The ABJM Amplituhedron

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Amplitudes 2023, CERN

Based on 2306.00951 (n-pt)
S. He, Y.-t. Huang

and related work 2204.08297 (4-pt)
S. He, Z. Li, Y.-Q. Zhang

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Planar N=4 SYM amplitudes correspond to positive geometry, the amplituhedron. In this talk, I would extend amplituhedron to another theory — **ABJM** theory.

Amplituhedron pictures for ABJM just emerges these years:

Group of Yutin, Song, Congkao,Tomasz, Johannes …

Today's talk

<table>
<tr><th>2021 yr</th><th>Last year</th><th>This year</th></tr>
<tr><td>Tree:

Orthogonal/ ABJM momentum amplituhedron
[Huang et al (2021); He et al. (2021)]</td><td colspan="2">Loop: / Tree + Loop :

ABJM amplituhedron
4-pt → n-pt
Define in 3d momentum twistor variables</td></tr>
</table>

Note: Loop geometry in momentum space see Stalknecht's poster. [Lukowski et al (2023)]

Outline

- Review ABJM amplitude

- Amplituhedron definition $\begin{cases} \text{Tree} \\ \\ \text{Loop} \end{cases}$

    1. Why it is right?

    2. Loop as fibration of tree ⇒ *Chamber*

- Negative geometry: bipartite structure

Review ABJM amplitude

In ABJM theory, the physical d.o.f. are encoded in the bi-fundamental matter fields, which can be arranged into supermultiplets:

$$\begin{aligned}\Phi &= X_4 + \eta_A \psi^A - \frac{1}{2}\epsilon^{ABC} \eta_A\eta_B X_C - \eta_1\eta_2\eta_3 \psi^4 \,, \\ \bar{\Psi} &= \bar{\psi}_4 + \eta_A \bar{X}^A - \frac{1}{2}\epsilon^{ABC} \eta_A\eta_B \bar{\psi}_C - \eta_1\eta_2\eta_3 \bar{X}^4 \,.\end{aligned}$$

The scattering of different field configuration is glued into a super-amplitude:

1. only even-pt
2. Grassmanian degree = $3(k+2)$ with $k = n/2 - 2$

Definition of tree geometry

The ABJM amplituhedron is defined as reducing the original amplituhedron's external twistors to 3d and flipping the overall positive conditions to negative conditions.

Tree 4d

$$\mathbb{T}_n : \quad \langle i i{+}1 j j{+}1 \rangle > 0, \quad \{\langle 123i \rangle\} \text{ has } k \text{ sign flips.}$$

flip the overall sign

3d

$$\mathbb{T}_n : \quad \begin{array}{c} \langle i i{+}1 j j{+}1 \rangle < 0, \quad \{\langle 123i \rangle\} \text{ has } k \text{ sign flips,} \\ \langle\langle i i{+}1 \rangle\rangle = 0 \text{ for } i, j = 1, 2, \ldots, n. \end{array} \;.$$

symplectic reduction

3d twistor:

conformal group $SU(4) \longrightarrow Sp(4)$

4d 3d

$$\langle\langle i\, i+1 \rangle\rangle := \Omega_{ab} Z_i^a Z_{i+1}^b = 0,$$

$$\langle ijkl \rangle := \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L$$

$$\mathbf{\Omega}_{IJ} = \begin{pmatrix} 0 & \epsilon_{2\times 2} \\ \epsilon_{2\times 2} & 0 \end{pmatrix}, \text{ where } \epsilon_{2\times 2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Tree geometry

1. With an overall sign flip, symplectic reduction kills all the odd-multiplicity:

$$\mathbb{T}_n : \quad \begin{array}{c} \langle ii{+}1jj{+}1\rangle < 0, \quad \{\langle 123i\rangle\} \text{ has } k \text{ sign flips}, \\ \langle\!\langle ii{+}1\rangle\!\rangle = 0 \text{ for } i,j = 1,2,\ldots,n. \end{array} \quad .$$

$$\langle abcd\rangle = -\langle\!\langle ab\rangle\!\rangle\langle\!\langle cd\rangle\!\rangle + \langle\!\langle bc\rangle\!\rangle\langle\!\langle da\rangle\!\rangle - \langle\!\langle ac\rangle\!\rangle\langle\!\langle db\rangle\!\rangle$$

ex $\langle 1234\rangle = \langle\!\langle 13\rangle\!\rangle\langle\!\langle 24\rangle\!\rangle$

$$\underset{-}{\langle n{-}2n{-}1n\hat{1}\rangle}\,\underset{-}{\langle n\hat{1}23\rangle} = \underset{-}{\langle n{-}1n12\rangle}\overset{(-)^{n-3}}{\left(\frac{\prod_{i=1}^{n-3}\langle ii{+}1i{+}2i{+}3\rangle}{\prod_{i=2}^{n-3}\langle\!\langle ii{+}2\rangle\!\rangle^2}\right)} .$$

2. Only allows the $k = \dfrac{n}{2} - 2$.

Tree geometry is only nontrivial for ABJM kinematic

ABJM momentum amplituhedron $\xrightarrow[\text{T-duality map}]{}$ $\mathbb{T}_n$

Definition of loop geometry

We can do the same reduction on the loop twistors $(AB)_a$ and flip the overall positive condition to negative. The reduced space defines the geometry for ABJM integrands.

Loop 4d

$$\mathbb{L}_n : \quad \begin{array}{r} \langle (AB)_a i i{+}1 \rangle > 0, \ \{\langle (AB)_a 1 i \rangle\} \text{ has } k+2 \text{ sign flip}, \\ \langle (AB)_a (AB)_b \rangle > 0, \ \text{for any two loop } a, b \in 1, 2, \ldots, L. \end{array}$$

3d

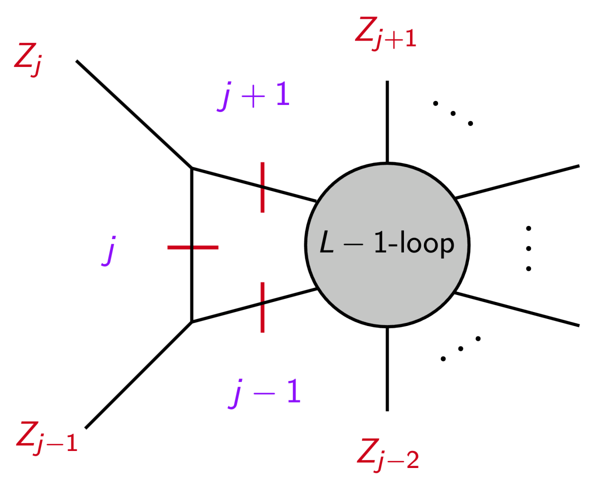
$$\mathbb{L}_n : \quad \begin{array}{r} \langle (AB)_a i i{+}1 \rangle < 0, \ \{\langle (AB)_a 1 i \rangle\} \text{ has } k+2 \text{ sign flip}, \\ \langle (AB)_a (AB)_b \rangle < 0, \ \text{for any two loop } a, b \in 1, 2, \ldots, L, \\ \langle\langle A_a B_a \rangle\rangle = 0 \text{ for } a \in 1, \ldots, L. \end{array}$$

$$\langle\langle A_a B_a \rangle\rangle = \Omega_{IJ} A_a^I B_a^J = 0 \qquad \langle (AB)_a i j \rangle := \epsilon_{IJKL} A_a^I B_a^J Z_i^K Z_j^L$$

Loop geometry

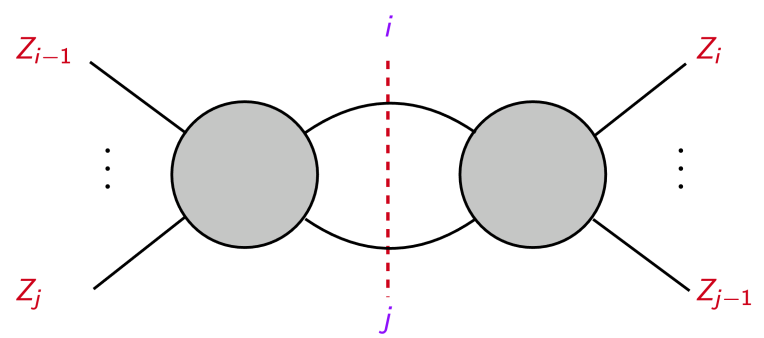
Loop geometries manifest many properties of ABJM integrands:

1. **Soft** cut:



$$A_n^{\ell-\text{loop}}\Big|_{\substack{\text{cut}\\ i-1,\,i,\,i+1}} = (-1)^i A_n^{(\ell-1)-\text{loop}}$$

2. **Double** cut:



$$\mathcal{A}_n^{L\text{-loop}}\Big|_{i,j} = \sum_{\text{state}} \mathcal{A}_{n_1}^{L_1\text{-loop}} \times \mathcal{A}_{n_2}^{L_2\text{-loop}}$$

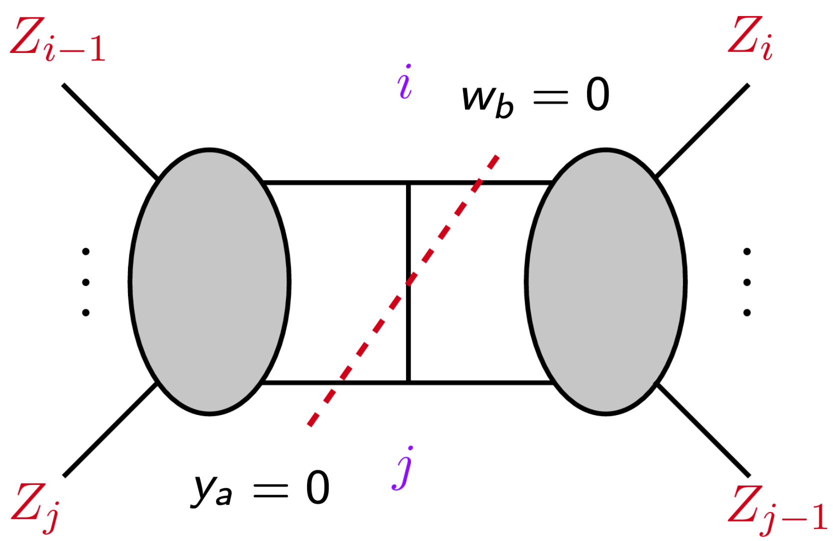
$$L = L_1 + L_2 + 1 \qquad n = n_1 + n_2 - 4 \qquad k = k_1 + k_2$$

3. **Vanishing** cut: factorized into odd-pt amplitudes.

Example: vanishing cut

$$\langle (AB)_a j{-}1 j\rangle = \langle (AB)_b i{-}1 i\rangle = 0$$

when i, j are both even or odd ⇒ odd-pt amplitude



$$(AB) = (Z_{i-1} + xZ_i - wZ_j,\ yZ_i + Z_{j-1} + zZ_j)$$

$$\frac{\langle (AB)_a (AB)_b\rangle}{\langle i{-}1\, i\, j{-}1\, j\rangle} = \underbrace{w_a y_b}_{+} - \underbrace{\left(1 - \frac{z_b}{z_a}\right)^2}_{+} \frac{\overset{-}{\langle (AB)_a j{-}2 j{-}1\rangle}}{\underset{-}{\langle (AB)_b i i{+}1\rangle}} \frac{\langle\langle i{-}1 i{+}1\rangle\rangle}{\langle\langle j{-}2\, j\rangle\rangle}$$

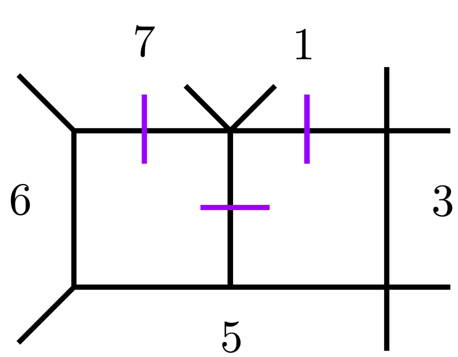
> 0 1

2

$+$: i, j different even/odd parity

$-$: i, j same even/odd parity

The vanishing of odd-pt cut is an amazing identity from the point of view of local integrands:



cancels with contributions from 21 other double-box and many double triangles.

Loops as a "fibration" of Trees

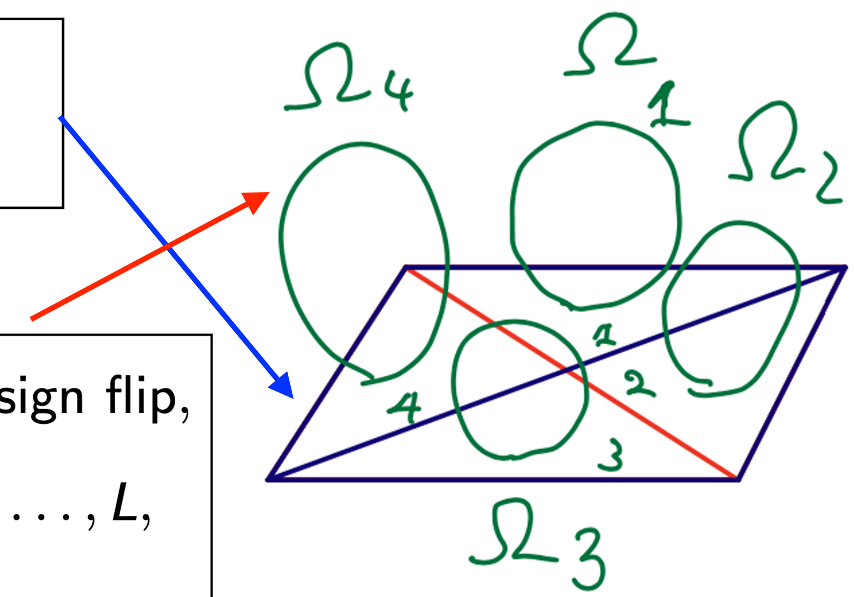
$\{Z_i\} \in \mathbb{T}_n : \quad \langle i\,i{+}1\,j\,j{+}1\rangle < 0, \quad \{\langle 123i\rangle\}$ has k sign flips, $\langle\langle i\,i{+}1\rangle\rangle = 0$ for $i, j = 1, 2, \ldots, n$.

Fiber $\equiv$ $\mathbb{L}_n :$

$$\langle (AB)_a\, i\,i{+}1\rangle < 0,\ \ \{\langle (AB)_a 1 i\rangle\} \text{ has } k+2 = \frac{n}{2} \text{ sign flip,}$$

$$\langle (AB)_a (AB)_b\rangle < 0,\ \text{ for any two loop } a, b \in 1, 2, \ldots, L,$$

$$\langle\langle A_a B_a\rangle\rangle = 0 \text{ for } a \in 1, \ldots, L.$$



Q: *1. Is loop geometry the "same" when $\{Z_i\}$ varies in the tree region?*

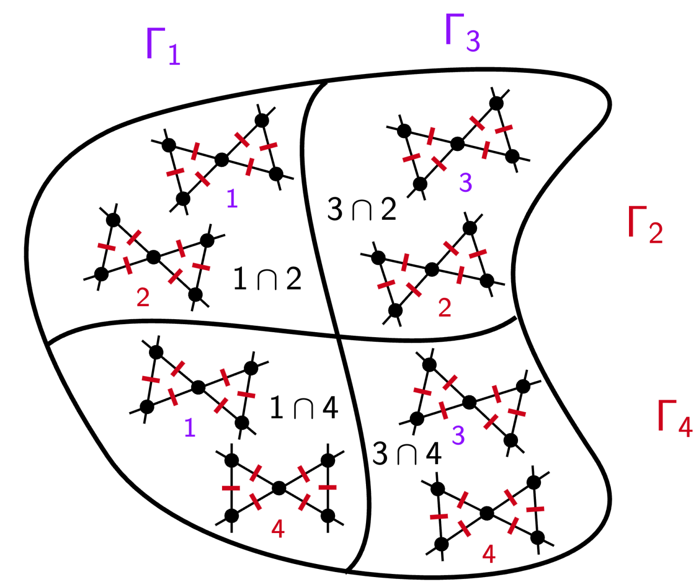
2. If not, then how does loop fibration triangulate tree geometry?

A: $\mathbb{T}_n$ *has natural triangulation in terms of Chambers, where in each chamber, the loop form is the same.*

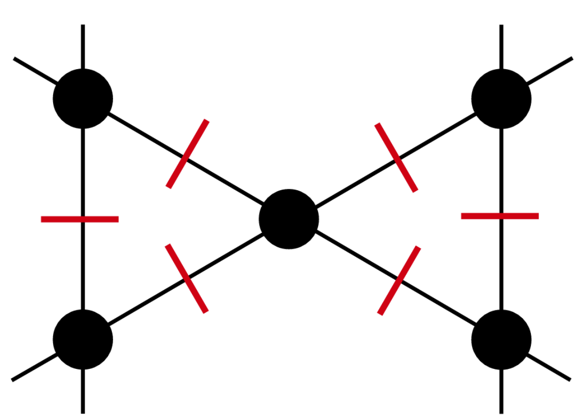
Chambers are subregions of $\mathbb{T}_n$, identified by

1. Subsets of maximal cuts are positive.

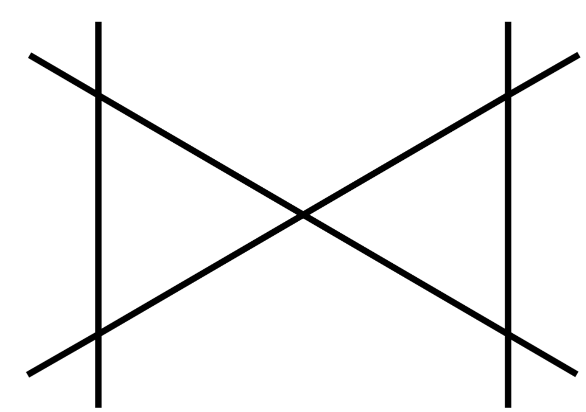
cut solutions AB reside in $\mathbb{L}_n$



2. Collection of (n-3) dim cells whose the image in Tn overlapped.



maximal cut

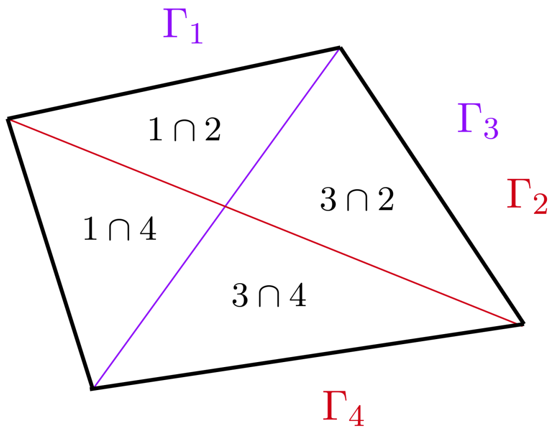


on-shell diagram

n=6: only single BCFW cell #chamber = 1

n=8: 4 (n-3) dim cells $\sum_{\text{sol.}} \Gamma_1 + \Gamma_3 = \sum_{\text{sol.}} \Gamma_2 + \Gamma_4 = A_8^{\text{tree}}$

$\Rightarrow$ #chamber = 4



n=10: 25 (n-3) dim cells $\Rightarrow$ #chamber = 50

degenerate at one-loop only 25 distinct region

dependency break at two-loop

n=8:

$$1\cap 2:\ \sum_{i=1}^{8}(-1)^i I_{box}(i-1,i,i+1)+\overbrace{I_{tri}(1,3,5)+I_{box}(1,3,5)+I_{tri}(5,7,1)+I_{box}(5,7,1)}^{1}$$

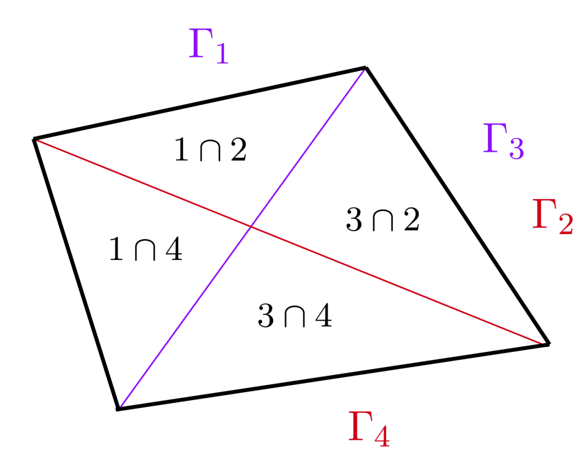
$$\underbrace{+I_{tri}(2,4,6)-I_{box}(2,4,6)+I_{tri}(6,8,2)-I_{box}(6,8,2)}_{2}.$$

$$[1\cap 2]_+\times\left(I_8^{L=1}(0)+I_8^{L=1}(1)+I_8^{L=1}(2)\right)$$

$$+$$

$$[3\cap 2]_+\times\left(I_8^{L=1}(0)+I_8^{L=1}(3)+I_8^{L=1}(2)\right)$$

$$\mathrm{LS}_2\times\left(I_8^{L=1}(0)+I_8^{L=1}(2)\right)$$

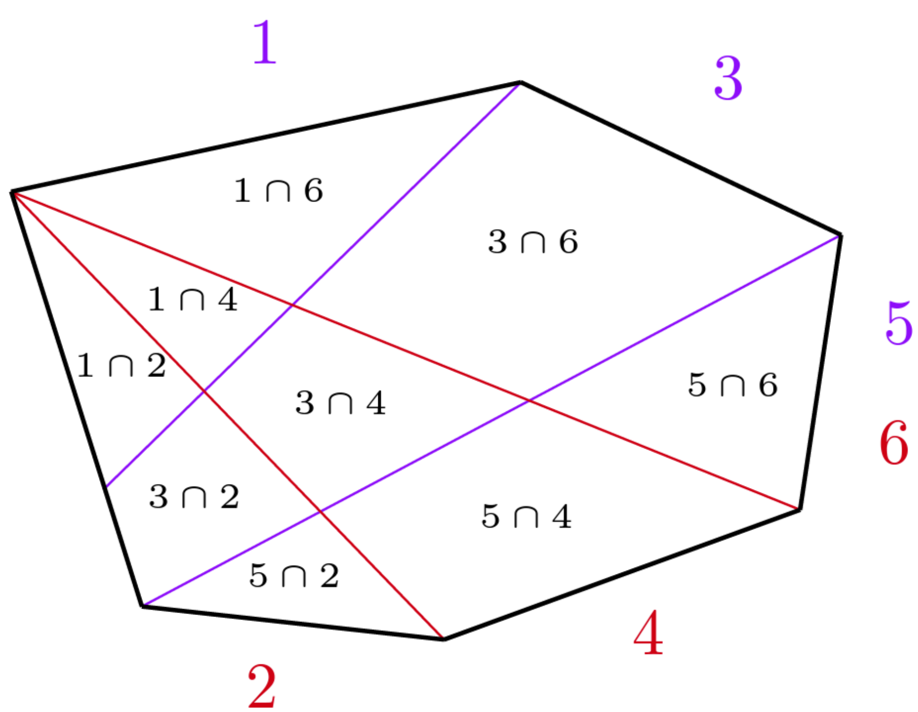


In N=4 SYM, it also can use chambers to calculate its loop integrand:

One-loop n=6 NMHV

[Ferro et al. (2023)]

In progress with Song, Yutin, and Matteo



$$[p \cap q] \times \left(\sum_{\substack{i=\{p,q\} \\ j=\{1,2\}}} I_j^{\text{1mb}}(i) + I_j^{\text{2mh}}(i) + \sum_{i=1}^{6} I^{\text{tri}}(i) \right)$$

Canonical form of chamber $1 \cap 2$:

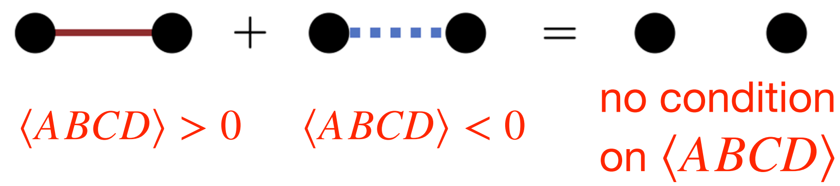
$$\frac{\langle 23456\rangle^2 \langle 34561\rangle^2}{\langle Y1345\rangle\langle Y1356\rangle\langle Y2346\rangle\langle Y2456\rangle\langle Y3456\rangle} \langle Y d^4 Y\rangle$$

Negative geometry

Arkani-Hamed et al (2022)]

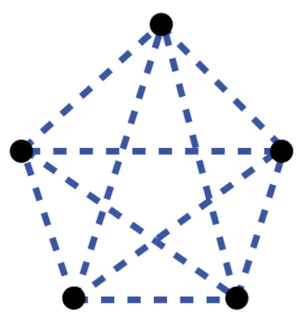
Also see
Jaroslav's talk.

We can also consider negative geometry of ABJM amplituhedron

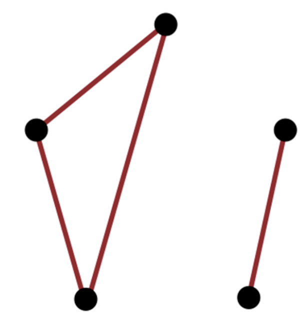


Express all positive links using negative and empty

$$\Omega(\mathcal{A}_L) =$$

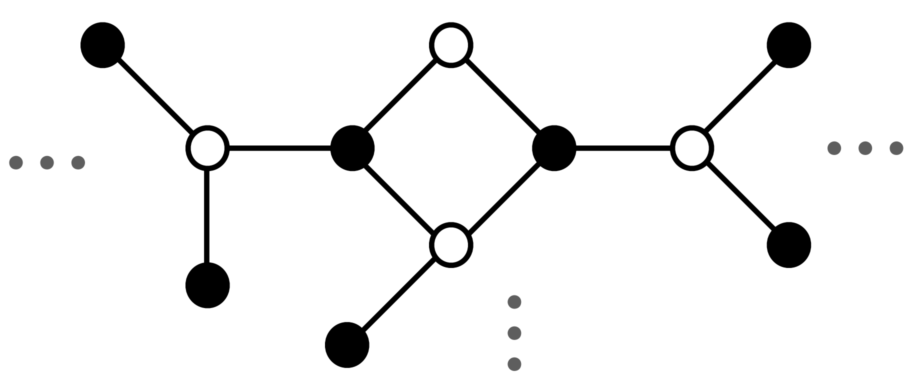


$$= \sum_{\text{all}\, G} (-1)^{E(G)}$$



only focus connected graphs enough

The negative geometry of ABJM is much simpler than N=4 SYM.
Connected graphs only could be bipartite graphs:

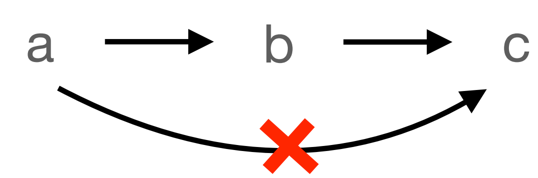


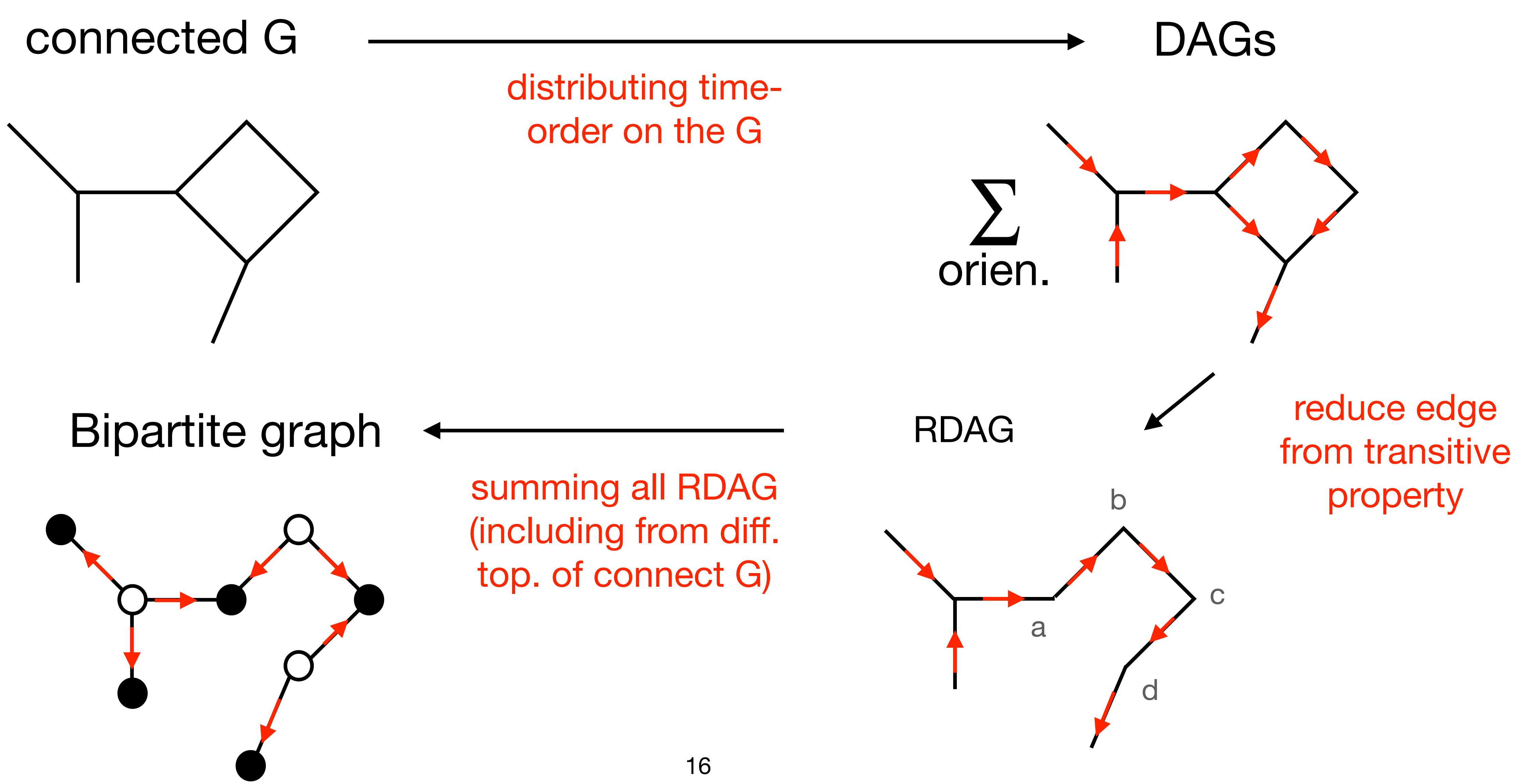
The emergence of bipartite graphs [He et al. (2022)]

Mutual negativity link implies two loop variables are time-like separated in $R^{1,2}$:

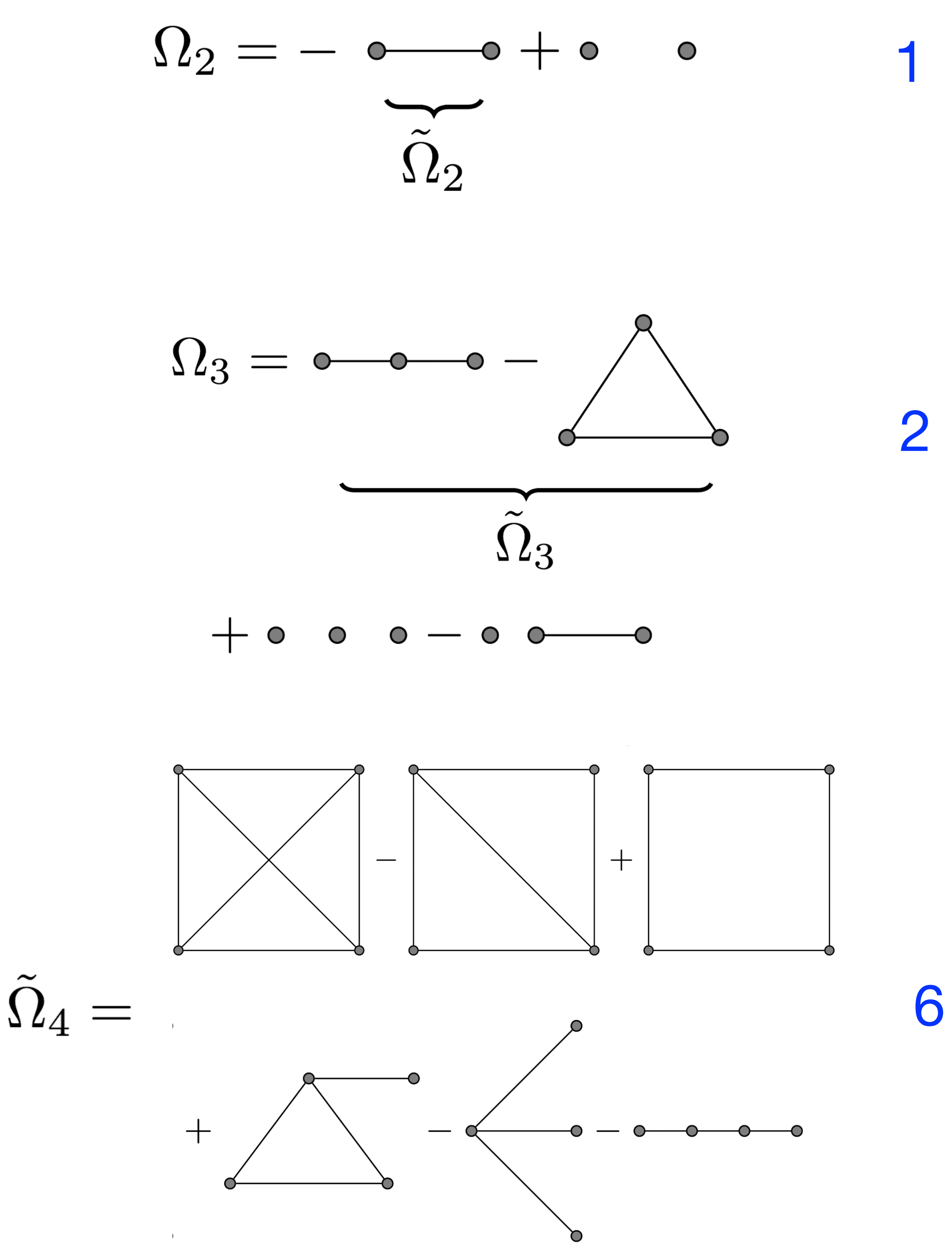
$$\langle (AB)_a (AB)_b \rangle > 0$$

Time-like separation is a transitive property:

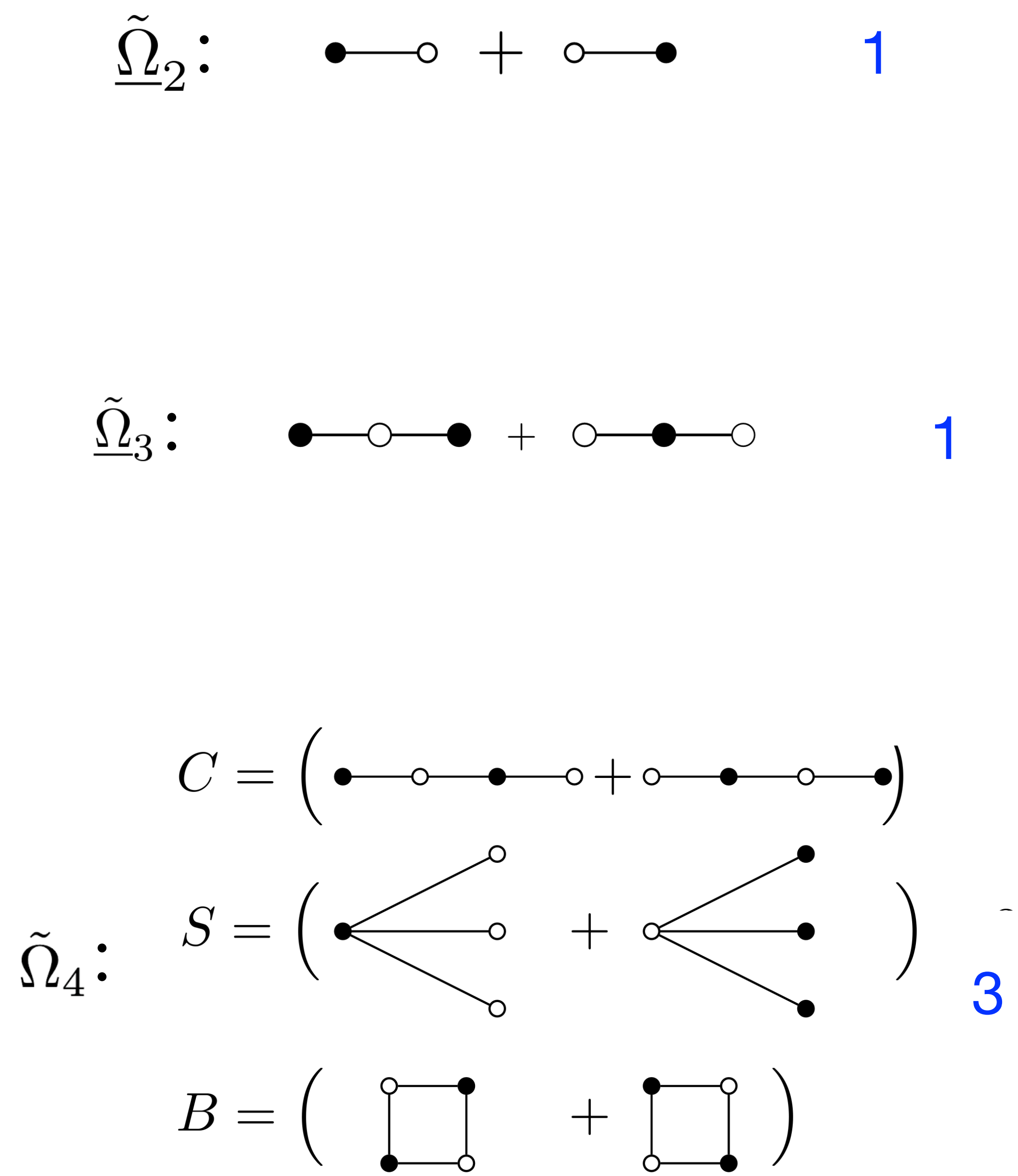




Connected graph



Bipartite graph



Connected graph | Bipartite graph

L	top. of G	top. of g
2	1	1
3	2	1
4	6	3
5	21	5
6	112	17
7	853	44

Example: two-loop 6-pt integrand and its bipartite representation

Form of **positive geometry**

$$\sum_{i=1}^{6} -\frac{1}{2} I^{\text{critter}}(i) + I^{\text{crab}}(i) + I_{+}^{\text{2mh}}(i) - \sum_{i=1,3,5} I_1^{bt}(i) + \sum_{i=2,4,6} I_2^{bt}(i)$$

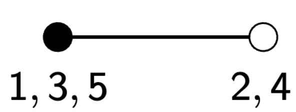
$$\updownarrow \quad \tilde{\Omega}_2 = \Omega_2 - \tfrac{1}{2}\Omega_1^2$$

Form of **negative geometry**

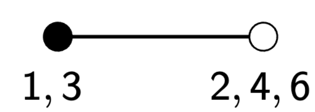
$$\begin{gathered} I_a(135,246) + [I_b(135,24) + (24 \to 46, 26) + I_b(13,246) + (13 \to 35, 15)] \\ +[I_c(13,24) + (24 \to 26) + (13 \to 35; 24 \to 24, 46) + (13 \to 15; 24 \to 26, 46) \\ +I_c(13,46) + (13 \to 35, 15; 46 \to 26, 24)] + (\ell_1 \leftrightarrow \ell_2)\,, \end{gathered}$$

with numerators

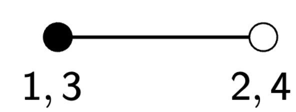
$$:= N_a(135,246) = \epsilon(\ell_1, 1, 3, 5, {}^{\mu})\epsilon(\ell_2, 2, 4, 6, {}_{\mu}) - (\ell_1 \cdot \ell_2)\sqrt{(1\cdot 3\cdot 5)}\sqrt{(2\cdot 4\cdot 6)} \\ -\epsilon(\ell_1, 1, 3, 5, \ell_2)\sqrt{(2\cdot 4\cdot 6)} - \epsilon(\ell_1, 2, 4, 6, \ell_2)\sqrt{(1\cdot 3\cdot 5)},$$



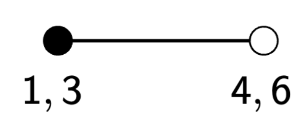
$$:= N_b(135,24) = (\ell_1 \cdot y_{4,(23)\bigcap(145)})(1\cdot 3) - \frac{\epsilon(\ell_1, 2, 3, 4, 5)\sqrt{(1\cdot 3\cdot 5)}}{(3\cdot 5)},$$



$$:= N_b(13,246) = (\ell_2 \cdot y_{3,(12)\bigcap(634)})(2\cdot 6) - \frac{\epsilon(\ell_2, 6, 1, 2, 3)\sqrt{(2\cdot 4\cdot 6)}}{(2\cdot 6)},$$



$$:= N_c(13,24) = -\frac{(1\cdot 3)(2\cdot 4)}{2},$$



$$:= N_c(13,46) = \frac{(1\cdot 3)(4\cdot 6)}{2}.$$

Example: two-loop 8-pt integrand and its bipartite representation

2211.01792 with S. H., Y.-t. Huang, Z.-Li.

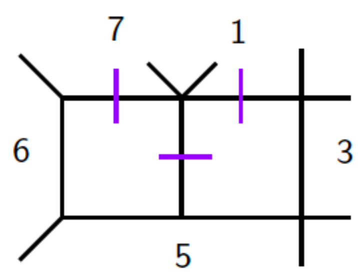
1. Containing sYM BDS.
2. The BDS piece has an integrand representation.

Parity-even letters: reduced from D=4.

$$A_8^{2-\text{loop}} = A_8^{\text{tree}} \times \left[\text{BDS}_8 + \pi^2 + R_8^{\text{even},A}\right] + \sum_{i=1}^{8} (-1)^i \left[\mathcal{D}_{i,i+2,i+4} R^{\text{even},\mathcal{D}}_{i,i+2,i+4} + \mathcal{B}_{i,i+2,i+4} R^{\text{odd}}_{i,i+2,i+4}\right] + \sum_{i=1}^{8} \bar{\mathcal{D}}_{i,i+2,i+4} R^{\text{even},\bar{\mathcal{D}}}_{i,i+2,i+4}$$

Very interesting parity-odd letters!

Elliptic integral appear:



$$\text{Fmb}(v,w) := \int \frac{dc}{4\pi\sqrt{c}} \frac{w-v}{(1+c)\sqrt{-4vw + (v+w-(1+c)vw)^2}} \times \left(\log(z\bar{z})\log(1+z) - \log(z\bar{z})\log(1+\bar{z}) + 2\text{Li}_2(-z) - 2\text{Li}_2(-\bar{z})\right)$$

Cancelled out finally!

Form of **positive geometry** of chamber $[1 \cap 2]$

$$1 \cap 2: \sum_{i=1}^{8} -I_A^{db}(i)+I_B^{db}(i)+\frac{1}{2}I_C^{db}(i)+\sum_{i=2,6} I_{\overline{D},+}^{db}(i)+I_{\overline{E},+}^{db}(i)+\sum_{i=4,8} I_{\underline{D},+}^{db}(i)+I_{\underline{E},+}^{db}(i)$$

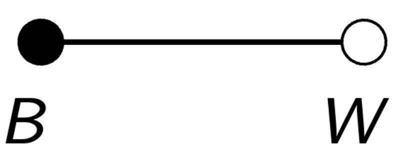
$$+\sum_{i=3,7} -I_{F,+,+}^{db}(i)+\frac{1}{2}I_{G,+,+}^{db}(i)-\sum_{i=1,5} I_{\overline{A}}^{bt}(i)+I_{\underline{A}}^{bt}(i)-\sum_{i=3,7} I_B^{bt}(i)+\sum_{i=3,7} I_{\overline{D},+}^{db}(i)+I_{\overline{E},+}^{db}(i)$$

$$+\sum_{i=1,5} I_{\underline{D},+}^{db}(i)+I_{\underline{E},+}^{db}(i)+\sum_{i=4,8} -I_{F,+,+}^{db}(i)+\frac{1}{2}I_{G,+,+}^{db}(i)+\sum_{i=2,6} I_{\overline{A}}^{bt}(i)+I_{\underline{A}}^{bt}(i)+\sum_{i=4,8} I_B^{bt}(i).$$

1

2

its bipartite representation



$B = \{1,3,5\}$ or $\{1,5,7\}$

$W = \{2,4,6\}$ or $\{2,6,8\}$

$$\begin{aligned}
I(135,246) =& I_a(135,246) + [I_b(135,24) + I_b[135,46] + I_b(13,246) + I_b(35,246)] \\
& + [I_c(13,24) + (13 \to 35) + (24 \to 46) + (13 \to 35; 24 \to 46)], \\
I(571,682) =& I(135,246)\Big|_{(135,246)\to(571,682)} \\
I(135,268) =& I_a(135,268) + [I_b(135,28) + I_b(135,68) + I_b(13,268) + I_b(35,268)] \\
& + [I_c(13,28) + I_c(35,68)], \\
I(157,246) =& I(135,628)\Big|_{(135,268)\to(571,624)}
\end{aligned}$$

Integrating all but one loop variable, the resulting function is infrared-finite. → extract cusp anomaly dimension

For n=4 : [He et al. (2023)] [Henn et al. (2023)] n=6, in progress

Define $$\mathcal{W}_L(\ell_1,1,2,3,4)=\int\prod_{i=2}^{L}d^3\ell_i\,\underline{\tilde{\Omega}}_L,$$

Even- and odd- loop are quiet different

$$\mathcal{W}_L=\begin{cases}\dfrac{\epsilon(\ell_1,1,2,3,4)}{(\ell_1\cdot 1)(\ell_1\cdot 2)(\ell_1\cdot 3)(\ell_1\cdot 4)}\,F_{L-1}(z)\,, & L\text{ odd}\\ \left(\dfrac{(1\cdot 3)(2\cdot 4)}{(\ell_1\cdot 1)(\ell_1\cdot 2)(\ell_1\cdot 3)(\ell_1\cdot 4)}\right)^{3/4}F_{L-1}(z), & L\text{ even}\end{cases}$$

$$z=\frac{(\ell_1\cdot 2)(\ell_1\cdot 4)(1\cdot 3)}{(\ell_1\cdot 1)(\ell_1\cdot 3)(2\cdot 4)}$$

ABJM

$$F_0(z) = 1 \qquad F_1(z) = -\pi(z^{1/4} + z^{-1/4}) \qquad F_2(z) = 4\left(f(z) + f\left(\frac{1}{z}\right) + \frac{\pi^2}{2}\right)$$

$$f(x) := \frac{t-1}{t+1}\left(\frac{\pi^2}{2} + \mathrm{Li}_2(1-t) + \log(t)\log(t-1) - \frac{1}{4}\log(t)^2\right) \text{ with } t := \frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}-\sqrt{x}}.$$

SYM

$$F^{(0)}(z) = -1 \qquad F^{(1)}(z) = \log^2 z + \pi^2$$

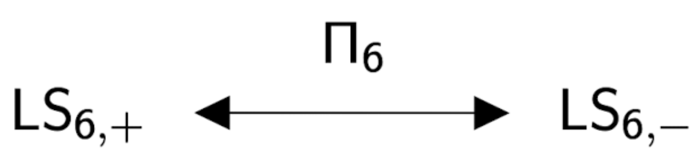
$$\begin{aligned} F^{(2)}(z) = &-\frac{1}{2}\log^4 z + \log^2 z\left[\frac{2}{3}\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \frac{2}{3}\mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{19\pi^2}{9}\right] \\ &+ \log z\left[4\mathrm{Li}_3\left(\frac{1}{z+1}\right) - 4\mathrm{Li}_3\left(\frac{z}{z+1}\right)\right] + \frac{2}{3}\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right]^2 \\ &+ \frac{8}{3}\pi^2\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right] + 8\mathrm{Li}_4\left(\frac{1}{z+1}\right) + 8\mathrm{Li}_4\left(\frac{z}{z+1}\right) - \frac{23\pi^4}{18} \end{aligned}$$

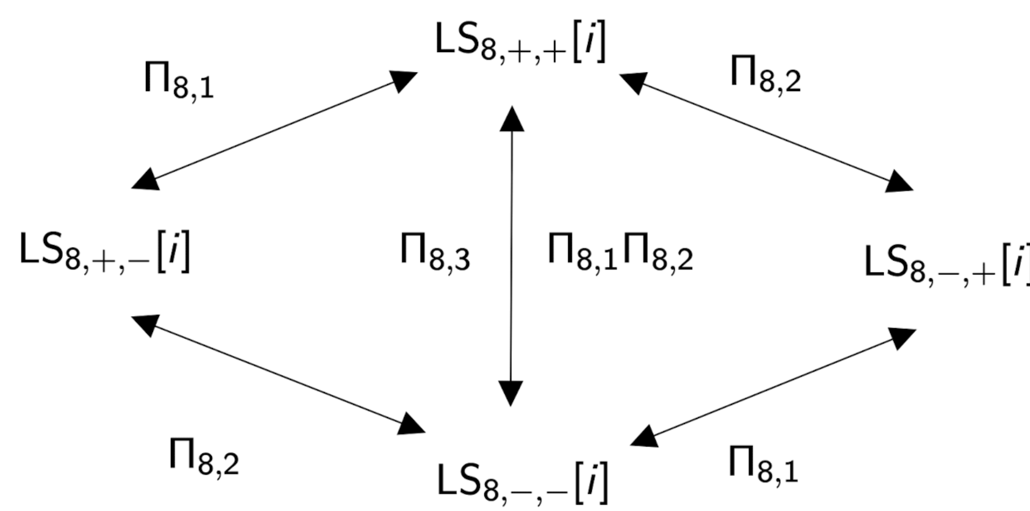
Outlook

1. Higher-pt (6-pt) negative geometry $F(Z_i)$.

2. What is Y=CZ?

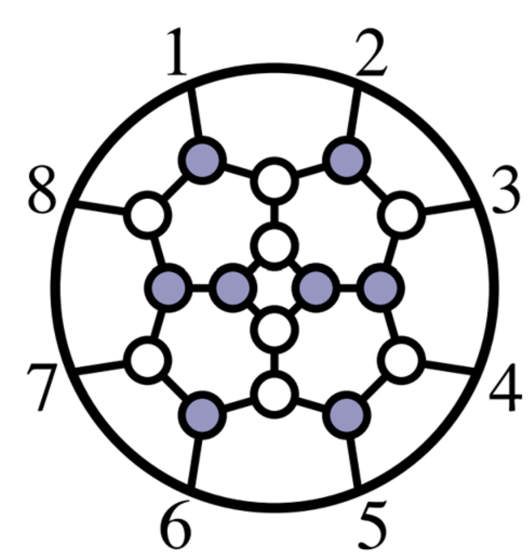
3. What about branches? Parity at n=6,8. n>8 ???





4. BDS geometry.

5. What are the chambers when $\Gamma(C) > 2$?
   8-pt N^2MHV SYM.



Thank you for listening!