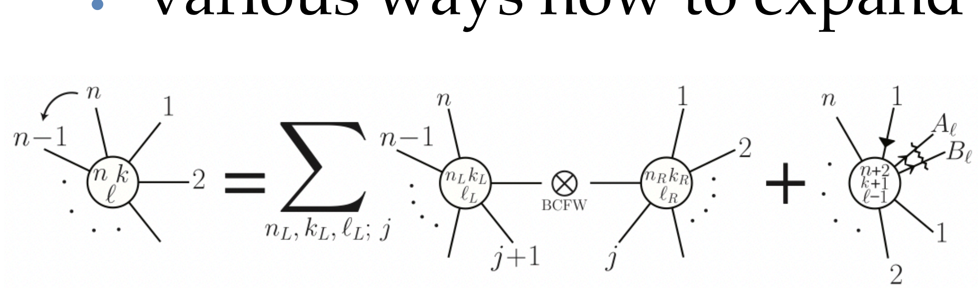
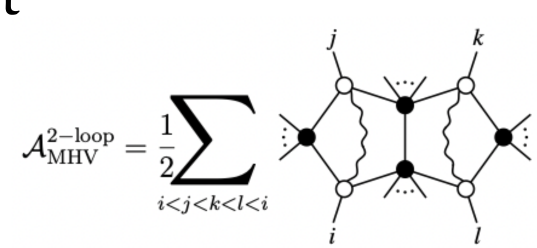
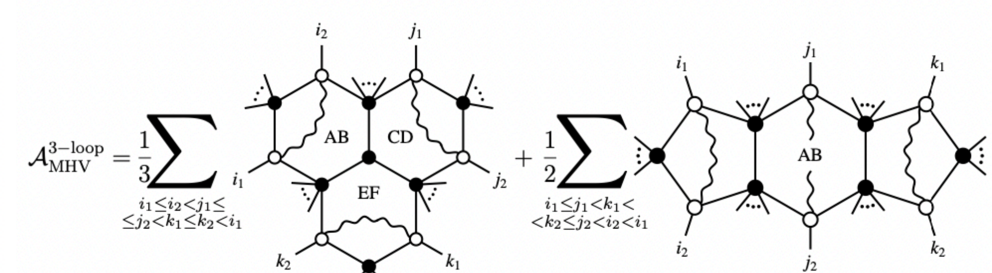
[b] *Department of Physics, Princeton University, Princeton, NJ 08544, USA*
[c] *Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2J W29, CA*

- various ways how to expand it



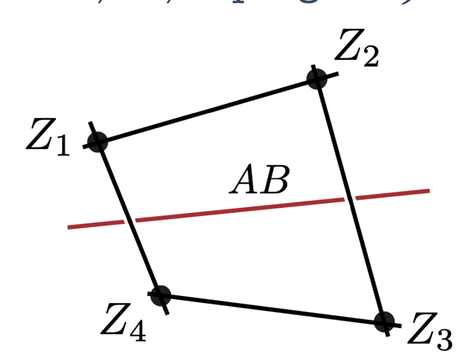




(Arkani-Hamed, Bourjaily, Cachazo, Trnka, 2010) (Bourjaily, Trnka, 2015)
(Bourjaily, Brown, Patatoukos, JT, in progress)

Properties of the loop integrand

- symmetric function of all loop lines AB_i
- the only poles for MHV are $\langle AB_i\, j\, j{+}1\rangle$ or $\langle AB_i AB_j\rangle$
- cuts in momentum twistor space: localizing AB_i - intersection with other lines



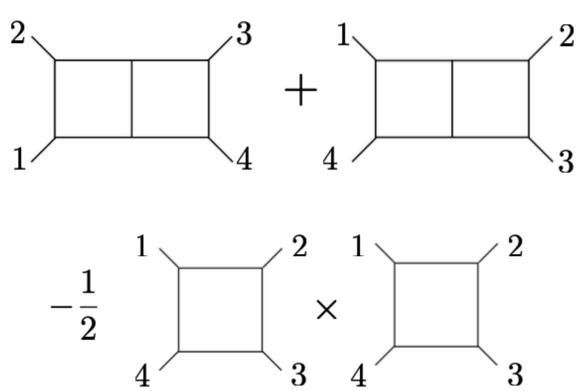
AMPLITUDE LOGARITHM

As we learnt from BDS ansatz **logarithm of the amplitude** is a special function with **mild IR divergence**

$$\ln \mathcal{M}_n = \sum_{l=1}^{\infty} a^l f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) = \frac{\gamma_{\text{cusp}}}{\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)$$

It makes sense to construct the integrand for the logarithm from products of amplitudes, which makes this property manifest

two-loop 4pt example: $\widetilde{\mathcal{I}}_4^{(2)}(AB_i, Z_j) = \mathcal{I}_4^{(2)}(AB, CD) - \mathcal{I}_4^{(1)}(AB) \times \mathcal{I}_4^{(1)}(CD)$



not a planar object!

$$\widetilde{\mathcal{I}}_4^{(2)} = \frac{\langle 1234\rangle^3(\langle AB13\rangle\langle CD24\rangle + \langle AB24\rangle\langle CD13\rangle)}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle\langle ABCD\rangle\langle CD12\rangle\langle CD23\rangle\langle CD34\rangle\langle CD41\rangle}$$

(Arkani-Hamed, Bourjaily, Cachazo, Trnka, 2010) (Drummond, Henn, 2010)

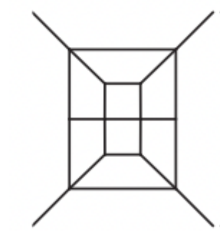
IR property: vanishing in collinear regions $Z_A \to Z_2, \quad Z_B \to Z_1 + \alpha Z_3$

in fact, only non-zero residue when we move all loop lines in the collinear region

(Arkani-Hamed, Bourjaily, Cachazo, JT, 2010) (Arkani-Hamed, Trnka, 2013)

AMPLITUDES UP TO 10 LOOPS

The 6-loop and 7-loop integrand was constructed using soft-collinear bootstrap method applying this IR property on the logarithm



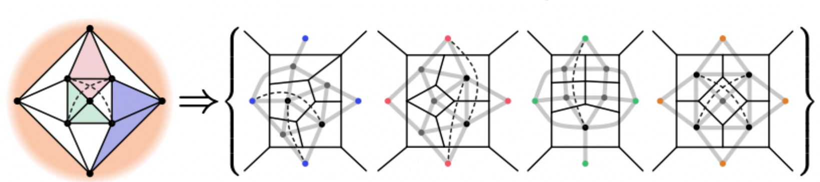
(Bourjaily, DiRe, Skaikh, Spradlin, Volovich, 2011) (Bourjaily, Heslop, Tran, 2015)

- new ideas needed at 8-loops: terms which vanish in the collinear regions

The integrand up to 10 loops using the four-point stress correlator

(Bourjaily, Heslop, Tran, 2016)

- hidden properties of "f-graphs", extraction of the amplitude in the light-like limit



ℓ	number of plane graphs	number of graphs admitting decoration	number of decorated plane graphs (f-graphs)	number of planar DCI integrands
1	0	0	0	1
2	1	1	1	1
3	1	1	1	2
4	4	3	3	8
5	14	7	7	34
6	69	31	36	284
7	446	164	220	3,239
8	3,763	1,432	2,709	52,033
9	34,662	13,972	43,017	1,025,970
10	342,832	153,252	900,145	24,081,425
11	3,483,075	1,727,655	22,097,035	651,278,237

For IR finite objects at higher points (remainder and ratio functions)

Powerful non-integrand methods: hexagon and heptagon bootstraps

(Dixon, McLeod, von Hippel, Caron-Huot, Drummond, Henn, Dulat, Papathanasiou, Gurdogan, Wilhelm, Goncharov, Spradlin, Vergu, Volovich,.....)

Flux tube S-matrix approach, OPE, Y-system, strong coupling, S-matrix bootstrap

(Basso, Sever, Vieira, Tumanov, Wilhelm, Alday, Maldacena, Correia, Zhiboedov,....)

LOOPS OF LOOPS IN THE AMPLITUHEDRON

AMPLITUHEDRON

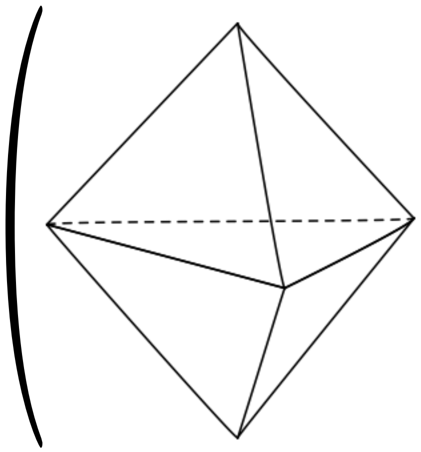
In 2013 together with Nima we found a new **geometric construction** for planar integrands in N=4 SYM

(Arkani-Hamed, JT, 2013) (Arkani-Hamed, Thomas, JT, 2017) (Ferro, Lukowski, 2022)

This is a generalization of our earlier work on the on-shell diagrams and positive Grassmannian and Hodges' polytopes

(Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT, 2012) (Hodges 2009) (Arkani-Hamed, Bourjaily, Cachazo, Hodges, JT, 2010)

$$\mathcal{A}_{n,k}^{\ell-\text{loop}} = \Omega \left(\vphantom{\int} \right)$$



Tree-level amplitudes and loop integrands

"**Volume**": differential form with logarithmic singularities on the boundaries of the space

Amplituhedron defined by a set of inequalities in the kinematical space

General Amplituhedron is curvy: complicated boundary structure

(Franco, Galloni, Mariotti, JT, 2014)
(Lukowski, Moerman, 2020)
(Dian, Heslop, Stewart, 2022)

How to calculate Ω ?

AMPLITUHEDRON

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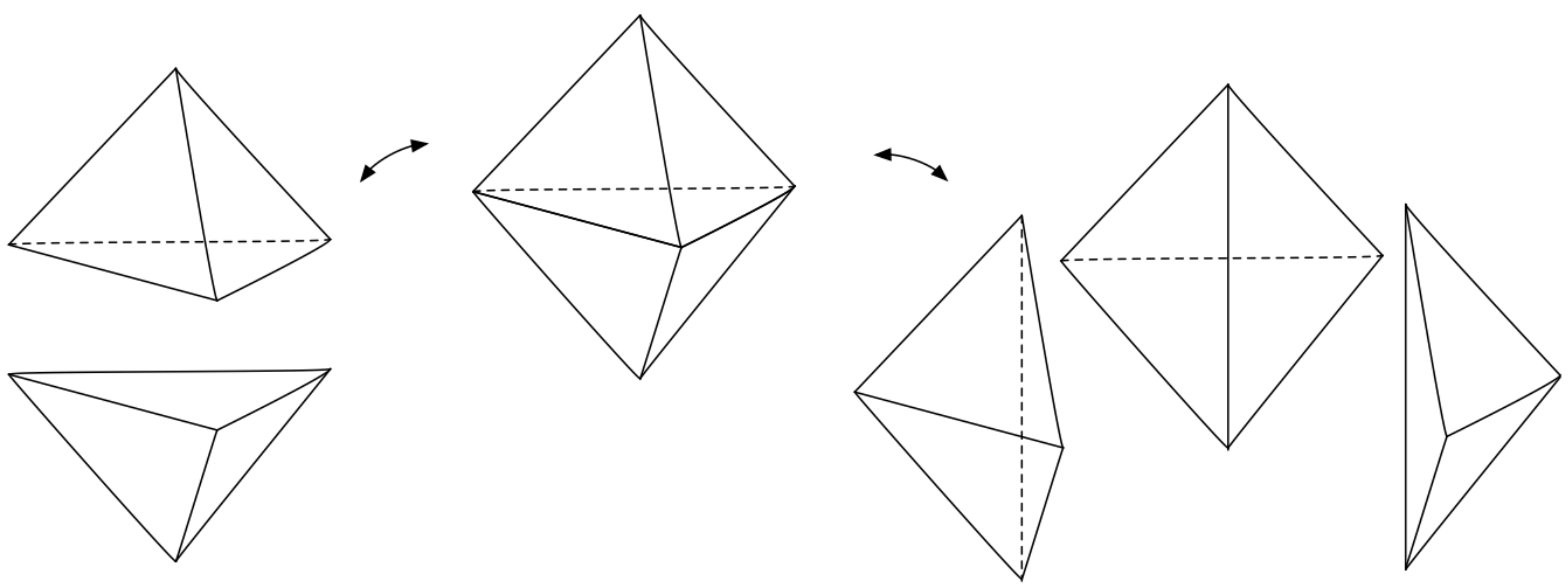


Triangulation in terms of "simplices": difficult to do in general

FOUR-POINT AMPLITUHEDRON

(Arkani-Hamed, JT, 2013)

Definition of the space to all loops: **mathematical problem**

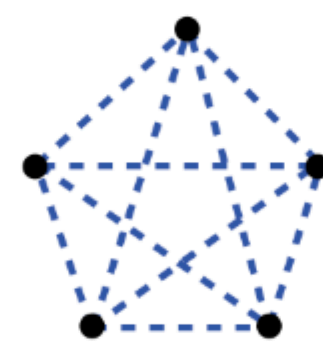
- Each loop described by a constrained line in momentum twistor space

$$\langle AB_i 12\rangle, \langle AB_i 23\rangle, \langle AB_i 34\rangle, \langle AB_i 14\rangle > 0, \quad \langle AB_i 13\rangle, \langle AB_i 24\rangle < 0 \qquad \langle IJKL\rangle = \det|Z_I Z_J Z_K Z_L|$$

- Mutual positive condition for any pair of lines $\langle AB_i AB_j\rangle > 0$

The L-loop integrand:
volume form on this space

$\Omega_L =$



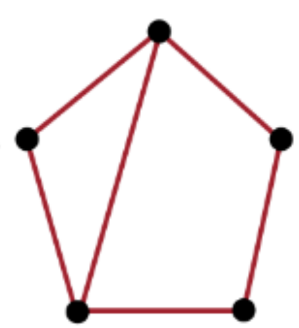
loop line in one-loop amplituhedron

mutual positivity $\langle AB_i AB_j\rangle > 0$

Loop integrand for the logarithm: **collection** of geometries

(Arkani-Hamed, JT, 2013) (Arkani-Hamed, Henn, JT, 2021)

$$\widetilde{\Omega}_L = \sum_G (-1)^{E(G)}$$



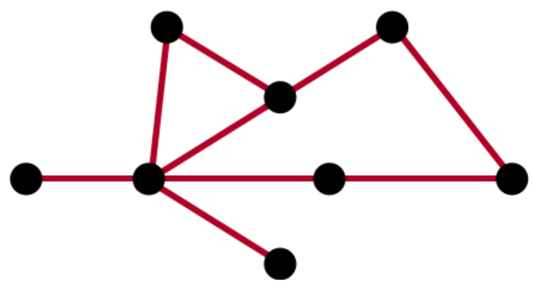
loop line in one-loop amplituhedron

sum over connected graphs with L vertices

mutual negativity $\langle AB_i AB_j\rangle < 0$

LOOPS OF LOOPS

General Amplituhedron geometry

$\Omega_G^{\text{tree}} =$  $= \dfrac{d\mu \cdot \mathcal{N}_G}{\prod_i D_i \cdot \prod_{\text{links}} \langle AB_i AB_j \rangle}$

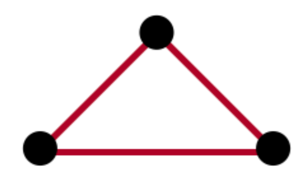
Reminder: the number of spacetime loops = number of vertices

Natural hierarchy of geometries: more "loops" = more complicated

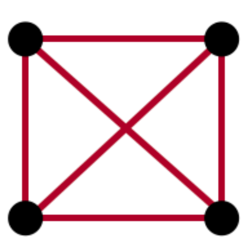
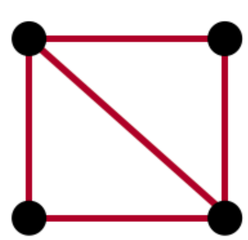
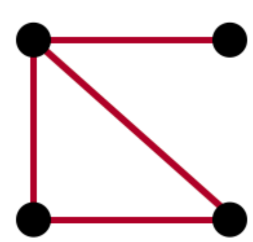
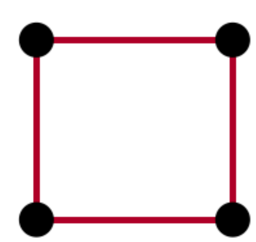
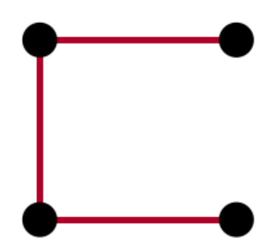
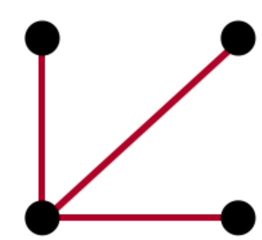
Lowest examples:

$\widetilde{\Omega}_2 =$ 

$\widetilde{\Omega}_2^{\text{tree}}$

$\widetilde{\Omega}_3 =$  $+$ 

$\widetilde{\Omega}_3^{1-\text{loop}}$ $\quad$ $\widetilde{\Omega}_3^{\text{tree}}$

$\widetilde{\Omega}_4 =$  $+$  $+$  $+$  $+$  $+$ 

$\widetilde{\Omega}_4^{3-\text{loop}}$ $\quad$ $\widetilde{\Omega}_4^{2-\text{loop}}$ $\quad$ $\widetilde{\Omega}_4^{1-\text{loop}}$ $\quad$ $\widetilde{\Omega}_4^{\text{tree}}$

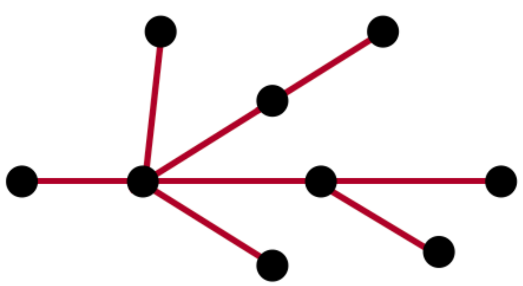
TREE-LEVEL

(Arkani-Hamed, Henn, JT, 2021)

Tree-level approximation: only keep geometries with tree graphs

We found a closed form for the numerator of any tree graph!

$$\Omega_G^{\text{tree}} =$$



$$= \frac{d\mu \cdot \mathcal{N}_G}{\prod_i D_i \cdot \prod_{\text{links}} \langle AB_i AB_j \rangle}$$

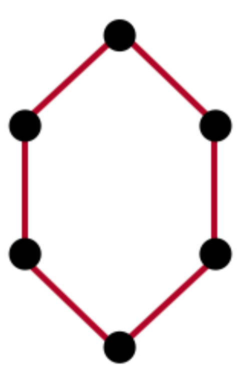
The numerator takes a factorized form

$$\mathcal{N}_G = \langle 1234 \rangle^{L+1} \times \prod_{\text{links}} N_{ij}^{(-)}$$

of the 2-loop numerators

$$N_{ij}^{(-)} = \langle AB_i 13 \rangle \langle AB_j 24 \rangle + \langle AB_i 24 \rangle \langle AB_j 13 \rangle$$

Same formula does not hold for a loop graph



$$\mathcal{N}_G = \langle 1234 \rangle^{L+1} \times \prod_{\text{links}} N_{ij}^{(-)}$$

satisfies many consistency constraints but not all (vanishing on double pole)

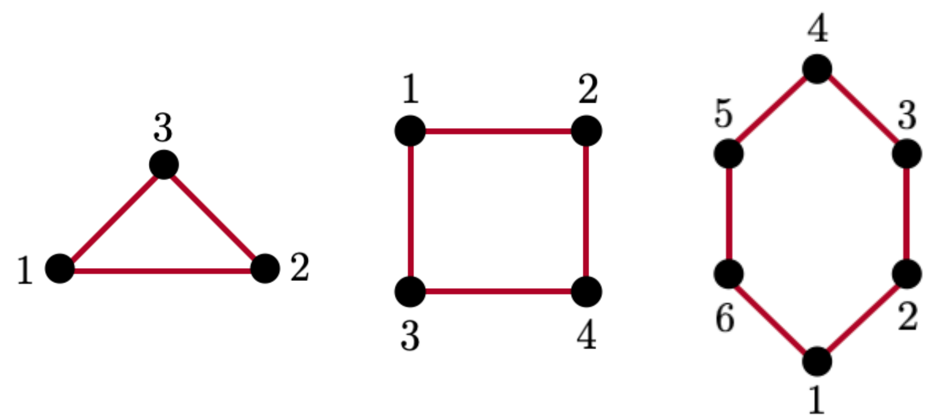
Need to find a specific correction which does not spoil any cuts we already matched

ONE-LOOP

(Brown, Oktem, Paranjape, JT, to appear)

One-loop: we found a numerator for a general one-loop graph

- First step: find an integrand for a graph with a closed loop



Write the numerator as

$$\mathcal{N}_G = \prod_{\text{links}} N_{ij}^{(-)} + R_{\text{loop}}$$

extremely constrained, write ansatz of all terms

For example: for 3-cycle we have two terms

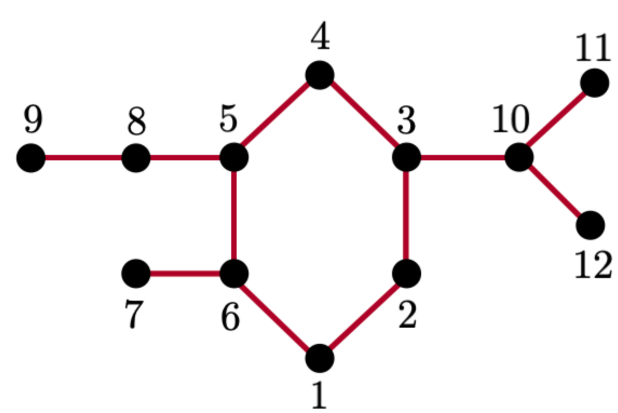
$$R^{(3)}_{\text{loop}} = c_1 \{ \langle AB_1 12\rangle\langle AB_1 34\rangle\langle AB_2 12\rangle\langle AB_2 34\rangle\langle AB_3 12\rangle\langle AB_3 34\rangle + \ldots \}$$
$$+ c_2 \{ \langle AB_1 12\rangle\langle AB_1 34\rangle(\langle AB_2 12\rangle\langle AB_3 34\rangle + \langle AB_3 12\rangle\langle AB_2 34\rangle)(\langle AB_2 13\rangle\langle AB_3 24\rangle + \langle AB_3 13\rangle\langle AB_2 24\rangle) + \ldots \}$$

Solve from the double pole cancelation: $c_1 = 4,\ c_2 = -1$

Generalized to any cycle

- Second step: add tree branches

$$\mathcal{N}_G = \left(\prod_{\text{loop links}} N_{ij}^{(-)} + R_{\text{loop}} \right) \times \prod_{\text{other links}} N_{ij}^{(-)}$$



Solved for any one-loop graph!

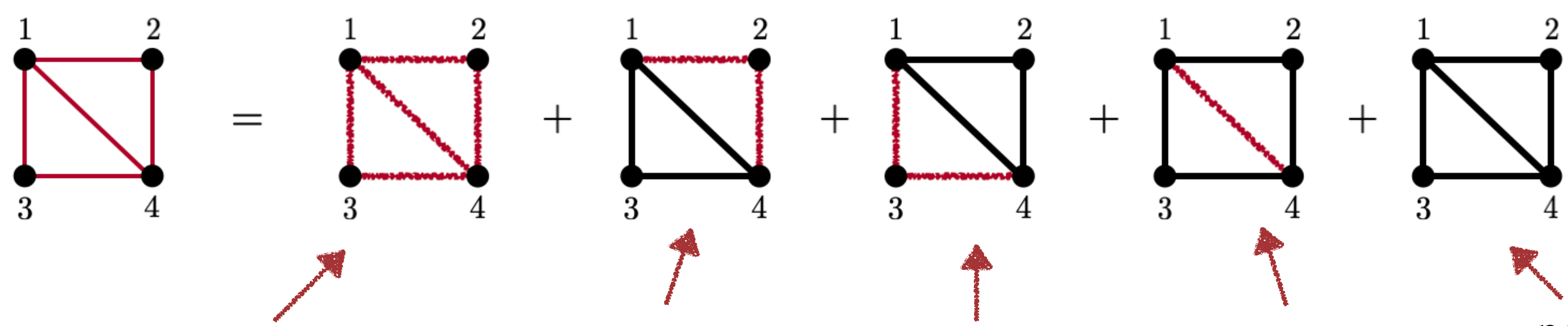
HIGHER LOOPS

(in progress)

We do not know how to find the numerator for a general higher-loop graph
= as hard as solving the four-point problem completely

Useful strategy: **loop decomposition**

denote: ●—● $= N_{ij}^{(-)}$



$N_{12}^{(-)} N_{24}^{(-)} N_{34}^{(-)} N_{13}^{(-)} N_{14}^{(-)}$ $\quad R_{\text{loop}}^{(134)} \times N_{12}^{(-)} N_{24}^{(-)}$ $\quad R_{\text{loop}}^{(124)} \times N_{13}^{(-)} N_{34}^{(-)}$ $\quad R_{\text{loop}}^{(1243)} \times N_{14}^{(-)}$ $\quad$ new two-loop object

tree objects

one-loop (and tree) objects

Only few cases solved at the moment, new ideas needed

IR FINITE FUNCTION AND RESUMMATION

BEYOND INTEGRAND

Why is the integrand so complicated while the amplitude takes a simple form (with only numbers unfixed, though divergent)?

Can we extract an **IR finite function** from the integrand?

$\widetilde{\Omega}_L(\ell_1, \ldots, \ell_L)$ ← rational function

$\downarrow$ integrate over ℓ_L

$\mathcal{I}_L(\ell_1, \ldots, \ell_{L-1})$ ← weight 2 function

$\downarrow$ integrate over ℓ_{L-1}

$\vdots$ $\quad\vdots$

$\downarrow$ integrate over ℓ_2

$\mathcal{I}_L(\ell_1)$

$\downarrow$ integrate over ℓ_1

$\ln M_L$ ← IR divergent, BDS ansatz

all are IR finite

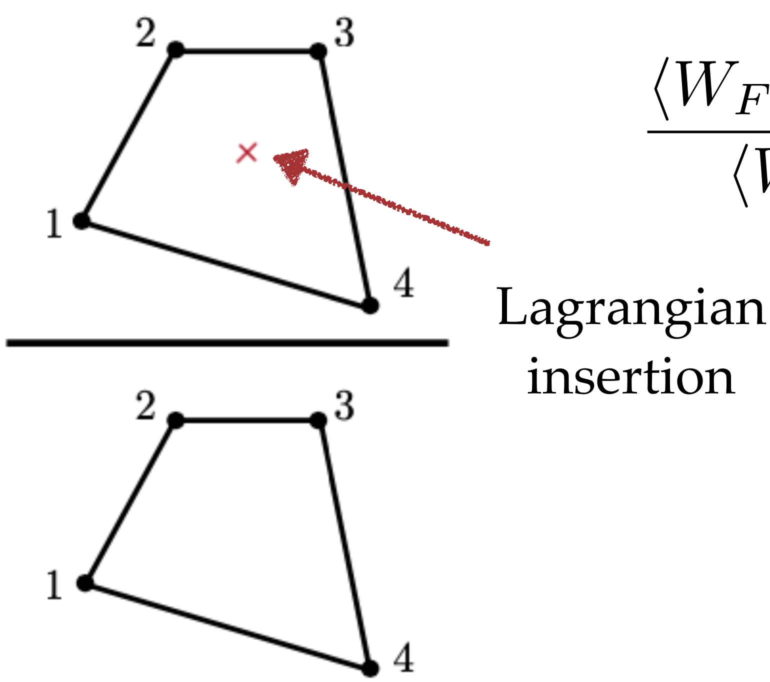
weight (2L-2) function of a single cross ratio

$$\mathcal{F}_{L-1}(z)$$

$$z = \frac{\langle AB12\rangle\langle AB34\rangle}{\langle AB23\rangle\langle AB14\rangle}$$

WILSON LOOPS

Same object appeared in the study of Wilson loops



$$\frac{\langle W_F(x_1,x_2,x_3,x_4)\mathcal{L}(x_0)\rangle}{\langle W_F(x_1,x_2,x_3,x_4)\rangle} = \frac{g^2}{\pi^2}\frac{\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle}\mathcal{F}(g,z)$$

- **weak coupling**: expansion in g^2

  (Alday, Heslop, Sikorowski, 2012) (Alday, Henn, Sikorowski, 2013)

- **strong coupling**: expansion in $1/g$ (string tension)

  (Alday, Buchbinder, Tseytlin, 2011) (Engelund, Roiban, 2011, 2012)

$$\mathcal{F}(g,z) = g\frac{z}{(z-1)^3}\left[2(1-z)+(z+1)\log z\right]+\cdots$$

We can extract $\Gamma_{\rm cusp}(g)$ from this function

(Alday, Henn, Sikorowski, 2013) (Henn, Korchemsky, Mistlberger, 2019) (Arkani-Hamed, Henn, JT, 2021)

$$g\frac{\partial}{\partial g}\Gamma_{\rm cusp}(g) = -\frac{1}{\pi}\int_{-\pi}^{\pi} d\phi\, F(g,z=e^{i\phi})$$

compare (or even try to derive) an exact formula from integrability

(Beisert, Eden, Staudacher, 2006)

NEGATIVE GEOMETRIES

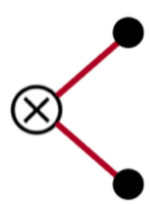
(Arkani-Hamed, Henn, JT, 2021)

Negative geometries: freezing one of the loops and integrate the rest

$$\boxed{\mathcal{F}_{L-1}(z) \;\; \text{from} \;\; \widetilde{\Omega}_L}$$

Simplest two-loop (one-loop integration) result:  $= [\pi^2 + \log^2 z]$

Three-loop result: three different contributions



$$= -\frac{1}{2}\left[\pi^2 + \log^2 z\right]^2$$

$$= -\frac{1}{12}\left[\pi^2 + \log^2 z\right] \times \left[5\pi^2 + \log^2 z\right]$$

Tree graphs

$$\begin{aligned} = &-\frac{1}{6}\log^4 z + \log^2 z\left[-\frac{2}{3}\mathrm{Li}_2\left(\frac{1}{z+1}\right) - \frac{2}{3}\mathrm{Li}_2\left(\frac{z}{z+1}\right) + \frac{\pi^2}{9}\right] \\ &+ \log z\left[4\mathrm{Li}_3\left(\frac{z}{z+1}\right) - 4\mathrm{Li}_3\left(\frac{1}{z+1}\right)\right] - \frac{2}{3}\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right]^2 \\ &- \frac{8}{3}\pi^2\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right] - 8\mathrm{Li}_4\left(\frac{1}{z+1}\right) - 8\mathrm{Li}_4\left(\frac{z}{z+1}\right) - \frac{\pi^4}{18} \end{aligned}$$

One-loop graph

Same loop order, tree graphs are simpler - consistent with simple integrand

RESUMMATION

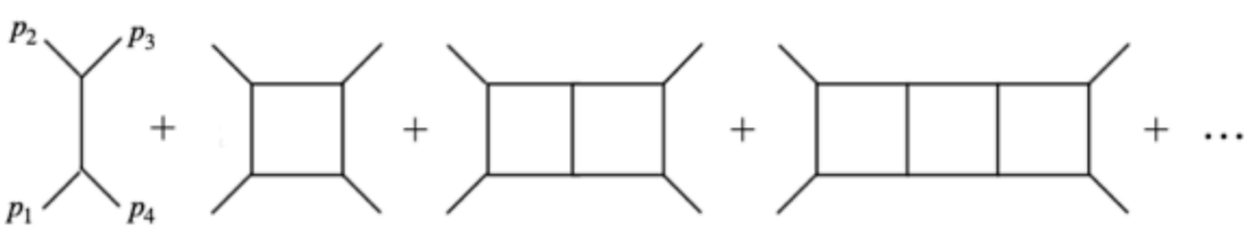
Planar N=4 SYM is special: no non-perturbative contributions

$$\mathcal{M}_4^{\text{exact}} = \sum_{L=0}^{\infty} g^{2L} \mathcal{M}_4^{(L)}$$

- approximate amplitude at each loop order and resum to all loops
- compare to strong coupling expansion (if available)

Only known (to me) example is the **ladder resummation**

(Broadhurst, Davydychev, 2010)



$\sim e^{-g}$ for $g \gg 1$

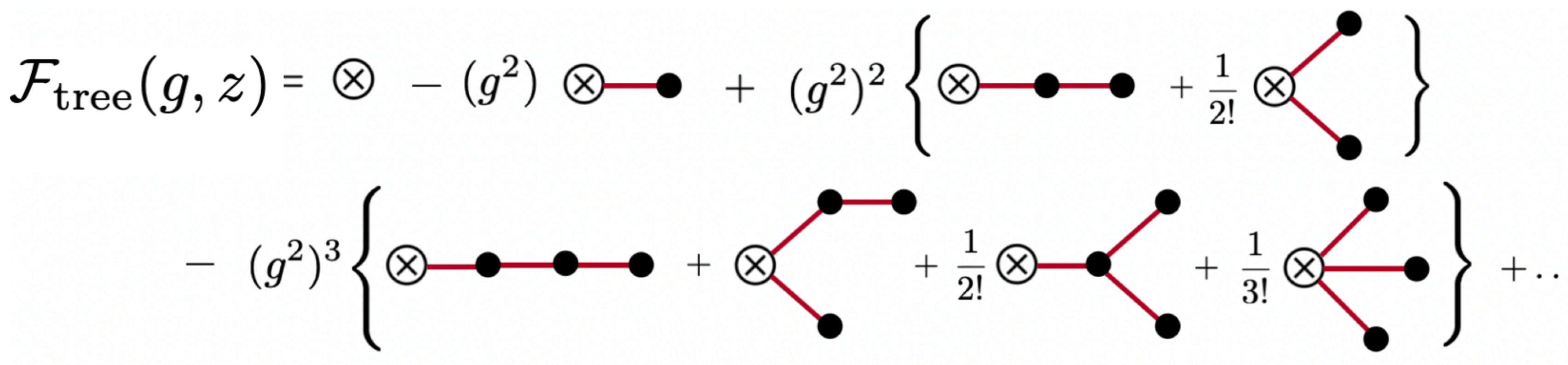
exponentially suppressed vs linear growth

We can try to resum our tree contributions (if possible to integrate all of them)

TREE-LEVEL CONTRIBUTION

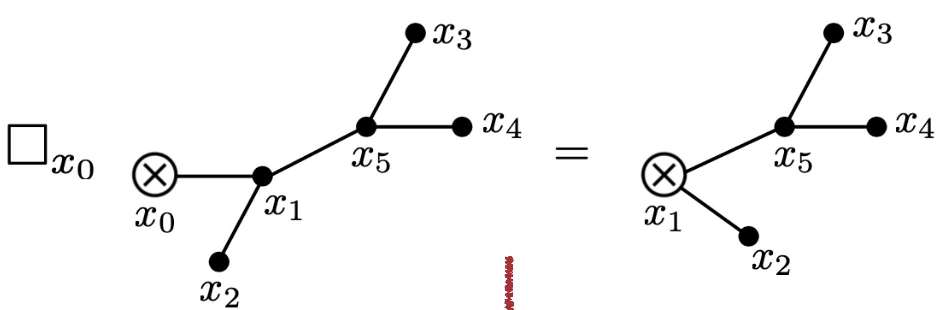
(Arkani-Hamed, Henn, JT, 2021)

Let us only consider **tree graphs**



Differential operator acting on the graphs (integrand)

trick to avoid integration



Same operator does not work for loop graphs: search for its generalization

differential equation for the generating function

$$\mathcal{F}_{\text{tree}}(g,z) = e^{\mathcal{H}_{\text{tree}}(g,z)}$$

$$\frac{1}{2}(z\partial_z)^2\mathcal{H}_{\text{tree}}(g,z) + g^2 e^{\mathcal{H}_{\text{tree}}(g,z)} = 0$$

solve with boundary conditions

$$\mathcal{F}_{\text{tree}}(g,z) = \frac{A^2}{g^2}\frac{z^A}{(z^A+1)^2}$$

where $\quad \dfrac{A}{2g\cos\frac{\pi A}{2}} = 1$

STRONG COUPLING

(Arkani-Hamed, Henn, JT, 2021)

$$\mathcal{F}_{\text{tree}}(g,z) = \frac{A^2}{g^2}\frac{z^A}{(z^A+1)^2}$$

How good is this approximation to the exact result?

Naively, we expect it to be very bad — for infinite L vanishing part of diagrams contribute

Easy to expand at strong coupling:

$$\mathcal{F}_{\text{tree}}(g,z) = -\frac{z}{(1+z)^2} + \mathcal{O}\left(\frac{1}{g}\right)$$

→ misses the leading term

→ has 1/g expansion

For $\Gamma_{\text{cusp}}(g)$ we get even more surprising result:

$$\Gamma_{\text{tree}}(g) \to \begin{cases} 2g - \frac{3\log 2}{2\pi} + \cdots \\ \frac{8}{\pi}g - 1 + \ldots \end{cases}$$

→ exact

→ our tree approximation

qualitatively correct behavior at strong coupling

OPEN QUESTIONS

How does the subleading (one-loop) contribution contribute to the $\mathcal{F}(g,z)$ function?

↓

How do we reconstruct $\mathcal{F}(g,z) \sim g$ behavior at strong coupling?

And what about cusp anomalous dimension?

$$\Gamma_{\text{tree}}(g) \to g\left(\frac{8}{\pi} + \gamma^{(1)} + \gamma^{(2)} + \dots\right) = 2g$$

$\frac{8}{\pi} \simeq 2.55$

Is it just a small negative correction?
Do tree graphs dominate or are there massive cancelations between various orders?

COMMENT ON LOOPS

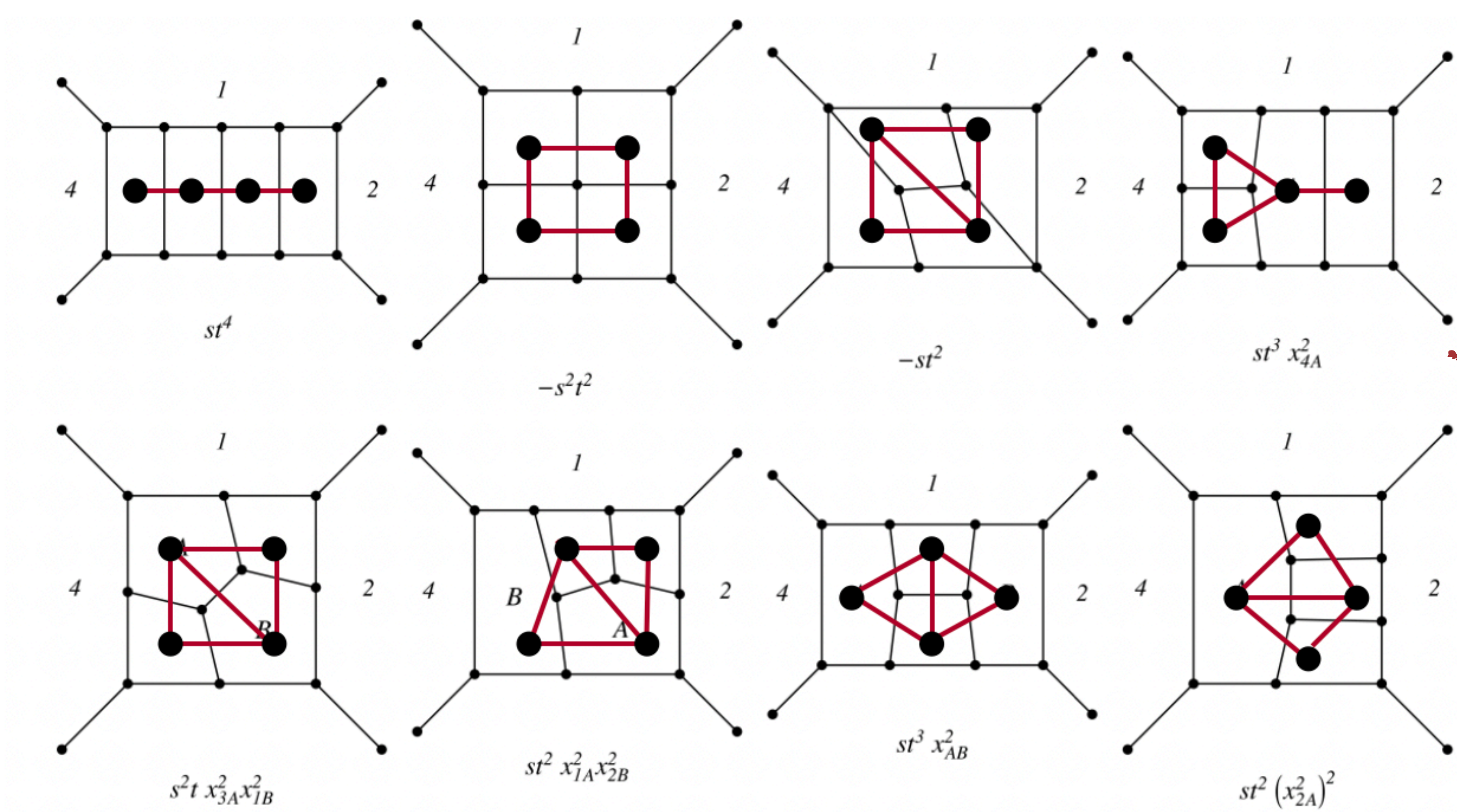
How is the loop of loops expansion related to summing certain classes of diagrams?

- Short answer: NO
- Long answer: some limited correspondence via # of internal propagators

more propagators = more complicated

$\langle AB_i AB_j \rangle$

Standard diagrams for four-loop integrand



Negative geometries

- Not planar objects
- No unphysical cuts

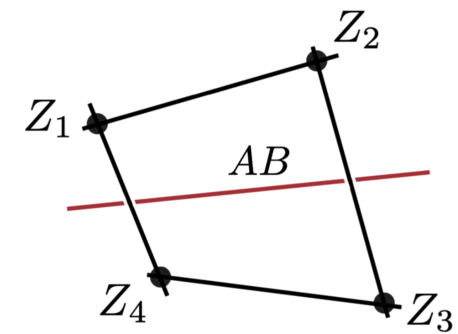
Most complicated internal graph is 1-loop (not 2-loop), one tree graph missing

No direct relation

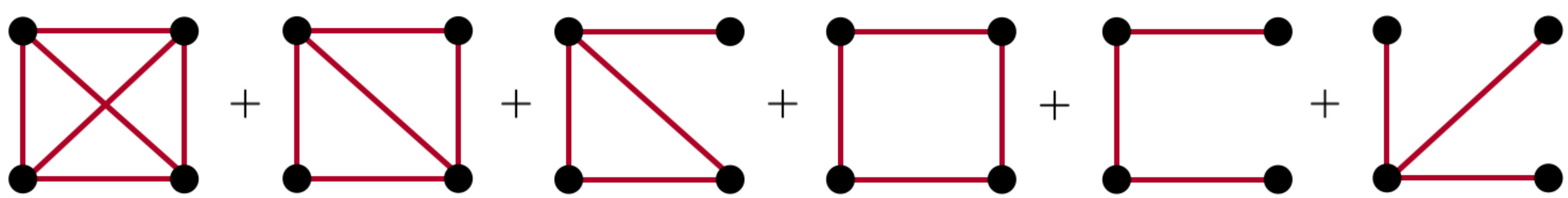
CONCLUSION

SUMMARY

New expansion of the planar N=4 SYM integrand using **negative geometries**: goal is to determine the four-point all-loop integrand (and amplitude)



Hierarchy of geometries: **loops of loops** expansion, all-loop integrand for tree-level and one-loop geometries



IR finite function: for trees found differential equation, integrated and resummed, interesting strong coupling behavior

OUTLOOK

Finish one-loop calculation, differential operator, higher loop integrands — organizing principle

with Taro Brown, Umut Oktem, Shruti Paranjape,.....

Higher point integrands / integrals from negative geometry

with Dmitry Chicherin, Johannes Henn, Antonela Matijasic, Julian Miczajka




Alternative IR finite object: deformed Amplituhedron

with Nima Arkani-Hamed, Wojciech Flieger, Johannes Henn, Anders Schreiber




poster at this conference

Amplituhedron and negative geometries for ABJM **next talk**

(Song He, Yutin Huang, Chia-Kai Kuo, 2023) (Lukowski, Stalknecht, 2023)

Thank you!